

The Local Cauchy Problem for Multicomponent Reactive Flows in Full Vibrational Non-equilibrium

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We consider reactive mixtures of dilute polyatomic gases in full vibrational non-equilibrium. The governing equations are derived from the kinetic theory and possesses an entropy. We recast this system of conservation laws into a symmetric conservative form by using entropic variables. Following a formalism developed by the authors in a previous paper, the system is then rewritten into a normal form, that is, in the form of a quasilinear symmetric hyperbolic–parabolic system. Using a result of Vol’pert and Hudjaev, we prove local existence and uniqueness of a bounded smooth solution to the Cauchy problem. © 1998 B. G. Teubner Stuttgart—John Wiley & Sons, Ltd.

1. Introduction

In this paper, we investigate the system of equations modelling multicomponent reactive gases in which polyatomic molecules are in full vibrational non-equilibrium. We prove the well-posedness of the Cauchy problem locally in time with smooth initial conditions.

We first present the set of governing equations for multicomponent reactive gaseous flows in full vibrational non-equilibrium. We express the conservation equations, the transport fluxes and the thermodynamic properties. In this model, there is a translational–rotational temperature and the species vibrational quantum state densities are independent unknowns. We also summarize the derivation of these relations from the kinetic theory. An important consequence of the kinetic framework is that the transport fluxes have their natural symmetry properties [8, 3, 2, 4].

We then apply the formalism developed by the authors in Reference 4 concerning symmetrizability. We exhibit a mathematical entropy function for this system of multicomponent reactive flows in full vibrational non-equilibrium. It is taken to be the opposite of the physical mixture entropy density per unit volume. The related entropic variables lead to a symmetric conservative system. Here we make a crucial use of the naturally symmetric form of the transport fluxes provided by the kinetic

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theory. For symmetrization problems arising when the natural symmetry of transport fluxes has been artificially destroyed [5, 1] we refer to [2–4].

We prove that the nullspace naturally associated with dissipation matrices of the symmetrized system satisfy an invariance property. This allows us to recast the system into a normal form [4]. This normal form is a symmetric hyperbolic–parabolic composite system which decouples the hyperbolic components from the parabolic components. This form is not unique as shown in [4] where all normal forms have been characterized for such systems.

For the resulting system of conservation laws in normal form, we next prove the existence of a unique solution to the Cauchy problem, locally in time, in a space of bounded smooth functions. More precisely, solutions are considered in the functional spaces $V_l(\mathbb{R}^d)$ constituted by the (classes of) functions of $L^\infty(\mathbb{R}^d)$ having derivatives of orders 1 to l in $L^2(\mathbb{R}^d)$, for values of l such that $l > d/2 + 3$ where d is the space dimension. Our method of proof relies on the results of Vol’pert and Hudjaev concerning the Cauchy problem for symmetric quasilinear hyperbolic–parabolic composite systems of partial differential equations [7].

The governing equations for multicomponent reactive flows in full vibrational non-equilibrium are presented in section 2. Symmetrization and normal forms are investigated in Section 3. Finally, the Cauchy problem is investigated in section 4.

2. Governing equations

2.1. Conservation equations

The system of conservation laws modelling multicomponent reactive flows in full vibrational non-equilibrium express the conservation of vibrationally excited species mass, momentum and energy. These equations are provided by the kinetic theory of gases following a formalism generalized from [8] and [2] which will be sketched in section 2.5. They can be written in the form

$$\partial_t U + \sum_{i \in C} \partial_i F_i + \sum_{i \in C} \partial_i \mathcal{F}_i = \Omega, \tag{2.1}$$

where ∂_t is the time derivative operator, U the conservative variables, ∂_i the space derivative operator in the i th direction, $C = \{1, 2, 3\}$ the set of direction indices, F_i the advective flux in the i th direction, \mathcal{F}_i the dissipative flux in the i th direction and Ω the source term.

The variables U and the advective fluxes $F_i, i \in C$, are given by

$$U = \begin{pmatrix} \rho_{1,1} \\ \vdots \\ \rho_{n_s, I_{n_s}} \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho e^{\text{tot}} \end{pmatrix}, \quad F_i = \begin{pmatrix} \rho_{1,1} v_i \\ \vdots \\ \rho_{n_s, I_{n_s}} v_i \\ \rho v_1 v_i + \delta_{i1} p \\ \rho v_2 v_i + \delta_{i2} p \\ \rho v_3 v_i + \delta_{i3} p \\ \rho e^{\text{tot}} v_i + p v_i \end{pmatrix}, \tag{2.2}$$

where $\rho_{k,K}$ is the density of the k th species in the K th vibrational quantum state, n_S the number of species, $S = [1, n_S]$ the set of species indices, I_k the number of vibrational quantum states of the k th species, $\mathcal{J}_k = [1, I_k]$ the corresponding set of vibrational state indices, $\rho = \sum_{k \in S, K \in \mathcal{J}_k} \rho_{k,K}$ the total density, v_i the mass averaged flow velocity in the i th direction, δ_{ij} the Kronecker symbol, e^{tot} the total energy per unit mass of the mixture, and p the thermodynamical pressure. The couple of indices (k, K) vary in the set $S^{\text{vib}} = \bigcup_{k \in S} k \times \mathcal{J}_k$ which cardinal is $n_T = \sum_{k \in S} I_k$.

For convenience, the dissipative flux \mathcal{F}_i is splitted between the mass and heat diffusion flux $\mathcal{F}_i^{D\lambda}$ and the viscous flux $\mathcal{F}_i^{K\eta}$ so that

$$\mathcal{F}_i = \mathcal{F}_i^{D\lambda} + \mathcal{F}_i^{K\eta}. \tag{2.3}$$

The fluxes $\mathcal{F}_i^{D\lambda}$ and $\mathcal{F}_i^{K\eta}$, and the source term Ω , are given by

$$\mathcal{F}_i^{K\eta} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Pi_{i1} \\ \Pi_{i2} \\ \Pi_{i3} \\ \sum_{j \in C} \Pi_{ij} v_j \end{pmatrix}, \quad \mathcal{F}_i^{D\lambda} = \begin{pmatrix} \rho_{1,1} (\mathcal{V}_{1,1})_i \\ \vdots \\ \rho_{n_S, I_{n_S}} (\mathcal{V}_{n_S, I_{n_S}})_i \\ 0 \\ 0 \\ 0 \\ q_i \end{pmatrix}, \quad \Omega = \begin{pmatrix} m_1 \omega_{1,1} \\ \vdots \\ m_{n_S} \omega_{n_S, I_{n_S}} \\ \rho g_1 \\ \rho g_2 \\ \rho g_3 \\ \rho g \cdot v \end{pmatrix}, \tag{2.4}$$

where $\Pi = (\Pi_{ij})_{i,j \in C}$ is the viscous stress tensor, $\mathcal{V}_{k,K} = ((\mathcal{V}_{k,K})_1, (\mathcal{V}_{k,K})_2, (\mathcal{V}_{k,K})_3)^t$ the diffusion velocity of the k th species in the K th vibrational quantum state, $q = (q_1, q_2, q_3)^t$ the heat flux vector, m_k the molar mass of the k th species, $\omega_{k,K}$ the molar production rate of the k th species in the K th vibrational quantum state, $g = (g_1, g_2, g_3)^t$ the external force per unit mass acting on the species, $v = (v_1, v_2, v_3)^t$ the velocity vector and t the transposition symbol.

These equations have to be completed by the relations expressing the transport fluxes Π , $\mathcal{V}_{k,K}$, $k \in S, K \in \mathcal{J}_k$, and q , the thermodynamic properties p and e^{tot} , the chemical/vibrational source terms $\omega_{k,K}$, $k \in S, K \in \mathcal{J}_k$, and the specific force g .

2.2. Transport fluxes

The expressions for the transport fluxes derived from the kinetic theory of dilute polyatomic gas mixture can be written

$$\mathcal{V}_{k,K} = - \sum_{(l,L) \in S^{\text{vib}}} D_{k,K,l,L}^{\text{rot}} d_{l,L} - \theta_{k,K}^{\text{rot}} \partial_x \log T, \quad (k, K) \in S^{\text{vib}}, \tag{2.5}$$

$$\Pi = - (\kappa^{\text{rot}} - \frac{2}{3} \eta^{\text{rot}}) (\partial_x \cdot v) I - \eta^{\text{rot}} (\partial_x v + (\partial_x v)^t), \tag{2.6}$$

$$q = \lambda^{\text{rot}} \partial_x T - p \sum_{(k,K) \in S^{\text{vib}}} \theta_{k,K}^{\text{rot}} d_{k,K} + \sum_{(k,K) \in S^{\text{vib}}} \rho_{k,K} h_{k,K} \mathcal{V}_{k,K}, \tag{2.7}$$

where $D = (D_{k,K,l,L}^{\text{rot}})_{(k,K),(l,L) \in S^{\text{vib}}}$ is the diffusion matrix, $d_{k,K}$ the diffusion driving force of the k th species in the K th vibrational quantum state, $(\theta_{k,K}^{\text{rot}})_{(k,K) \in S^{\text{vib}}} \in \mathbb{R}^{n_r}$ the thermal diffusion vector, $\partial_x = (\partial_1, \partial_2, \partial_3)^t$ the usual differential operator, T the translational-rotational absolute temperature, κ^{rot} the volume viscosity, η^{rot} the shear viscosity, λ^{rot} the partial thermal conductivity and $h_{k,K}$ the enthalpy per unit mass of the k th species in the K th vibrational quantum state. The vectors $d_{k,K}$, $(k, K) \in S^{\text{vib}}$, take into account the effects of various state variable gradients and are given by

$$d_{k,K} = \partial_x X_{k,K} + X_{k,K} \frac{\partial_x p}{p}, \tag{2.8}$$

where $X_{k,K}$ denotes the mole fraction of the k th species in the K th vibrational quantum state. Alternate expressions for the diffusion velocities and the heat flux vector are

$$v_{k,K} = - \sum_{(l,L) \in S^{\text{vib}}} D_{k,K,l,L}^{\text{rot}} (d_{l,L} + \chi_{l,L}^{\text{rot}} \partial_x \log T), \quad (k, K) \in S^{\text{vib}}, \tag{2.9}$$

$$q = - \lambda^{\text{rot}} \partial_x T + p \sum_{(l,L) \in S^{\text{vib}}} \chi_{l,L}^{\text{rot}} v_{l,L} + \sum_{(l,L) \in S^{\text{vib}}} \rho_{l,L} h_{l,L} v_{l,L}, \tag{2.10}$$

where $(\chi_{k,K}^{\text{rot}})_{(k,K) \in S^{\text{vib}}} \in \mathbb{R}^{n_r}$ is the thermal diffusion ratio vector and λ^{rot} the thermal conductivity. Both expressions (2.5) and (2.9) for the diffusion velocities and (2.7) and (2.10) for the heat flux vector will be used in the following.

