Universal Regularity Theorems for Optimal Open-loop Controls

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HAPPY

BIRTHDAY

ANDREI !!!

A universal regularity theorem for (open-loop) optimal controls is a theorem of the form:

THEOREM 1. For every optimal control problem of the form

$$\text{minimize } \int_a^b L(x,u)dt \text{ subject to } \dot{x} = f(x,u)\,,$$

with $f, L \in C^\infty$. every optimal open-loop control is smooth,

or

THEOREM 2. For every optimal control problem of the form

$$\text{minimize } \int_a^b L(x,u)dt \text{ subject to } \dot{x} = f(x,u)\,,$$

with $f, L \in C^\omega$. every optimal open-loop control is smooth.

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These theorems aren't true of course. We want theorems like these that are true.

From now on, I will deal with **minimum time problems**.

This is just for simplicity. Everything carries over after a few changes to general optimal control problems of the form described above.

A TRUE THEOREM

Preliminary definitions

DEFINITION: Let h be a function defined on an interval $[a, b]$ with values in $\mathbb{R}$ or $\mathbb{R}^n$ or some real-analytic manifold H. Then the **good set for real-analyticity** for h is the set

$$G_{C^\omega}(h) = \{t \in [a, b] : h \text{ is } C^\omega \text{ on a neighborhood of } t\}\,.$$

The **bad set for real-analyticity** for h is the set

$$B_{C^\omega}(h) = [a, b] \backslash G_{C^\omega}(h)\,.$$

REMARK:

◇ $G_{C^\omega}(h)$ is relatively open on $[a, b]$.

◇ $B_{C^\omega}(h)$ is closed.

DEFINITION: A function $h : [a, b] \mapsto H$ is **nice** (*for the purposes of this talk*) if the set $G_{C^\omega}(h)$ is dense in $[a, b]$, i.e., if the set $B_{C^\omega}(h)$ is nowhere dense in $[a, b]$.

A TRUE THEOREM

An almost correct statement

MAIN THEOREM. Let Σ be a control system of the form

$$\dot{x} = f(x,u)\,, \;\; x \in X\,, \;\; u \in U\,,$$

where

(1) X is a real analytic manifold,

(2) U is a compact subanalytic subset of a real analytic manifold Y,

(3) f is a real analytic map from $X \times U$ to the tangent bundle TX of X. (Precisely: f is the restriction to $X \times U$ of a real analytic map F defined on an open neighborhhod of $X \times U$ in $X \times Y$.)

Then: **every time-optimal control is nice.**

It depends on what the meaning of the word "is" is. President Bill Clinton

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The main theorem is true as stated except for a minor detail:

There exist optimal control problems where

(*) every control is optimal.

For example, for the system

$$\dot{x} = u\,,\;\; \dot{y} = 1\,,\;\; x \in \mathbb{R}\,,\;\; y \in \mathbb{R}\,,\;\; u \in [-1, 1]\,,$$

the state variable y keep track of time, so (*) holds.

To take care of this, we need to make a slight change in the meaning of "is".

A TRUE THEOREM

The correct statement

MAIN THEOREM. Let Σ be a control system of the form

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Then: **for every time-optimal trajectory-control pair (ξ,η), either η is nice or there is some other time-optimal trajectory-control pair $(\tilde{\xi},\tilde{\eta})$, such that $\tilde{\xi}$ has the same endpoints as ξ and $\tilde{\eta}$ is nice.**

WHAT DOES "SUBANALYTIC" MEAN?

ANSWER 1. For the purposes of this talk, it doesn't matter, as long as all you want is to understand the statement of the main result. For simplicity, you could regard the theorem as talking about a control set U which is a cube or a ball.

However, the more general assumption that U is "compact subanalytic" is needed if you want to understand the proof: in the proof we use an inductive argument in which, even if you start with a cube or a ball, you are lead immediately to more general compact subanalytic sets.

For example, in the inductive argument we "reduce" our problem to one with a "smaller" control set $\widehat{U}$, given as the set of zeros of a real analytic function on U. Such a set can be quite awful, even if U is a cube or a ball, but it is still subanalytic.

