

ASYMPTOTIC STABILITY OF EQUILIBRIUM STATES FOR AMBIPOLAR PLASMAS

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We investigate a system of partial differential equations modeling ambipolar plasmas. The ambipolar — or zero current — model is obtained from general plasmas equations in the limit of vanishing Debye length. In this model, the electric field is expressed as a linear combination of macroscopic variable gradients. We establish that the governing equations can be written as a symmetric form by using entropic variables. The corresponding dissipation matrices satisfy the null space invariant property and the system of partial differential equations can be written as a normal form, i.e. in the form of a symmetric hyperbolic–parabolic composite system. By properly modifying the chemistry source terms and/or the diffusion matrices, asymptotic stability of equilibrium states is established and decay estimates are obtained. We also establish the continuous dependence of global solutions with respect to vanishing electron mass.

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1. Introduction

Ionized gas mixtures — or plasmas — with chemical reactions are related to a wide range of practical applications such as laboratory plasmas, high-speed gas flows or atmospheric phenomena. This is a strong motivation for investigating the structure and properties of the corresponding systems of partial differential equations.

The equations governing high density low temperature plasmas can be derived from the kinetic theory of ionized gas mixtures. Different systems can be obtained depending on the various characteristic lengths and times of the phenomena under investigation. Assuming that there is a single temperature in the mixture — this is the case for numerous practical applications — the corresponding governing equations are derived in Ferziger and Kaper⁵ and Giovangigli and Graille⁹ for general reactive polyatomic gas mixtures.

The ambipolar approximation is often used in the modeling of laboratory and space plasmas and is obtained for vanishing Debye length.^{2,17} The corresponding model is a quasi-neutral model where the conduction current is set to zero. In this approximation, there is no magnetic field and the (internal) electric field is eliminated through the use of zero current constraint. The electric field can then be expressed in terms of macroscopic variable gradients and the resulting transport fluxes involve new effective ambipolar transport coefficients.

The governing equations for reactive ionized gas mixture in the ambipolar limit constitute a second-order quasilinear system of conservation laws. The asymptotic stability of equilibrium states for this quasilinear system of partial differential equations is by itself an important question. This system, however, also depends on numerous parameters such as thermal conductivity and chemical reaction constants.

One of these parameters, often used in the physical modeling, is the electron mass, which is usually assumed to be zero. In order to investigate this limit, the dependence of the system coefficients on the electron mass must be clarified since electron diffusivities become infinite as the electron mass vanishes. In order to do so, we explicit the dependence of multicomponent diffusion matrices on binary diffusion coefficients and establish the smoothness of the system coefficients with respect to the electron mass.

Next we consider an abstract system of conservation laws depending smoothly on a parameter. We investigate symmetrizability, asymptotic stability of equilibrium states, and continuous dependence of solutions with respect to the parameter. We first establish continuous dependence of solutions locally in time and then globally in time around equilibrium states under appropriate norms. Decay estimates are also established globally with respect to the parameter.

We then apply these results to the system of partial differential equations modeling ambipolar plasmas. We first establish that the system can be written as a symmetric form and admits an entropy in the mathematical sense.^{13,14} The resulting dissipation matrices are shown to satisfy the null space invariance property introduced by Kawashima.¹³ The system of partial differential equation is next written into a normal form, that is, in the form of a symmetric hyperbolic–parabolic composite system with two hyperbolic components, with smooth dependence of the system coefficients on the electron mass.

The structure and properties of the equations in the ambipolar limit are first insufficient to establish asymptotic stability. This problem, however, is shown to be artificial and due to the lack of dissipativity properties associated with the electric charge equation, which must guarantee that the charge remains zero for physical solutions. Two modified forms are then introduced for the system of governing equations, i.e. such that regular physical solutions coincide. These reformulations guarantee asymptotic stability and continuous dependence of global solutions with respect to the electron mass.

One can first modify chemistry production rates in the direction of the charge vector and orthogonally to chemical reaction vectors. This yields a consumption term in the charge equation ensuring enough dissipativity. A second modification,

which has interesting numerical consequences, consists in modifying the diffusion coefficients in the direction of the dyadic product of the charge vector.

Our paper is organized as follows. In Sec. 2, we present the governing equation for reactive ionized gas mixtures in the ambipolar limit. In Sec. 3, we investigate the dependence of the system coefficients on the electron mass. Symmetrization and local existence for an abstract system depending on a parameter is considered in Sec. 4. Global existence around equilibrium states with continuous dependence on a parameter is established in Sec. 5. Symmetrization for the quasilinear system modeling ambipolar plasmas is obtained in Sec. 6. Finally, in Sec. 7, we establish asymptotic stability of equilibrium states for ambipolar plasmas with continuous dependence on the electron mass.

2. Ambipolar Reactive Gas Mixtures

We consider a reactive ionized gas mixture composed of n^s chemical species in the presence of an electric field. The general governing equations — derived from the kinetic theory of gases — can be split between conservation equations, transport fluxes, thermodynamics, chemical production rates, and are completed by Maxwell’s equations for the electric field.^{5,9} The full system of partial differential equations has a complex structure and is out of the scope of the present paper. These equations are simplified here in the ambipolar — or zero current — approximation where the conduction current vanishes.

2.1. Conservation equations

We denote by $\mathfrak{S} = \{1, \dots, n^s\}$ the species indexing set, n^s the number of species, ρ_k the mass per unit volume of the k th species, γ_k the number of mole per unit volume of the k th species, \varkappa_k the molar charge of the k th species, and m_k the molar mass of the k th species so that $\rho_k = m_k \gamma_k$. In contrast with previous work,¹¹ we will use molar quantities like $(\gamma_1, \dots, \gamma_{n^s})$ in order to describe the state of the mixture. This molar formulation is, of course, strictly equivalent to a mass formulation using mass densities like $(\rho_1, \dots, \rho_{n^s})$ because the species mass m_k , $k \in \mathfrak{S}$, are strictly positive. However, we will ultimately investigate the asymptotic limit of vanishing electron mass, and, therefore, we need to work in a molar framework.

The mole conservation equation for the k th species reads

$$\partial_t \gamma_k + \partial_{\mathbf{x}} \cdot (\gamma_k \mathbf{v}) + \partial_{\mathbf{x}} \cdot \mathcal{F}_k = \omega_k, \quad k \in \mathfrak{S}, \tag{2.1}$$

where \mathbf{v} is the macroscopic velocity of the mixture, \mathcal{F}_k the molar diffusion flux and ω_k the molar production rate of the k th species. Bold symbols are used for vector or tensor quantities in the physical space \mathbb{R}^3 so that for instance $\partial_{\mathbf{x}} = (\partial_1, \partial_2, \partial_3)^t$.

In the absence of magnetic field, the momentum conservation equation can be written in the form

$$\partial_t(\rho \mathbf{v}) + \partial_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p \mathbb{I}) + \partial_{\mathbf{x}} \cdot \mathbf{\Pi} = q \mathbf{E}, \tag{2.2}$$

where p is the pressure, \mathbb{I} the unit tensor, $\mathbf{\Pi}$ the viscous tensor, q the total charge per unit volume, and \mathbf{E} the (internal) electric field.

Finally, the energy conservation equation reads

$$\begin{aligned} \partial_t \left(\mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right) + \partial_{\mathbf{x}} \cdot \left(\left(\mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + p \right) \mathbf{v} \right) + \partial_{\mathbf{x}} \cdot (\mathbf{Q} + \mathbf{\Pi} \cdot \mathbf{v}) \\ = (q \mathbf{E} + \mathbf{j}) \cdot \mathbf{E}, \end{aligned} \tag{2.3}$$

where \mathcal{E} is the internal energy per unit volume, \mathbf{Q} the heat flux, and \mathbf{j} the conduction current vector. In the following, the (internal) electric field is eliminated from the governing equations by using the ambipolar constraint.

2.2. Transport fluxes

The molar diffusion flux \mathcal{F}_k , $k \in \mathfrak{S}$, can be written in the form

$$\mathcal{F}_k = \gamma_k \mathbf{V}_k, \quad k \in \mathfrak{S}, \tag{2.4}$$

where the diffusion velocity of the k th species \mathbf{V}_k is given by

$$\mathbf{V}_k = - \sum_{l \in \mathfrak{S}} D_{kl} (\mathbf{d}_l + \chi_l \partial_{\mathbf{x}} \log T), \quad k \in \mathfrak{S}, \tag{2.5}$$

and where the diffusion driving force \mathbf{d}_k reads

$$\mathbf{d}_k = \frac{1}{p} (\partial_{\mathbf{x}} p_k - \gamma_k \mathcal{Z}_k \mathbf{E}). \tag{2.6}$$

In these relations, D_{kl} , $k, l \in \mathfrak{S}$, are the multicomponent diffusion coefficients, χ_k , $k \in \mathfrak{S}$, the thermal diffusion ratios, and T the absolute temperature. The expression of the heat flux is

$$\mathbf{Q} = -\lambda \partial_{\mathbf{x}} T + \sum_{k \in \mathfrak{S}} (p \chi_k + \gamma_k H_k) \mathbf{V}_k, \tag{2.7}$$

where H_k is the enthalpy per unit mole of the k th species. Finally, the viscous tensor is given by

$$\mathbf{\Pi} = -\kappa (\partial_{\mathbf{x}} \cdot \mathbf{v}) \mathbb{I} - \eta \mathbf{S}, \tag{2.8}$$

where \mathbf{S} is the usual strain symmetric traceless tensor

$$\mathbf{S} = \partial_{\mathbf{x}} \mathbf{v} + \partial_{\mathbf{x}} \mathbf{v}^t - \frac{2}{3} (\partial_{\mathbf{x}} \cdot \mathbf{v}) \mathbb{I}.$$

Remark 2.1. In the presence of magnetic fields, the transport fluxes in multicomponent gas mixtures are nonisotropic.^{9,10} In this situation, the viscous tensor involve all symmetric tensors constructed from \mathbf{S} and the antisymmetric rotation tensor associated with the magnetic field. More general structure for the stresses are also obtained in field-dependent media as in electrorheological fluids investigated by Rajagopal and Růžiča,¹⁶ in ferrofluids investigated by Eringen and Maugin,³ or in granular materials investigated by Massoudi and Boyle.¹⁵

2.3. Zero current constraint

In ionized mixtures, there is a Coulomb screening by mobile charges over distances of the order of the Debye length. For small Debye length, the mixture can be considered as quasi-neutral, so that $q = 0$, although the electric field is nonzero. In the absence of external electric field, and for small Debye length, it is also natural to assume that positive ions and electrons diffuse as a team^{1,2,17} so that the conduction current \mathbf{j} vanishes. This is the origin of the terminology ambipolar and this approximation is consistent with the charge equation

$$\partial_t q + \partial_{\mathbf{x}} \cdot (q\mathbf{v}) + \partial_{\mathbf{x}} \cdot \mathbf{j} = 0.$$

In other words, the (internal) polarization electric field \mathbf{E} insure that the conduction current \mathbf{j} vanish. In this situation, provided that the initial charge is zero, we recover that the charge q remains zero at all time.

We must now eliminate the electric field \mathbf{E} by using the zero conduction current constraint. The conduction current $\mathbf{j} = \sum_{k \in \mathfrak{S}} \varkappa_k \mathcal{F}_k = \sum_{k \in \mathfrak{S}} \varkappa_k \gamma_k \mathbf{V}_k$ can conveniently be written in the compact form

$$\mathbf{j} = \langle z, \mathbf{V} \rangle,$$

where $z = (z_1, \dots, z_{n^s})^t$, $z_k = \gamma_k \varkappa_k$, $k \in \mathfrak{S}$, $\mathbf{V} = (\mathbf{V}_1, \dots, \mathbf{V}_{n^s})^t$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product between quantities in \mathbb{R}^{n^s} or $(\mathbb{R}^3)^{n^s}$. On the other hand, thanks to isotropy of diffusive processes, the relations expressing the diffusion velocities can be recast in the vector form

$$\mathbf{V} = -D \left(\mathbf{d}^0 + \chi \partial_{\mathbf{x}} \log T + z \frac{\mathbf{E}}{p} \right),$$

where $\mathbf{d}^0 = (\mathbf{d}_1^0, \dots, \mathbf{d}_{n^s}^0)^t$, $\mathbf{d}_k^0 = (\partial_{\mathbf{x}} p_k)/p$, $k \in \mathfrak{S}$, and $\chi = (\chi_1, \dots, \chi_{n^s})^t$. Therefore, the constraint $\mathbf{j} = 0$ implies that

$$\frac{\mathbf{E}}{p} = \frac{\langle z, D(\mathbf{d}^0 + \chi \partial_{\mathbf{x}} \log T) \rangle}{\langle z, Dz \rangle}.$$

Defining the square matrix $\widehat{D} = (\widehat{D}_{kl})_{k,l}$ by

$$\widehat{D} = D - \frac{Dz \otimes Dz}{\langle z, Dz \rangle}, \tag{2.9}$$

it is readily seen that

$$\mathbf{V} = -\widehat{D}(\mathbf{d}^0 + \chi \partial_{\mathbf{x}} \log T), \tag{2.10}$$

i.e. $\mathbf{V}_k = -\sum_{l \in \mathfrak{S}} \widehat{D}_{kl}(\mathbf{d}_l^0 + \chi_l \log T)$, $k \in \mathfrak{S}$. These expressions now guarantee that the conduction current \mathbf{j} vanishes independently of the state variables and their gradients.

2.4. Thermodynamics

The pressure p , the mass density ρ and the charge per unit volume q can be expressed as

$$p = \sum_{k \in \mathfrak{S}} RT\gamma_k, \quad \rho = \sum_{k \in \mathfrak{S}} m_k\gamma_k, \quad q = \sum_{k \in \mathfrak{S}} \varkappa_k\gamma_k,$$

where m_k is the mass per unit mole of the k th species, \varkappa_k the charge per unit mole of the k th species, and R the perfect gas constant. The internal energy \mathcal{E} and the enthalpy \mathcal{H} per unit volume can be decomposed into

$$\mathcal{E} = \sum_{k \in \mathfrak{S}} \gamma_k E_k, \quad \mathcal{H} = \sum_{k \in \mathfrak{S}} \gamma_k H_k,$$

where E_k and $H_k = E_k + RT$ are the internal energy and the internal enthalpy per unit mole of the k th species and T the absolute temperature. The internal energy E_k can be written in the form

$$E_k(T) = E_k^{\text{st}} + \int_{T^{\text{st}}}^T C_{v,k}(\tau) d\tau,$$

where $E_k^{\text{st}} = E_k(T^{\text{st}})$ is the formation energy per unit mole of the k th species at the positive standard temperature T^{st} and $C_{v,k}$ is the constant-volume molar specific heat of the k th species.

The entropy \mathcal{S} and Gibbs function \mathcal{G} per unit volume can be expressed in terms of the species entropies per unit mole S_k , $k \in \mathfrak{S}$, and Gibbs functions per unit mole G_k , $k \in \mathfrak{S}$, from the relations

$$\mathcal{S} = \sum_{k \in \mathfrak{S}} \gamma_k S_k, \quad \mathcal{G} = \sum_{k \in \mathfrak{S}} \gamma_k G_k,$$

where

$$S_k(T, \gamma_k) = S_k^{\text{st}} + \int_{T^{\text{st}}}^T \frac{C_{v,k}(\tau)}{\tau} d\tau - R \log \left(\frac{\gamma_k}{\gamma^{\text{st}}} \right),$$

S_k^{st} is the formation entropy at the standard temperature T^{st} and standard pressure p^{st} , $\gamma^{\text{st}} = p^{\text{st}}/RT^{\text{st}}$ is the standard concentration, and where $G_k = H_k - TS_k$. We also define the species reduced chemical potential $\mu_k = G_k/RT$, $k \in \mathfrak{S}$. Finally, the species Gibbs functions G_k and the species reduced chemical potential μ_k , $k \in \mathfrak{S}$, are functions of γ_k and T , which can be written

$$G_k(\gamma_k, T) = G_k^u(T) + RT \log \gamma_k, \quad \mu_k(\gamma_k, T) = \mu_k^u(T) + \log \gamma_k,$$

where G_k^u , $k \in \mathfrak{S}$, are the species unitary Gibbs functions per unit mole and μ_k^u , $k \in \mathfrak{S}$, are the species unitary reduced chemical potentials.

2.5. Chemical source terms

We consider n^r elementary reactions among the n^s species which can be formally written as

$$\sum_{k \in \mathfrak{S}} \nu_{kr}^f \mathfrak{M}_k \rightleftharpoons \sum_{k \in \mathfrak{S}} \nu_{kr}^b \mathfrak{M}_k, \quad r \in \mathfrak{R},$$

where \mathfrak{M}_k is the chemical symbol of the k th species, ν_{kr}^f and ν_{kr}^b are the forward and backward stoichiometric coefficients of the k th species in the r th reaction, respectively and $\mathfrak{R} = \{1, \dots, n^r\}$ is the set of reaction indices.

The Maxwellian production rates given by the kinetic theory can be written

$$\omega_k = \sum_{r \in \mathfrak{R}} (\nu_{kr}^b - \nu_{kr}^f) \tau_r, \quad k \in \mathfrak{S}, \tag{2.11}$$

where τ_r is the rate of progress of the r th reaction. The rates of progress are given by the symmetric expression⁸

$$\tau_r = \mathcal{K}_r^s (\exp\langle \nu_r^f, \mu \rangle - \exp\langle \nu_r^b, \mu \rangle), \tag{2.12}$$

where $\nu_r^f = (\nu_{1r}^f, \dots, \nu_{n^s r}^f)^\mathfrak{t}$, $\nu_r^b = (\nu_{1r}^b, \dots, \nu_{n^s r}^b)^\mathfrak{t}$, $\mu = (\mu_1, \dots, \mu_{n^s})^\mathfrak{t}$, and \mathcal{K}_r^s is the symmetric reaction constant. We define $\nu_{kr} = \nu_{kr}^b - \nu_{kr}^f$, $k \in \mathfrak{S}$, $r \in \mathfrak{R}$, and the reaction vectors $\nu_r = (\nu_{1r}, \dots, \nu_{n^s r})^\mathfrak{t}$, $r \in \mathfrak{R}$, so that $\nu_r = \nu_r^b - \nu_r^f$, and we denote by $\mathcal{R} = \text{span}\{\nu_r, r \in \mathfrak{R}\}$ the linear space spanned by the vectors ν_r , $r \in \mathfrak{R}$.

2.6. Mathematical assumptions

We describe in this section the mathematical assumptions concerning thermochemistry, and, partly, the assumptions concerning transport coefficients.