Note that we have used the superscript rot since the transport coefficients only involve translational and rotational energy exchanges. In addition, the diffusion matrix D^{rot} associated with these fluxes is symmetric as specified in Section 2.6 where the properties of the various transport coefficients are expressed.

2.3. Thermodynamic properties

From the kinetic theory, the state law expressing the pressure p is

$$p = \rho r T, \tag{2.11}$$

where

$$\rho r = R_g \sum_{(k,K) \in S^{\text{vib}}} \frac{\rho_{k,K}}{m_k}. \tag{2.12}$$

In these expressions, $\rho = \sum_{(k,K) \in S^{\text{vib}}} \rho_{k,K}$ is the total mass density, r the specific gas constant of the mixture, and R_g the universal gas constant. The specific total energy e^{tot} and the specific internal energy e of the mixture are given by

$$e^{\text{tot}} = e + \frac{1}{2} v \cdot v, \tag{2.13}$$

where

$$\rho e = \sum_{(k,K) \in S^{\text{vib}}} \rho_{k,K} e_{k,K}. \tag{2.14}$$

The quantity $e_{k,K}$ is the internal energy per unit mass of the k th species in the vibrational quantum state K and can be written

$$e_{k,K}(T) = e_{k,K}^0 + \int_{T_0}^T c_{v^{k,K}}^{\text{rot}}(T') dT', \tag{2.15}$$

where $e_{k,K}^0$ is the energy of formulation of the k th species in the K th vibrational quantum state at the positive reference temperature T_0 and $c_{v^{k,K}}^{\text{rot}}$ is the rotational specific heat at constant volume of the k th species in the K th vibrational quantum state. The mixture rotational specific heat at constant volume c_v^{rot} is also defined by

$$\rho c_v^{\text{rot}} = \sum_{(k,K) \in S^{\text{vib}}} \rho_{k,K} c_{v^{k,K}}^{\text{rot}}. \tag{2.16}$$

Similarly, the specific total enthalpy h^{tot} and specific enthalpy h are written

$$h^{\text{tot}} = h + \frac{1}{2} v \cdot v, \quad \rho h = \sum_{(k,K) \in S^{\text{vib}}} \rho_{k,K} h_{k,K}, \tag{2.17}$$

where $h_{k,K}$, the enthalpy per unit mass of the k th species in the K th vibrational quantum state, reads

$$h_{k,K}(T) = e_{k,K}(T) + r_k T, \tag{2.18}$$

and where $r_k = R_g/m_k$ is the specific gas constant of the k th species. It is also convenient to denote by $e_{k,K}^{\text{tot}}$ the total energy per unit mass and by $h_{k,K}^{\text{tot}}$ the total enthalpy per unit mass of the k th species in the K th vibrational quantum state

$$e_{k,K}^{\text{tot}} = e_{k,K} + \frac{1}{2} v \cdot v, \quad h_{k,K}^{\text{tot}} = h_{k,K} + \frac{1}{2} v \cdot v. \tag{2.19}$$

The kinetic theory also yields the specific (physical) entropy of the k th species in the K th vibrational quantum state

$$s_{k,K}(\rho_{k,K}, T) = s_{k,K}^{00} + \int_{T_0}^T \frac{c_{v^{k,K}}^{\text{rot}}(T')}{T'} dT' - r_k \log\left(\frac{\rho_{k,K}}{\gamma_0 m_k}\right), \tag{2.20}$$

where $s_{k,K}^{00}$ is the standard formation entropy at the positive reference temperature T_0 and positive reference pressure $\gamma_0 R_g T_0$ where γ_0 is a reference molar concentration. Finally, we will also need the expression of the chemical/vibrational potential $\mu_{k,K}$ of the k th species in the K th vibrational quantum state

$$\mu_{k,K}(\rho_{k,K}, T) = e_{k,K} + r_k T - s_{k,K} T. \tag{2.21}$$

2.4. Source terms

The detailed description of chemical/vibrational source terms $\omega_{k,K}$, $(k, K) \in S^{\text{vib}}$, is beyond the scope of this paper. It would only be needed for investigating global existence results in the neighbourhood of constant equilibrium states and the

asymptotic stability of these states. In the following sections, we only require that the chemical/vibrational source terms $\omega_{k,K}$, $(k, K) \in S^{\text{vib}}$, are functions of the natural variables $Y = (\rho_{1,1}, \dots, \rho_{n_s, I_{n_s}}, v_1, v_2, v_3, T)^t$

$$\omega_{k,K} = \omega_{k,K}(Y), \tag{2.22}$$

with a similar assumption for the specific force term g

$$g = g(Y). \tag{2.23}$$

Remark 2.1. In this paper, for sake of simplicity, we only consider a species independent specific force, as gravity for instance. For the structure of transport fluxes arising when the specific forces are species dependent we refer to [6] and [4].

2.5. Derivation from the kinetic theory

In this section we briefly discuss the derivation of the preceding equations from the semi-classical kinetic theory of reactive polyatomic gas mixtures [8, 2]. This derivation requires that the chemical and vibrational characteristic times are larger than the translational and rotational characteristic times of the mixtures [2].

More specifically, the Boltzmann equations describing reactive mixtures take the form

$$\mathcal{D}_k(f_k) = \frac{1}{\varepsilon} \mathfrak{T}_k^{\text{fast}}(f_1, \dots, f_{n_s}) + \varepsilon^\alpha \mathfrak{T}_k^{\text{slow}}(f_1, \dots, f_{n_s}), \quad k \in S, \tag{2.24}$$

where f_k is the distribution function of the k th species, \mathfrak{D}_k the transport differential operator of the k th species, $\mathfrak{T}_k^{\text{fast}}$ is the fast collision operator and $\mathfrak{T}_k^{\text{slow}}$ the slow collision operator. In addition, ε is the formal expansion parameter associated with the Enskog procedure and α depends on the regime under consideration. Accordingly, the species distribution functions are expanded in the form

$$f_k = f_k^0(1 + \varepsilon\phi_k + O(\varepsilon^2)), \quad k \in S,$$

where f_k^0 is the local Maxwellian distribution related to the fast collision operators.

For polyatomic molecules, the energy can be divided into several contributions coming from translational, rotational and vibrational modes. The electronic energy transfers are neglected in a very wide range of temperature, efficient collisions being very seldom because of the gap of energy between two electronic levels. In the previous study of Ern and Giovangigli [2], the internal modes were at equilibrium such that we had

$$\mathfrak{T}_k^{\text{fast}}(f_1, \dots, f_{n_s}) = \sum_{l \in S} \mathfrak{C}_{kl}^{\text{tr, rot, vib}}(f_k, f_l),$$

$$\mathfrak{T}_k^{\text{slow}}(f_1, \dots, f_{n_s}) = \mathfrak{S}_k(f_1, \dots, f_{n_s}),$$

where $\mathfrak{C}_{kl}^{\text{tr,rot,vib}}$ is the non-reactive collision operator for the species pair (k, l) involving translational, rotational and vibrational energy exchanges and \mathfrak{S}_k the reactive collision operator for the k th species. These reactive and non-reactive collision operators have complex expressions and we refer to [2] for more details.

In this paper, we assume that the characteristic times associated with the exchange of vibrational energy are of the same order of magnitude as the chemical times but are longer than the characteristic times associated with translational and rotational energy exchanges. In this situation, the polyatomic molecules of the mixture are in full vibrational non-equilibrium, and we have

$$\begin{aligned}\mathfrak{T}_k^{\text{fast}}(f_1, \dots, f_{n_s}) &= \sum_{l \in S} \mathfrak{C}_{kl}^{\text{tr,rot}}(f_k, f_l), \\ \mathfrak{T}_k^{\text{slow}}(f_1, \dots, f_{n_s}) &= \sum_{l \in S} \mathfrak{C}_{kl}^{\text{vib}}(f_k, f_l) + \mathfrak{S}_k(f_1, \dots, f_{n_s}),\end{aligned}$$

where $\mathfrak{C}_{kl}^{\text{tr,rot}}$ is the non-reactive collision operator for the species pair (k, l) involving translational and rotational energy exchanges, $\mathfrak{C}_{kl}^{\text{vib}}$ is the non-reactive collision operator for the species pair (k, l) translational–vibrational, rotational–vibrational and vibrational–vibrational energy exchanges.

It is then possible to distinguish a number of different regimes for the mixture, depending on the relative order of magnitude of the fast and slow collision terms. The ‘slow regime’ corresponds to $\alpha = 1$ and produces expressions (2.5)–(2.10) for the transport fluxes. In this situation, the chemical source terms are evaluated from the Maxwellian collisional terms. On the other hand, the ‘tempered regime’ corresponds to $\alpha = 0$. In this situation, the pressure tensor and the Maxwellian source terms are perturbed, because of the reactive part of the distribution functions ϕ_k . Such perturbations only occur at these two places and are known to be small, so that we recover the set of equations described above by neglecting them [2].

Finally, the macroscopic quantities are obtained in terms of the distribution functions by integrating over the velocity space and summing over the rotational quantum numbers. Similarly, the macroscopic governing equations are obtained by multiplying the Boltzmann equations by the fast collisional invariants, integrating with respect to the particle velocity and summing with respect to the rotational quantum states of the particles.

Remark 2.2. In the model investigated in this paper, the vibrational non-equilibrium is a global non-equilibrium such that each molecule in a given vibrational state can be considered as an independent species. The modelling of vibrational transfers and of chemical reactions is then contained in the source term. As a consequence, the structure that we obtain for our system is formally equivalent to the one obtained with vibrational equilibrium.