So here is **ANSWER 2:** A semianalytic subset of a real analytic manifold Y is a subset S of Y such that every point y_* of Y has an open neighborhood V such that $V \cap S$ belongs to the Boolean algebra of subsets of V generated by the sets $Z(f)$, $Z_+(f)$, for $f \in C^\omega(U, \mathbb{R})$, where $Z(f) = \{y \in V : f(y) = 0\}$ and $Z_+(f) = \{y \in V : f(y) > 0\}$.

A subanalytic subset of a real analytic manifold Y is a subset S of Y which is locally the image of a semianalytic subset under a proper real-analytic map. (Precisely: for every $y_* \in Y$ there exist an open neighborhood V of Y, a real analytic manifold W, a semianalytic subset T of W, and a real-analytic map $f : W \mapsto V$ such that (1) f is proper on $\text{Clos}_W(T)$ and (2) $f(W) = S \cap V$.)

WHY DOES REAL-ANALYTICITY MATTER?

Because for C^∞ systems there is no such "universal regularity" result.

REASON: Given any positive T and any measurable function $\eta : [0,T] \mapsto [-1,1]$ one can easily construct a pair of C^∞ vector fields f, g on $\mathbb{R}^3$ and endpoint conditions $x_{in}, x_{term} \in \mathbb{R}^3$ such that

The problem of driving x_{in} to x_{term} by means of trajectory of the system

$$\dot{x} = f(x) + ug(x)\,, \quad u \in [-1,1]$$

has a unique solution, and that solution is η.

In particular, η is time-optimal, and no other control can drive x_{in} to x_{term}.

THE PROOF IN A SIMPLE CASE

The proof of the Main Theorem in the general case is very complicated, and makes extensive use of stratification theorems about subanalytic sets, including some new results on parametric stratifications.

However, a much simpler proof can be given for systems of the form

$$\dot{x} = f(x) + ug(x)\,,\ \ u \in [-1, 1],$$

So we will do that proof first and, if there is time (which, of course, will not happen) we will sketch the proof of the general case.

Naturally, we assume that

(*) the state variable x takes values in a real-analytic manifold X,

(**) f and g are real-analytic vector fields on X.

Step 1 of the proof

(Simple but essential)

Let $L(f,g)$ be the Lie algebra of vector fields generated by f and g. (This means that $L(f,g)$ is the linear span over $\mathbb{R}$ of f, g and all the iterated brackets $[f,g]$, $[f,[f,g]]$, $[g,[f,g]]$, $[f,[f,[f,g]]]$, $[g,[f,[f,g]]]$, $[f,[g,[f,g]]]$, etc.)

Then $L(f,g)$ is a Lie algebra of real-analytic vector fields, so it is well know that $L(f,g)$ has a maximal integral manifold (MIM) S_x through every point x of X.

Furthermore, the MIMs of $L(f,g)$ form a partition of X, and every trajectory of our system is entirely contained in a MIM.

So, to study a time-optimal trajectory-control pair (ξ, η), it suffices to restrict oneself to a MIM. That is, we may assume without loss of generality that

(#) X itself is a MIM of $L(f, g)$,

that is,

(##) for every $x \in X$, the equality

$$L(f, g)(x) = T_x X$$

holds, where $T_x X$ is the tangent space of X at x and

$$L(f, g)(x) = \{V(x) : V \in L(f, g)\}.$$

(In other words, "the vector field pair (f, g) has the accessibility property", so the reachable set $\mathcal{R}(x)$ from any initial point x has nonempty interior.)

Step 2 of the proof.

From now one we assume that the time-optimal control η that we are trying to study and its corresponding trajectory ξ are defined on an interval $[0, T]$.

Since our trajectory-control pair (ξ, η) is time-optimal, we apply the Pontryagin Maximum Principle and get a "nontrivial minimizing adjoint vector" λ.

Then λ is a function defined on $[0, T]$ such that the value $\lambda(t)$ **belongs to** $T^{\#}_{\xi(t)}X$ (the cotangent space to X at $\xi(t)$) for every $t \in [0, T]$.

Furthermore, λ is **nontrivial**, i.e.,

$$\lambda(t) \neq 0 \qquad \text{for all } t \in [0, T]\,.$$

.