2.6.1. Assumption on thermochemistry

The species of the mixture are assumed to be constituted by neutral atoms and electrons. We denote by $\mathfrak{A} = \{1, \dots, n^a\}$ the atom indexing set, n^a the number of atoms in the mixture, \tilde{m}_l , $l \in \mathfrak{A}$, the atom masses and \mathfrak{a}_{kl} the number of l th atoms in the k th species. We also introduce the atomic vectors \mathfrak{a}_l , $l \in \mathfrak{A}$, defined by $\mathfrak{a}_l = (\mathfrak{a}_{1l}, \dots, \mathfrak{a}_{n^s l})^\mathfrak{t}$, $l \in \mathfrak{A}$. We define \mathfrak{a}_{k0} as the number of electrons in the k th species and for notational convenience, we define $\overline{\mathfrak{A}} = \{0\} \cup \mathfrak{A} = \{0, \dots, n^a\}$. We also assume that the electron species is present in the mixture as well as one neutral species and one positively charged species. For notational convenience, we assume that the last species in the mixture is the electron species. Since we will ultimately investigate the limit of vanishing electron mass, we will only assume that the electron mass m_{n^s} is non-negative. We define the mass vector m and the charge vector \varkappa by

$$m = (m_1, \dots, m_{n^s})^\mathfrak{t}, \quad \varkappa = (\varkappa_1, \dots, \varkappa_{n^s})^\mathfrak{t},$$

and the unit vector \mathbf{u} by $\mathbf{u} = (1, \dots, 1)^\top$. We also define the mole fraction of the k th species \mathbf{x}_k by $x_k = \gamma_k / \sum_{l \in \mathfrak{S}} \gamma_l$, and the mass fractions of the k th species by $y_k = \rho_k / \sum_{l \in \mathfrak{S}} \rho_l$ so that $y_k = m_k \gamma_k / \sum_{l \in \mathfrak{S}} m_l \gamma_l$. We correspondingly define the mole fractions vector $\mathbf{x} = (x_1, \dots, x_{n^s})^\top$ and the mass fraction vector $\mathbf{y} = (y_1, \dots, y_{n^s})^\top$.

(Th₁) *The nonelectron species molar masses m_k , $k \in \mathfrak{S}$, $k \neq n^s$, and the gas constant R are positive constants. The electron molar mass $m_{n^s} = \tilde{m}_0$ is non-negative. The formation energies E_k^{st} , $k \in \mathfrak{S}$, and the formation entropies S_k^{st} , $k \in \mathfrak{S}$, are constants. The molar specific heats $C_{v,k}$, $k \in \mathfrak{S}$, are \mathcal{C}^∞ functions of $T \geq 0$ and there exist positive constants \underline{c}_v and \bar{c}_v with $0 < \underline{c}_v \leq C_{v,k}(T) \leq \bar{c}_v$, for $T \geq 0$ and $k \in \mathfrak{S}$.*

(Th₂) *The atom molar masses \tilde{m} , $l \in \mathfrak{A}$, are positive constants and the species molar masses m_k , $k \in \mathfrak{S}$, are given by*

$$m_k = \sum_{l \in \mathfrak{A}} \tilde{m}_l \mathbf{a}_{kl} + \tilde{m}_0 \mathbf{a}_{k0}, \quad k \in \mathfrak{S}.$$

We also have the proportionality relation $\varkappa_k = -\alpha \mathbf{a}_{k0}$, $k \in \mathfrak{S}$, where α is a positive constant which represents the absolute value of electron charge per unit mole.

(Th₃) *The stoichiometric coefficients ν_{kr}^f and ν_{kr}^b , $k \in \mathfrak{S}$, $r \in \mathfrak{R}$, and the atomic coefficients \mathbf{a}_{kl} , $k \in \mathfrak{S}$, $l \in \mathfrak{A}$, are non-negative integers. The numbers of electrons \mathbf{a}_{k0} , $k \in \mathfrak{S}$, $l \in \mathfrak{A}$, are integers. The atomic vectors \mathbf{a}_l , $l \in \overline{\mathfrak{A}}$, and the reaction vectors ν_r , $r \in \mathfrak{R}$, satisfy the conservation relations $\langle \nu_r, \mathbf{a}_l \rangle = 0$, $r \in \mathfrak{R}$, $l \in \overline{\mathfrak{A}}$. This relation expresses atom conservation for $l \in \mathfrak{A}$ and charge conservation for $l = 0$.*

(Th₄) *The rate constants \mathcal{K}_r^s , $r \in \mathfrak{R}$, are \mathcal{C}^∞ positive functions of $T > 0$.*

(Th₅) *There exists at least a positive ionized species such that $\varkappa_k > 0$, a neutral species such that $\varkappa_k = 0$, and we assume that the last species is constituted by electrons so that $\varkappa_{n^s} < 0$.*

These assumptions imply in particular the vector properties $\mathbf{a}_l \in \mathcal{R}^\perp$, $l \in \mathfrak{A}$, and $\mathbf{a}_0 \in \mathcal{R}^\perp$, where $\mathcal{R} = \text{span}\{\nu_r, r \in \mathfrak{R}\}$. In addition, we have the vector relations $m = \sum_{l \in \mathfrak{A}} \tilde{m}_l \mathbf{a}_l + \tilde{m}_0 \mathbf{a}_0$, so that $m \in \mathcal{R}^\perp$, and $\varkappa = -\alpha \mathbf{a}_0$, so that $\varkappa \in \mathcal{R}^\perp$. Note that with (Th₅) the vectors $\rho \mathbf{y} = (\rho_1, \dots, \rho_{n^s})^\top$ and $z = (z_1, \dots, z_{n^s})^\top$ are linearly independent, as are the vectors m and \varkappa . Defining $m'_k = \sum_{l \in \mathfrak{A}} \tilde{m}_l \mathbf{a}_{kl}$ and $\rho' = \sum_{1 \leq k \leq n^s - 1} m'_k \gamma_k$ we also have $\rho = \rho' - \tilde{m}_0 q / \alpha$. Finally, the presence of a neutral species in the model is not strictly needed, but somewhat simplifies the presentation, especially for deriving explicit normal forms.

2.6.2. Assumptions on transport coefficients

We introduce a first set of assumptions concerning the transport coefficients which is only valid for positive electron mass. These assumptions will be generalized in order to encompass the limiting case of zero electron mass in the next section.

- (tr₁) The multicomponent diffusion coefficients D_{kl} , $k, l \in \mathfrak{S}$, the thermal diffusion ratios χ_k , $k \in \mathfrak{S}$, the volume viscosity κ , the shear viscosity η and the thermal conductivity λ are C^∞ functions of (T, γ) , where $\gamma = (\gamma_1, \dots, \gamma_{n^s})^\dagger$, for $T > 0$ and $\gamma > 0$.
- (tr₂) The thermal conductivity λ and the shear viscosity η are positive functions. The volume viscosity κ is a non-negative function.
- (tr₃) For $\gamma > 0$ and $T > 0$, the matrix $D = (D_{kl})_{k,l}$ is real symmetric positive semidefinite and its null space is spanned by the vector $\mathbf{y} = (y_1, \dots, y_{n^s})^\dagger$. The thermal diffusion ratios χ_k , $k \in \mathfrak{S}$, verify the relation $\langle \chi, \mathbf{u} \rangle = 0$.

These properties have important consequences for the matrix \widehat{D} of effective diffusion coefficients defined in (2.9).

Lemma 2.1. *Under Assumptions (Th₁)–(Th₅) and (tr₁)–(tr₃) the matrix \widehat{D} is symmetric positive semidefinite. Its null space is spanned by the vectors \mathbf{y} and \mathbf{z} , where $z_k = \gamma_k \chi_k$, $k \in \mathfrak{S}$, so that $N(\widehat{D}) = \mathbb{R}\mathbf{y} \oplus \mathbb{R}\mathbf{z}$ and $R(\widehat{D}) = \text{span}(\mathbf{y}, \mathbf{z})^\perp$.*

Proof. First note that \mathbf{y} and \mathbf{z} are nonzero since $\gamma > 0$ and are not proportional since there exists positively as well as negatively charged species. This implies that $\langle D\mathbf{z}, \mathbf{z} \rangle > 0$ so that \widehat{D} is well defined. After a little algebra, we obtain that

$$\langle \widehat{D}x, x \rangle = \left\langle D \left(x - \frac{\langle Dx, \mathbf{z} \rangle}{\langle Dz, \mathbf{z} \rangle} \mathbf{z} \right), x - \frac{\langle Dx, \mathbf{z} \rangle}{\langle Dz, \mathbf{z} \rangle} \mathbf{z} \right\rangle.$$

The properties of \widehat{D} are then directly deduced from the properties of D . □

2.7. Quasilinear formulation

The relations (2.9), (2.10) imply that conduction current \mathbf{j} vanishes, so that the charge q remains zero if it is initially zero. As a consequence, the momentum and energy conservation equations can be simplified into

$$\partial_t(\rho\mathbf{v}) + \partial_{\mathbf{x}} \cdot (\rho\mathbf{v} \otimes \mathbf{v} + p\mathbb{I}) + \partial_{\mathbf{x}} \cdot \mathbf{\Pi} = 0, \tag{2.13}$$

$$\partial_t \left(\mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right) + \partial_{\mathbf{x}} \cdot \left(\left(\mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + p \right) \mathbf{v} \right) + \partial_{\mathbf{x}} \cdot (\mathbf{Q} + \mathbf{\Pi} \cdot \mathbf{v}) = 0. \tag{2.14}$$

Whenever neutrality holds, one could further express γ_{n^s} in terms of the heavy species molar densities, and eliminate completely the electrons from the governing equations. This simplification, however, will not be used in this paper, since it forbids symmetrization of the resulting system of partial differential equations. Similarly, since the mass density can be written as $\rho = \rho' - \tilde{m}_0 q / \alpha$ where $\rho' = \sum_{K=1}^{n^s-1} m'_K \gamma_K$, one could use ρ' instead of ρ in the governing equations, but this simpler formulation is not needed in the following.

We introduce a compact notation that will be used in the following. We define the conservative variable \mathbf{U} by

$$\mathbf{U} = \left(\gamma_1, \dots, \gamma_{n^s}, \rho v_1, \rho v_2, \rho v_3, \mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right)^\mathbf{t}, \tag{2.15}$$

and the natural variable \mathbf{Z} by

$$\mathbf{Z} = (\gamma_1, \dots, \gamma_{n^s}, v_1, v_2, v_3, T)^\mathbf{t}. \tag{2.16}$$

The components of \mathbf{U} naturally appear as conserved quantities in the molar formulation of the system of partial differential equations governing ambipolar plasmas. On the other hand, the components of the natural variable \mathbf{Z} are more practical to use in actual calculations of differential identities.

The conservation equations can be written in the compact form

$$\partial_t \mathbf{U} + \sum_{i \in \mathcal{C}} \partial_i \mathbf{F}_i + \sum_{i \in \mathcal{C}} \partial_i \mathbf{F}_i^{\text{diss}} = \Omega, \tag{2.17}$$

where \mathcal{C} denotes the set $\{1, 2, 3\}$, \mathbf{F}_i the convective flux in the i th direction, $\mathbf{F}_i^{\text{diss}}$ the dissipative flux in the i th direction, and Ω is the source term. The source term Ω is given by

$$\Omega = (\omega_1, \dots, \omega_{n^s}, 0, 0, 0, 0)^\mathbf{t}, \tag{2.18}$$

and the convective flux \mathbf{F}_i by

$$\begin{aligned} \mathbf{F}_i = & \left(\gamma_1 v_i, \dots, \gamma_{n^s} v_i, \rho v_1 v_i + \delta_{i1} p, \rho v_2 v_i \right. \\ & \left. + \delta_{i2} p, \rho v_3 v_i + \delta_{i3} p, \left(\mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + p \right) v_i \right)^\mathbf{t}. \end{aligned} \tag{2.19}$$

The dissipative flux $\mathbf{F}_i^{\text{diss}}$ can be decomposed into

$$\mathbf{F}_i^{\text{diss}} = \mathbf{F}_i^{\text{diff}} + \mathbf{F}_i^{\text{visc}}, \tag{2.20}$$

where $\mathbf{F}_i^{\text{visc}}$, the viscous flux, and $\mathbf{F}_i^{\text{diff}}$, the diffusion flux, are defined by

$$\begin{aligned} \mathbf{F}_i^{\text{visc}} = & \left(0, \dots, 0, \Pi_{i1}, \Pi_{i2}, \Pi_{i3}, \sum_{j \in \mathcal{C}} \Pi_{ij} v_j \right)^\mathbf{t} \\ \mathbf{F}_i^{\text{diff}} = & (\mathcal{F}_{1i}, \dots, \mathcal{F}_{n^s i}, 0, 0, 0, \mathcal{Q}_i)^\mathbf{t}. \end{aligned}$$

The convective and dissipative fluxes are naturally given in terms of the natural variable \mathbf{Z} . In order to relate the natural variable \mathbf{Z} to the conservative variable \mathbf{U} , we investigate the map $\mathbf{Z} \mapsto \mathbf{U}$. We introduce the open sets \mathcal{O}_Z and \mathcal{O}_U defined by

$$\begin{aligned} \mathcal{O}_Z = & (\mathbb{R}_+^*)^{n^s} \times \mathbb{R}^3 \times \mathbb{R}_+^*, \\ \mathcal{O}_U = & \{ u \in \mathbb{R}^{n^s+4} : u_1, \dots, u_{n^s} > 0, u_{n^s+4} > f(u_i) \}, \end{aligned}$$

where f is the map from $(\mathbb{R}_+^*)^{n^s} \times \mathbb{R}^3$ in \mathbb{R} given by

$$f(u) = \frac{1}{2} \frac{u_{n^s+1}^2 + u_{n^s+2}^2 + u_{n^s+3}^2}{\sum_{i \in \mathfrak{S}} m_i u_i} + \sum_{i \in \mathfrak{S}} u_i E_i^0, \tag{2.21}$$

and E_i^0 is the internal energy per unit mole at zero temperature. The following lemma is easily established as in the neutral case.¹¹

Lemma 2.2. *Assume that properties (Th₁)–(Th₅) hold. The map $Z \mapsto U$ is a C^∞ diffeomorphism from the open set \mathcal{O}_Z onto the open convex set \mathcal{O}_U .*

As a consequence of this lemma, and from the expressions of convective and dissipative fluxes in terms of Z , we can rewrite (2.17) as a quasilinear form in the conservative variable U

$$\partial_t U + \sum_{i \in \mathcal{C}} A_i(U) \partial_i U = \sum_{i,j \in \mathcal{C}} \partial_i (B_{ij}(U) \partial_j U) + \Omega(U), \tag{2.22}$$

where $A_i = \partial_U F_i$, $F_i^{\text{diss}} = -\sum_{j \in \mathcal{C}} B_{ij}(U) \partial_j U$, $i \in \mathcal{C}$, and where A_i , $i \in \mathcal{C}$, B_{ij} , $i, j \in \mathcal{C}$ and Ω are smooth.

3. Vanishing Electron Mass

The asymptotic stability of equilibrium states for the quasilinear system of partial differential equations modeling ambipolar plasmas (2.22) is, in itself, an important question. We note, however, that this system also depends on numerous parameters such as thermal conductivity and chemical reaction rate constants. This is a strong motivation for further investigating the dependence of solutions on the system parameters. This will be done in the next sections under the assumption that the system coefficients depend smoothly on the parameters under consideration.

One of these parameters, often used in the physical modeling, is the electron mass $m_{n^s} = \tilde{m}_0$, which is generally assumed to be zero. In order to investigate this limit, the dependence of the system coefficients on the electron mass must be clarified. The thermodynamic assumptions (Th₁)–(Th₅) can be used for any non-negative electron mass and need not be modified. However, the transport properties (th₁)–(th₃) are only valid for positive electron mass and must be replaced.

In order to do so, we need to express the dependence of multicomponent diffusion matrices on binary diffusion coefficients explicitly and investigate the limit of the diffusion coefficients \widehat{D} for vanishing electron mass.

3.1. Definition of D as a generalized inverse

We introduce the matrix Δ defined by

$$\begin{cases} \Delta_{kk} = \sum_{\substack{l \in \mathfrak{S} \\ l \neq k}} \frac{x_k x_l}{\mathcal{D}_{kl}^{\text{bin}}}, & k \in \mathfrak{S}, \\ \Delta_{kl} = -\frac{x_k x_l}{\mathcal{D}_{kl}^{\text{bin}}}, & k, l \in \mathfrak{S}, k \neq l, \end{cases} \tag{3.1}$$

where $\mathcal{D}_{kl}^{\text{bin}}$ is the binary diffusion coefficient for the species pair (k, l) and x_k the mole fraction of the k th species given by $x_k = \gamma_k / \sum_{l \in \mathfrak{S}} \gamma_l$. These coefficients $\mathcal{D}_{kl}^{\text{bin}}$ are only defined for positive species masses. In a first-order theory, $\mathcal{D}_{kl}^{\text{bin}}$ only depends on pressure and temperature $\mathcal{D}_{kl}^{\text{bin}} = \mathcal{D}_{kl}^{\text{bin}}(T, p)$. More generally, for more accurate multicomponent diffusion coefficients, the quantities $\mathcal{D}_{kl}^{\text{bin}}, k, l \in \mathfrak{S}$, are Schur complements from transport linear systems of size larger than n^s , and are then functions of T, p and γ , but have similar properties.⁴ The following properties of the matrix Δ are easily established.^{7,8}

Proposition 3.1. *Assume that the coefficients $\mathcal{D}_{kl}^{\text{bin}}, k, l \in \mathfrak{S}, k \neq l$, are positive and symmetric, and that $\gamma > 0$. Then Δ is symmetric positive semidefinite, $N(\Delta) = \mathbb{R}\mathbf{u}$ where $\mathbf{u} = (1, \dots, 1)^t$, $R(\Delta) = \mathbf{u}^\perp$, Δ is irreducible and a singular M -matrix.*

We define the mass fractions by $y_k = \rho_k / \sum_{l \in \mathfrak{S}} \rho_l$ so that $y_k = m_k \gamma_k / \sum_{l \in \mathfrak{S}} m_l \gamma_l$ and the mass fraction vector $\mathbf{y} = (y_1, \dots, y_{n^s})^t$. The multicomponent diffusion matrix D can then be defined as a proper generalized inverse of Δ .^{7,8}

Proposition 3.2. *Keeping the assumptions of Proposition 3.1 there exists a unique generalized inverse D of Δ with prescribed range \mathbf{y}^\perp and null space $\mathbb{R}\mathbf{y}$, i.e. a the unique matrix D such that $D\Delta D = D$, $\Delta D \Delta = \Delta$, $R(D) = \mathbf{y}^\perp$, and $N(D) = \mathbb{R}\mathbf{y}$. This matrix D is positive semidefinite, we have $\Delta D = I - \mathbf{y} \otimes \mathbf{u}$, $D \Delta = I - \mathbf{u} \otimes \mathbf{y}$, and for a, b positive with $ab = 1$, we have $D = (\Delta + a\mathbf{y} \otimes \mathbf{y})^{-1} - b\mathbf{u} \otimes \mathbf{u}$. The coefficients of D are smooth functions of (T, γ) for $T > 0, \gamma > 0$, provided that the binary diffusion coefficients are smooth functions of (T, γ) so that the assumptions concerning D in (tr_1) – (tr_3) hold.*

These results can easily be extended to the case of the matrix \widehat{D} . More specifically, when neutrality holds, i.e. $q = \langle z, \mathbf{u} \rangle = 0$, there exists a unique \mathbf{w} such that $\Delta \mathbf{w} = z$ and $\langle \mathbf{w}, \mathbf{y} \rangle = 0$. Introducing the matrix $\widehat{\Delta} = \Delta - z \otimes z / \langle \mathbf{w}, z \rangle$, one can establish that \widehat{D} is the generalized inverse of $\widehat{\Delta}$ with prescribed null space $\mathbb{R}\mathbf{y} \oplus \mathbb{R}z$ and range $\text{span}(\mathbf{y}, z)^\perp$. However, these results will not be needed in the following where we will directly use the relation (2.9) defining \widehat{D} from D .