2.6. Mathematical assumptions

We introduce here the mathematical assumptions concerning the transport coefficients, the thermodynamic properties, the chemical/vibrational source terms and the

specific force term. We assume that the natural variables

$$Y = (\rho_{1,1}, \dots, \rho_{n_s, I_{n_s}}, v_1, v_2, v_3, T)^t,$$

takes its values in the open convex set \mathcal{O}_Y

$$\mathcal{O}_Y = (0, \infty)^{n_r} \times \mathbb{R}^3 \times (T_0, \infty), \tag{2.25}$$

where T_0 is positive and we assume the following dependence and regularity properties.

- (H₁) The transport coefficients $(D_{k,K,l,L}^{\text{rot}})_{(k,K),(l,L) \in S^{\text{vib}}}$, $(\theta_{k,K}^{\text{rot}})_{(k,K) \in S^{\text{vib}}}$ as well as $(\chi_{k,K}^{\text{rot}})_{(k,K) \in S^{\text{vib}}}$, the scalars κ^{rot} , η^{rot} , λ'^{rot} , and λ^{rot} are C^∞ functions of $(\rho_{1,1}, \dots, \rho_{n_s, I_{n_s}}, T) \in (0, \infty)^{n_r} \times (T_0, \infty)$.
- (H₂) The specific heats $c_{v,k,K}^{\text{rot}}$, $(k,K) \in S^{\text{vib}}$, are C^∞ functions of $T \in [T_0, \infty)$. Moreover there exists a positive constant \mathcal{A} with $0 < \mathcal{A} \leq c_{v,k,K}^{\text{rot}}(\xi)$, for $\xi \geq T_0$ and $(k,K) \in S^{\text{vib}}$.
- (H₃) The force term g is a C^∞ function of $Y \in \mathcal{O}_Y$.
- (H₄) The chemical/vibrational production rate vector $\omega = (\omega_{1,1}, \dots, \omega_{n_s, I_{n_s}})^t$ is a C^∞ function of $Y \in \mathcal{O}_Y$.
- (H₅) The shear viscosity η^{rot} , the thermal conductivity λ^{rot} , and the partial thermal conductivity λ'^{rot} are positive and the volume viscosity κ^{rot} is non-negative.
- (H₆) The matrix $D^{\text{rot}} = (D_{k,K,l,L}^{\text{rot}})_{(k,K),(l,L) \in S^{\text{vib}}}$ is symmetric and positive semidefinite and its nullspace is spanned by $\varrho = (\rho_{1,1}, \dots, \rho_{n_s, I_{n_s}})^t$. We have in particular the mass constraints

$$\sum_{k \in S, K \in \mathcal{I}_k} \rho_{k,K} D_{k,K,l,L}^{\text{rot}} = 0, \quad l \in S, L \in \mathcal{I}_l.$$

- (H₇) The thermal diffusion vector $\theta^{\text{rot}} \in \mathbb{R}^{n_r}$ satisfies the mass constraint

$$\sum_{k \in S, K \in \mathcal{I}_k} \rho_{k,K} \theta_{k,K}^{\text{rot}} = 0.$$

- (H₈) The thermal diffusion ratio vector $\chi^{\text{rot}} \in \mathbb{R}^{n_r}$ satisfies the relations

$$\sum_{(l,L) \in S^{\text{vib}}} D_{k,K,l,L}^{\text{rot}} \chi_{l,L}^{\text{rot}} = \theta_{k,K}^{\text{rot}}, \quad (k,K) \in S^{\text{vib}}, \quad \sum_{(k,K) \in S^{\text{vib}}} \chi_{k,K}^{\text{rot}} = 0.$$

The partial thermal conductivity λ'^{rot} is given by

$$\lambda^{\text{rot}} = \lambda'^{\text{rot}} - \frac{p}{T} \sum_{(l,L) \in S^{\text{vib}}} \chi_{l,L}^{\text{rot}} \theta_{l,L}^{\text{rot}}.$$

- (H₉) The chemical/vibrational production rate vector $\omega = (\omega_{1,1}, \dots, \omega_{n_s, I_{n_s}})^t$ satisfies the mass conservation relation

$$\sum_{(k,K) \in S^{\text{vib}}} m_k \omega_{k,K} = 0.$$

We point out that all these assumptions are suggested by the semi-classical kinetic theory of dilute polyatomic reactive gas mixtures. As previously mentioned, the diffusion coefficients considered here are symmetric and, therefore, are consistent with Onsager reciprocal relations. The mass constraints of the diffusion matrix and the thermal diffusion vector also imply the mass conservation relation $\sum_{(k, K) \in S^{vib}} \rho_{k, K} \mathcal{V}_{k, K} = 0$. In addition, the positivity properties of the transport coefficients are associated with the positivity of the entropy production quadratic form as it was shown in the case of vibrational equilibrium [3, 2].

Further note that the gas species specific heats—and therefore the energies and enthalpies—obtained from the kinetic theory, could also be extended—from a mathematical point of view—up to zero temperature, but not the gas entropy which explodes like $\log T$. However, since the basic assumptions of the kinetic theory of dilute gas mixtures are not valid at low temperatures, where the gases are ultimately transformed into liquids and then into solids, we have chosen to restrict the temperature domain to $[T_0, \infty)$, where T_0 is positive, for modelling gas mixtures.

2.7. The quasilinear form

By expressing the natural variables Y in terms of the conservative variables U , we now rewrite the system of conservation equations (2.1) in a quasilinear form. For this purpose, we first investigate the map $Y \rightarrow U$ and its range.

Proposition 2.3. *The map $Y \mapsto U$ is a C^∞ diffeomorphism from the open set $\mathcal{O}_Y = (0, \infty)^{n_T} \times \mathbb{R}^3 \times (T_0, \infty)$ onto an open set \mathcal{O}_U . The open set \mathcal{O}_U is convex and given by*

$$\mathcal{O}_U = \{z \in \mathbb{R}^{n_T+4}; z_{k, K} > 0, (k, K) \in S^{vib}, z_{n_T+4} - \phi(z_{1,1}, \dots, z_{n_S, I_{n_S}}, z_{n_T+1}, z_{n_T+2}, z_{n_T+3}) > 0\}, \tag{2.26}$$

where

$$\begin{aligned} \phi(z_{1,1}, \dots, z_{n_S, I_{n_S}}, z_{n_T+1}, z_{n_T+2}, z_{n_T+3}) &= \frac{1}{2} \frac{z_{n_T+1}^2 + z_{n_T+2}^2 + z_{n_T+3}^2}{\sum_{(k, K) \in S^{vib}} z_{k, K}} \\ &+ \sum_{(k, K) \in S^{vib}} z_{k, K} e_{k, K}^0. \end{aligned}$$

The proof is almost straightforward using hypothesis (H_2) , expression (2.2) and the triangular structure of $\partial_Y U$ on \mathcal{O}_Y .

From (2.5) to (2.8) and Proposition 2.3, the dissipation fluxes can then be written as linear combinations of the conservative variables gradients

$$\mathcal{F}_i = - \sum_{j \in C} G_{ij}(U) \partial_j U, \tag{2.27}$$

where the proportionality coefficients are the dissipation matrices, $G_{ij}(U)$, $i, j \in C$, which are functions of the conservative variable U . These matrices are square matrices

of dimension $n_T + 4$, and, from (2.3), they admit the following decomposition:

$$G_{ij} = G_{ij}^{K\eta} + G_{ij}^{D\lambda}, \quad (2.28)$$

where $\mathcal{F}_i^{K\eta} = -\sum_{j \in C} G_{ij}^{K\eta}(U) \partial_j U$ and $\mathcal{F}_i^{D\lambda} = -\sum_{j \in C} G_{ij}^{D\lambda}(U) \partial_j U$. We may further introduce the Jacobian matrices $A_i, i \in C$, of the advection fluxes $F_i, i \in C$,

$$A_i = \partial_U F_i, \quad (2.29)$$

and finally rewrite the system into the quasi-linear form

$$\partial_t U + \sum_{i \in C} A_i \partial_i U = \sum_{i, j \in C} \partial_i (G_{ij} \partial_j U) + \Omega(U), \quad (2.30)$$

where the matrix coefficients are defined on the open convex set \mathcal{O}_U . The detailed form of the coefficient matrices $A_i, i \in C$, and $G_{ij}(U), i, j \in C$, will only be needed for some explicit calculations that are not described in the following, and, therefore, will not be given. We refer to [6] for more details in the case of vibrational equilibrium.

3. Symmetrization and normal forms

In this section, the results of Giovangigli and Massot [4] about symmetrizability are restated and applied to the multicomponent flows in full vibrational non-equilibrium governing equations. A conservative symmetric form of the system is given.

We show that the symmetrized equations are such that the nullspace naturally associated with dissipation matrices is a fixed subspace. We then know that this symmetrized system admits a set of normal form characterized by Theorem 3.8 of [4]. These normal forms are symmetric hyperbolic–parabolic composite systems. We here choose to present the one that decouples as much as possible the parabolic components even if it also perturbs the structure of the chemical/vibrational source term. Any normal form could be used for the local existence theorem obtained in the last part of the paper.

3.1. A conservative symmetric form for multicomponent flows

We now consider the system (2.30) and apply the general results obtained in a previous paper [4]. We first note that the smoothness of the matrix coefficients is a direct consequence of assumptions (H_1) and (H_2) . We next define the mathematical entropy function \mathcal{H} as the opposite of the physical mixture entropy density per unit volume

$$\mathcal{H} = - \sum_{(k, K) \in S^{\text{vib}}} \rho_{k, K} S_{k, K}. \quad (3.1)$$

We then consider the associated entropic variables

$$V = (\partial_U \mathcal{H})^t, \quad (3.2)$$

for which a straightforward calculation yields

$$V = \frac{1}{T} (\mu_{1,1} - \frac{1}{2} v \cdot v, \dots, \mu_{n_s, I_{n_s}} - \frac{1}{2} v \cdot v, v_1, v_2, v_3, -1), \tag{3.3}$$

where $\mu_{k,K}$ is the chemical/vibrational potential of the k th species in the K th vibrational quantum state [4, 6].