In addition, λ satisfies the **Hamiltonian maximization condition:**

(AC) for almost all $t \in [0,T]$, the function

$$U \ni u \mapsto H(\Xi(t),u) \in \mathbb{R}$$

is maximized by $u = \eta(t)$, where

(a) H is the Hamiltonian, i.e. the function $H : T^{\#}X \times U \mapsto \mathbb{R}$ given by

$$H(p,x,u) = \langle p, f(x,u)\rangle \qquad \text{for } x \in X,\, p \in T^{\#}_x X\,,\, u \in U\,,$$

and $\Xi : [0,T] \mapsto T^{\#}X$ is the "adjoint lift" of ξ, I.e., the curve given by

$$\Xi(t) = \langle \lambda(t), \xi(t)\rangle\,.$$

Finally, λ satisfies the **adjoint equation**. This means that

(AE) If V is any smooth vector field on X, then

$$\frac{d}{dt}\langle\lambda(t), V(\xi(t))\rangle = \langle\lambda(t), [f, V](\xi(t))\rangle + u(t)\langle\lambda(t), [g, V](\xi(t))\rangle \text{ a.e..}$$

That is, if we write

$$\mu_V(p, x) = \langle p, V(x)\rangle$$

(so $\mu_V : T^{\#}X \mapsto \mathbb{R}$ is the **momentum function** or **Hamiltonian function** or **switching function** corresponding to the vector field V), then

$$\dot{\mu}_V = \mu_{[f,V]} + \eta\mu_{[g,V]}$$

along the curve Ξ.

Step 3

From now on we write $\mu_V(t)$ rather than $\mu_V(\Xi(t))$ for any switching function μ_V.

We observe that the Hamiltonian Maximization condition implies that

$$\eta(t) \equiv 1 \quad \text{on any interval where} \quad \mu_g > 0\,,$$
$$\eta(t) \equiv -1 \quad \text{on any interval where} \quad \mu_g < 0\,.$$

We conclude that

η is smooth on the open set $\{t \in [0,T] : \mu_V(t) \neq 0\}$.

Step 4

TRIVIAL LEMMA: Let $h : [0,T] \mapsto \mathbb{R}$ be continuous. Let

$$W(h) = \{t \in [0,T] : h(t) \neq 0 \text{ or } h(s) \equiv 0 \text{ for all } s \text{ near } t\}.$$

(Naturally, "for all s near t" means "for all $s \in]t-\varepsilon, t+\varepsilon[\cap [0,T]$ for some positive ε".)

Then $W(h)$ **is relatively open and dense in** $[0,T]$.

PROOF: Trivial.

Step 5

We study the set $W(\mu_g)$. This set is open and dense, and is the union

$$W(\mu_g) = \bigcup_{k=1}^{\infty} I_k(g)$$

of a sequence of pairwise disjoint relatively open intervals $I_k(g)$, such that on each such interval either μ_g never vanishes or μ_g vanishes identically. Relabel the intervals of the first kind as $I_i^1(g)$ and those of the second kind as $I_j^2(g)$, so

$$W(\mu_g) = \Big(\bigcup_{i=1}^{\infty} I_i^1(g)\Big) \cup \Big(\bigcup_{j=1}^{\infty} I_j^2(g)\Big)\,.$$

Step 6

Now, on the intervals $I_j^2(g)$, where μ_g vanishes identically, differentiate μ_g and get

$$0 = \dot{\mu}_g = \mu_{[f,g]}\,,$$

so $\mu_{[f,g]}$ vanishes identically on $I_j^2(g)$.

Differentiate $\mu_{[f,g]}$ on each $I_j^2(g)$, and get

$$0 = \dot{\mu}_{[f,g]} = \mu_{[f,[f,g]]} + \eta\mu_{[g,[f,g]]}\,.$$

Then get a set $W_j^2(\mu_{[g,[f,g]]})$ which is open and dense in $I_j^2(g)$ and can be written as a union

$$W_j^2(\mu_{[g,[f,g]]}) = \Big(\bigcup_{i=1}^{\infty} I_{j,i}^{2,1}([g,[f,g]])\Big) \cup \Big(\bigcup_{j_1=1}^{\infty} I_{j,j_1}^{2,2}([g,[f,g]])\Big)\,,$$

where

(1) $\mu_{[g,[f,g]]}$ never vanishes on $I^{2,1}_{j,i}([g,[f,g]])$,

(2) $\mu_{[g,[f,g]]}$ vanishes identically on $I^{2,2}_{j,i}([g,[f,g]])$,

Then on the intervals $I^{2,1}_{j,i}([g,[f,g]])$ we have

$$\eta(t) = -\frac{\mu_{[f,[f,g]]}}{\mu_{[g,[f,g]]}},$$

because $\mu_{[f,[f,g]]} + \eta\mu_{[g,[f,g]]} \equiv 0$.