On the other hand, from the kinetic theory of gases we have

$$\mathcal{D}_{kl}^{\text{bin}} = O\left(\frac{1}{\sqrt{m_k m_l}}\right), \tag{3.2}$$

and the quantity $\mathcal{D}_{kl}^{\text{bin}} \sqrt{m_k m_l}$ can be assumed to be smooth. Therefore, electron diffusivities $\mathcal{D}_{kn^s}^{\text{bin}}, k \in \mathfrak{S}, k \neq n^s$, explode for vanishing electron mass $m_{n^s} \rightarrow 0$.

3.2. Diffusion matrices for vanishing electron mass

In this section, we specify the assumptions concerning the asymptotic limit of vanishing electron mass. We define the small parameter ε by

$$\varepsilon = (m_{n^s}/\bar{m})^{1/2},$$

where \bar{m} is a characteristic mass of heavy species, i.e. nonelectron species, and we investigate the behavior of the system coefficients as $\varepsilon \rightarrow 0$. We will denote by $[0, \bar{\varepsilon}]$ an interval of relevant values for ε such that heavy species masses stay away from zero, so that (Th₁)–(Th₅) are satisfied, where $\bar{\varepsilon} > 0$ is fixed. From relations (3.2), electron diffusivities $\mathcal{D}_{kn^s}^{\text{bin}}, k \in \mathfrak{S}, k \neq n^s$, goes to infinity as $\varepsilon \rightarrow 0$ and we set

$$\begin{cases} \mathcal{D}_{kl}^{\text{bin}} = \check{\mathcal{D}}_{kl}^{\text{bin}}, & k \neq l, k \neq n^s \text{ and } l \neq n^s, \\ \mathcal{D}_{kl}^{\text{bin}} = \frac{1}{\varepsilon} \check{\mathcal{D}}_{kl}^{\text{bin}}, & k \neq l, k = n^s \text{ or } l = n^s, \end{cases} \quad (3.3)$$

where the coefficients $\check{\mathcal{D}}_{kl}^{\text{bin}}$ are assumed to be smooth functions of the state variables $T > 0, \gamma > 0$ and of the parameter $\varepsilon \in [0, \bar{\varepsilon}]$. The properties (tr₁)–(tr₃) only hold for positive species mass, that is for $\varepsilon > 0$ and we have now to establish that \hat{D} depends smoothly on the reduced mass ε and to identify its limit as $\varepsilon \rightarrow 0$, thereby removing the singular behavior at $\varepsilon = 0$.

In order to investigate the limit of \hat{D} as $\varepsilon \rightarrow 0$, it is convenient to introduce a partitioning of the species $\mathfrak{S} = \{1, \dots, n^s\}$ between heavy species $h = \{1, \dots, n^s - 1\}$ and electrons $e = \{n^s\}$. Correspondingly, there is a block decomposition of vectors $x \in \mathbb{R}^{n^s}$ in the form $x = (x^h, x^e)^\top$ and of matrices $M \in \mathbb{R}^{n, n}$ such that $y = Mx$ if and only if $y^h = M^{\text{hh}}x^h + M^{\text{he}}x^e$ and $y^e = M^{\text{eh}}x^h + M^{\text{ee}}x^e$. The matrix Δ admits in particular the block decomposition

$$\Delta = \begin{pmatrix} \Delta^{\text{hh}} & \Delta^{\text{he}} \\ \Delta^{\text{eh}} & \Delta^{\text{ee}} \end{pmatrix} = \begin{pmatrix} \check{\Delta}^{\text{hh}} & \varepsilon \check{\Delta}^{\text{he}} \\ \varepsilon \check{\Delta}^{\text{eh}} & \varepsilon \check{\Delta}^{\text{ee}} \end{pmatrix}, \quad (3.4)$$

where the coefficients of $\check{\Delta}$ are smooth functions of $T > 0, \gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$ and are defined as in (3.1) with $\mathcal{D}_{kl}^{\text{bin}}$ replaced by $\check{\mathcal{D}}_{kl}^{\text{bin}}$. We will denote by Δ_0^{hh} the matrix $\Delta^{\text{hh}} = \check{\Delta}^{\text{hh}}$ obtained for $\varepsilon = 0$, and by y_0^h the vector y^h obtained for $\varepsilon = 0$, keeping in mind that $m_k, k \in \mathfrak{S}$, depend on ε from (Th₂).

Proposition 3.3. *Assume that the coefficients $\check{\mathcal{D}}_{kl}^{\text{bin}}, k, l \in \mathfrak{S}, k \neq l$, are positive and symmetric, and smooth functions of $T > 0, \gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$. There exists coefficients \check{D} which are smooth functions of $T > 0, \gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$ such that for any $\varepsilon > 0$ we have*

$$D = \begin{pmatrix} \check{D}^{\text{hh}} & \check{D}^{\text{he}} \\ \check{D}^{\text{eh}} & \frac{1}{\varepsilon} \check{D}^{\text{ee}} \end{pmatrix}.$$

Moreover, the matrix \check{D}_0^{hh} obtained for $\varepsilon = 0$ is the diffusion matrix between heavy species in the absence of electrons, i.e. \check{D}_0^{hh} is the generalized inverse of Δ_0^{hh} with

null space $\mathbb{R}y_0^h$ and range $(y_0^h)^\perp$. Finally the scalar coefficient \check{D}_0^{ee} obtained for $\varepsilon = 0$ if a positive function of $T > 0$ and $\gamma > 0$.

Proof. From (3.4) we can introduce the matrices

$$\Delta^{up} = \begin{bmatrix} \check{\Delta}^{hh} & \varepsilon \check{\Delta}^{he} \\ \check{\Delta}^{eh} & \check{\Delta}^{ee} \end{bmatrix}, \quad \Delta^{lo} = \begin{bmatrix} \check{\Delta}^{hh} & \check{\Delta}^{he} \\ \varepsilon \check{\Delta}^{eh} & \check{\Delta}^{ee} \end{bmatrix},$$

which are smooth functions of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$. From the properties of Δ , it is easily obtained that for $\varepsilon \in [0, \bar{\varepsilon}]$ we have $N(\Delta^{up}) = \mathbb{R}u$, $R(\Delta^{up}) = \tilde{u}^\perp$, $N(\Delta^{lo}) = \mathbb{R}\tilde{u}$, $R(\Delta^{lo}) = u^\perp$, where we have defined $\tilde{u} = (u^h, \varepsilon)^\top$. From the definition of ε we also have $y = (y^h, y^e)^\top$ with $y^e = \varepsilon^2 \check{y}^e$ where \check{y}^e is independent of ε , and we define $\check{y} = (y^h, \varepsilon \check{y}^e)^\top$. We now introduce the generalized inverse D^{up} of Δ^{up} with range $R(D^{up}) = y^\perp$ and null space $N(D^{up}) = \mathbb{R}\check{y}$, and the generalized inverse D^{lo} of Δ^{lo} with range $R(D^{lo}) = \check{y}^\perp$ and null space $N(D^{lo}) = \mathbb{R}y$. These matrices D^{up} and D^{lo} are well defined for any $\varepsilon \in [0, \bar{\varepsilon}]$ since $N(\Delta^{up}) \oplus y^\perp = \mathbb{R}^{n^s}$, $R(\Delta^{up}) \oplus \mathbb{R}\check{y} = \mathbb{R}^{n^s}$, $N(\Delta^{lo}) \oplus \check{y}^\perp = \mathbb{R}^{n^s}$, $R(\Delta^{lo}) \oplus \mathbb{R}y = \mathbb{R}^{n^s}$, thanks to $\langle y, u \rangle = \langle \check{y}, \tilde{u} \rangle = 1$. In addition, these matrices D^{up} and D^{lo} are smooth functions of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$ since for any positive α, β with $\alpha\beta = 1$ we have

$$(D^{lo} + \alpha \tilde{u} \otimes u)(\Delta^{lo} + \beta y \otimes \check{y}) = (D^{up} + \alpha u \otimes \tilde{u})(\Delta^{up} + \beta \check{y} \otimes y) = I.$$

Denoting by M the diagonal matrix $M = \text{diag}(1, \dots, 1, \varepsilon)$, it is easily seen that for any positive ε we have $\Delta = M\Delta^{up} = \Delta^{lo}M$ and $D = D^{up}M^{-1} = M^{-1}D^{lo}$. These relations yield in particular that $D^{up} = (D^{lo})^\top$ so that $D^{hh} = (D^{up})^{hh} = (D^{lo})^{hh}$, $D^{he} = (D^{up})^{he}/\varepsilon = (D^{lo})^{he}$, $D^{eh} = (D^{up})^{eh} = (D^{lo})^{eh}/\varepsilon$, and $D^{ee} = (D^{up})^{ee}/\varepsilon = (D^{lo})^{ee}/\varepsilon$. This shows in particular that D^{hh} , D^{he} , and D^{eh} are smooth functions of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$. The identification of Δ_0^{hh} results from simple algebraic manipulations making use of the properties of D and Δ , thanks to $m_{n^s} = \bar{m}\varepsilon^2$. In addition, we have for $\varepsilon = 0$ that $(D_0^{up})^{ee} = (D_0^{lo})^{ee} = 1/\Delta_0^{ee}$ and it is easily deduced from the general properties of diagonal diffusion coefficients⁸ that $(\Delta^{up})^{ee}$ is a positive function of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$. □

We can now establish that \hat{D} is a smooth function of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$.

Proposition 3.4. *Assume that the coefficients \check{D}_{kl}^{bin} , $k, l \in \mathfrak{S}$, $k \neq l$, are positive, symmetric, and smooth functions of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$. Then the matrix \hat{D} is a smooth function of $T > 0$, $\gamma > 0$, $\varepsilon \in [0, \bar{\varepsilon}]$. Moreover, defining $z_k = z_k/z_{n^s}$ for $k \in \{1, \dots, n^s - 1\}$, its limit as $\varepsilon \rightarrow 0$ is given by*

$$\lim_{\varepsilon \rightarrow 0} \hat{D}(\varepsilon) = \begin{bmatrix} \check{D}_0^{hh} & -\check{D}_0^{hh}z^h \\ -(\check{D}_0^{hh}z^h)^\top & \check{D}_0^{hh}z^h, z^h \end{bmatrix}.$$

Proof. The smoothness of \hat{D} for positive ε is a direct consequence of the smoothness of D . The only nontrivial part of this proposition concerns the behavior for

small ε . We first note that

$$\langle Dz, z \rangle = \frac{(z^e)^2}{\varepsilon} (\check{D}^{ee} + 2\varepsilon\check{D}^{eh}z^h + \varepsilon\langle \check{D}^{hh}z^h, z^h \rangle),$$

so that for the block \widehat{D}^{hh} it is easily obtained that

$$\widehat{D}^{hh} = \check{D}^{hh} - \varepsilon \frac{(\check{D}^{hh}z^h + \check{D}^{he}) \otimes (\check{D}^{hh}z^h + \check{D}^{he})}{\check{D}^{ee} + 2\varepsilon\check{D}^{eh}z^h + \varepsilon\langle \check{D}^{hh}z^h, z^h \rangle}.$$

This expression shows that \widehat{D}^{hh} is smooth up to $\varepsilon = 0$ and converges to \widehat{D}_0^{hh} as $\varepsilon \rightarrow 0$. For the term \widehat{D}^{he} we can write that

$$\widehat{D}^{he} = \check{D}^{he} - \frac{(\check{D}^{hh}z^h + \check{D}^{he})(\check{D}^{ee} + \varepsilon\check{D}^{eh}z^h)}{\check{D}^{ee} + 2\varepsilon\check{D}^{eh}z^h + \varepsilon\langle \check{D}^{hh}z^h, z^h \rangle},$$

so that \widehat{D}^{he} is smooth up to $\varepsilon = 0$ and converge to $-\widehat{D}_0^{hh}z_0^h$ as $\varepsilon \rightarrow 0$. Thanks to the symmetry of \widehat{D} , the term \widehat{D}^{eh} is similar. Finally, the last term reads

$$\widehat{D}^{ee} = \frac{1}{\varepsilon} \left(\check{D}^{ee} - \frac{(\check{D}^{ee} + \varepsilon\check{D}^{eh}z^h)^2}{\check{D}^{ee} + 2\varepsilon\check{D}^{eh}z^h + \varepsilon\langle \check{D}^{hh}z^h, z^h \rangle} \right)$$

and can be recast in the form

$$\widehat{D}^{ee} = \frac{\check{D}^{ee}\langle \check{D}^{hh}z^h, z^h \rangle - \varepsilon(\check{D}^{eh}z^h)^2}{\check{D}^{ee} + 2\varepsilon\check{D}^{eh}z^h + \varepsilon\langle \check{D}^{hh}z^h, z^h \rangle},$$

and there is a cancellation of singularity. This term is thus smooth up to $\varepsilon = 0$ and converges to $\langle \check{D}_0^{hh}z^h, z^h \rangle$ as $\varepsilon \rightarrow 0$. □

Remark 3.1. The limiting transport coefficients obtained with \widehat{D}_0^{hh} can also be obtained by letting $\mathbf{d}_{n^s} = 0$, so that $\mathbf{E} = p\mathbf{d}_{n^s}^0/z_{n^s}$, and by substituting this relation in the expression of the diffusion velocities (2.5), (2.6). This relation can also be obtained by letting the electron mass to go to zero in an electron momentum conservation equation.

3.3. Assumptions for vanishing electron mass

As a consequence of the results obtained in the preceding sections, we can reformulate the assumptions on transport coefficients as follows:

- (Tr₁) The effective multicomponent diffusion coefficients \widehat{D}_{kl} , $k, l \in \mathfrak{S}$, the thermal diffusion ratios χ_k , $k \in \mathfrak{S}$, the volume viscosity κ , the shear viscosity η , and the conductivity λ are \mathcal{C}^∞ functions of $T > 0$, $\gamma > 0$ and $\varepsilon \in [0, \bar{\varepsilon}]$.
- (Tr₂) The thermal conductivity λ and the shear viscosity η are positive functions. The volume viscosity κ is a non-negative function.
- (Tr₃) For $\gamma > 0$, $T > 0$, and $\varepsilon \in [0, \bar{\varepsilon}]$, the matrix $\widehat{D} = (\widehat{D})_{k,l}$ is real symmetric positive semidefinite and its null space is spanned by the vectors \mathbf{y} and \mathbf{z} . The thermal diffusion ratios χ_k , $k \in \mathfrak{S}$, verify the relation $\langle \chi, \mathbf{u} \rangle = 0$.

We can finally rewrite the quasilinear system (2.22) in the form

$$\partial_t \mathbf{U} + \sum_{i \in \mathcal{C}} \mathbf{A}_i(\mathbf{U}, \varepsilon) \partial_i \mathbf{U} = \sum_{i, j \in \mathcal{C}} \partial_i (\mathbf{B}_{ij}(\mathbf{U}, \varepsilon) \partial_j \mathbf{U}) + \Omega(\mathbf{U}, \varepsilon), \tag{3.5}$$

where we have emphasized the dependence of the coefficients on the reduced electron mass parameter ε . The system coefficients of (3.5) are naturally defined in the open domain $(\mathbf{U}, \varepsilon) \in \mathcal{O}_{(\mathbf{U}, \varepsilon)}$ where

$$\mathcal{O}_{(\mathbf{U}, \varepsilon)} = \{(\mathbf{U}, \varepsilon) \in \mathbb{R}^{n^s+4} : \mathbf{U}_1, \dots, \mathbf{U}_{n^s} > 0, \varepsilon > 0, \mathbf{U}_{n^s+4} > f(\mathbf{U}_1, \dots, \mathbf{U}_{n^s+3}, \varepsilon)\},$$

where f is the map introduced in (2.21) which depends on ε through the species mass m_k , $k \in \mathfrak{S}$. We have seen, in addition, that the system coefficients can be smoothly extended up to $\varepsilon = 0$.

4. Local Existence for an Abstract System

In this section, we investigate symmetrization and local existence of solutions for hyperbolic–parabolic systems depending on a parameter.

4.1. Conservative symmetrization

We consider an abstract second-order quasilinear system depending on a parameter in the form

$$\partial_t \mathbf{U}^* + \sum_{i \in \mathcal{C}^*} \mathbf{A}_i^*(\mathbf{U}^*, \varepsilon^*) \partial_i \mathbf{U}^* = \sum_{i, j \in \mathcal{C}^*} \partial_i (\mathbf{B}_{ij}^*(\mathbf{U}^*, \varepsilon^*) \partial_j \mathbf{U}^*) + \Omega^*(\mathbf{U}^*, \varepsilon^*), \tag{4.1}$$

where $(\mathbf{U}^*, \varepsilon^*) \in \mathcal{O}_{(\mathbf{U}^*, \varepsilon^*)}$, $\mathcal{O}_{(\mathbf{U}^*, \varepsilon^*)}$ is an open set of $\mathbb{R}^{n^*} \times \mathbb{R}^{m^*}$, and $\mathcal{C}^* = \{1, \dots, d\}$ denotes the direction indices of \mathbb{R}^d . Note that the superscript $*$ is used to distinguish between the abstract second-order system (4.1) of size n^* in \mathbb{R}^d with ε^* of size m^* and the particular ambipolar plasmas system (3.5) of size $n^s + 4$ in \mathbb{R}^3 with ε the reduced electron mass. All quantities associated with the abstract system have the corresponding superscript $*$, so that, for instance, the unknown vector is \mathbf{U}^* . We consider open domains $\mathcal{O}_{(\mathbf{U}^*, \varepsilon^*)}$, for the sake of simplicity, and assume that the slices $\mathcal{O}_{\mathbf{U}^*}^{\varepsilon^*} = \{\mathbf{U}^* \in \mathbb{R}^{n^*}; (\mathbf{U}^*, \varepsilon^*) \in \mathcal{O}_{(\mathbf{U}^*, \varepsilon^*)}\}$ are convex for all ε^* . We assume that the following properties hold for system (4.1).

(Edp₁) *The convective fluxes \mathbf{F}_i^* , $i \in \mathcal{C}^*$, dissipation matrices \mathbf{B}_{ij}^* , $i, j \in \mathcal{C}^*$, and source term Ω^* are smooth functions of the variable $(\mathbf{U}^*, \varepsilon^*) \in \mathcal{O}_{(\mathbf{U}^*, \varepsilon^*)}$.*

The following definition of a symmetric (conservative) form for the system (4.1) is adapted from Kawashima and Shizuta.¹⁴

Definition 4.1. Consider a \mathcal{C}^∞ diffeomorphism $(\mathbf{U}^*, \varepsilon^*) \rightarrow (\mathbf{V}^*, \varepsilon^*)$ from the open domain $\mathcal{O}_{(\mathbf{U}^*, \varepsilon^*)}$ onto an open domain $\mathcal{O}_{(\mathbf{V}^*, \varepsilon^*)}$ and consider the system in the \mathbf{V}^* variable

$$\tilde{\mathbf{A}}_0^*(\mathbf{V}^*, \varepsilon^*) \partial_t \mathbf{V}^* + \sum_{i \in \mathcal{C}^*} \tilde{\mathbf{A}}_i^*(\mathbf{V}^*, \varepsilon^*) \partial_i \mathbf{V}^* = \sum_{i, j \in \mathcal{C}^*} \partial_i (\tilde{\mathbf{B}}_{ij}^*(\mathbf{V}^*, \varepsilon^*) \partial_j \mathbf{V}^*) + \tilde{\Omega}^*(\mathbf{V}^*, \varepsilon^*), \tag{4.2}$$

where

$$\begin{cases} \tilde{A}_0^* = \partial_{V^*} U^*, & \tilde{A}_i^* = A_i^* \partial_{V^*} U^* = \partial_{V^*} F_i^*, \\ \tilde{B}_{ij}^* = B_{ij}^* \partial_{V^*} U^*, & \tilde{\Omega}^* = \Omega^*. \end{cases} \tag{4.3}$$

The system is said of symmetric form if the matrices $\tilde{A}_0^*, \tilde{A}_i^*, i \in C^*$, and $\tilde{B}_{ij}^*, i, j \in C^*$, verify the following properties (S₁)–(S₄).