Proposition 2.3, expressions (2.15), (2.20), (2.21), the smoothness assumption (H₂), the triangular structure of $\partial_Y V$ over \mathcal{O}_Y and finally the fact that $-s_{k,K}$ is an increasing function of $\rho_{k,K}$ at fixed T lead to the following proposition.

Proposition 3.1. *The change of variable $U \mapsto V$ from the open convex set \mathcal{O}_U onto the open set $\mathcal{O}_V = \mathbb{R}^{n_r+3} \times (-1/T_0, 0)$ is a C^∞ diffeomorphism.*

Before enunciating Theorem 3.2 where we investigate the conservative symmetric form, let us introduce the explicit form of the matrix coefficients we are going to work with. For the sake of clarity, we here adopt a compact notation. In the following, a and b denote the couples (k, K) and (l, L) varying in the set S^{vib} .

The matrix \tilde{A}_0 is defined by

$$\tilde{A}_0 = \begin{pmatrix} \left(\frac{\rho a}{r_a} \delta_{ab} \right)_{a,b \in S^{\text{vib}}} & & \text{Sym} \\ \left(\frac{\rho b}{r_b} v_i \right)_{i \in C, b \in S^{\text{vib}}} & (\Sigma_\rho v_i v_j + \rho T \delta_{ij})_{i,j \in C} & \\ \left(\frac{\rho b}{r_b} e_b^{\text{tot}} \right)_{b \in S^{\text{vib}}} & (\Sigma_e v_j + \rho T v_j)_{j \in C} & Y_e \end{pmatrix}, \tag{3.4}$$

where

$$r_a = r_k \quad \text{for } a = (k, K)$$

and

$$\Sigma_\rho = \sum_{a \in S^{\text{vib}}} \frac{\rho a}{r_a} = \sum_{k \in S, K \in \mathcal{J}_k} \frac{\rho_{k,K}}{r_k}.$$

Similarly,

$$\Sigma_e = \sum_{(k,K) \in S^{\text{vib}}} \frac{\rho_{k,K}}{r_k} e_{k,K}^{\text{tot}}, \quad Y_e = \sum_{(k,K) \in S^{\text{vib}}} \frac{\rho_{k,K}}{r_k} e_{k,K}^2 + \rho T (v \cdot v + c_v^{\text{tot}} T).$$

Since this matrix is symmetric, we only give its block lower triangular part and write ‘Sym’ in the upper triangular part. On the other hand, denoting by $\zeta = (\zeta_1, \zeta_2, \zeta_3)^t$ an

arbitrary vector of \mathbb{R}^3 , the matrices $\tilde{A}_i, i \in C$, are defined by

$$\sum_{i \in C} \tilde{A}_i \xi_i = \begin{pmatrix} \left(\delta_{ab} \frac{\rho_b}{r_b} v \cdot \xi \right)_{a,b \in S^{vib}} & & & \text{Sym} \\ \left(\rho_b T \xi_i + \frac{\rho_b}{r_b} v_i v \cdot \xi \right)_{i \in C, b \in S^{vib}} & (\sum_{\rho} v_i v_j v \cdot \xi + \rho T(v_i \xi_j + v_j \xi_i + v \cdot \xi \delta_{ij}))_{i,j \in C} & & \\ \left(h_b^{tot} \frac{\rho_b}{r_b} v \cdot \xi \right)_{b \in S^{vib}} & (\sum_h v_j v \cdot \xi + \rho T v_j v \cdot \xi + \rho T h^{tot} \xi_j)_{j \in C} & & Y_h v \cdot \xi \end{pmatrix} \tag{3.5}$$

where

$$\Sigma_h = \sum_{(k,K) \in S^{vib}} \frac{\rho_{k,K}}{r_k} h_{k,K}^{tot}, \quad Y_h = \sum_{(k,K) \in S^{vib}} \frac{\rho_{k,K}}{r_k} h_{k,K}^2 + \rho T(v \cdot v + (c_v^{rot} + r)T).$$

Furthermore, concerning the dissipation matrices, we have the usual decomposition

$$\tilde{G}_{ij} = \tilde{G}_{ij}^{\kappa\eta} + \tilde{G}_{ij}^{D\lambda}. \tag{3.6}$$

The viscous matrices $\tilde{G}_{ij}^{\kappa\eta}, i, j \in C$, are defined by

$$\tilde{G}_{ij}^{\kappa\eta} = \begin{pmatrix} 0_{n_r \times n_r} & 0_{n_r \times 4} \\ 0_{4 \times n_r} & \tilde{\mathcal{H}}_{ij} \end{pmatrix}, \tag{3.7}$$

so that we only need the expressions of $\tilde{\mathcal{H}}_{ij}, i, j \in C$. For sake of brevity, we only define $\tilde{\mathcal{H}}_{11}$ and $\tilde{\mathcal{H}}_{12}$:

$$\tilde{\mathcal{H}}_{11} = T \begin{pmatrix} (\kappa^{rot} + \frac{4}{3}\eta^{rot}) & 0 & 0 & (\kappa^{rot} + \frac{4}{3}\eta^{rot})v_1 \\ 0 & \eta^{rot} & 0 & \eta^{rot}v_2 \\ 0 & 0 & \eta^{rot} & \eta^{rot}v_3 \\ (\kappa^{rot} + \frac{4}{3}\eta^{rot})v_1 & \eta^{rot}v_2 & \eta^{rot}v_3 & (\kappa^{rot} + \frac{1}{3}\eta^{rot})v_1^2 + \eta^{rot}v \cdot v \end{pmatrix},$$

$$\tilde{\mathcal{H}}_{12} = T \begin{pmatrix} 0 & (\kappa^{rot} - \frac{2}{3}\eta^{rot}) & 0 & (\kappa^{rot} - \frac{2}{3}\eta^{rot})v_2 \\ \eta^{rot} & 0 & 0 & \eta^{rot}v_1 \\ 0 & 0 & 0 & 0 \\ \eta^{rot}v_2 & (\kappa^{rot} - \frac{2}{3}\eta^{rot})v_1 & 0 & (\kappa^{rot} + \frac{1}{3}\eta^{rot})v_1v_2 \end{pmatrix},$$

the other matrices being obtained by circular permutation and using the relations

$$\tilde{\mathcal{H}}_{12} = \tilde{\mathcal{H}}_{21}^t, \quad \tilde{\mathcal{H}}_{13} = \tilde{\mathcal{H}}_{31}^t, \quad \tilde{\mathcal{H}}_{23} = \tilde{\mathcal{H}}_{32}^t.$$

On the other hand, the heat and mass diffusion matrices $\tilde{G}_{ij}^{D\lambda}$, $i, j \in C$, satisfy

$$\tilde{G}_{11}^{D\lambda} = \tilde{G}_{22}^{D\lambda} = \tilde{G}_{33}^{D\lambda} = \tilde{G}^{D\lambda}, \quad \tilde{G}_{ij}^{D\lambda} = 0, i \neq j,$$

where $\tilde{G}^{D\lambda}$ is defined by

$$\tilde{G}^{D\lambda} = \begin{pmatrix} (\mathcal{D}_{a,b}^{\text{rot}})_{a,b \in S^{\text{vib}}} & & \text{Sym} \\ & 0_{3 \times n_T} & 0_{3 \times 3} \\ \left(\sum_{a \in S^{\text{vib}}} \mathcal{D}_{a,b}^{\text{rot}} h_a + \rho_b \theta_b^{\text{rot}} T \right)_{b \in S^{\text{vib}}} & 0_{1 \times 3} & \lambda^{\text{rot}} T^2 + 2 \sum_{a \in S^{\text{vib}}} \rho_a \theta_a^{\text{rot}} h_a T + \sum_{a,b \in S^{\text{vib}}} \mathcal{D}_{a,b}^{\text{rot}} h_a h_b \end{pmatrix} \tag{3.8}$$

and the symmetric matrix \mathcal{D} has been defined from the multicomponent diffusion matrix D by

$$\mathcal{D}_{k,K,l,L}^{\text{rot}} = \rho_{k,K} \rho_{l,L} D_{k,K,l,L}^{\text{rot}} / r \rho. \tag{3.9}$$

Finally, the source term $\tilde{\Omega}$ is defined by

$$\tilde{\Omega} = \Omega. \tag{3.10}$$

Theorem 3.2. *The system associated with the entropic variables $V \in \mathcal{O}_V$ can then be written*

$$\tilde{A}_0 \partial_t V + \sum_{i \in C} \tilde{A}_i \partial_i V = \sum_{i,j \in C} \partial_i (\tilde{G}_{ij} \partial_j V) + \tilde{\Omega}, \tag{3.11}$$

and is of the symmetric form in the sense that Properties (S₁)–(S₄) are satisfied [4]:

- (S₁) *The matrix $\tilde{A}_0(V)$ is symmetric and positive definite for $V \in \mathcal{O}_V$.*
- (S₂) *The matrices $\tilde{A}_i(V)$, $i \in C$, are symmetric for $V \in \mathcal{O}_V$.*
- (S₃) *We have $\tilde{G}_{ij}(V)^t = \tilde{G}_{ji}(V)$ for $i, j \in C$ and $V \in \mathcal{O}_V$.*
- (S₄) *The matrix $\tilde{B}(V, w) = \sum_{i,j \in C} \tilde{G}_{ij}(V) w_i w_j$ is symmetric and positive semi-definite, for $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$, where \mathcal{S}^2 is the unit sphere in 3 dimensions.*

The function \mathcal{H} is an entropy for system (2.30), that is, \mathcal{H} satisfies Properties (E₁)–(E₄) [4]:

- (E₁) *The function \mathcal{H} is a strictly convex function on \mathcal{O}_U in the sense that the Hessian matrix is positive definite on \mathcal{O}_U .*
- (E₂) *There exists real-valued smooth functions $\sigma_i = \sigma_i(U)$ such that*

$$(\partial_U \mathcal{H}) A_i = \partial_U \sigma_i, \quad i \in C, U \in \mathcal{O}_U. \tag{3.12}$$

(E₃) We have the property

$$(\partial_U^2 \mathcal{H}(U))^{-1}(G_{ij})^t = G_{ji}(\partial_U^2 \mathcal{H}(U))^{-1}, \quad i, j \in C, \quad U \in \mathcal{O}_U. \tag{3.13}$$

(E₄) The matrix $\tilde{B}(V, w) = \sum_{i, j \in C} G_{ij}(U)(\partial_U^2 \mathcal{H}(U))^{-1} w_i w_j$ is symmetric positive semi-definite for $U \in \mathcal{O}_U$ and $w \in \mathcal{S}^2$.