So the curve $\Xi = (\lambda, \xi)$ and the control η on the intervals $I^{2,1}_{j,i}([g,[f,g]])$, satisfy

$$\begin{aligned}
\dot{\xi} &= f(\xi) + \eta g(\xi)\,, \\
\dot{\lambda} &= -\lambda\frac{\partial f}{\partial x}(\xi) - \eta\lambda\frac{\partial g}{\partial x}(\xi)\,, \\
\eta &= -\frac{\mu_{[f,[f,g]]}(\lambda,\xi)}{\mu_{[g,[f,g]]}(\lambda,\xi)}\,,
\end{aligned}$$

so that

$$\begin{aligned}\dot{\xi} &= f(\xi) - \frac{\mu_{[f,[f,g]]}(\lambda,\xi)}{\mu_{[g,[f,g]]}(\lambda,\xi)} g(\xi)\,,\\ \dot{\lambda} &= -\lambda \frac{\partial f}{\partial x}(\xi) + \frac{\mu_{[f,[f,g]]}(\lambda,\xi)}{\mu_{[g,[f,g]]}(\lambda,\xi)} \lambda \frac{\partial g}{\partial x}(\xi)\,.\end{aligned}$$

Therefore ξ and λ are real-analytic functions of t on $I^{2,1}_{j,i}([g,[f,g]])$, and η is real-analytic as well, since

$$\eta = -\frac{\mu_{[f,[f,g]]}(\lambda,\xi)}{\mu_{[g,[f,g]]}(\lambda,\xi)}\,.$$

Hence:

(1) η **is smooth on each of the intervals** $I^{2,1}_{j,i}([g,[f,g]])$**.**

(2) The switching function $\mu_{[g,[f,g]]}$ **vanishes identically on each of the intervals** $I^{2,2}_{j,j_1}([g,[f,g]])$**.**

(3) The switching function $\mu_{[f,[f,g]]}$ **also vanishes identically on the** $I^{2,2}_{j,j_1}([g,[f,g]])$**, because**

$$\mu_{[f,[f,g]]} + \eta\mu_{[g,[f,g]]} \equiv 0$$

there.

So, on the intervals $I^{2,2}_{j,j_1}([g,[f,g]])$, **the switching functions μ_g, $\mu_{[f,g]}$, $\mu_{[f,[f,g]]}$, $\mu_{[g,[f,g]]}$, vanish identically.**

That is **all the switching functions of order ≤ 3 vanish identically.**

Step 7

We now repeat the procedure. We differentiate the switching functions $\mu_{[f,[f,g]]}$ and $\mu_{[g,[f,g]]}$ on $I^{2,2}_{j,j_1}([g,[f,g]])$ and get the equations

$$\begin{aligned}\dot\mu_{[f,[f,g]]} &= \mu_{[f,[f,[f,g]]]} + \eta\mu_{[g,[f,[f,g]]]}\\ \dot\mu_{[g,[f,g]]} &= \mu_{[f,[g,[f,g]]]} + \eta\mu_{[g,[g,[f,g]]]}\,.\end{aligned}$$

Applying the "trivial lemma" to the function

$$h^{(4)} = \mu^2_{[g,[f,[f,g]]]} + \mu^2_{[g,[g,[f,g]]]} \,,$$

we find open dense subsets $W^{2,2}_{j,j_1}(h^{(4)})$ of $I^{2,2}_{j,j_1}([g,[f,g]])$ that can be written as a union

$$W^{2,2}_{j,j_1}(h^{(4)}) = (\bigcup_{i=1}^{\infty} I^{2,2,1}_{j,j_1,i}([g,[f,[f,g]]],[g,[g,[f,g]]]])$$

$$\cup(\bigcup_{j_2=1}^{\infty} I^{2,2,2}_{j,j_1,j_2}([g,[f,[f.g]]],[g,[g,[f,g]]])) \,,$$

where

(1) $h^{(4)}$ never vanishes on $I^{2,2,1}_{j,j_1,i}([g,[f,[f.g]]],[g,[g,[f,g]]])$,

(2) $h^{(4)}$ vanishes identically on $I^{2,2,2}_{j,j_1,j_2}([g,[f,[f.g]]],[g,[g,[f,g]]])$.