- (S₁) The matrix $\tilde{A}_0^*(V^*, \varepsilon^*)$ is symmetric positive definite for $(V^*, \varepsilon^*) \in \mathcal{O}_{(V^*, \varepsilon^*)}$.
- (S₂) The matrices $\tilde{A}_i^*(V^*, \varepsilon^*), i \in C^*$, are symmetric for $(V^*, \varepsilon^*) \in \mathcal{O}_{(V^*, \varepsilon^*)}$.
- (S₃) We have $\tilde{B}_{ij}^*(V^*, \varepsilon^*)^\dagger = \tilde{B}_{ji}^*(V^*, \varepsilon^*)$ for $i, j \in C^*$, and $(V^*, \varepsilon^*) \in \mathcal{O}_{(V^*, \varepsilon^*)}$.
- (S₄) The matrix $\tilde{B}^*(V^*, \varepsilon^*; w) = \sum_{i,j \in C^*} \tilde{B}_{ij}^*(V^*, \varepsilon^*) w_i w_j$ is symmetric and positive semidefinite, for $(V^*, \varepsilon^*) \in \mathcal{O}_{(V^*, \varepsilon^*)}$, and $w \in \Sigma^{d-1}$, where Σ^{d-1} is the unit sphere in d dimensions.

The following generalized definition of an entropy function is adapted from Kawashima¹³ and Kawashima and Shizuta.¹⁴

Definition 4.2. Consider a C^∞ function $\sigma^*(U^*, \varepsilon^*)$ defined over the open domain $\mathcal{O}_{(U^*, \varepsilon^*)}$ such that the slices $\mathcal{O}_{U^*}^{\varepsilon^*} = \{U^* \in \mathbb{R}^{n^*}; (U^*, \varepsilon^*) \in \mathcal{O}_{(U^*, \varepsilon^*)}\}$ are convex. The function σ^* is said to be an entropy function for the system (4.1) if the following properties hold.

- (E₁) The function σ^* is a strictly convex function of $U^* \in \mathcal{O}_{U^*}^{\varepsilon^*}$ in the sense that the Hessian matrix is positive definite in each slice $\mathcal{O}_{U^*}^{\varepsilon^*}$.
- (E₂) There exists real-valued C^∞ functions $q_i^* = q_i^*(U^*, \varepsilon^*)$ such that

$$(\partial_{U^*} \sigma^*) A_i^* = \partial_{U^*} q_i^*, \quad i \in C^*, \quad (U^*, \varepsilon^*) \in \mathcal{O}_{(U^*, \varepsilon^*)}.$$

- (E₃) We have the property that, for any $(U^*, \varepsilon^*) \in \mathcal{O}_{(U^*, \varepsilon^*)}$

$$(\partial_{U^*}^2 \sigma^*)^{-1} (B_{ij}^*)^\dagger = B_{ji}^* (\partial_{U^*}^2 \sigma^*)^{-1}, \quad i, j \in C^*.$$

- (E₄) The matrix $\tilde{B}^* = \sum_{i,j \in C^*} B_{ij}^*(U^*, \varepsilon^*) (\partial_{U^*}^2 \sigma^*(U^*, \varepsilon^*))^{-1} w_i w_j$ is symmetric positive semidefinite for any $(U^*, \varepsilon^*) \in \mathcal{O}_{(U^*, \varepsilon^*)}$ and $w \in \Sigma^{d-1}$.

Kawashima and Shizuta have established^{8,14} the equivalence between conservative symmetrizability and the existence of an entropy function. For systems depending on a parameter ε^* , some limitations on the domains $\mathcal{O}_{(U^*, \varepsilon^*)}$ seem necessary, like the smoothness of the slices $\varepsilon^* \rightarrow \mathcal{O}_{U^*}^{\varepsilon^*}$, using local charts. In order to avoid such technicalities, we only give a simplified version of an equivalence theorem, sufficient for our application to ambipolar plasmas.

Theorem 4.3. Assume that the system (4.1) admits an entropy function σ^* defined over $\mathcal{O}_{(U^*, \varepsilon^*)}$. Then, the system can be symmetrized over $\mathcal{O}_{(U^*, \varepsilon^*)}$ with the symmetrizing variable $V^* = (\partial_{U^*} \sigma^*)^\dagger$. Conversely, assume that the system can be symmetrized, and that, for the sake of simplicity, the open $\mathcal{O}_{(V^*, \varepsilon^*)}$ is in the form $\mathcal{O}_{(V^*, \varepsilon^*)} = \mathcal{O}_{V^*} \times \mathcal{O}_{\varepsilon^*}$ where $\mathcal{O}_{V^*} \subset \mathbb{R}^{n^*}$ is independent of ε^* and $\mathcal{O}_{\varepsilon^*} \subset \mathbb{R}^{m^*}$

independent of V^* . Then there exists an entropy defined on $\mathcal{O}_{(U^*, \varepsilon^*)}$ such that $V^* = (\partial_{U^*} \sigma^*)^\dagger$.

4.2. Normal form

We assume that the abstract quasilinear system (4.1) satisfies

(Edp₂) *The system (4.1) admits an entropy function σ^* on the open set $\mathcal{O}_{(U^*, \varepsilon^*)}$ and the slices $\mathcal{O}_{U^*}^{\varepsilon^*} = \{U^* \in \mathbb{R}^{n^*}; (U^*, \varepsilon^*) \in \mathcal{O}_{(U^*, \varepsilon^*)}\}$ are convex.*

Introducing the symmetrizing variable $V^* = (\partial_{U^*} \sigma^*)^\dagger$, the corresponding symmetric system (4.2) then satisfies Properties (S₁)–(S₄). However, depending on the range of the dissipation matrices \tilde{B}_{ij}^* , this system lies between the two limit cases of a hyperbolic system and a strongly parabolic system. In order to split the variables between hyperbolic and parabolic variables, we have to put the system into a normal form, in the form of a symmetric hyperbolic–parabolic composite system.

Introducing a new variable W^* , associated with a diffeomorphism from $\mathcal{O}_{(V^*, \varepsilon^*)}$ onto $\mathcal{O}_{(W^*, \varepsilon^*)}$, and multiplying the conservative symmetric form (4.2) on the left side by the transpose of the matrix $\partial_{W^*} V^*$, we then get a new system in the variable W^* and have the following definition of a normal form.¹⁴

Definition 4.4. Consider a system in symmetric form, as in Definition 4.1, and a diffeomorphism $(V^*, \varepsilon^*) \rightarrow (W^*, \varepsilon^*)$ from $\mathcal{O}_{(V^*, \varepsilon^*)}$ onto an open set $\mathcal{O}_{(W^*, \varepsilon^*)}$. The system in the new variable W^*

$$\begin{aligned} \bar{A}_0^*(W^*, \varepsilon^*) \partial_t W^* + \sum_{i \in C^*} \bar{A}_i^*(W^*, \varepsilon^*) \partial_i W^* &= \sum_{i, j \in C^*} \partial_i (\bar{B}_{ij}^*(W^*, \varepsilon^*) \partial_j W^*) \\ &+ \bar{T}^*(W^*, \varepsilon^*; \partial_x W^*) + \bar{\Omega}^*(W^*, \varepsilon^*), \end{aligned} \tag{4.4}$$

where

$$\begin{cases} \bar{A}_0^* = (\partial_{W^*} V^*)^\dagger \tilde{A}_0^* (\partial_{W^*} V^*), & \bar{B}_{ij}^* = (\partial_{W^*} V^*)^\dagger \tilde{B}_{ij}^* (\partial_{W^*} V^*), \\ \bar{A}_i^* = (\partial_{W^*} V^*)^\dagger \tilde{A}_i^* (\partial_{W^*} V^*), & \bar{\Omega}^* = (\partial_{W^*} V^*)^\dagger \tilde{\Omega}^*, \\ \bar{T}^* = - \sum_{i, j \in C^*} \partial_i (\partial_{W^*} V^*)^\dagger \tilde{B}_{ij}^* (\partial_{W^*} V^*) \partial_j W^*, \end{cases} \tag{4.5}$$

satisfies properties (S₁)–(S₄) rewritten in terms of overbar quantities. This system is then said to be of the normal form if there exists a partition of $\{1, \dots, n^*\}$ into $I = \{1, \dots, n_0^*\}$ and $II = \{n_0^* + 1, \dots, n^*\}$ such that the following properties hold.

(Nor₁) *The matrices \bar{A}_0^* and \bar{B}_{ij}^* have the block structure*

$$\bar{A}_0^* = \begin{pmatrix} \bar{A}_0^{*I,I} & 0 \\ 0 & \bar{A}_0^{*II,II} \end{pmatrix}, \quad \bar{B}_{ij}^* = \begin{pmatrix} 0 & 0 \\ 0 & \bar{B}_{ij}^{*II,II} \end{pmatrix}.$$

(Nor₂) The matrix $\overline{B}^{*\text{II,II}}(W^*, \varepsilon^*; w) = \sum_{i,j \in C^*} \overline{B}_{ij}^{*\text{II,II}}(W^*, \varepsilon^*) w_i w_j$ is positive definite, for $(W^*, \varepsilon^*) \in \mathcal{O}_{(W^*, \varepsilon^*)}$, and $w \in \Sigma^{d-1}$.

(Nor₃) Denoting $\partial_x = (\partial_1, \dots, \partial_d)^\dagger$, we have

$$\overline{T}^*(W^*, \varepsilon^*, \partial_x W^*) = (\overline{T}_I^*(W^*, \varepsilon^*, \partial_x W_{\text{I}}^*), \overline{T}_{\text{II}}^*(W^*, \varepsilon^*, \partial_x W^*))^\dagger,$$

where we have used the vector and matrix block structures induced by the partitioning of $\{1, \dots, n^*\}$ into $I = \{1, \dots, n_0^*\}$ and $\text{II} = \{n_0^* + 1, \dots, n^*\}$, so that we have $W^* = (W_{\text{I}}^*, W_{\text{II}}^*)^\dagger$, for instance.

A sufficient condition for system (4.2) to be recast into a normal form is that, for any fixed value of ε^* , the null space naturally associated with dissipation matrices is a fixed subspace of \mathbb{R}^{n^*} . This is Condition N introduced by Kawashima and Shizuta, which is now assumed to hold. We strengthen this condition by assuming that there exists a smooth explicit representation of this null space in terms of ε^* .

(Edp₃) The null space of the matrix

$$\tilde{B}^*(V^*, \varepsilon^*; w) = \sum_{i,j \in C^*} \tilde{B}_{ij}^*(V^*, \varepsilon^*) w_i w_j,$$

does not depend on V^* and $w \in \Sigma^{d-1}$, we denote by n_0^* its dimension $n_0^* = \dim(N(\tilde{B}^*))$, and we have $\tilde{B}_{ij}^*(V^*, \varepsilon^*) N(\tilde{B}^*) = 0$, $i, j \in C^*$. Furthermore, there exists a C^∞ application $\varepsilon^* \rightarrow P(\varepsilon^*)$ such that the first n_0^* columns of $P(\varepsilon^*)$ span the null space $N(\tilde{B}^*)$.

In order to characterize more easily normal forms for symmetric systems of conservation laws satisfying (Edp₁)–(Edp₃) we introduce the auxiliary variables^{8,11} $U^{*'} = P^\dagger U^*$ and $V^{*'} = P^{-1}V$. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right block of size $n^* - n_0^*$, where $n_0^* = \dim(N(\tilde{B}^*))$. Normal symmetric forms are then equivalently — and more easily — obtained from the $V^{*'}$ symmetric equation.^{8,11} A careful examination of the proof in Giovangigli and Massot reveals that the following theorem holds.

Theorem 4.5. Consider a system of conservation laws (4.2) that is symmetric in the sense of Definition 4.1 and assume that the null space invariance property (Edp₃) is satisfied. Denoting by $U^{*'} = P^\dagger U^*$ and $V^{*'} = P^{-1}V$, the usual auxiliary variable, any normal form of the system (4.2) is given by a change of variable in the form

$$W^* = (\phi_{\text{I}}(U_{\text{I}}^{*'}, \varepsilon^*), \phi_{\text{II}}(V_{\text{II}}^{*'}, \varepsilon^*))^\dagger,$$

where ϕ_{I} and ϕ_{II} are two diffeomorphisms of $\mathbb{R}^{n_0^*} \times \mathbb{R}^{m_0^*}$ and $\mathbb{R}^{n^* - n_0^*} \times \mathbb{R}^{m_0^*}$, respectively, and we have

$$\overline{T}^*(W^*, \varepsilon^*, \partial_x W^*) = (0, \overline{T}_{\text{II}}^*(W^*, \varepsilon^*, \partial_x W_{\text{II}}^*))^\dagger.$$

4.3. Local existence

In this section we investigate local existence of solutions around equilibrium states and the continuous dependence of solutions with respect to a parameter. We consider a system of conservation laws satisfying (Edp₁)–(Edp₃) and the additional property:

(Edp₄) *The system (4.1) admits an equilibrium point U^{*e} independent of ε^* .*

We will denote by V^{*e} and W^{*e} the equilibrium point in the V^* and W^* variables respectively. We assume for convenience that the domain $\mathcal{O}_{(W^*, \varepsilon^*)}$ contains a subset in the form $\mathcal{O}_{W^*} \times K_{\varepsilon^*}$ where \mathcal{O}_{W^*} is an open set of \mathbb{R}^{n^*} independent of ε^* , such that $W^{*e} \in \mathcal{O}_{W^*}$, and K_{ε^*} is a compact set of \mathbb{R}^{m^*} . In the following, we investigate the dependence of local solutions on the parameter ε^* . We will denote by $\|\bullet\|_l$ the norm in the Sobolev space $W_2^l(\mathbb{R}^d)$ and otherwise $\|\bullet\|_A$ in the functional space A .

Theorem 4.6. *Let $d \geq 1$ and $l \geq [d/2] + 2$ be integers and let $b > 0$ be given. Let \mathcal{O}_0 be given such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{W^*}$, d_1 such that $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{W^*})$, and define $\mathcal{O}_1 = \{W^* \in \mathcal{O}_{W^*}; d(W^*, \overline{\mathcal{O}}_0) < d_1\}$. Then there exists $\bar{\tau} > 0$ small enough, which only depend on \mathcal{O}_1 , b and K_{ε^*} , such that for any W^{*0} with $\|W^{*0} - W^{*e}\|_l < b$ and $W^{*0} \in \overline{\mathcal{O}}_0$, and any $\varepsilon^* \in K_{\varepsilon^*}$, there exists a unique local solution W^* to the system*

$$\overline{A}_0^* \partial_t W^* + \sum_{i \in \mathcal{C}^*} \overline{A}_i^* \partial_i W^* = \sum_{i,j \in \mathcal{C}^*} \partial_i (\overline{B}_{ij}^* \partial_j W^*) + \overline{T}^* + \overline{\Omega}^*,$$

with initial condition

$$W^*(0, x) = W^{*0}(x),$$

such that

$$W^*(t, x) \in \mathcal{O}_1$$

and

$$W_I^* - W_I^{*e} \in C^0([0, \bar{\tau}], W_2^l(\mathbb{R}^d)) \cap C^1([0, \bar{\tau}], W_2^{l-1}(\mathbb{R}^d)),$$

$$W_{II}^* - W_{II}^{*e} \in C^0([0, \bar{\tau}], W_2^l(\mathbb{R}^d)) \cap C^1([0, \bar{\tau}], W_2^{l-2}(\mathbb{R}^d)) \cap L^2((0, \bar{\tau}), W_2^{l+1}(\mathbb{R}^d)).$$

In addition, there exists $C > 0$ which only depend on \mathcal{O}_1 , b and K_{ε^*} , such that

$$\sup_{0 \leq \tau \leq \bar{\tau}} \|W^*(\tau) - W^{*e}\|_l^2 + \int_0^{\bar{\tau}} \|W_{II}^*(\tau) - W_{II}^{*e}\|_{l+1}^2 d\tau \leq C \|W^{*0} - W^{*e}\|_l^2. \tag{4.6}$$

Finally, if W^* is the solution corresponding to the initial state $W^{*0}(x)$ and parameter ε^* and \widehat{W}^* is the solution corresponding to the initial state $\widehat{W}^{*0}(x)$ and parameter $\widehat{\varepsilon}^*$, we have the estimate

$$\begin{aligned} & \sup_{0 \leq \tau \leq \bar{\tau}} \|W^*(\tau) - \widehat{W}^*(\tau)\|_{l-1}^2 + \int_0^{\bar{\tau}} \|W_{II}^*(\tau) - \widehat{W}_{II}^*(\tau)\|_l^2 d\tau \\ & \leq C (\|W^{*0} - \widehat{W}^{*0}\|_{l-1}^2 + \delta_{l-1}^2(\varepsilon^*, \widehat{\varepsilon}^*)), \end{aligned} \tag{4.7}$$

where $C > 0$ only depends \mathcal{O}_1 , b and K_{ε^*} , and where

$$\begin{aligned} \delta_{l-1}(\varepsilon^*, \hat{\varepsilon}^*) &= |\bar{A}_0^*(\cdot, \varepsilon^*) - \bar{A}_0^*(\cdot, \hat{\varepsilon}^*)|_{C^{l-1}(\bar{\mathcal{O}}_1)} + \sum_{i \in \mathcal{C}^*} |\bar{A}_i^*(\cdot, \varepsilon^*) - \bar{A}_i^*(\cdot, \hat{\varepsilon}^*)|_{C^{l-1}(\bar{\mathcal{O}}_1)} \\ &+ \sum_{i,j \in \mathcal{C}^*} |\bar{B}_{ij}^*(\cdot, \varepsilon^*) - \bar{B}_{ij}^*(\cdot, \hat{\varepsilon}^*)|_{C^{l-1}(\bar{\mathcal{O}}_1)} + |\bar{\Omega}^*(\cdot, \varepsilon^*) - \bar{\Omega}^*(\cdot, \hat{\varepsilon}^*)|_{C^{l-1}(\bar{\mathcal{O}}_1)}. \end{aligned}$$