Proof. The calculation of the matrices $\tilde{A}_0, \tilde{A}_i, i \in C$, and $\tilde{G}_{ij}, i, j \in C$, is lengthy but straightforward and, therefore, is omitted. This calculation is easily conducted by using the natural variable Y as an intermediate variable. The symmetry properties of $\tilde{A}_0, \tilde{A}_i, i \in C$, and $\tilde{G}_{ij}, i, j \in C$, required in (S₁)–(S₄), are then obtained. We also have the identity $\tilde{\Omega} = \Omega$, since (3.11) is derived by a change of variable.

Consider then a vector $x \in \mathbb{R}^{n_r+4}$, with components $(x_{1,1}, \dots, x_{n_s, l_{n_s}}, x_{n_r+1}, \dots, x_{n_r+4})$. After a little algebra, we obtain that

$$\begin{aligned} x^t \tilde{A}_0 x &= \rho T ((x_{n_r+1} + v_1 x_{n_r+4})^2 + (x_{n_r+2} + v_1 x_{n_r+4})^2 + (x_{n_r+3} + v_1 x_{n_r+4})^2 \\ &+ \sum_{a \in S^{\text{vib}}} \frac{\rho_a}{r_a} (x_a + v_1 x_{n_r+1} + v_2 x_{n_r+2} + v_3 x_{n_r+3} + e_a^{\text{tot}} x_{n_r+4})^2 \\ &+ \rho c_v^{\text{tot}} T^2 x_{n_r+4}^2, \end{aligned}$$

so that from (H₂) and the positivity of $\rho_a, a \in S^{\text{vib}}$, and T , we deduce that \tilde{A}_0 is positive definite.

On the other hand, by using (H₆)–(H₈), one can establish that

$$\begin{aligned} x^t \tilde{G}^{D\lambda} x &= \sum_{a, b \in S^{\text{vib}}} \mathcal{D}_{a,b}^{\text{rot}} \left(x_a + \left(h_a + \frac{p\chi_a^{\text{rot}}}{\rho_a} \right) x_{n_r+4} \right) \\ &\times \left(x_b + \left(h_b + \frac{p\chi_b^{\text{rot}}}{\rho_b} \right) x_{n_r+4} \right) + \lambda^{\text{rot}} T^2 x_{n_r+4}^2, \end{aligned} \tag{3.14}$$

which shows that $\tilde{G}^{D\lambda}$ is positive semi-definite, thanks to (H₆).

Furthermore, a straightforward calculation leads to the following expression for the quadratic form associated with $\tilde{B}(V, w)$:

$$\begin{aligned} x^t \tilde{B}(V, w) x &= T(\kappa^{\text{rot}} + \frac{1}{3}\eta^{\text{rot}})(o_1 w_1 + o_2 w_2 + o_3 w_3)^2 \\ &+ T\eta^{\text{rot}}(o_1^2 + o_2^2 + o_3^2) + x^t \tilde{G}^{D\lambda} x, \end{aligned} \tag{3.15}$$

where $o_i = x_{n_r+i} + v_i x_{n_r+4}, i = 1, 2, 3$, and where $w_1^2 + w_2^2 + w_3^2 = 1$. We thus obtain that the matrix \tilde{B} is symmetric, because it is the sum of symmetric matrices, and is positive semi-definite for $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$, thanks to the positivity assumptions (H₅) and (H₆). Finally, \mathcal{H} also satisfies (E₁)–(E₄) of [4] as is easily checked and is strictly convex since \tilde{A}_0 is positive definite over the open convex set \mathcal{O}_U . □

3.2. Normal forms for multicomponent flows

In this section we first establish that the symmetric system (3.11) satisfies an invariance property. More precisely, we show that the nullspace naturally associated with the dissipation matrices stays in a fixed subspace. This property allows us to take a first step to decouple the hyperbolic part of the system from the parabolic one. We make use of the auxiliary variables introduced in [4] which lead to a new symmetric conservative form of the system. At this point, following the authors in [4], we know the whole set of normal forms, that is, of symmetric hyperbolic–parabolic composite forms of the system. We finally chose to present the one which decouples the parabolic variables as much as possible.

Proposition 3.3. *The nullspace of the matrix \tilde{B} associated with system (3.11) is one dimensional and is given by*

$$N(\tilde{B}) = \text{span}(1, \dots, 1, 0, 0, 0, 0)^t. \tag{3.16}$$

Proof. According to equations (3.14) and (3.15), the matrix \tilde{B} is positive semi-definite, so that its nullspace is constituted by the vectors x of \mathbb{R}^{n_r+4} such that $x^t \tilde{B} x = 0$. On the other hand, we have

$$\begin{aligned} x^t \tilde{B}(V, w)x &= (\kappa^{\text{rot}} + \frac{1}{3}\eta^{\text{rot}})T(o_1 w_1 + o_2 w_2 + o_3 w_3)^2 \\ &\quad + \eta^{\text{rot}}T(o_1^2 + o_2^2 + o_3^2) + \sum_{a,b \in S^{\text{vib}}} \mathcal{D}_{a,b}^{\text{rot}} \left(x_a + \left(h_a + \frac{p\lambda_a^{\text{rot}}}{\rho_a} \right) x_{n_r+4} \right) \\ &\quad \times \left(x_b + \left(h_b + \frac{p\lambda_b^{\text{rot}}}{\rho_b} \right) x_{n_r+4} \right) + \lambda^{\text{rot}} T^2 x_{n_r+4}^2, \end{aligned}$$

where $o_i = x_{n_r+i} + v_i x_{n_r+4}$, $i = 1, 2, 3$. As a consequence, $x^t \tilde{B} x = 0$ implies that $x_{n_r+4} = 0$ and that $x_{n_r+i} = 0$, $i = 1, 2, 3$, thanks to (H₅). Therefore, x is in the nullspace of $\tilde{B}(V, w)$ if and only if we have

$$\sum_{(k, K), (l, L) \in S^{\text{vib}}} \mathcal{D}_{k,K,l,L}^{\text{rot}} x_{k,K} x_{l,L} = 0. \tag{3.17}$$

Using (H₆) and (3.9) we then obtain that the nullspace of $\tilde{B}(V, w)$ is one-dimensional and spanned by the vector $(1, \dots, 1, 0, 0, 0, 0)^t$ and is thus independent of $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$. □

Since the system of equations governing multicomponent reacting flows satisfies the previous invariance property, we can now obtain from Lemma 3.7 of [4] the auxiliary variables U' and V' which lead to a new symmetric conservative form equivalent to the first one obtained, but where the dissipation matrices only act on the parabolic part of the new entropic variables V' . From Lemma 3.7 of [4] and Proposition 3.3 we

introduce the matrix P

$$P = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.18}$$

in such a way that the new conservative variables U' are given by

$$U' = P^t U,$$

and read

$$U' = (\rho, \rho_{1,2}, \rho_{n_s, I_{n_s}}, \rho v_1, \rho v_2, \rho v_3, \rho e^{\text{tot}})^t. \tag{3.19}$$

The associated entropic variables are then $V' = P^{-1}V$ where V is given by (3.3), and the corresponding symmetric system is easily obtained from (3.11) in the following proposition.

Proposition 3.4. *The system in the new dependent variables V' ,*

$$V' = \frac{1}{T}(\mu_{1,1} - \frac{1}{2}v \cdot v, \mu_{1,2} - \mu_{1,1}, \dots, \mu_{n_s, I_{n_s}} - \mu_{1,1}, v_1, v_2, v_3 - 1)^t, \tag{3.20}$$

can be written

$$\tilde{A}'_0 \partial_t V' + \sum_{i \in C} \tilde{A}'_i \partial_i V' = \sum_{i,j \in C} \partial_i (\tilde{G}'_{ij} \partial_j V') + \tilde{\Omega}', \tag{3.21}$$

where $V' \in \mathcal{O}_{V'} = \mathbb{R}^{n_T+3} \times (-1/T_0, 0)$, $\tilde{A}'_0 = P^t \tilde{A}_0 P$, $\tilde{A}'_i = P^t \tilde{A}_i P$, $i = 1, 2, 3$, $\tilde{G}'_{ij} = P^t \tilde{G}_{ij} P$, $i, j = 1, 2, 3$, and where $\tilde{\Omega}' = P^t(\Omega) = (0, \Omega_{II})^t$.

In particular, properties (S₁)–(S₄) which express the fact that the system is symmetric, are satisfied and the dissipation matrices are given by

$$\tilde{G}'_{ij} = \begin{pmatrix} 0 & 0_{1 \times (n_T+3)} \\ 0_{(n_T+3) \times 1} & \tilde{G}^{II,II}_{ij} \end{pmatrix}, \tag{3.22}$$

where $\tilde{G}^{II,II}_{ij}$ is the lower right block of size $n_T + 3$ of \tilde{G} .

We now investigate normal forms for the system (3.11), or, equivalently, for the system (3.21). We have a comprehensive description of this set of normal forms as was shown in [4]. We use the possibility of mixing parabolic components—the V'_{II} components—established in Theorem 3.8 of [4], in order to simplify, as much as possible, the analytic expression of the normal variables, and, consequently, of the matrix coefficients appearing in the normal form. More specifically, we consider the variables

$$W = (\rho, \log(\rho_{1,2}^{r_1} / \rho_{1,1}^{r_1}), \dots, \log(\rho_{n_s, I_{n_s}}^{r_{n_s}} / \rho_{1,1}^{r_1}), v_1, v_2, v_3, T)^t,$$

easily obtained by combining the V'_{II} components, and derive the corresponding normal form of the governing equations. We could also obtain the normal form associated with the ‘natural’ normal variables $\hat{W} = (U'_I, V'_{II})^t$ which guarantees a conservative form for the dissipative terms of the system, leaves invariant the source term Ω_{II} , but has a more complex expression.