Then **the control η is smooth on the intervals**

$$I^{2,2,1}_{j,j_1,i}([g,[f,[f,g]]],[g,[g,[f,g]]])$$

and

all the switching functions of order ≤ 4 vanish identically on the intervals $I^{2,2,2}_{j,j_1,j_2}([g,[f,[f,g]]],[g,[g,[f,g]]])$.

Step 8

The procedure continues indefinitely:

At Step k, we get an open dense set which is a countable union of two kinds of intervals:

(1) intervals where the control is real-analytic,

and

(2) intervals I_j where all the switching functions of order $\leq n$ vanish identically.

Then we find for each I_j an open dense subset D_j that splits into a union of intervals where the control is real-analyic and intervals where all the switching functions of order $\leq n+1$ vanish identically.

Eventually, the procedure will stop and we will get a real-analytic control η on an open dense subset of $[0, T]$, provided that

(*) for every $t \in [0, T]$ there exists an n and an n-order switching function that does not vanish at t.

Indeed, an elementary compactness argument shows that if (*) holds then there exists an n such that for every t some switching function of order $\leq n$ is nonzero at t, which means that the "bad" intervals I_j become empty.

So we have to ask:

When does condition (*) hold?

Condition (*) does not hold if

(#) for some $t \in [0, T]$ all the switching functions

$\mu_g, \mu_{[f,g]}, \mu_{[f,[f,g]]}, \mu_{[g,[f,g]]}, \mu_{[f,[f,[f,g]]]}, \mu_{[g,[f,[f,g]]]}, \mu_{[f,[g,[f,g]]]},$

$\mu_{[g,[g,[f,g]]]}, \cdots$

vanish at $\Xi(t)$.

This means that the covector $\lambda(t)$ annihilates all the vectors

$(\%) g(\xi(t)), [f, g](\xi(t)), [f, [f, g]](\xi(t)), [g, [f, g]](\xi(t)), [f, [f, [f, g]]](\xi(t)),$

$[g, [f, [f, g]]](\xi(t)), [f, [g, [f, g]]](\xi(t)), [g, [g, [f, g]]](\xi(t)), \ldots$

These vectors span the space $L_0(f, g)(\xi(t))$, where $L_0(f, g)$ is the **strong accessibility Lie algebra**, i.e., the ideal of $L(f, g)$ generated by g. (In simpler words, $L_0(f, g)$ is the span of all the iterated brackets of f and g **except** f).

It is well known that, on an integral manifold of $L(f, g)$, the dimension of the space $L_0(f, g)(x)$ is constant (i.e., independent of x) and equal to $\dim L(f, g)(x)$ or to $\dim L(f, g)(x) - 1$. The former case is impossible if (#) holds, because λ is nontrivial, so if $\lambda(t)$ annihilates all the vectors of (%), i.e., the space $L_0(f, g)(\xi(t))$, then $L_0(f, g)(\xi(t))$ must have codimension 1 in $L(f, g)(\xi(t))$.

So we have proved:

If η is not real-analytic on an open dense subset of $[0, T]$, then

$$\dim L_0(f,g)(x) = \dim L(f,g)(x) - 1 \qquad \text{for all } x \in X\,.$$

In this case, we can prove that **locally, our system is degenerate, in the sense that every trajectory is time-optimal.**

Proof: Define a 1-form ω on X by letting

$$\langle \omega, f\rangle = 1\,; \qquad \langle \omega, V\rangle = 0 \text{ for } V \in L_0(f,g)\,.$$

Then for every trajectory $\gamma : [a,b] \mapsto X$ of our system

$$\int_\gamma \omega = \int_a^b \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt = \int_a^b \langle \omega, f\rangle + \eta\langle \omega, g\rangle = \int_a^b 1\, dt = b - a\,.$$

On the other hand, **ω is closed**. Hence, locally, the integral of ω is independent from the path. So all trajectories from a given point x_{in} to a given point x_{term} take exactly the same time.

Now we need to prove that in this degenerate case we can **choose** a different control $\tilde{\eta}$ that steers the initial point of ξ to the terminal point of ξ.

This will be done in the second hour of this talk.

Finally, we have to prove the general case of the main theorem. This will be done in Hours 3 and 4 of the talk.