Proof. Solutions to the nonlinear system (4.4) are fixed points $\tilde{W} = W$ of the linear equations¹³

$$\begin{cases} \bar{A}_0^{*I,I}(W^*, \varepsilon^*) \partial_t \tilde{W}_I^* + \sum_{i \in \mathcal{C}^*} \bar{A}_i^{*I,I}(W^*, \varepsilon^*) \partial_i \tilde{W}_I^* = f_I^*(W^*, \partial_x W_{II}^*, \varepsilon^*), \\ \bar{A}_0^{*II,II}(W^*, \varepsilon^*) \partial_t \tilde{W}_{II}^* - \sum_{i,j \in \mathcal{C}^*} \bar{B}_{ij}^{*II,II}(W^*, \varepsilon^*) \partial_i \partial_j \tilde{W}_{II}^* = f_{II}^*(W^*, \partial_x W^*, \varepsilon^*), \end{cases} \quad (4.8)$$

with

$$\begin{aligned} f_I^* &= - \sum_{i \in \mathcal{C}^*} \bar{A}_i^{*I,II}(W^*, \varepsilon^*) \partial_i W_{II}^* + \bar{\Omega}_I^*(W^*, \varepsilon^*), \\ f_{II}^* &= - \sum_{i \in \mathcal{C}^*} \bar{A}_i^{*II,II}(W^*, \varepsilon^*) \partial_i W_{II}^* - \sum_{i \in \mathcal{C}^*} \bar{A}_i^{*II,I}(W^*, \varepsilon^*) \partial_i W_I^* \\ &+ \bar{T}_{II}^*(W^*, \varepsilon^*, \partial_x W^*) + \bar{\Omega}_I^*(W^*, \varepsilon^*) + \sum_{i,j \in \mathcal{C}^*} \partial_i (\bar{B}_{ij}^{*II,II}) \partial_j W_{II}^*, \end{aligned}$$

which are hyperbolic in \tilde{W}_I^* and strongly parabolic in \tilde{W}_{II}^* . Fixed points are investigated in the space $W^* \in X_{\bar{\tau}}(\mathcal{O}_1, M, M_1)$ defined by $W_I^* - W_I^{*e} \in C^0([0, \bar{\tau}], W_2^l(\mathbb{R}^d))$, $\partial_t W_I^* \in C^0([0, \bar{\tau}], W_2^{l-1}(\mathbb{R}^d))$, $W_{II}^* - W_{II}^{*e} \in C^0([0, \bar{\tau}], W_2^l(\mathbb{R}^d)) \cap L^2((0, \bar{\tau}), W_2^{l+1}(\mathbb{R}^d))$, $\partial_t W_{II}^* \in C^0([0, \bar{\tau}], W_2^{l-2}(\mathbb{R}^d)) \cap L^2((0, \bar{\tau}), W_2^{l-1}(\mathbb{R}^d))$, $W^*(t, x) \in \mathcal{O}_1$,

$$\sup_{0 \leq \tau \leq \bar{\tau}} \|W^*(\tau) - W^{*e}\|_l^2 + \int_0^{\bar{\tau}} \|W_{II}^*(\tau) - W_{II}^{*e}\|_{l+1}^2 d\tau \leq M^2$$

and

$$\int_0^{\bar{\tau}} \|\partial_t W^*(\tau)\|_{l-1}^2 d\tau \leq M_1^2.$$

For W^* in $X_{\bar{\tau}}(\mathcal{O}_1, M, M_1)$, and $2 \leq k \leq l$, we have the estimates¹³

$$\begin{aligned} \|\tilde{W}^*(t) - W^{*e}\|_k^2 + \int_0^t \|\tilde{W}_{II}^*(\tau) - W_{II}^{*e}\|_{k+1}^2 d\tau &\leq C_1^2 \exp(C_2(t + M_1\sqrt{t})) \\ &\times \left(\|W^{*0} - W^{*e}\|_k^2 + C_2 t \int_0^t \|f_I^*(\tau)\|_k^2 d\tau + C_2 \int_0^t \|f_{II}^*(\tau)\|_{k-1}^2 d\tau \right), \end{aligned} \quad (4.9)$$

where $C_1 = C_1(\mathcal{O}_1, K_{\varepsilon^*})$ depends on \mathcal{O}_1 and K_{ε^*} and $C_2 = C_2(\mathcal{O}_1, M, K_{\varepsilon^*})$ depends on \mathcal{O}_1 , M and K_{ε^*} , and is an increasing function of M . From the classical estimates

$$\|f(\phi) - f(0)\|_l \leq C_0 \|f\|_{C^l(|\cdot| \leq \|\phi\|_{L^\infty})} (1 + \|\phi\|_{L^\infty})^{l-1} \|\phi\|_l,$$

where C_0 is a universal constant, we also obtain upper bounds in the form

$$\|f_I^*(t)\|_{l-1}^2 + \|f_{II}^*(t)\|_{l-1}^2 \leq C_2 M^2, \quad \int_0^t \|f_I^*(\tau)\|_l^2 d\tau \leq C_2(1+t)M^2. \tag{4.10}$$

From the governing equations, we also obtain

$$\int_0^{\bar{\tau}} \|\partial_t \tilde{W}^*(\tau)\|_{l-1}^2 d\tau \leq C_3^2(\tilde{M}^2 + t(M^2 + \tilde{M}^2)), \tag{4.11}$$

where \tilde{M} is defined for \tilde{W}^* as M for W^* and C_3 depends on \mathcal{O}_1 , M and K_{ε^*} , and is an increasing function of M . We now define for any $\alpha \in (0, b]$

$$M_\alpha = 2C_1(\mathcal{O}_1, K_{\varepsilon^*})\alpha, \quad M_{1\alpha} = 2C_3(\mathcal{O}_1, M_b, K_{\varepsilon^*})2C_1(\mathcal{O}_1, K_{\varepsilon^*})\alpha.$$

Then $\bar{\tau} \leq 3/2$ let small enough such that

$$\begin{aligned} \exp(C_2(\mathcal{O}_1, M_b, K_{\varepsilon^*})(\bar{\tau} + M_{1b}\sqrt{\bar{\tau}})) &\leq 2, \\ C_2^2(\mathcal{O}_1, M_b, K_{\varepsilon^*})\bar{\tau}(1 + \bar{\tau})(2C_1(\mathcal{O}_1, K_{\varepsilon^*}))^2 &\leq 1, \\ C_0 M_{1b}\sqrt{\bar{\tau}} &< d_1, \end{aligned}$$

where $\|\phi\|_{L^\infty} \leq C_0\|\phi\|_{l-1}$. Then, for any $\alpha \in (0, b]$, any $W^* \in X_{\bar{\tau}}(\mathcal{O}_1, M_\alpha, M_{1\alpha})$ any $W^{*0}(x)$, such that $W^{*0} - W^e \in W_2^l(\mathbb{R}^d)$, $W^{*0} \in \bar{\mathcal{O}}_0$, and $\|W^{*0} - W^{*e}\|_l < \alpha$, and any $\varepsilon^* \in K_{\varepsilon^*}$, the solution \tilde{W}^* to the linearized equations stays in the same space $X_{\bar{\tau}}(\mathcal{O}_1, M_\alpha, M_{1\alpha})$. More specifically, we obtain from (4.9) and (4.10) that

$$\tilde{M}^2 \leq 2C_1^2\alpha^2(1 + 4C_1^2C_2^2t(1 + 2t)) \leq 4C_1^2\alpha^2 = M_\alpha^2$$

and from (4.11) we deduce that $\tilde{M}_1^2 \leq C_3^2(2C_1\alpha)^2(1 + 2t) \leq M_{1\alpha}^2$, and finally that $\|\tilde{W}^* - W^{*e}\|_{L^\infty} \leq C_0 M_{1\alpha}\sqrt{\bar{\tau}} < d_1$.

In order to obtain fixed points, we establish that for $\bar{\tau}$ small enough, the map $W^* \rightarrow \tilde{W}^*$ is a contraction in all the spaces $X_{\bar{\tau}}(\mathcal{O}_1, M_\alpha, M_{1\alpha})$, $\alpha \in (0, b]$, and we establish simultaneously inequality (4.7). Let W^* and \hat{W}^* be in $X_{\bar{\tau}}(\mathcal{O}_1, M_b, M_{1b})$, let $W^{*0}(x)$ and $\hat{W}^{*0}(x)$ such that $W^{*0} - W^e \in W_2^l(\mathbb{R}^d)$, $\hat{W}^{*0} - W^e \in W_2^l(\mathbb{R}^d)$, $W^{*0}, \hat{W}^{*0} \in \bar{\mathcal{O}}_0$, $\|W^{*0} - W^{*e}\|_l < \alpha$, $\|\hat{W}^{*0} - W^{*e}\|_l < \alpha$, let $\varepsilon^*, \hat{\varepsilon}^* \in K_{\varepsilon^*}$, and define $\delta W^* = W^* - \hat{W}^*$ and $\delta \tilde{W}^* = \tilde{W}^* - \tilde{\hat{W}}^*$. Forming the difference between the linearized equations, we obtain that

$$\begin{cases} \bar{A}_0^{*I,I}(W^*, \varepsilon)\partial_t \delta \tilde{W}_I^* + \sum_{i \in \mathcal{C}^*} \bar{A}_i^{*I,I}(W^*, \varepsilon^*)\partial_i \delta \tilde{W}_I^* = \delta_{w^*} f_I^* + \delta_{\varepsilon^*} f_I^*, \\ \bar{A}_0^{*II,II}(W^*, \varepsilon^*)\partial_t \delta \tilde{W}_{II}^* - \sum_{i,j \in \mathcal{C}^*} \bar{B}_{ij}^{*II,II}(W^*, \varepsilon^*)\partial_i \partial_j \delta \tilde{W}_{II}^* = \delta_{w^*} f_{II}^* + \delta_{\varepsilon^*} f_{II}^*, \end{cases} \tag{4.12}$$

where

$$\begin{aligned} \delta_{\mathbf{w}^*} \mathbf{f}_{\mathbf{I}}^* &= \bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \mathbf{f}_{\mathbf{I}}^*(\mathbf{W}^*, \partial_x \mathbf{W}_{\mathbf{II}}^*, \varepsilon^*) - \mathbf{f}_{\mathbf{I}}^*(\widehat{\mathbf{W}}^*, \partial_x \widehat{\mathbf{W}}_{\mathbf{II}}^*, \varepsilon^*) \\ &\quad - \sum_{i \in \mathcal{C}^*} \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \bar{\mathbf{A}}_i^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) - \bar{\mathbf{A}}_i^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \partial_i \mathbf{W}_{\mathbf{I}}^*, \end{aligned}$$

$$\begin{aligned} \delta_{\mathbf{w}^*} \mathbf{f}_{\mathbf{II}}^* &= \bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \mathbf{f}_{\mathbf{II}}^*(\mathbf{W}^*, \partial_x \mathbf{W}^*, \varepsilon^*) - \mathbf{f}_{\mathbf{II}}^*(\widehat{\mathbf{W}}^*, \partial_x \widehat{\mathbf{W}}^*, \varepsilon^*) \\ &\quad + \sum_{i,j \in \mathcal{C}^*} \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \bar{\mathbf{B}}_{ij}^{\mathbf{II},\mathbf{II}}(\mathbf{W}^*, \varepsilon^*) - \bar{\mathbf{B}}_{ij}^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \partial_i \partial_j \mathbf{W}_{\mathbf{II}}^*, \end{aligned}$$

$$\begin{aligned} \delta_{\varepsilon^*} \mathbf{f}_{\mathbf{I}}^* &= \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) - \bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \mathbf{f}_{\mathbf{I}}^*(\mathbf{W}^*, \partial_x \mathbf{W}_{\mathbf{II}}^*, \varepsilon^*) \\ &\quad - \sum_{i \in \mathcal{C}} \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) - \bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \left(\bar{\mathbf{A}}_0^{\mathbf{I},\mathbf{I}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \bar{\mathbf{A}}_i^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \partial_i \mathbf{W}_{\mathbf{I}}^* \\ &\quad - \sum_{i \in \mathcal{C}} \left(\bar{\mathbf{A}}_i^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) - \bar{\mathbf{A}}_i^{\mathbf{I},\mathbf{I}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \partial_i \mathbf{W}_{\mathbf{I}}^* + \mathbf{f}_{\mathbf{I}}^*(\widehat{\mathbf{W}}^*, \partial_x \widehat{\mathbf{W}}_{\mathbf{II}}^*, \hat{\varepsilon}^*) \\ &\quad - \mathbf{f}_{\mathbf{I}}^*(\widehat{\mathbf{W}}^*, \partial_x \widehat{\mathbf{W}}_{\mathbf{II}}^*, \varepsilon^*) \end{aligned}$$

and

$$\begin{aligned} \delta_{\varepsilon^*} \mathbf{f}_{\mathbf{II}}^* &= \left(\bar{\mathbf{A}}_0^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) - \bar{\mathbf{A}}_0^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \left(\bar{\mathbf{A}}_0^{\mathbf{II},\mathbf{II}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \mathbf{f}_{\mathbf{II}}^*(\mathbf{W}^*, \partial_x \mathbf{W}^*, \varepsilon^*) \\ &\quad + \sum_{i,j \in \mathcal{C}} \left(\bar{\mathbf{A}}_0^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) - \bar{\mathbf{A}}_0^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \left(\bar{\mathbf{A}}_0^{\mathbf{II},\mathbf{II}}(\mathbf{W}^*, \varepsilon^*) \right)^{-1} \bar{\mathbf{B}}_{ij}^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \partial_i \partial_j \mathbf{W}_{\mathbf{II}}^* \\ &\quad + \sum_{i,j \in \mathcal{C}} \left(\bar{\mathbf{A}}_i^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) - \bar{\mathbf{A}}_i^{\mathbf{II},\mathbf{II}}(\widehat{\mathbf{W}}^*, \varepsilon^*) \right) \partial_i \partial_j \mathbf{W}_{\mathbf{II}}^* + \mathbf{f}_{\mathbf{II}}^*(\widehat{\mathbf{W}}^*, \partial_x \widehat{\mathbf{W}}^*, \hat{\varepsilon}^*) \\ &\quad - \mathbf{f}_{\mathbf{II}}^*(\widehat{\mathbf{W}}^*, \partial_x \widehat{\mathbf{W}}^*, \varepsilon^*). \end{aligned}$$

These expressions now imply that

$$\|\delta_{\mathbf{w}^*} \mathbf{f}_{\mathbf{I}}^*\|_{l-1}^2 + \|\delta_{\mathbf{w}^*} \mathbf{f}_{\mathbf{II}}^*\|_{l-2}^2 \leq C_4 (\|\delta \mathbf{W}_{\mathbf{I}}^*\|_{l-1}^2 + \|\delta \mathbf{W}_{\mathbf{II}}^*\|_l^2),$$

$$\|\delta_{\varepsilon^*} \mathbf{f}_{\mathbf{I}}^*\|_{l-1}^2 + \|\delta_{\varepsilon^*} \mathbf{f}_{\mathbf{II}}^*\|_{l-2}^2 \leq C_5 \delta_{l-1}^2(\varepsilon^*, \hat{\varepsilon}^*),$$

so that

$$\begin{aligned} \sup_{0 \leq \tau \leq \bar{\tau}} \|\delta \widetilde{\mathbf{W}}^*(\tau)\|_{l-1}^2 + \int_0^{\bar{\tau}} \|\delta \widetilde{\mathbf{W}}_{\mathbf{II}}^*(\tau)\|_l^2 d\tau &\leq C_6 (\|\mathbf{W}^{*0} - \widehat{\mathbf{W}}^{*0}\|_{l-1}^2 + \delta_{l-1}^2(\varepsilon^*, \hat{\varepsilon}^*)) \\ &\quad + C_7 t(1+t) \left(\sup_{0 \leq \tau \leq \bar{\tau}} \|\delta \mathbf{W}^*(\tau)\|_{l-1}^2 + \int_0^{\bar{\tau}} \|\delta \mathbf{W}_{\mathbf{II}}^*(\tau)\|_l^2 d\tau \right), \end{aligned} \tag{4.13}$$

where all constants C_4, C_5, C_6, C_7 , depend on \mathcal{O}_1, b and K_{ε^*} . Now if $\bar{\tau}$ is small enough so that $C_7\bar{\tau}(1+\bar{\tau}) < 1/2$, by letting $W^{*0} = \widetilde{W}^{*0}$ and $\varepsilon^* = \hat{\varepsilon}^*$, we obtain, that the map $W^* \rightarrow \widetilde{W}^*$ is a contraction in all the spaces $X_{\bar{\tau}}(\mathcal{O}_1, M_\alpha, M_{1\alpha})$, $\alpha \in (0, b]$. Introducing the iterates W^{*n} starting at the initial condition W^{*0} and such that $W^{*(n+1)} = \widetilde{W}^{*n}$, that is, $W^{*(n+1)}$ is obtained as the solution of linearized equations, then the sequence $\{W^{*n}\}_{n \geq 0}$ is easily shown to be convergent to a local solution of the nonlinear equations satisfying the estimates (4.6) at order $l - 1$. Inequality (4.7) is then obtained by letting $\delta\widetilde{W}^* = \delta W^*$ in (4.13). Finally, the estimates (4.6) at order l are recovered since for any $\alpha \in (0, b]$, the space $X_{\bar{\tau}}(\mathcal{O}_1, M_\alpha, M_{1\alpha})$ is invariant, and the proof is complete. \square

5. Global Existence and Asymptotic Stability for an Abstract System

In this section we investigate asymptotic stability of equilibrium states for an abstract system of conservation law in normal form and the continuous dependence of solutions with respect to a parameter. We consider a system of conservation laws satisfying (Edp₁)–(Edp₄) and assume for convenience either that the domain $\mathcal{O}_{(W^*, \varepsilon^*)}$ contains a subset in the form $\mathcal{O}_{W^*} \times K_{\varepsilon^*}$, where \mathcal{O}_{W^*} is an open set of \mathbb{R}^{n^*} independent of ε^* and K_{ε^*} a compact set of \mathbb{R}^{m^*} independent of W^* , or that there is a smooth extension of the system coefficients to such a domain.

5.1. Local dissipativity

If we linearize system (4.4) around the constant stationary state W^{*e} , we obtain a linear system in the variable $w^* = W^* - W^{*e}$

$$\bar{A}_0^*(W^{*e}, \varepsilon^*)\partial_t w^* + \sum_{i \in C} \bar{A}_i^*(W^{*e}, \varepsilon^*)\partial_i w^* = \sum_{i, j \in C^*} \bar{B}_{ij}^*(W^{*e}, \varepsilon^*)\partial_i \partial_j w^* - \bar{\Gamma}^*(W^{*e}, \varepsilon^*)w^*,$$

where $\bar{\Gamma}^*$ is defined by $\bar{\Gamma}^* = -\partial_{W^*}\bar{\Omega}^*$. By Fourier transform, the spectral problem associated with this linear system reads

$$\lambda \bar{A}_0^*(W^{*e}, \varepsilon^*)\phi + (\zeta \bar{A}^*(W^{*e}, w, \varepsilon^*) - \zeta^2 \bar{B}^*(W^{*e}, w, \varepsilon^*) + \bar{\Gamma}^*(W^{*e}, \varepsilon^*))\phi = 0, \tag{5.1}$$

where $\zeta \in i\mathbb{R}$, $i^2 = -1$, $w \in \Sigma^{d-1}$, $\varepsilon^* \in K_{\varepsilon^*}$, $\bar{A}^*(W^{*e}, w, \varepsilon^*) = \sum_{i \in C^*} \bar{A}_i^*(W^{*e}, \varepsilon^*)w_i$, and $\bar{B}^*(W^{*e}, w, \varepsilon^*) = \sum_{i, j \in C^*} \bar{B}_{ij}^*(W^{*e}, \varepsilon^*)w_i w_j$. We will denote by $\Lambda(\zeta, w, \varepsilon^*)$ the complex numbers λ such that there exists $\phi \in \mathbb{C}^{n^*}$, $\phi \neq 0$, satisfying (5.1).