Let us first define the matrix coefficients of the normal form for the variable W , before we enounce Theorem 3.5. The matrix \bar{A}_0 is defined by

$$\bar{A}_0 = \begin{pmatrix} \frac{1}{\Sigma_\rho} & & & 0 \\ & \mathcal{X} & & \\ & & \frac{\rho}{T} I_3 & \\ 0 & & & \frac{\rho c_v^{\text{rot}}}{T^2} \end{pmatrix},$$

where \mathcal{X} is a square matrix of dimension $n_T - 1$ given by

$$\mathcal{X}_{k,K,l,L} = \delta_{k,l} \delta_{K,L} \frac{\rho_{k,K}}{r_k} - \frac{\rho_{k,K} \rho_{l,L}}{r_k r_l} \frac{1}{\Sigma_\rho}, \quad (k, K), (l, L) \in S_{\bullet}^{\text{vib}}, \tag{3.23}$$

where $S_{\bullet}^{\text{vib}} = S^{\text{vib}} \setminus \{1, 1\}$. Denoting by $\zeta = (\zeta_1, \zeta_2, \zeta_3)^t$ an arbitrary vector of \mathbb{R}^3 , the matrices $\bar{A}_i, i = 1, 2, 3$, are defined by

$$\sum_{i \in C} \bar{A}_{iu} \zeta_i = \begin{pmatrix} \frac{v \cdot \zeta}{\Sigma_\rho} & & & & \text{Sym} \\ 0_{(n_T-1) \times 1} & \mathcal{X} v \cdot \zeta & & & \\ \frac{\rho}{\Sigma_\rho} \zeta & \zeta \otimes \mathcal{Z} & \frac{\rho}{T} v \cdot \zeta I_3 & & \\ 0 & 0_{1 \times (n_T-1)} & \frac{\rho r}{T} \zeta^t & \frac{\rho c_v^{\text{rot}}}{T^2} v \cdot \zeta \end{pmatrix},$$

where \mathcal{Z} is a vector of dimension $n_T - 1$ given by

$$\mathcal{Z}_b = \rho_b - \frac{\rho_b \rho}{r_b \Sigma_\rho}, \quad b \in S_{\bullet}^{\text{vib}}.$$

Note that the exponents for the matrix coefficients and the indices for the variables are related to the bloc decomposition associated with the partitioning of the variable $W = (W_I, W_{II})^t$.

Theorem 3.5. *The system in the variables $W = (W_I, W_{II})^t$, on the open convex set $\mathcal{O}_W = (0, \infty) \times \mathbb{R}^{n_r-1} \times \mathbb{R}^3 \times (T_0, \infty)$, with hyperbolic variable $W_I = (\rho)$ and parabolic variables $W_{II} = (\log(\rho_{1,2}^r/\rho_{1,1}^r), \dots, \log(\rho_{n_s, I_{n_s}}^r/\rho_{1,1}^r), v_1, v_2, v_3, T)^t$, can be written*

$$\bar{A}_0^{I,I} \partial_t W_I + \sum_{i \in C} \bar{A}_i^{I,I} \partial_i W_I + \sum_{i \in C} \bar{A}_i^{I,II} \partial_i W_{II} = 0, \tag{3.24}$$

$$\begin{aligned} &\bar{A}_0^{II,II} \partial_t W_{II} + \sum_{i \in C} \bar{A}_i^{II,I} \partial_i W_I + \sum_{i \in C} \bar{A}_i^{II,II} \partial_i W_{II} \\ &= \sum_{i,j \in C} \partial_i (\bar{G}_{ij}^{II,II} \partial_j W_{II}) + \bar{H}_{II} + \bar{\Omega}_{II}, \end{aligned} \tag{3.25}$$

and is of the normal form.

Proof. The calculations are lengthy but straightforward and make use of Theorem 3.2, Proposition 3.4 and assumptions (H₁)–(H₉). □

Remark 3.6. Note that when the source term Ω remains in a fixed subspace of $\mathbb{R}^{n_r} \times 0_{\mathbb{R}^4}$, the source term $\bar{\Omega}$ is no longer in a fixed subspace of \mathbb{R}^{n_r+4} of the same dimension because of the coefficients $e_{k,K}/T^2$ in the term $\sum_{(k,K) \in S^{vb}} e_{k,K} m_k \omega_{k,K}/T^2$ which introduce an explicit dependence on the state variables.

4. The Cauchy problem

In this section we first restate an existence theorem of Vol’pert and Hudjaev concerning symmetric hyperbolic–parabolic composite systems. We only present a simplified quasilinear version of their existence result [7]. We then apply this existence result to the equations governing multicomponent reactive flows in full vibrational non-equilibrium, using the normal form obtained in the previous section.

4.1. The mathematical framework

We consider the Cauchy problem for an abstract system of partial differential equations in the form

$$\begin{aligned} \bar{A}_0^{*I,I} \partial_t W_I^* &= - \sum_{i \in C^*} \bar{A}_i^{*I,I} \partial_i W_I^* + \bar{\Gamma}_I^*, \\ \bar{A}_0^{*II,II} \partial_t W_{II}^* &= - \sum_{i \in C^*} \bar{A}_i^{*II,I} \partial_i W_I^* + \sum_{i,j \in C^*} \partial_i (\bar{G}_{ij}^{*II,II} \partial_j W_{II}^*) + \bar{\Gamma}_{II}^* \end{aligned} \tag{4.1}$$

where $I = \{1, \dots, n_0\}$ and $II = \{n_0 + 1, \dots, n\}$ form a partition of $\{1, \dots, n\}$, $W^* = (W_I^*, W_{II}^*)$ is the corresponding decomposition of the unknown vector W^* ,

and where $C^* = \{1, \dots, d\}$ is the indexing set of spatial coordinates. Note that the superscript $*$ is used in order to distinguish between the abstract second order system (4.1) of size n and \mathbb{R}^d and the particular multicomponent reactive flows system (3.24), (3.25) of size $n_T + 4$ in \mathbb{R}^3 . These equations are considered in the strip \bar{Q}_Θ where Θ is positive and $Q_t = (0, t) \times \mathbb{R}^d$ for $t > 0$. The matrix and vector coefficients are assumed to be in the form

$$\begin{aligned} \bar{A}_0^{*I,I} &= \bar{A}_0^{*I,I}(W_I^*, W_{II}^*), \\ \bar{A}_0^{*II,II} &= \bar{A}_0^{*II,II}(W_I^*, W_{II}^*), \\ \bar{A}_i^{*I,I} &= \bar{A}_i^{*I,I}(W_I^*, W_{II}^*, \partial_x W_{II}^*), \quad i \in C^*, \\ \bar{A}_i^{*II,I} &= \bar{A}_i^{*II,I}(W_I^*, W_{II}^*, \partial_x W_{II}^*), \quad i \in C^*, \\ \bar{G}_{ij}^{*II,II} &= \bar{G}_{ij}^{*II,II}(W_I^*, W_{II}^*), \quad i, j \in C^*, \\ \bar{\Gamma}_I^* &= \bar{\Gamma}_I^*(W_I^*, W_{II}^*, \partial_x W_{II}^*), \\ \bar{\Gamma}_{II}^* &= \bar{\Gamma}_{II}^*(W_I^*, W_{II}^*, \partial_x W_{II}^*). \end{aligned} \tag{4.2}$$

We assume that the matrices $\bar{A}_0^{*I,I}(z)$, $\bar{A}_0^{*II,II}(z)$, and $\bar{G}_{ij}^{*II,II}(z)$, $i, j \in C^*$, are smooth functions of $z \in \mathcal{O}_{W^*}$, where \mathcal{O}_{W^*} is a convex open set of \mathbb{R}^n . Similarly, we assume that the matrices $\bar{A}_i^{*I,I}(z, \zeta)$, $i \in C^*$, the matrices $\bar{A}_i^{*II,I}(z, \zeta)$, $i \in C^*$, and the vectors $\bar{\Gamma}_I(z, \zeta)$ and $\bar{\Gamma}_{II}(z, \zeta)$ are smooth functions of $z \in \mathcal{O}_{W^*}$, and $\zeta \in \mathbb{R}^{d \times (n-n_0)}$.

We will use the classical functional spaces $L^p(\mathbb{R}^d)$ with norm

$$\|\phi\|_{0,p} = |\phi|_{0,p} = \left(\int_{\mathbb{R}^d} |\phi(x)|^p dx \right)^{1/p}, \quad p \geq 1,$$

the Sobolev spaces $W_p^l(\mathbb{R}^d)$, $1 \leq p \leq \infty$, with norm

$$\|\phi\|_{l,p} = \sum_{k \in [0,l]} |\phi|_{k,p}, \quad |\phi|_{k,p} = \sum_{|\beta|=k} \|\partial^\beta \phi\|_{0,p},$$

and the functional spaces $V_l(\mathbb{R}^d)$ with norm [7]

$$\|\phi\|_l = |\phi|_{0,\infty} + \sum_{k \in [1,l]} |\phi|_{k,2}.$$

We extend these definition to vector functions by using the Euclidian norm of \mathbb{R}^d . According to the Sobolev inequalities, there is an imbedding of $W_2^l(\mathbb{R}^d)$ into $W_\infty^k(\mathbb{R}^d)$ for $l > d/2 + k$, and an imbedding of $W_2^l(\mathbb{R}^d)$ into $V_l(\mathbb{R}^d)$ for $l > d/2$. In the following, \mathcal{L} denotes an arbitrary fixed positive continuous convex function, on the open convex set \mathcal{O}_{W^*} , which grows without bound as any finite point of the boundary of \mathcal{O}_{W^*} is approached.

4.2. An existence theorem in $V_l(\mathbb{R}^d)$

We consider the Cauchy problem for the system (4.1), (4.2), with smooth initial conditions

$$W^*(0, x) = W^{*0}(x). \tag{4.3}$$

The following theorem of Vol’pert and Hudjaev shows that, in a certain strip, there exists a solution which preserves the smoothness of the initial condition [7].