The results of Shizuta and Kawashima¹⁸ can directly be generalized to parameter dependent situations. The smoothness of compensating matrices is indeed a consequence of their explicit representation using matrix operational calculus.¹⁸

Theorem 5.1. *The following properties are equivalent*

(Spe₁) *There exists a compensating matrix K defined and C^∞ over $\Sigma^{d-1} \times K_{\varepsilon^*}$. For any $w \in \Sigma^{d-1}$ and any $\varepsilon^* \in K_{\varepsilon^*}$, the matrix $K(w, \varepsilon^*)$ is real, the product*

$K(w, \varepsilon^*) \bar{A}_0^*(W^{*e}, \varepsilon^*)$ is skew-symmetric, $K(-w, \varepsilon^*) = -K(w, \varepsilon^*)$, and the matrix

$$K(w, \varepsilon^*) \bar{A}^*(W^{*e}, w, \varepsilon^*) + \bar{B}^*(W^{*e}, w, \varepsilon^*) + \bar{L}^*(W^{*e}, \varepsilon^*),$$

is positive definite.

- (Spe₂) Let $\zeta \in i\mathbb{R}$, $\zeta \neq 0$, $w \in \Sigma^{d-1}$, and $\varepsilon^* \in K_{\varepsilon^*}$. Then all eigenvalues λ of $\Lambda(\zeta, w, \varepsilon^*)$ have a negative real part.
- (Spe₃) Let $\Psi \in \mathbb{R}^{n^*} \setminus \{0\}$ such that $\bar{B}^*(W^{*e}, w, \varepsilon^*)\Psi = \bar{L}^*(W^{*e}, \varepsilon^*)\Psi = 0$ for some $w \in \Sigma^{d-1}$, $\varepsilon^* \in K_{\varepsilon^*}$. Then we have $\zeta \bar{A}_0^*(W^{*e}, \varepsilon^*)\Psi + \bar{A}^*(W^{*e}, w, \varepsilon^*)\Psi \neq 0$ for any $\zeta \in \mathbb{R}$.
- (Spe₄) There exists $\delta > 0$ such that for any $\zeta \in i\mathbb{R}$, $w \in \Sigma^{d-1}$, $\varepsilon^* \in K_{\varepsilon^*}$, and any eigenvalue λ of $\Lambda(\zeta, w, \varepsilon^*)$, we have

$$\Re(\lambda) \leq -\delta \frac{|\zeta|^2}{1 + |\zeta|^2}.$$

Remark 5.1. It is not known if the matrix $K(w, \varepsilon^*)$ is of the form $\sum_{j \in \mathcal{C}} K^j(\varepsilon^*)w_j$. Nevertheless, in practical applications, it is generally possible to obtain compensating matrices in this form.

5.2. Global existence and asymptotic stability

We now investigate the existence of solutions globally in time around equilibrium states. We assume that the system is strictly dissipative in the sense of (5.1) and the source term is dissipative in the following sense.¹¹

- (Dis₁) For any $\varepsilon^* \in K_{\varepsilon^*}$, the matrix $\bar{A}_0^*(W^{*e}, \varepsilon^*)$ is symmetric positive definite, the matrices $\bar{A}_i^*(W^{*e}, \varepsilon^*)$, $i \in \mathcal{C}$, are symmetric, we have the reciprocity relations $\bar{B}_{i,j}^*(W^{*e}, \varepsilon^*)^\dagger = \bar{B}_{j,i}^*(W^{*e}, \varepsilon^*)$, $i, j \in \mathcal{C}$, and the matrix $\bar{L}^*(W^{*e}, \varepsilon^*)$ is symmetric positive semidefinite.
- (Dis₂) The linearized system is strictly dissipative in the sense of Theorem 5.1.
- (Dis₃) The smallest linear subspace containing the source term vector $\tilde{\Omega}^*(V^*, \varepsilon^*)$, for all $(V^*, \varepsilon^*) \in \mathcal{O}_{(V^*, \varepsilon^*)}$, is included in the range of $\tilde{L}^*(V^*, \varepsilon^*)$, with $\tilde{L}^* = (\partial_{V^*} W^*)^\dagger \bar{L}^*(V^*, \varepsilon^*) \partial_{V^*} W^*$.
- (Dis₄) For any $\varepsilon^* \in K_{\varepsilon^*}$, there exists a neighborhood of (V^*, ε^*) , in $\mathcal{O}_{(V^*, \varepsilon^*)}$, and a positive constant $\delta > 0$ such that, for any (V^*, ε^*) in this neighborhood, we have

$$\delta |\tilde{\Omega}^*(V^*, \varepsilon^*)|^2 \leq -\langle V^* - V^{*e}, \tilde{\Omega}^*(V^*, \varepsilon^*) \rangle.$$

Theorem 5.2. Let $d \geq 1$ and $l \geq [d/2] + 2$ be integers and consider the system (4.4). Then there exists $b > 0$ small enough such that if W^{*0} satisfies $\|W^{*0} - W^{*e}\|_l < b$, there exists a unique global solution W^* for any $\varepsilon^* \in K_{\varepsilon^*}$ to the Cauchy problem

$$\bar{A}_0^* \partial_t W^* + \sum_{i \in \mathcal{C}^*} \bar{A}_i^* \partial_i W^* = \sum_{i, j \in \mathcal{C}^*} \partial_i (\bar{B}_{ij}^* \partial_j W^*) + \bar{T}^* + \bar{\Omega}^*,$$

with initial condition

$$W^*(0, x) = W^{*0}(x)$$

and

$$W_I^* - W_I^{*e} \in C^0([0, \infty), W_2^l(\mathbb{R}^d)) \cap C^1([0, \infty), W_2^{l-1}(\mathbb{R}^d)) \cap L^2((0, \infty), W_2^l(\mathbb{R}^d)),$$

$$W_{II}^* - W_{II}^{*e} \in C^0([0, \infty), W_2^l(\mathbb{R}^d)) \cap C^1([0, \infty), W_2^{l-2}(\mathbb{R}^d)) \cap L^2((0, \infty), W_2^{l+1}(\mathbb{R}^d)).$$

Furthermore, W^* satisfies the estimate

$$\|W^*(t) - W^{*e}\|_l^2 + \int_0^t (\|\partial_x W_I^*(\tau)\|_{l-1}^2 + \|\partial_x W_{II}^*(\tau)\|_l^2) d\tau \leq C \|W^{*0} - W^{*e}\|_l^2,$$

uniformly in ε^* where C is a positive constant and $\sup_{x \in \mathbb{R}^3} |W^*(t) - W^e|$ goes to zero as $t \rightarrow \infty$ uniformly in ε^* . Finally, emphasizing the dependence on ε^* by denoting $W^*(t, x, \varepsilon^*)$ the solution obtained for $\varepsilon^* \in K_{\varepsilon^*}$ we have for any $\alpha^* \in K_{\varepsilon^*}$

$$\lim_{\substack{\varepsilon^* \rightarrow \alpha^* \\ \varepsilon^* \in K_{\varepsilon^*}}} \sup_{t \geq 0} \|W^*(t, \cdot, \varepsilon^*) - W^*(t, \cdot, \alpha^*)\|_{C^{l-(d/2)+2}} = 0.$$

The main idea is that all usual estimates can be made uniform with respect to the parameter ε^* since we are considering a compact set K_{ε^*} . Thanks to the local existence theorem and to uniform estimates, global solutions are obtained for all $\varepsilon^* \in K_{\varepsilon^*}$. Continuity with respect to the parameter ε^* is then a consequence of the continuity over finite time interval and of uniform asymptotic stability. We define for convenience $N_l(t) = N_l(0, t)$ where

$$N_l(t_1, t_2)^2 = \sup_{t_1 \leq \tau \leq t_2} \|W^*(\tau) - W^{*e}\|_l^2 + \int_{t_1}^{t_2} (\|\partial_x W_I^*(t)\|_{l-1}^2 + \|\partial_x W_{II}^*(t)\|_l^2) dt.$$

Lemma 5.1. *Let $\bar{\sigma}^*$ denotes the modified entropy*

$$\bar{\sigma}^*(W^*, \varepsilon^*) = \sigma^*(W^*, \varepsilon^*) - \sigma^*(W^{*e}, \varepsilon^*) - (\partial_{U^*} \sigma^*(W^{*e}, \varepsilon^*))(W^* - W^{*e}).$$

There exists a neighborhood \mathfrak{B} of W^{*e} and constants \underline{c} and \bar{c} such that

$$\forall W^* \in \mathfrak{B} \quad \forall \varepsilon^* \in K_{\varepsilon^*}, \quad \underline{c} |W^* - W^{*e}|^2 \leq \bar{\sigma}^*(W^*, \varepsilon^*) \leq \bar{c} |W^* - W^{*e}|^2.$$

Lemma 5.2. *Let $d \geq 2, l \geq [d/2] + 1$ and \mathfrak{B} a bounded neighborhood of W^{*e} . There exists a constant $\beta_0(\mathfrak{B})$ independent of ε^* such that*

$$\forall \varepsilon^* \in K_{\varepsilon^*}, \quad N_l(\tau) \leq \beta_0(\mathfrak{B}) \Rightarrow W^* \in \mathfrak{B}, \quad (t, x) \in [0, \tau] \times \mathbb{R}^d.$$

Proposition 5.1. *Let $d \geq 2, l \geq [d/2] + 2$ and assume that $W^{*0}(x)$ is such that $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d)$. Assume that W^* is a solution over $[0, \tau]$ such that*

$$W_I^* - W_I^{*e} \in C^0([0, \tau], W_2^l(\mathbb{R}^d)) \cap C^1([0, \tau], W_2^{l-1}(\mathbb{R}^d)),$$

$$W_{II}^* - W_{II}^{*e} \in C^0([0, \tau], W_2^l(\mathbb{R}^d)) \cap C^1([0, \tau], W_2^{l-2}(\mathbb{R}^d)) \cap L^2((0, \tau), W_2^{l+1}(\mathbb{R}^d)),$$

and that $N_l(\tau) \leq \beta_0(\mathfrak{B})$. There exists constants $b' \leq \beta_0(\mathfrak{B})$ and $C' > 1$ independent of ε^* such that

$$N_l(\tau) \leq b' \Rightarrow N_l(\tau) \leq C' \|W^{*0} - W^{*e}\|_l.$$

Proof. The proofs of Lemmas 5.1, 5.2 and Proposition 5.1 are similar to the situation without parameter,^{11,13} thanks to the compactity of the set K_{ε^*} .

We now apply the local existence Theorem 4.6 with $\mathcal{O}_0 = \mathfrak{B}$, any $d_1 > 0$ such that $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{W^*})$, and $b_0 = \beta_0(\mathfrak{B})$. There exists a local solution defined over $[0, \bar{\tau}]$ for any $\varepsilon^* \in K_{\varepsilon^*}$, whenever $\|W^{*0} - W^{*e}\|_l < b_0$, and from Theorem 4.6 we also have estimates in the form

$$N_l(\bar{\tau}) \leq C\|W^{*0} - W^{*e}\|_l,$$

where $C > 1$ depends on \mathcal{O}_1 , $b_0 = \beta_0(\mathfrak{B})$ and K_{ε^*} . Let then

$$\bar{b} = \inf \left(\frac{b'}{C}, \frac{b'}{C'(1+C^2)^{1/2}} \right),$$

where b' and C' are given by Proposition 5.1 and assume that $\|W^{*0} - W^{*e}\|_l \leq \bar{b}$. For any $\varepsilon^* \in K_{\varepsilon^*}$, we first have a solution defined on $[0, \bar{\tau}]$ such that

$$N_l(\bar{\tau}) \leq C\|W^{*0} - W^{*e}\|_l \leq C\bar{b} \leq b' \leq b_0 = \beta_0(\mathfrak{B}).$$

Since $N_l(\bar{\tau}) \leq b'$, we also have $N_l(\bar{\tau}) \leq C'\|W^{*0} - W^{*e}\|_l \leq C'\bar{b}$. We can now start again from $W^*(\bar{\tau})$ at $\bar{\tau}$ since $\|W^*(\bar{\tau}) - W^{*e}\|_l \leq N_l(\bar{\tau}) \leq b_0$ and we have a solution defined on $[\bar{\tau}, 2\bar{\tau}]$ with $N_l(\bar{\tau}, 2\bar{\tau}) \leq CN_l(\bar{\tau})$. As a consequence, we obtain that

$$N_l(2\bar{\tau}) \leq (1+C^2)^{1/2}N_l(\bar{\tau}) \leq (1+C^2)^{1/2}C'\bar{b} \leq b' \leq b_0,$$

so that from Proposition 5.1 with $\tau = 2\bar{\tau}$ we obtain

$$N_l(2\bar{\tau}) \leq C'\bar{b} \leq b' \leq b_0.$$

We can start again from $W^*(2\bar{\tau})$ at $2\bar{\tau}$ and an easy induction shows that the solution is defined for all time and that for any $t \geq 0$ we have $N_l(t) \leq C'\|W^{*0} - W^{*e}\|_l$ uniformly for $\varepsilon^* \in K_{\varepsilon^*}$.

We emphasize now the dependence on ε^* by denoting $W^*(t, x, \varepsilon^*)$ the solution obtained for $\varepsilon^* \in K_{\varepsilon^*}$. We introduce $\Phi(t, \varepsilon^*) = \|\partial_x W^*(t, \cdot, \varepsilon^*)\|_{l-2}^2$ and it is easily established that for any $\varepsilon^* \in K_{\varepsilon^*}$

$$\int_0^\infty |\Phi(t, \varepsilon^*)| dt + \int_0^\infty |\partial_t \Phi(t, \varepsilon^*)| dt \leq C\|W^{*0} - W^{*e}\|_l^2,$$

where C is independent ε^* so that $\lim_{t \rightarrow \infty} \|\partial_x W^*(t, \cdot, \varepsilon^*)\|_{l-2} = 0$ uniformly in $\varepsilon^* \in K_{\varepsilon^*}$. Let then $\alpha^* \in K_{\varepsilon^*}$ and let $a > 0$ be given. From these estimates, we can find a time τ_a such that $\|\partial_x W^*(t, \cdot, \varepsilon^*)\|_{l-2} \leq a/2$ for $t \geq \tau_a$ and $\varepsilon^* \in K_{\varepsilon^*}$. This implies that $\|\partial_x(W^*(t, \cdot, \varepsilon^*) - W^*(t, \cdot, \alpha^*))\|_{l-2} \leq a$ for any $t \geq \tau_a$ and any $\varepsilon^*, \alpha^* \in K_{\varepsilon^*}$. On the other hand, we have $(I_a - 1)\bar{\tau} < \tau_a \leq I_a\bar{\tau}$ for I_a large enough and we can divide the time interval $[0, I_a\bar{\tau}]$ into the union of intervals in the form $[i\bar{\tau}, (i+1)\bar{\tau}]$, for $i = 0, I_a - 1$. We can now apply the estimates (4.7) to deduce that

$$\sup_{0 \leq \tau \leq I_a\bar{\tau}} \|W^*(\tau, \cdot, \varepsilon^*) - W^*(\tau, \cdot, \alpha^*)\|_{l-1} \leq (1+C)^{I_a} \delta_{l-1}(\varepsilon^*, \alpha^*) = 0,$$

so that as $\varepsilon^* \rightarrow \alpha^*$ in K_{ε^*} , $W^*(t, \cdot, \varepsilon^*)$ converges uniformly in $t \in [0, I_a \bar{r}]$ to $W^*(t, \cdot, \alpha^*)$ in the W_2^{l-1} norm. We have thus established that

$$\lim_{\substack{\varepsilon^* \rightarrow \alpha^* \\ \varepsilon^* \in K_{\varepsilon^*}}} \sup_{t \geq 0} \|\partial_x(W^*(t, \cdot, \varepsilon^*) - W^*(t, \cdot, \alpha^*))\|_{l-2} = 0,$$

and using the interpolation inequality $\|\phi\|_{C^{l-(\lfloor d/2 \rfloor + 2)}} \leq C_0 \|\partial_x^{l-1} \phi\|_0^a \|\phi\|_0^{1-a}$, we conclude that $\lim_{\varepsilon^* \rightarrow \alpha^*} \sup_{t \geq 0} \|W^*(t, \cdot, \varepsilon^*) - W^*(t, \cdot, \alpha^*)\|_{C^{l-(\lfloor d/2 \rfloor + 2)}} = 0$. \square

5.3. Decay estimates

Uniform decay estimates can be obtained, thanks to the compacity of K_{ε^*} . These estimates can then be used to improve the the continuous dependence on the parameter by using the space W_2^{l-1} instead of $C^{l-(\lfloor d/2 \rfloor + 2)}$.

Theorem 5.3. *Let $d \geq 2$, $l \geq \lfloor d/2 \rfloor + 3$ and $W^{*0}(x)$ be given, such that*

$$W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d),$$

*with $p \in [1, 2)$. Then, if $\|W^{*0} - W^{*e}\|_l$ and $\|W^{*0} - W^{*e}\|_{L^p}$ are small enough, the unique global solution to the Cauchy problem satisfies the decay estimate*

$$\|W^*(t) - W^{*e}\|_{l-2} \leq C(1+t)^{-\gamma} (\|W^{*0} - W^{*e}\|_{l-2} + \|W^{*0} - W^{*e}\|_{L^p}), \quad 0 \leq t,$$

uniformly in $\varepsilon^ \in K_{\varepsilon^*}$, where C is a positive constant and $\gamma = d(1/2p - 1/4)$. Finally, for any $\alpha^* \in K_{\varepsilon^*}$ we have*

$$\lim_{\substack{\varepsilon^* \rightarrow \alpha^* \\ \varepsilon^* \in K_{\varepsilon^*}}} \sup_{t \geq 0} \|W^*(t, \cdot, \varepsilon^*) - W^*(t, \cdot, \alpha^*)\|_{l-1} = 0.$$

Proof. The proof of decay estimates is similar to the case without parameter,^{11,13} thanks to the compacity of the set K_{ε^*} . These estimates combined with those of Theorem 5.2, implies that

$$\lim_{t \rightarrow \infty} \|W^*(t, \cdot, \varepsilon^*) - W^{*e}\|_{l-1} = 0,$$

uniformly for $\varepsilon^* \in K_{\varepsilon^*}$, and we can proceed as in the proof of Theorem 5.2. \square

Remark 5.2. Decay estimates can also be obtained uniformly for $d = 1$ provided that estimates¹³ about the exponential of $(\bar{A}_0^*)^{-1/2}(\zeta \bar{A}^* - \zeta^2 \bar{B}^* + \bar{\Gamma}^*)(\bar{A}_0^*)^{-1/2}$ at (W^{*e}, ε^*) can be obtained around $\zeta = 0$ uniformly in ε^* .

6. Symmetrization for Ambipolar Plasmas

We investigate in this section symmetric forms for the system of partial differential equations modeling ambipolar plasmas (3.5).

6.1. Entropy and symmetric conservative form

We define the mathematical entropy σ by

$$\sigma = - \sum_{k \in \mathfrak{S}} \frac{\gamma_k S_k}{R}, \tag{6.1}$$

where the $1/R$ factor is introduced for convenience, and the corresponding entropic variables V reads

$$V = (\partial_U \sigma)^t = \frac{1}{RT} \left(G_1 - \frac{1}{2} m_1 \mathbf{v} \cdot \mathbf{v}, \dots, G_{n^s} - \frac{1}{2} m_{n^s} \mathbf{v} \cdot \mathbf{v}, v_1, v_2, v_3, -1 \right)^t. \tag{6.2}$$

Theorem 6.1. *The function σ is a mathematical entropy for the system (3.5). The map $(U, \varepsilon) \rightarrow (V, \varepsilon)$ is a C^∞ diffeomorphism from $\mathcal{O}_{(U, \varepsilon)}$ onto $\mathcal{O}_{(V, \varepsilon)} = \mathcal{O}_V \times \mathcal{O}_\varepsilon$ where $\mathcal{O}_V = \mathbb{R}^{n^s+3} \times (-\infty, 0)$ is independent of ε and $\mathcal{O}_\varepsilon = (0, \bar{\varepsilon})$ is independent of V . In addition, this diffeomorphism admits a smooth extension up to $\varepsilon = 0$ and $\varepsilon = \bar{\varepsilon}$. The system written in terms of the entropic variable V*

$$\tilde{A}_0 \partial_t V + \sum_{i \in \mathcal{C}} \tilde{A}_i \partial_i V = \sum_{i, j \in \mathcal{C}} \partial_i (\tilde{B}_{ij} \partial_j V) + \tilde{\Omega},$$

with

$$\tilde{A}_0 = \partial_V U, \quad \tilde{A}_i = A_i \partial_V U, \quad \tilde{B}_{ij} = B_{ij} \partial_V U \quad \text{and} \quad \tilde{\Omega} = \Omega$$

is of the symmetric form, i.e. the matrices $\tilde{A}_0, \tilde{A}_i, i \in \mathcal{C}$ and $\tilde{B}_{ij}, i, j \in \mathcal{C}$, verify properties (S₁)–(S₄). The matrix \tilde{A}_0 is given by

$$\tilde{A}_0 = \begin{bmatrix} (\gamma_k \delta_{kl})_{k, l \in \mathfrak{S}} & & \text{Sym} \\ (\gamma_l m_l v_i)_{i \in \mathcal{C}, l \in \mathfrak{S}} & (\rho RT \delta_{ij} + \Sigma_{m^2} v_i v_j)_{i \in \mathcal{C}, j \in \mathcal{C}} & \\ (\gamma_l E_l^{\text{tot}})_{l \in \mathfrak{S}} & (\rho RT v_j + \Sigma_{me} v_j)_{j \in \mathcal{C}} & \Upsilon_e \end{bmatrix},$$

where

$$\Sigma_{m^2} = \sum_{k \in \mathfrak{S}} \gamma_k m_k^2, \quad \Sigma_{me} = \sum_{k \in \mathfrak{S}} \gamma_k m_k E_k^{\text{tot}},$$

$$\Upsilon_e = \sum_{k \in \mathfrak{S}} \gamma_k (E_k^{\text{tot}})^2 + RT(\rho \mathbf{v} \cdot \mathbf{v} + C_v T).$$

Since this matrix is symmetric, we only give its left lower triangular part and write “Sym” in the upper triangular part. Denoting by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^t$ an arbitrary vector

of \mathbb{R}^3 , the matrices $\tilde{\mathbf{A}}_i$, $i \in \mathcal{C}$, are given by

$$\sum_{i \in \mathcal{C}} \xi_i \tilde{\mathbf{A}}_i = \begin{bmatrix} (\gamma_k \delta_{kl} \mathbf{v} \cdot \boldsymbol{\xi})_{k,l \in \mathfrak{S}} & \text{Sym} \\ (\gamma_l m_l v_i \mathbf{v} \cdot \boldsymbol{\xi} + \gamma_l RT \xi_i)_{i \in \mathcal{C}, l \in \mathfrak{S}} & \Sigma_{\gamma, v} \\ (\gamma_l H_l^{\text{tot}} \mathbf{v} \cdot \boldsymbol{\xi})_{l \in \mathfrak{S}} & \Sigma_{h, v} \quad \Upsilon_h \mathbf{v} \cdot \boldsymbol{\xi} \end{bmatrix},$$

with

$$\begin{aligned} \Sigma_{mh} &= \sum_{k \in \mathfrak{S}} \gamma_k m_k H_k^{\text{tot}} \quad \Upsilon_h = \sum_{k \in \mathfrak{S}} \gamma_k (H_k^{\text{tot}})^2 + RT(\rho \mathbf{v} \cdot \mathbf{v} + C_p T). \\ \Sigma_{\gamma, v} &= \Sigma_{m^2} \mathbf{v} \otimes \mathbf{v} \mathbf{v} \cdot \boldsymbol{\xi} + \rho RT(\mathbf{v} \cdot \boldsymbol{\xi} \mathbb{I}_{3,3} + \mathbf{v} \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes \mathbf{v}) \\ \Sigma_{h, v} &= (\Sigma_{mh} + \rho RT) \mathbf{v} \cdot \boldsymbol{\xi} \mathbf{v}^t + RTH^{\text{tot}} \boldsymbol{\xi}^t. \end{aligned}$$

Moreover, we have the decomposition

$$\tilde{\mathbf{B}}_{ij} = \delta_{ij} RT \tilde{\mathbf{B}}^D + \kappa RT \tilde{\mathbf{B}}_{ij}^\kappa + \eta RT \tilde{\mathbf{B}}_{ij}^\eta,$$

with

$$\tilde{\mathbf{B}}^D = \frac{1}{p} \begin{bmatrix} (\hat{D}_{kl} \gamma_k \gamma_l)_{k,l \in \mathfrak{S}} & \text{Sym} \\ 0_{3, n^s} & 0_{3,3} \\ \left(\sum_{k \in \mathfrak{S}} \gamma_l \hat{D}_{kl} (p \chi_k + \gamma_k H_k) \right)_{l \in \mathfrak{S}} & 0_{1,3} \quad \Upsilon_D \end{bmatrix},$$

where

$$\Upsilon_D = \lambda p T + \sum_{k,l \in \mathfrak{S}} \hat{D}_{kl} (p \chi_k + \gamma_k H_k) (p \chi_l + \gamma_l H_l),$$

and denoting by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^t$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^t$ arbitrary vectors of \mathbb{R}^3 , the matrices $\tilde{\mathbf{B}}_{ij}^\kappa$ and $\tilde{\mathbf{B}}_{ij}^\eta$, $i, j \in \mathcal{C}$, are given by

$$\sum_{i,j \in \mathcal{C}} \xi_i \zeta_j \tilde{\mathbf{B}}_{ij}^\kappa = \begin{bmatrix} 0_{n^s, n^s} & 0_{n^s, 3} & 0_{n^s, 1} \\ 0_{3, n^s} & \boldsymbol{\xi} \otimes \boldsymbol{\zeta} & \mathbf{v} \cdot \boldsymbol{\xi} \boldsymbol{\zeta} \\ 0_{1, n^s} & \mathbf{v} \cdot \boldsymbol{\xi} \boldsymbol{\zeta}^t & \mathbf{v} \cdot \boldsymbol{\xi} \mathbf{v} \cdot \boldsymbol{\zeta} \end{bmatrix},$$

and

$$\sum_{i,j \in \mathcal{C}} \xi_i \zeta_j \tilde{\mathbf{B}}_{ij}^\eta = \begin{bmatrix} 0_{n^s, n^s} & 0_{n^s, 3} & 0_{n^s, 1} \\ 0_{3, n^s} & \boldsymbol{\xi} \cdot \boldsymbol{\zeta} \mathbb{I}_{3,3} + \boldsymbol{\zeta} \otimes \boldsymbol{\xi} - \frac{2}{3} \boldsymbol{\xi} \otimes \boldsymbol{\zeta} & \boldsymbol{\xi} \cdot \boldsymbol{\zeta} \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\zeta} \boldsymbol{\xi} - \frac{2}{3} \mathbf{v} \cdot \boldsymbol{\xi} \boldsymbol{\zeta} \\ 0_{1, n^s} & \boldsymbol{\xi} \cdot \boldsymbol{\zeta} \mathbf{v}^t + \mathbf{v} \cdot \boldsymbol{\zeta} \boldsymbol{\xi}^t - \frac{2}{3} \mathbf{v} \cdot \boldsymbol{\xi} \boldsymbol{\zeta}^t & \boldsymbol{\xi} \cdot \boldsymbol{\zeta} \mathbf{v} \cdot \mathbf{v} + \frac{1}{3} \mathbf{v} \cdot \boldsymbol{\xi} \mathbf{v} \cdot \boldsymbol{\zeta} \end{bmatrix}.$$

Proof. The matrices $\tilde{\mathbf{A}}_0$, $\tilde{\mathbf{A}}_i$, $i \in \mathcal{C}$, and $\tilde{\mathbf{B}}_{ij}$, $i, j \in \mathcal{C}$, are easily evaluated by using the natural variable \mathbf{Z} . These matrices are symmetric, and we note that $\tilde{\mathbf{A}}_0$ is positive definite since for any vector x of \mathbb{R}^{n^s+4}

$$\begin{aligned} \langle \tilde{\mathbf{A}}_0 x, x \rangle &= RC_v T^2 x_{n^s+4}^2 + \rho RT \sum_{\mu \in \mathcal{C}} (x_{n^s+\mu} + v_\mu x_{n^s+4})^2 \\ &\quad + \sum_{k \in \mathfrak{S}} \gamma_k \left(x_k + m_k \sum_{\mu \in \mathcal{C}} v_\mu x_{n^s+\mu} + E_k^{\text{tot}} x_{n^s+4} \right)^2. \end{aligned}$$

Similarly $\tilde{\mathbf{B}}$ is positive semidefinite since we have

$$\begin{aligned} \frac{\langle \tilde{\mathbf{B}}x, x \rangle}{RT} &= \frac{1}{p} \sum_{k,l \in \mathfrak{S}} \hat{D}_{kl} (\gamma_k x_k + (\gamma_k H_k + p\chi_k) x_{n^s+4}) (\gamma_l x_l + (\gamma_l H_l + p\chi_l) x_{n^s+4}) \\ &\quad + \lambda T x_{n^s+4}^2 + \eta \sum_{\nu \in \mathcal{C}} (x_{n^s+\nu} + v_\nu x_{n^s+4})^2 \\ &\quad + \left(\kappa + \frac{1}{3}\eta \right) \left(\sum_{\nu \in \mathcal{C}} \xi_\nu (x_{n^s+\nu} + v_\nu x_{n^s+4}) \right)^2. \end{aligned}$$

From the equivalence Theorem 4.3 we deduce that σ is a mathematical entropy. □

6.2. Normal variable

In this section we investigate normal forms for system (3.5). We first establish the null space invariance property.

Lemma 6.1. *The null space of the matrix*

$$\tilde{\mathbf{B}}(\mathbf{V}, \varepsilon, \boldsymbol{\xi}) = \sum_{i,j \in \mathcal{C}} \tilde{\mathbf{B}}_{ij}(\mathbf{V}) \xi_i \xi_j$$

is independent of $\mathbf{V} \in \mathcal{O}_{\mathbf{V}}$ and $\boldsymbol{\xi} \in \Sigma^2$, where Σ^2 is the unit sphere in three dimensions. For any $\varepsilon \in [0, \bar{\varepsilon}]$ this null space is given by

$$N(\tilde{\mathbf{B}}) = \mathbb{R}\hat{m} \oplus \mathbb{R}\hat{z},$$

where $\hat{m} = (m, 0, 0, 0, 0)^\text{t}$, $\hat{z} = (z, 0, 0, 0, 0)^\text{t}$, and we have $\tilde{\mathbf{B}}_{ij}(\mathbf{V})N(\tilde{\mathbf{B}}) = 0$, $i, j \in \mathcal{C}$, for $\mathbf{V} \in \mathcal{O}_{\mathbf{V}}$, $\varepsilon \in [0, \infty)$.

Proof. The expression of $\langle \tilde{\mathbf{B}}x, x \rangle$ in the proof of Theorem 6.1 yields that $\langle \tilde{\mathbf{B}}x, x \rangle = 0$ if and only if $x_{n^s+1} = 0$, $x_{n^s+2} = 0$, $x_{n^s+3} = 0$, $x_{n^s+4} = 0$, and $(x_k)_{k \in \mathfrak{S}} \in N(\hat{D})$. By using Lemma 2.1, we deduce that $N(\tilde{\mathbf{B}})$ is spanned by $\hat{m} = (m_1, \dots, m_{n^s}, 0, 0, 0, 0)^\text{t}$ and $\hat{z} = (z_1, \dots, z_{n^s}, 0, 0, 0, 0)^\text{t}$. It is then easily checked that $\tilde{\mathbf{B}}_{ij}(\mathbf{V})N(\tilde{\mathbf{B}}) = 0$, $i, j \in \mathcal{C}$, for $\mathbf{V} \in \mathcal{O}_{\mathbf{V}}$ and $\varepsilon \in [0, \infty)$. □

Making use of the explicit basis of $N(\tilde{B})$ we define the matrix P from

$$P = \begin{pmatrix} m_1 & \varkappa_1 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ m_2 & \varkappa_2 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ m_3 & \varkappa_3 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ m_{n^s} & \varkappa_{n^s} & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.3}$$

From Lemma 6.1 and from assumptions (Th₁)–(Th₅), assuming for instance that the first species is neutral and the second has a positive charge, it is easily checked that the matrix P is always nonsingular, that the first two columns are spanning $N(\tilde{B})$, and that P is a smooth function of $\varepsilon \in [0, \bar{\varepsilon}]$.

We may then introduce the auxiliary variable $U' = P^t U$ and the corresponding entropic variable $V' = P^{-1} V$ given by

$$U' = \left(\rho, q, \gamma_3, \dots, \gamma_{n^s}, \rho v_1, \rho v_2, \rho v_3, \mathcal{E} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right)^t$$

and

$$V' = \frac{1}{RT} \left(\frac{\varkappa_2 G_1 - \varkappa_1 G_2}{\varkappa_2 m_1 - \varkappa_1 m_2} - \frac{1}{2} \mathbf{v} \cdot \mathbf{v}, \frac{m_2 G_1 + m_1 G_2}{\varkappa_2 m_1 - \varkappa_1 m_2}, V'_3, \dots, V'_{n^s}, v_1, v_2, v_3, -1 \right)^t,$$

where

$$V'_k = G_k - r_k G_1 - s_k G_2, \quad 3 \leq k \leq n^s,$$

and

$$r_k = \frac{\varkappa_2 m_k - \varkappa_k m_2}{\varkappa_2 m_1 - \varkappa_1 m_2}, \quad s_k = \frac{\varkappa_k m_1 - \varkappa_1 m_k}{\varkappa_2 m_1 - \varkappa_1 m_2}, \quad 3 \leq k \leq n^s.$$

From Theorem 4.5, normal variables are in the form $W = (\phi_I(U'_I, \varepsilon), \phi_{II}(V'_{II}, \varepsilon))^t$, where U'_I is the first two components of U' and V'_{II} the last $n^s + 2$ components of V' . For convenience, we choose the variable W given by

$$W = (\rho, q, \log(\gamma_3/\gamma_1^{r_3} \gamma_2^{s_3}), \dots, \log(\gamma_{n^s}/\gamma_1^{r_{n^s}} \gamma_2^{s_{n^s}}), v_1, v_2, v_3, T)^t. \tag{6.4}$$

Theorem 6.2. *The map $(V, \varepsilon) \rightarrow (W, \varepsilon)$ is a C^∞ diffeomorphism from $\mathcal{O}_{(V, \varepsilon)}$ onto $\mathcal{O}_{(W, \varepsilon)} = \mathcal{O}_{\rho, q} \times \mathbb{R}^{n^s+1} \times (0, \infty)$, where*

$$\mathcal{O}_{\rho, q} = \left\{ (u_1, u_2) \in \mathbb{R}^2 : u_1 > 0, \min_{k \in \mathfrak{S}} \frac{\varkappa_k}{m_k} u_1 < u_2 < \max_{k \in \mathfrak{S}} \frac{\varkappa_k}{m_k} u_1 \right\}.$$

This diffeomorphism admits a smooth extension up to $\varepsilon = 0$ and $\varepsilon = \bar{\varepsilon}$. The system written in the W variable

$$\bar{A}_0 \partial_t W + \sum_{i \in \mathcal{C}} \bar{A}_i \partial_i W = \sum_{i,j \in \mathcal{C}} \partial_i (\bar{B}_{ij} \partial_j W) + \bar{T} + \bar{\Omega}, \tag{6.5}$$

where $\bar{A}_0 = \partial_W V^t \tilde{A}_0 \partial_W V$, $\bar{A}_i = \partial_W V^t \tilde{A}_i \partial_W V$, $i \in \mathcal{C}$, $\bar{B}_{ij} = \partial_W V^t \tilde{B}_{ij} \partial_W V$, $i, j \in \mathcal{C}$, $\bar{T} = -\sum_{i,j \in \mathcal{C}} \partial_i (\partial_W V^t \tilde{B}_{ij} \partial_W V \partial_j W)$, and $\bar{\Omega} = \partial_W V^t \Omega$, is in the normal form. The matrix \bar{A}_0 is given by

$$\bar{A}_0 = \begin{bmatrix} \bar{A}_0^{I,I} & \text{Sym} \\ 0_{n^s+2,2} & \bar{A}_0^{II,II} \end{bmatrix},$$

with

$$\bar{A}_0^{I,I} = \frac{1}{\Sigma_{\mathcal{X}^2} \Sigma_{m^2} - \Sigma_{m\mathcal{X}}^2} \begin{bmatrix} \Sigma_{\mathcal{X}^2} & -\Sigma_{m\mathcal{X}} \\ -\Sigma_{m\mathcal{X}} & \Sigma_{m^2} \end{bmatrix}, \quad \bar{A}_0^{II,II} = \begin{bmatrix} \bar{A}^{II,II} & \text{Sym} \\ 0_{3,n^s-2} & \frac{\rho}{RT} \mathbb{I}_{3,3} \\ 0_{1,n^s-2} & 0_{1,3} & \frac{C_v}{RT^2} \end{bmatrix},$$

$$\Sigma_{m^2} = \sum_{k \in \mathcal{S}} \gamma_k m_k^2, \quad \Sigma_{m\mathcal{X}} = \sum_{k \in \mathcal{S}} \gamma_k m_k \varkappa_k, \quad \Sigma_{\mathcal{X}^2} = \sum_{k \in \mathcal{S}} \gamma_k \varkappa_k^2,$$

and $\bar{A}^{II,II}$ is the square matrix of dimension $n^s - 2$ whose coefficients are

$$\bar{A}_{k,l}^{II,II} = \gamma_k \delta_{kl} - \gamma_k \gamma_l \frac{m_k m_l \Sigma_{\mathcal{X}^2} - (m_k \varkappa_l + m_l \varkappa_k) \Sigma_{m\mathcal{X}} + \varkappa_k \varkappa_l \Sigma_{m^2}}{\Sigma_{\mathcal{X}^2} \Sigma_{m^2} - \Sigma_{m\mathcal{X}}^2}, \quad 3 \leq k, l \leq n^s.$$