Theorem 4.1. *Suppose that the system (4.1)–(4.3) satisfies the following assumptions where l denotes an integer such that $l > d/2 + 3$.*

- (Ex₁) *The initial condition W^{*0} satisfies $\sup_{x \in \mathbb{R}^d} \mathcal{L}(W^{*0}(x)) < +\infty$ and W^{*0} is in the space $V_l(\mathbb{R}^d)$.*
- (Ex₂) *The matrix coefficients $\bar{A}_0^{*I,I}(z)$, $\bar{A}_0^{*II,II}(z)$, and $\bar{G}_{ij}^{*II,II}(z)$, $i, j \in C^*$, have continuous derivative of order $l > d/2 + 3$ with respect to $z \in \mathcal{O}_{W^*}$.*
- (Ex₃) *The matrix coefficients $\bar{A}_i^{*I,I}(z, \zeta)$ and $\bar{A}_i^{*II,I}(z, \zeta)$, $i \in C^*$, and the vectors $\bar{\Gamma}_I^*(z, \zeta)$ and $\bar{\Gamma}_{II}^*(z, \zeta)$, have continuous derivative of order $l > d/2 + 3$ with respect to $z \in \mathcal{O}_{W^*}$ and $\zeta \in \mathbb{R}^{d \times (n-n_0)}$.*
- (Ex₄) *The matrix coefficients $\bar{A}_0^{*I,I}$ and $\bar{A}_0^{*II,II}$ are symmetric and positive definite for $z \in \mathcal{O}_{W^*}$.*
- (Ex₅) *The matrix coefficients $\bar{A}_i^{*I,I}(z, \zeta)$, $i \in C^*$, are symmetric for $z \in \mathcal{O}_{W^*}$ and $\zeta \in \mathbb{R}^{d \times (n-n_0)}$.*
- (Ex₆) *The matrices $\bar{A}_0^{*I,I}$ and $\bar{A}_0^{*II,II}$ and the vectors $\bar{\Gamma}_I^*(z, 0)$ and $\bar{\Gamma}_{II}^*(z, 0)$ have continuous derivatives to order $l + 3$ in z .*
- (Ex₇) *For any compact subset K of \mathcal{O}_{W^*} , there exists $\alpha = \alpha(K)$ such that for any smooth function z from \mathbb{R}^d to \mathbb{R}^n with values in K we have*

$$\int_{\mathbb{R}^d} \sum_{i, j \in C^*} (\partial_i \phi_{II})^t \bar{G}_{ij}^{*II,II}(z(x)) (\partial_j \phi_{II}) \, dx \geq \alpha \int_{\mathbb{R}^d} \sum_{i \in C^*} (\partial_i \phi_{II})^t (\partial_i \phi_{II}) \, dx, \tag{4.4}$$

where ϕ_{II} is any function in $W^{1/2}(\mathbb{R}^d)$ with $n - n_0$ components.

Then there exists t_0 , $0 < t_0 \leq \Theta$, such that the Cauchy problem (4.1), (4.2), admits a unique solution $W^* = (W_I^*, W_{II}^*)^t$ defined on $[0, t_0] \times \mathbb{R}^d$, which is continuous with its derivatives of first order in t and second order in x , and for which the following quantities are finite:

$$\sup_{0 \leq t \leq t_0} \|W^*(t)\|_t, \quad \sup_{\bar{Q}_0} \mathcal{L}(W^*), \tag{4.5}$$

$$\sup_{0 \leq t \leq t_0} \|\partial_t W_I^*(t)\|_{l-1}, \quad \int_0^{t_0} (\|\partial_t W_{II}^*(\tau)\|_{l-1}^2 + \|W_{II}^*(\tau)\|_{l+1}^2) \, d\tau. \tag{4.6}$$

Moreover, either $t_0 = \Theta$, or there exists t_1 such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \rightarrow t_1^-$, at least one of the quantities

$$\|W_I^*(t_0)\|_{1,\infty} + \|W_{II}^*(t_0)\|_{2,\infty}, \quad \sup_{\bar{Q}_0} \mathcal{L}(W^*), \tag{4.7}$$

grows without bound, that is to say, the solution can be extended as long as quantities (4.7) remain finite.

Remark 4.2. Theorem 4.1 is a simplified quasilinear version of the results obtained by Vol’pert and Hudjaev [7]. The dependence on (t, x) of the various coefficients can also be included as detailed in [7] and [6]. In this situation, the assumptions (Ex₂), (Ex₃) and (Ex₆) become

- (Ex₂) The matrix coefficients $\bar{A}_0^{*I,I}(t, x, z)$, $\bar{A}_0^{*II,II}(t, x, z)$, and $\bar{G}_{ij}^{*II,II}(t, x, z)$, $i, j \in C^*$, have continuous and bounded derivatives until order $l > d/2 + 3$ with respect to $(t, x) \in \bar{Q}_\Theta$ and $z \in \mathcal{O}_{W^*}$.
- (Ex₃) The matrix coefficients $\bar{A}_i^{*I,I}(t, x, z, \zeta)$ and $\bar{A}_i^{*II,I}(t, x, z, \zeta)$, $i \in C^*$, and the vectors $\bar{\Gamma}_I^*(t, x, z, \zeta)$ and $\bar{\Gamma}_{II}^*(t, x, z, \zeta)$, have continuous and bounded derivatives until order $l > d/2 + 3$ with respect to $(t, x) \in \bar{Q}_\Theta$, $z \in \mathcal{O}_{W^*}$ and $\zeta \in \mathbb{R}^{d \times (n-n_0)}$.
- (Ex₆) The matrices $\bar{A}_0^{*I,I}(t, x, z)$ and $\bar{A}_0^{*II,II}(t, x, z)$ and the vectors $\bar{\Gamma}_I^*(t, x, z, 0)$ and $\bar{\Gamma}_{II}^*(t, x, z, 0)$, have continuous and bounded derivatives to order $l + 3$ in (t, x, z) .

In addition, the following equi-integrability assumption is also needed and was overlooked in [7]:

- (Ex₈) For any compact set K of \mathcal{O}_W , the functions $(\partial_x^\nu(\bar{A}_0^{*-1}))(t, \cdot, z(t, \cdot), 0)$ and $(\partial_x^\nu \bar{\Gamma}^*)(t, \cdot, z(t, \cdot), 0)$, where z is any smooth function on Q_Θ taking its values in K , are uniformly bounded in L^2 for $1 \leq |\nu| \leq l + 2$.

Note that this property is automatically satisfied with the assumptions of Theorem 4.1 since these functional are independent of (t, x) .

Remark 4.3. We have kept the second-order terms in divergence form for convenience.

4.3. Application to multicomponent flows

We now apply Theorem 4.1 to the system modelling multicomponent reactive flows in \mathbb{R}^d with $d \in \{1, 2, 3\}$.

Theorem 4.4. Consider the Cauchy problem for the system (3.24), (3.25) in \mathbb{R}^d with $d \in \{1, 2, 3\}$

$$\bar{A}_0 \partial_t W + \sum_{i \in C} \bar{A}_i \partial_i W = \sum_{i, j \in C} \partial_i (\bar{G}_{ij} \partial_j W) + \bar{H} + \bar{\Omega}, \tag{4.8}$$

with initial conditions

$$W(0, x) = W^0(x), \tag{4.9}$$

where

$$W^0 \in V_l(\mathbb{R}^d), \quad \inf_{\mathbb{R}^d} \rho^0(x) > 0, \quad \inf_{\mathbb{R}^d} T^0(x) > T_0. \tag{4.10}$$

Then there exists $t_0, 0 < t_0 < \infty$ such that (4.8), (4.9) admit a unique solution $W = (W_I, W_{II})^t$ with $W(t, x) \in \mathcal{O}_W$ in the strip $\bar{Q}_{t_0} = [0, t_0] \times \mathbb{R}^d$, continuous in \bar{Q}_{t_0} with its derivatives of first order in t and second order in x , and for which the following inequalities hold;

$$\begin{aligned} & \sup_{0 \leq t \leq t_0} \left(\|\rho(t)\|_l + \sum_{(k, K) \in S_{\bullet}^{\text{vib}}} \|\log(\rho_{k, K}^r / \rho_{1, 1}^r)(t)\|_l + \sum_{i \in C} \|v_i(t)\|_l + \|T(t)\|_l \right) < +\infty, \\ & \inf_{\bar{Q}_{t_0}} \rho(t, x) > 0, \quad \inf_{\bar{Q}_{t_0}} T(t, x) > T_0, \\ & \sup_{0 \leq t \leq t_0} \|\partial_t \rho(t)\|_{l-1} < +\infty, \\ & \int_0^{t_0} \left(\sum_{(k, K) \in S_{\bullet}^{\text{vib}}} (\partial_t \log(\rho_{k, K}^r / \rho_{1, 1}^r)(\tau))^2_{l-1} + \sum_{i \in C} \|\partial_t v_i(\tau)\|_{l-1}^2 + \|\partial_t T(\tau)\|_{l-1}^2 \right. \\ & \quad \left. + \sum_{(k, K) \in S_{\bullet}^{\text{vib}}} \|\log(\rho_{k, K}^r / \rho_{1, 1}^r)(\tau)\|_{l+1}^2 + \sum_{i \in C} \|v_i(\tau)\|_{l+1}^2 + \|T(\tau)\|_{l+1}^2 \right) d\tau < +\infty. \end{aligned}$$

Moreover, either $t_0 = +\infty$, or there exists t_1 such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \rightarrow t_1^-$, either the following quantity,

$$\|\rho(t_0)\|_{1, \infty} + \sum_{(k, K) \in S_{\bullet}^{\text{vib}}} \|\log(\rho_{k, K}^r / \rho_{1, 1}^r)(t_0)\|_{2, \infty} + \sum_{i \in C} \|v_i(t_0)\|_{2, \infty} + \|T(t_0)\|_{2, \infty}, \tag{4.11}$$

grows without bound or $\inf_{Q_{t_0}} T \rightarrow T_0$.

Proof. Assume first that κ^{rot} is a positive function and rewrite the system (3.24), (3.25) of Theorem 3.5 in the form (4.1), (4.2) with

$$\begin{aligned} \bar{\Gamma}_I &= - \sum_{j \in C} \bar{A}_j^{I, II} \partial_j W_{II}, \\ \bar{\Gamma}_{II} &= \bar{H}_{II} + \bar{Q}_{II} - \sum_{j \in C} \bar{A}_j^{I, II} \partial_j W_{II}. \end{aligned}$$

We can then apply Theorem 4.1 to the resulting system with $n = n_T + 1 + d$ and $n_0 = 1$, where $d \in \{1, 2, 3\}$ is the particular dimension of interest.

Indeed, Properties (EX₁)–(EX₆) are easily checked from Theorem 3.5, the assumptions, and Properties (H₁)–(H₉). On the other hand, in order to establish (EX₇), we consider separately the contributions of the matrices $\bar{G}_{ij}^{D\lambda}$, $i, j \in C$, and $\bar{G}_{ij}^{\kappa\eta}$, $i, j \in C$. From Theorem 3.5, we have $\bar{G}^{D\lambda} = \bar{G}^{D\lambda}$, $i \in C$, whereas $\bar{G}_{ij}^{D\lambda} = 0$ for $i \neq j$. After a little algebra, we obtain for $\zeta_{II} = (\zeta_{1, 2}, \dots, \zeta_{n_s, I_{n_s}}, \zeta_{n_T+1}, \dots, \zeta_{n_T+4})^t \in \mathbb{R}^{n-n_0}$ that

$$\begin{aligned} \zeta_{II}^t \bar{G}^{D\lambda, II, II}(W) \zeta_{II} &= \sum_{a, b \in S^{\text{vib}}} \mathcal{D}_{a, b}^{\text{rot}} \left(\zeta_a + \left(\frac{r_a}{T} + \frac{p\chi_a^{\text{rot}}}{T^2 \rho_a} \right) \zeta_{n_T+4} \right) \\ &\quad \times \left(\zeta_b + \left(\frac{r_b}{T} + \frac{p\chi_b^{\text{rot}}}{T^2 \rho_b} \right) \zeta_{n_T+4} \right) + \frac{\lambda^{\text{rot}}}{T^2} \zeta_{n_T+4}^2, \end{aligned}$$

where we have extended ξ_{II} into $\xi = (0, \xi_{II})^t$. This identity shows that the matrix $\bar{G}^{D\lambda II, II}$ is positive definite on the subspace $\xi_{1,1} = \xi_{n_r+1} = \xi_{n_r+2} = \xi_{n_r+3} = 0$ for any $W \in \mathcal{O}_W$. As a consequence, for any function ϕ_{II} in $W^1_2(\mathbb{R}^d)$ with $n - n_0$ components, denoted by $\phi_{II} = (\phi_{1,2}, \dots, \phi_{n_s, I_{n_s}}, \phi_{n_r+1}, \dots, \phi_{n_r+4})$, we have

$$\partial_i \phi_{II}^t \bar{G}_{ii}^{D\lambda II, II}(W) \partial_i \phi_{II} \geq \alpha (\partial_i \phi_{1,2}^2 + \dots + \partial_i \phi_{n_s, I_{n_s}}^2 + \partial_i \phi_{n_r+4}^2), \tag{4.12}$$

for $i \in C$, uniformly for W in a given compact of $\mathcal{O}(W)$. On the other hand, after a little algebra, we also obtain from Theorem 3.5 that

$$\begin{aligned} \sum_{i,j \in C} \partial_i \phi_{II}^t \bar{G}_{ij}^{\kappa \eta II, II}(W) \partial_j \phi_{II} &= \frac{\eta^{\text{rot}}}{T} \sum_{i,j \in C} \left(\partial_i \mathfrak{v}_j + \partial_j \mathfrak{v}_i - \frac{2}{3} \left(\sum_{k \in C} \partial_k \mathfrak{v}_k \right) \delta_{ij} \right)^2 \\ &\quad + \frac{\kappa^{\text{rot}}}{T} \left(\sum_{k \in C} \partial_k \mathfrak{v}_k \right)^2, \end{aligned} \tag{4.13}$$

where $\mathfrak{v}_i = \phi_{n_r+i}$, $i \in C$. Since η^{rot} and κ^{rot} are positive, we have

$$\sum_{i,j \in C} \partial_i \phi_{II}^t \bar{G}_{ij}^{\kappa \eta II, II}(W) \partial_j \phi_{II} \geq \alpha \sum_{i,j \in C} (\partial_i \mathfrak{v}_j + \partial_j \mathfrak{v}_i)^2, \tag{4.14}$$

where α is a positive constant, uniformly for W in a compact of \mathcal{O}_W . Combining the estimates (4.12) and (4.14) with the identity

$$\int_{\mathbb{R}^d} \sum_{i,j \in C} (\partial_i \mathfrak{v}_j + \partial_j \mathfrak{v}_i)^2 dx = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j \in C} (\partial_i \mathfrak{v}_j)^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} \left(\sum_{i \in C} \partial_i \mathfrak{v}_i \right)^2 dx,$$

valid for $\mathfrak{v}_i \in W^1_2(\mathbb{R}^d)$, $i \in C$, we obtain (Ex₇). Finally, we note that from the conservation of ρ

$$\rho(t, x) \geq \inf_{\mathbb{R}^d} \rho^0(x) \exp \left(- \int_0^t \|\partial_x \cdot v(s)\|_{0, \infty} ds \right),$$

and thus $\inf_{\mathbb{R}^d} \rho(t, x) > 0$ as long as (4.11) remains finite, so that only T may reach the boundary of \mathcal{O}_W .

On the other hand, when κ^{rot} is only a non-negative function, we rewrite equation (3.25) in a different form. We first note that

$$\sum_{i,j \in C} \partial_i (\bar{G}_{ij}^{\kappa \eta II, II} \partial_j W_{II}) = \partial_x \cdot \left(\frac{\eta^{\text{rot}}}{T} (\partial_x v + (\partial_x v)^t) + \frac{\kappa^{\text{rot}} - \frac{2}{3} \eta^{\text{rot}}}{T} \partial_x \cdot v I \right),$$

and we have the identity [7]

$$\begin{aligned} \partial_x \cdot \left(\frac{\eta^{\text{rot}}}{T} (\partial_x v + (\partial_x v)^t) + \frac{\kappa^{\text{rot}} - \frac{2}{3} \eta^{\text{rot}}}{T} \partial_x \cdot v I \right) &= \sum_{i \in C} \partial_i \left(\frac{\eta^{\text{rot}}}{T} \partial_i v \right) \\ &\quad + \partial_x \cdot \left(\frac{\kappa^{\text{rot}} + \frac{1}{3} \eta^{\text{rot}}}{T} \partial_x \cdot v I \right) + \sum_{i \in C} \partial_i \left(\frac{\eta^{\text{rot}}}{T} \right) \partial_x v_i - (\partial_x \cdot v) \partial_x \left(\frac{\eta^{\text{rot}}}{T} \right). \end{aligned}$$

We can thus rewrite equation (3.25) in the form

$$\bar{A}_0^{II,II} \partial_t W_{II} = - \sum_{i \in C} \bar{\mathcal{A}}_i^{II,I} \partial_i W_I + \sum_{i,j \in C} \partial_i (\bar{\mathcal{G}}_{ij}^{II,II} \partial_j W_{II}) + \bar{\Gamma}_{II},$$

with new coefficients $\bar{\mathcal{A}}_i^{II,I}, i \in C$, and $\bar{\mathcal{G}}_{ij}^{II,II}, i, j \in C$, such that

$$\sum_{i,j \in C} \partial_i (\bar{\mathcal{G}}_{ij}^{II,II} \partial_j W_{II}) = \sum_{i \in C} \partial_i \left(\frac{\eta^{\text{rot}}}{T} \partial_i v \right) + \partial_x \cdot \left(\frac{\kappa^{\text{rot}} + \frac{1}{3} \eta^{\text{rot}}}{T} \partial_x \cdot v I \right),$$

and with the lower order factors $\sum_{i \in C} \partial_i (\eta^{\text{rot}}/T) \partial_x v_i$ and $-(\partial_x \cdot v) \partial_x (\eta^{\text{rot}}/T)$ dispatched in the terms $\bar{\mathcal{A}}_i^{II,I} \partial_i W_I, i \in C$, and $\bar{\Gamma}_{II}$.

More specifically, we define $\bar{\mathcal{G}}_{ij}^{II,II} = \bar{\mathcal{G}}_{ij}^{D\lambda II,II} + \bar{\mathcal{G}}_{ij}^{\kappa\eta II,II}, i, j \in C$, where $\bar{\mathcal{G}}_{ij}^{D\lambda II,II} = \bar{G}_{ij}^{D\lambda II,II}, i, j \in C$, whereas $\bar{\mathcal{G}}_{ij}^{\kappa\eta II,II}, i, j \in C$, are given by

$$\bar{\mathcal{G}}_{ii}^{\kappa\eta II,II} = \bar{G}_{ii}^{\kappa\eta II,II}, \quad i \in C$$

and

$$\bar{\mathcal{G}}_{ij}^{\kappa\eta II,II} = \begin{pmatrix} 0_{n_r \times n_r} & 0_{n_r \times 4} \\ 0_{4 \times n_r} & (\kappa^{\text{rot}} + \frac{1}{3} \eta^{\text{rot}}) (\delta_{(n_r+k)(n_r+i)} \delta_{(n_r+l)(n_r+j)})_{k,l \in [1,4]} \end{pmatrix}$$

for $i, j \in C, i \neq j$.

Furthermore, we define $\bar{\mathcal{A}}_i^{II,I}, i \in C$, and $\bar{\Gamma}_{II}$ by

$$\bar{\mathcal{A}}_i^{II,I} = \bar{A}_i^{II,I} + (\partial_x \cdot v) \partial_\rho \left(\frac{\eta^{\text{rot}}}{T} \right)$$

and

$$\begin{aligned} \bar{\Gamma}_{II} = & \sum_{i \in C} \partial_i \left(\frac{\eta^{\text{rot}}}{T} \right) \partial_x v_i - (\partial_x \cdot v) \left(\partial_{W_{II}} \left(\frac{\eta^{\text{rot}}}{T} \right) \right) \partial_x W_{II} + \bar{H}_{II} + \bar{\Omega}_{II} \\ & - \sum_{j \in C} \bar{A}_j^{I,II} \partial_j W_{II}, \end{aligned}$$

and the end of the proof is similar to that of the preceding case. □

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