Denoting by $\xi = (\xi_1, \xi_2, \xi_3)^t$ an arbitrary vector of \mathbb{R}^3 , the matrices \bar{A}_i , $i \in \mathcal{C}$, are given by

$$\sum_{i \in \mathcal{C}} \xi_i \bar{A}_i = \bar{A}_0 v \cdot \xi + \begin{bmatrix} 0_{n^s, n^s} & \text{Sym} \\ \bar{A}^\alpha & 0_{3,3} \\ 0_{1, n^s} & \frac{p}{RT^2} \xi^t & 0 \end{bmatrix},$$

where \bar{A}^α has its columns given by

$$\begin{aligned} \bar{A}_{1\bullet}^\alpha &= \frac{\rho \Sigma_{\mathcal{X}^2} - q \Sigma_{m\mathcal{X}}}{\Sigma_{\mathcal{X}^2} \Sigma_{m^2} - \Sigma_{m\mathcal{X}}^2} \xi, & \bar{A}_{2\bullet}^\alpha &= \frac{q \Sigma_{m^2} - \rho \Sigma_{m\mathcal{X}}}{\Sigma_{\mathcal{X}^2} \Sigma_{m^2} - \Sigma_{m\mathcal{X}}^2} \xi, \\ \bar{A}_{l\bullet}^\alpha &= \gamma_l \left(1 - \frac{\rho m_l \Sigma_{\mathcal{X}^2} - (\rho \varkappa_l + m_l q) \Sigma_{m\mathcal{X}} + q \varkappa_l \Sigma_{m^2}}{\Sigma_{\mathcal{X}^2} \Sigma_{m^2} - \Sigma_{m\mathcal{X}}^2} \right) \xi, & 3 \leq l \leq n^s. \end{aligned}$$

The matrices \bar{B}_{ij} have the structure $\bar{B}_{ij} = \delta_{ij} \bar{B}^D + \bar{B}_{ij}^\kappa + \bar{B}_{ij}^\eta$ and denoting by $\xi = (\xi_1, \xi_2, \xi_3)^t$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)^t$ arbitrary vectors of \mathbb{R}^3 , the matrices \bar{B}_{ij}^κ , \bar{B}_{ij}^η ,

$i, j \in \mathcal{C}$, are given by

$$\sum_{i,j \in \mathcal{C}} \xi_i \zeta_j (\bar{\mathbb{B}}_{ij}^\kappa + \bar{\mathbb{B}}_{ij}^\eta) = \frac{p}{R} \begin{bmatrix} 0_{n^s, n^s} & 0_{n^s, 3} & 0_{n^s, 1} \\ 0_{3, n^s} & \eta \xi \cdot \zeta \mathbb{I}_{3,3} + \eta \zeta \otimes \xi + \left(\kappa - \frac{2}{3} \eta \right) \xi \otimes \zeta & 0_{3, 1} \\ 0_{1, n^s} & 0_{1, 3} & 0_{1, 1} \end{bmatrix},$$

and the matrix $\bar{\mathbb{B}}^D$ is given by

$$\bar{\mathbb{B}}^D = \frac{1}{pT} \begin{bmatrix} 0_{2,2} & & & \text{Sym} \\ 0_{n^s-2,2} & RT^2 (\hat{D}_{kl} \gamma_k \gamma_l)_{3 \leq k, l \leq n^s} & & \\ 0_{3,2} & 0_{3, n^s-2} & & 0_{3,3} \\ 0_{1,2} & \left(\sum_{k \in \mathfrak{S}} \gamma_l \hat{D}_{kl} (\gamma_k RT + \chi_k p) \right)_{3 \leq l \leq n^s} & & 0_{1,3} \quad \Upsilon_{\bar{D}} \end{bmatrix},$$

$$\Upsilon_{\bar{D}} = \frac{1}{RT^2} \left(\lambda p T + \sum_{k, l \in \mathfrak{S}} \hat{D}_{kl} (p \chi_k + RT \gamma_k) (p \chi_l + RT \gamma_l) \right),$$

and finally

$$\bar{\Omega} = \left(0, 0, \omega_3, \dots, \omega_{n^s}, 0, 0, 0, -\frac{1}{T^2} \sum_{k \in \mathfrak{S}} E_k \omega_k \right)^\mathfrak{t}.$$

Proof. These are consequences of lengthy calculations and of Theorem 4.5. □

7. Asymptotic Stability for Ambipolar Plasmas

In this section, we investigate the asymptotic stability of equilibrium states for the system (6.5) modeling ambipolar plasmas as well as the limit of vanishing electron mass.

7.1. Main result

We consider the system (6.5) written in the $W = (W_I^\mathfrak{t}, W_{II}^\mathfrak{t})^\mathfrak{t}$ variable, with the hyperbolic variable

$$W_I = (\rho, q)^\mathfrak{t},$$

and parabolic variable

$$W_{II} = (\log(\gamma_3 / \gamma_1^{r_3} \gamma_2^{s_3}), \dots, \log(\gamma_{n^s} / \gamma_1^{r_{n^s}} \gamma_2^{s_{n^s}}), v_1, v_2, v_3, T)^\mathfrak{t}.$$

The following result is a direct consequence of the axiomatic structure of thermochemistry.^{8,11}

Proposition 7.1. *Let a temperature $T^e > 0$, a velocity $\mathbf{v}^e \in \mathbb{R}^3$, a mole density vector $\gamma^f > 0$ be given, and assume that properties (Th₁)–(Th₅) hold. Then there exists a unique constant equilibrium state U^e such that*

$$\Omega(U^e) = 0, \tag{7.1}$$

in the form $U^e = (\gamma_1^e, \dots, \gamma_{r^e}^e, \rho^e v_1^e, \rho^e v_2^e, \rho^e v_3^e, \rho^e e(T^e) + \frac{1}{2} \rho^e \mathbf{v}^e \cdot \mathbf{v}^e)^\dagger$ and such that $\gamma^e \in (\gamma^f + \mathcal{R}) \cap (0, +\infty)^{n^e}$.

Note that this equilibrium state is independent of the reduced electron mass ε . In addition, whenever γ^f is such that $q^f = \langle \gamma^f, \varkappa \rangle = 0$, we obtain $q^e = \langle \gamma^e, \varkappa \rangle = 0$, since $\varkappa \in \mathcal{R}^\perp$ and $\gamma^e \in \gamma^f + \mathcal{R}$. The equilibrium state corresponding to the various variables are also denoted with the superscript “e”, so that the equilibrium states in the variables V and W , for instance, are denoted by V^e and W^e , respectively.

Theorem 7.1. *Let $d \geq 1$ and $l \geq [d/2] + 2$ be integers and consider the system (6.5). There exists $b > 0$ small enough such that if $\|W^0 - W^e\|_l < b$, there exists a unique global solution W for any $\varepsilon \in [0, \bar{\varepsilon}]$ to the Cauchy problem*

$$\bar{A}_0 \partial_t W + \sum_{i \in \mathcal{C}} \bar{A}_i \partial_i W = \sum_{i, j \in \mathcal{C}} \partial_i (\bar{B}_{ij} \partial_j W) + \bar{T} + \bar{\Omega},$$

with initial condition

$$W(0, x) = W^0(x),$$

such that

$$W_I - W_I^e \in \mathcal{C}^0([0, \infty), W_2^l(\mathbb{R}^d)) \cap \mathcal{C}^1([0, \infty), W_2^{l-1}(\mathbb{R}^d)) \cap L^2((0, \infty), W_2^l(\mathbb{R}^d)),$$

$$W_{II} - W_{II}^e \in \mathcal{C}^0([0, \infty), W_2^l(\mathbb{R}^d)) \cap \mathcal{C}^1([0, \infty), W_2^{l-2}(\mathbb{R}^d)) \cap L^2((0, \infty), W_2^{l+1}(\mathbb{R}^d)).$$

Furthermore, W satisfies the estimate

$$\begin{aligned} & \|W(t) - W^e\|_l^2 + \int_0^t (\|\partial_x \rho(\tau)\|_{l-1}^2 + \|\partial_x q(\tau)\|_{l-1}^2 + \|\partial_x \mathbf{v}(\tau)\|_l^2 + \|\partial_x T(\tau)\|_l^2) d\tau \\ & + \sum_{3 \leq k \leq n^s} \int_0^t \|\partial_x \log(\gamma_k / \gamma_1^{T_k} \gamma_2^{S_k})(\tau)\|_l^2 d\tau \leq C \|W^0 - W^e\|_l^2, \end{aligned}$$

where C is a positive constant and $\sup_{x \in \mathbb{R}^3} |W(t) - W^e|$ goes to zero as $t \rightarrow \infty$. Finally, emphasizing the dependence on ε by denoting $W(t, x, \varepsilon)$ the solution obtained for $\varepsilon \in [0, \bar{\varepsilon}]$, we have for any $\alpha \in [0, \bar{\varepsilon}]$

$$\limsup_{\varepsilon \rightarrow \alpha} \sup_{t \geq 0} \|W(t, \cdot, \varepsilon) - W(t, \cdot, \alpha)\|_{C^{l-([d/2]+2)}} = 0.$$

Physically relevant solutions correspond to initial conditions such that $q^0 = 0$ and equilibrium states such that $q^e = 0$, since in this situation we easily recover that $q(t, x) = 0$ for any $t \geq 0$ and $x \in \mathbb{R}^d$.

Theorem 7.2. *Let $d \geq 2$, $l \geq [d/2] + 3$ and $W^0(x)$ be given, such that*

$$W^0 - W^e \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d),$$

with $p \in [1, 2)$. Then, if $\|W^0 - W^e\|_l$ and $\|W^0 - W^e\|_{L^p}$ are small enough, the unique global solution to the Cauchy problem satisfies the decay estimate

$$\|W(t) - W^e\|_{l-2} \leq C(1+t)^{-\gamma} (\|W^0 - W^e\|_{l-2} + \|W^0 - W^e\|_{L^p}), \quad t \in [0, +\infty),$$

uniformly in $\varepsilon \in [0, \bar{\varepsilon}]$ where C is a positive constant and $\gamma = d(1/2p - 1/4)$. Finally, for any $\alpha \in [0, \bar{\varepsilon}]$ we have

$$\limsup_{\varepsilon \rightarrow \alpha} \limsup_{t \geq 0} \|W(t, \cdot, \varepsilon) - W(t, \cdot, \alpha)\|_{l-1} = 0.$$

7.2. Proof

The system of partial differential equations modeling ambipolar plasmas has been written into a normal form in Theorem 6.2. The coefficients of this normal form are smooth functions of W and of the parameter $\varepsilon \in [0, \bar{\varepsilon}]$. Moreover, the equilibrium state is independent of ε . As a consequence, we only have to establish that properties (Dis₁)–(Dis₄) are satisfied.

The linearized system around the constant state W^e reads

$$\bar{A}_0(W^e; \varepsilon) \partial_t w + \sum_{i \in \mathcal{C}} \bar{A}_i(W^e; \varepsilon) \partial_i w = \sum_{i, j \in \mathcal{C}} \bar{B}_{ij}(W^e; \varepsilon) \partial_i \partial_j w - \bar{\Gamma}(W^e; \varepsilon) w,$$

where $\bar{\Gamma} = -\partial_W \bar{\Omega}$ and $w = W - W^e$. Property (Dis₁) is a direct consequence of the following expression of $\bar{\Gamma}(W^e; \varepsilon)$ at an equilibrium point

$$\bar{\Gamma}(W^e; \varepsilon) = \sum_{r \in \mathfrak{R}} \hat{\mathcal{K}}_r^s \bar{\nu}_r \otimes \bar{\nu}_r,$$

where $\bar{\nu}_r = (0, 0, \nu_{r3}, \dots, \nu_{rns}, 0, 0, 0, -\sum_{l \in \mathfrak{S}} \nu_{rl} E_l / RT^2)$ and $\hat{\mathcal{K}}_r^s = \mathcal{K}_r^s \exp\langle \nu_r^f, \mu \rangle$, obtained directly from $\bar{\Omega}$ or from the expression of $\tilde{\Gamma}(W^e; \varepsilon)$ given in Giovangigli and Massot.^{8,11} Properties (Dis₃) and (Dis₄) are also established in Giovangigli and Massot.^{8,11} In order to examine if (Dis₂) holds, the most convenient way is to use property (Spe₃) of Theorem 5.1.

Proposition 7.2. *For any equilibrium state W^e we have*

$$N(\bar{B}(W^e; \xi, \varepsilon)) = \mathbb{R}e^1 \oplus \mathbb{R}e^2 \subset N(\bar{\Gamma}(W^e; \varepsilon)),$$

and if the equilibrium point W^e is such that $q^e = 0$, there exist nonzero vectors Ψ of $\mathbb{R}e^1 \oplus \mathbb{R}e^2$ such that $\zeta \bar{\mathcal{A}}_0(W^e; \varepsilon) \Psi + \bar{A}(W^e; \xi, \varepsilon) \Psi = 0$ where ζ is real.

Proof. From the normal form established in Theorem 6.2, introducing the coordinates $(\mu'_1, \mu'_2, \mu'_3, \dots, \mu'_{ns})$ of the vector $(\gamma_k H_k + p \chi_k)_{k \in \mathfrak{S}}$ with respect to the basis

$(m, \varkappa, e^3, \dots, e^{n^s})$, we have

$$\frac{\langle \bar{\mathbf{B}}x, x \rangle}{RT} = \frac{1}{p} \sum_{k,l \geq 3} \hat{D}_{kl}(\gamma_k x_k + \mu'_k x_{n^s+4})(\gamma_l x_l + \mu'_l x_{n^s+4}) + \lambda T x_{n^s+4}^2 + \eta \sum_{\nu \in \mathcal{C}} (x_{n^s+\nu} + v_\nu x_{n^s+4})^2 + \left(\kappa + \frac{1}{3}\eta \right) \left(\sum_{\nu \in \mathcal{C}} \xi_\nu (x_{n^s+\nu} + v_\nu x_{n^s+4}) \right)^2,$$

and this yields $N(\bar{\mathbf{B}}(\mathbf{W}^e; \boldsymbol{\xi}, \varepsilon)) = \mathbb{R}e^1 \oplus \mathbb{R}e^2$. From the expression of $\bar{\mathbf{L}}(\mathbf{W}^e; \varepsilon)$ it is easily checked that $\mathbb{R}e^1 \oplus \mathbb{R}e^2 \subset N(\bar{\mathbf{L}}(\mathbf{W}^e; \varepsilon))$. A direct calculation yields

$$\begin{aligned} & (\zeta \bar{\mathbf{A}}_0 + \bar{\mathbf{A}})(\alpha_1 e^1 + \alpha_2 e^2) \\ &= \left((\zeta + \mathbf{v} \cdot \boldsymbol{\xi}) \frac{\alpha_1 \Sigma_{\varkappa^2} - \alpha_2 \Sigma_{m\varkappa}}{\Sigma_{\varkappa^2} \Sigma_{m^2} - \Sigma_{m\varkappa}^2}, (\zeta + \mathbf{v} \cdot \boldsymbol{\xi}) \frac{\alpha_2 \Sigma_{m^2} - \alpha_1 \Sigma_{m\varkappa}}{\Sigma_{\varkappa^2} \Sigma_{m^2} - \Sigma_{m\varkappa}^2}, \right. \\ & \left. 0, \dots, 0, \frac{\rho(\alpha_1 \Sigma_{\varkappa^2} - \alpha_2 \Sigma_{m\varkappa}) + q(\alpha_2 \Sigma_{m^2} - \alpha_1 \Sigma_{m\varkappa})}{\Sigma_{\varkappa^2} \Sigma_{m^2} - \Sigma_{m\varkappa}^2} \boldsymbol{\xi}^t, 0 \right), \end{aligned}$$

and selecting $\zeta = -\mathbf{v} \cdot \boldsymbol{\xi}$, $\alpha_1 = \Sigma_{m\varkappa}$, $\alpha_2 = \Sigma_{\varkappa^2}$, $\Psi = \alpha_1 e^1 + \alpha_2 e^2$, it is easily checked that $\zeta \bar{\mathbf{A}}_0 \Psi + \bar{\mathbf{A}} \Psi = 0$ when $q = 0$. □

This problem, however, is artificial and due to the lack of dissipativity properties associated with the electric charge equation, which must guarantee that the charge remains zero. Two equivalent form can be introduced for the system governing ambipolar plasmas, that is, such that regular solutions coincide, and which guarantee strict dissipativity.

One can first modify chemistry production rates $\bar{\boldsymbol{\Omega}}$ in the form

$$\bar{\boldsymbol{\Omega}} = \bar{\boldsymbol{\Omega}}^{(1)} + \bar{\boldsymbol{\Omega}}^{(2)},$$

where $\bar{\boldsymbol{\Omega}}^{(1)}$ is the previous source term given in Theorem 6.2 and $\bar{\boldsymbol{\Omega}}^{(2)}$ is defined by $\bar{\boldsymbol{\Omega}}^{(2)} = \bar{\mathbf{L}}^{(2)} \mathbf{W}$, with

$$\bar{\mathbf{L}}^{(2)} = \begin{bmatrix} 0 & 0 & 0_{1,n^s+2} \\ 0 & \alpha & 0_{1,n^s+2} \\ 0_{n^s+2,1} & 0_{n^s+2,1} & 0_{n^s+2,n^s+2} \end{bmatrix},$$

where $\alpha > 0$ is a positive parameter. In this situation, the null space of $\bar{\mathbf{B}}(\mathbf{W}^e; \boldsymbol{\xi}, \varepsilon)$ is unchanged, but e^2 is no longer in the null space $\bar{\mathbf{L}}(\mathbf{W}^e; \varepsilon)$ so that

$$N(\bar{\mathbf{B}}(\mathbf{W}^e; \boldsymbol{\xi}, \varepsilon)) \cap N(\bar{\mathbf{L}}(\mathbf{W}^e; \varepsilon)) = \mathbb{R}e^1,$$

and strict dissipativity is then easily established. Note that the corresponding charge equation reads

$$\partial_t q + \boldsymbol{\partial}_x \cdot (q\mathbf{v}) = -\alpha q,$$

and contains a consumption term $-\alpha q$. This equation, of course, guarantee that the charge remains zero if $q^0 = q^e = 0$, so that physical solutions of the modified system coincide with physical solutions of the original system.

A second modification, which has interesting numerical consequences,⁶ consists in modifying the diffusion coefficients. The resulting charge equation then contains a diffusion term and only one hyperbolic component remains. More specifically, we modify the matrices \bar{B}_{ij} , $i, j \in \mathcal{C}$, in the form

$$\bar{B}_{ij} = \bar{B}_{ij}^{(1)} + \delta_{ij} \bar{B}^{(2)}, \quad i, j \in \mathcal{C},$$

where $\bar{B}_{ij}^{(1)}$ is the previous matrix given in Theorem 6.2 and $\bar{B}^{(2)}$ is defined by

$$\bar{B}^{(2)} = \begin{bmatrix} 0 & 0 & 0_{1,n^s+2} \\ 0 & \alpha & 0_{1,n^s+2} \\ 0_{n^s+2,1} & 0_{n^s+2,1} & 0_{n^s+2,n^s+2} \end{bmatrix},$$

where $\alpha > 0$ is a positive parameter. In this situation, the null space of $\bar{B}(W^e, \xi, \varepsilon)$ do not contains e^2 and the null space $\bar{L}(W^e, \varepsilon)$ is unchanged so that

$$N(\bar{B}(W^e, \xi, \varepsilon)) \cap N(\bar{L}(W^e, \varepsilon)) = \mathbb{R}e^1,$$

and strict dissipativity is obtained. The corresponding charge equation now reads

$$\partial_t q + \partial_x \cdot (qv) = \partial_x \cdot (\alpha \partial_x q),$$

and the diffusion term $\partial_x \cdot (\alpha \partial_x q)$ as a stabilizing effect.⁶ This equation guarantees again that the charge remains zero if $q^0 = q^e = 0$, so that physical solutions of the modified system coincide with physical solutions of the original system. Of course, both modifications could also be combined.

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