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Revisiting the Greeks for European and American options.

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REVISITING THE GREEKS FOR EUROPEAN AND AMERICAN OPTIONS*

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In this paper, we address the problem of the Greeks' evaluation for European and American options, when the model is defined by a general stochastic differential equation. We represent the Greeks as expectations, in order to allow their computations using Monte Carlo simulations. We avoid the use of Malliavin calculus techniques since in general, it leads to random variables whose simulations are costly in terms of computational time. We take advantage of the Markovian structure to derive simple formulas in a great generality. Moreover, they appear to be efficient in practice.

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1. Statement of the problem

A main issue related to the option pricing is the determination of sensitivities of the option price with respect to the parameters defining the model of the underlying asset. These quantities are called the Greeks and they are computed as appropriate derivatives of the option price. The purpose of this work is to revisit some classic issues on the subject, using a point of view which differs from the previous ones. Firstly, we rederive known formulas with a simplified proof, that is without Malliavin calculus techniques, but with only the standard Itô's calculus. Secondly, we establish new representations for the Greeks, still by using simple arguments.

Now, let us specify our model. We consider a frictionless financial market, where we can trade d risky assets, with price process $[S_t = (S_t^1, \cdots, S_t^d)^*]_{0\leq t\leq T}$, and a non risky asset $(S_t^0)_{0\leq t\leq T}$. We assume $S_t^0 = 1$, which is still possible using a change of numéraire [MR98].

The log-price process $[X_t = (\log(S_t^1), \cdots, \log(S_t^d))^*]_{0\leq t\leq T}$ is assumed to be the solution of the stochastic differential equation

$$X_t = x + \int_0^t b(s, X_s)ds + \sum_{i=1}^{d} \int_0^t \sigma_i(s, X_s)dW_s^i. \tag{1}$$

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In the equation above, $W = (W^1, \cdots, W^d)^*$ is a standard d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$: we assume that the filtration is the one generated by W augmented by the null sets. Smoothness conditions on b and $\sigma = (\sigma_1, \cdots, \sigma_d)$ will be stated later on. Moreover, we will assume the uniform ellipticity of σ. As a consequence, the market is complete and we suppose that the dynamics (1) is directly given under the risk-neutral probability.

The price at time 0 of an European option with maturity T and payoff[a] F, is thus given by $P(0, S_0) = \mathbf{E}(F|S_0)$. Essentially[b], the hedging strategy Δ_0 at time 0 is given by $\partial_{S_0} P(0, S_0)$. A Monte Carlo approach to evaluate $\mathbf{E}(F)$ is usual. In particular, it is more efficient compared to a PDE approach, either because the dimension d is not small (more than 2), either because the payoff F is highly path-dependent and simulations are more suitable. In that case, it is useful to also evaluate Δ_0 using simulations. Three approaches are possible.

1. The *re-sampling method* or *finite difference method* [GY92 LP94] is based on the computation of $P(0, S)$ for different values of S close to S_0. Then, appropriate differences are formed in order to approximate the gradient by a discrete derivative. Unfortunately, it provides biased estimators and moreover, it is costly when d is large.
2. The *path-wise method* [BG96] consists in putting the differentiation inside the expectation. It gives $\Delta_0 = \mathbf{E}(\partial_{S_0} F)$ provided that $\partial_{S_0} F$ is meaningful, and direct Monte Carlo simulations can be performed to complete the evaluation of Δ_0. The limitation of this method is due to the possible lack of regularity of F: for instance, it cannot be applied to digital or barrier options.
3. The *likelihood method* or *score method* (introduced by Glynn [Gly86,Gly87], Reiman et al.[RW86], and later in finance by Broadie et al.[BG96]) differentiates the law of the payoff instead of the payoff itself. It writes $\Delta_0 = \mathbf{E}(F\,H)$ for some random variable[c] H. When the underlying asset has an explicit law (like in the geometric Brownian motion case), it is possible for some payoff to get an explicit expression for H, using a finite-dimensional integration by parts formula [BG96] (ibp in short). During the past five years, a lot of attention has been paid to the derivation of suitable weights H for various payoffs and for general models of type (1). The basic tool is the Malliavin calculus, and in particular the ibp formula on the infinite-dimensional Wiener space. For the case of vanilla, see [FLLLT99]; for Asian options, see [FLLLT99] and [Ben00]; for barrier and lookback options, see [GKH03] and [BGKH03].

Consequently, there is a lot of interest for the Malliavin calculus in finance (we refer

[a]the random variable F is square integrable and $\mathcal{F}_T$-measurable.
[b]this is not a general result, but a common feature for usual options in a Markovian setting. For a recent study on the Δ-hedging, see the paper by Bermin [Ber03].
[c]H is not unique since any random variable orthogonal to F can be added.

to [KHMar] for a nice expository work) but we may formulate two major criticisms to the use of these tools. Firstly, it is difficult to control, and the non academics may face some serious difficulties to correctly manipulate these concepts, with the inherent risk to devise wrong numerical procedures. The second reason has to do with the efficiency of the method. In general, the weights H are given by Skorohod integrals, whose simulation is far to be easy. For log-normal dynamics, there are many simplifications but for less specific models, it requires the simulation of many auxiliary processes, which significantly increases the computational time.

This work is aimed at deriving representations for H, in terms of explicit stochastic integrals. It leads to simple and efficient numerical procedures as experiments will illustrate. For this, the Malliavin calculus techniques are replaced by more standard tools, like martingales or Itô's calculus. In fact, the usual ibp of Malliavin calculus [Nua95] is intrinsically a **static** operation, i.e. it focuses on the law of a random variable such as X_T for instance. Our alternative approach is different and it leverages the **dynamic** structure of the process $(X_t)_{0\leq t\leq T}$. This type of idea dates back to Bismut [Bis84]. We do not assert that the Malliavin calculus is no more useful for the Greeks, but it appears to be unnecessary in many situations.

The outline of the paper is the following. In section 2, we focus on the Delta for European vanilla options: we obtain the same formula than in [FLLLT99] but our proof is different. This simple case enables us to introduce the methodology which will be used later on. Then, we handle general European barrier and lookback options, and also American vanilla options: the derived representations are new to our knowledge. In section 3, we address the sensitivity w.r.t. the volatility (called the Vega index), which is useful when the impact of a model misspecification needs to be measured. The given formula has been already proved in [GM02], but here we provide a simplified proof. Actually, it appears to be a particular case of a nice ibp formula, which has the advantage to use only the first derivatives of the coefficients b and σ, whereas the usual ibp formula needs two derivatives. A recent study related to the sensitivity analysis w.r.t. a boundary [CKG03] is also reported and the connection with American options is discussed. A numerical experiment illustrates the efficiency of our approach. Finally, we conclude in section 4, by listing some open problems.

To simplify the notations, sensitivities are computed relatively to the parameters defining X in (1), that is we skip the influence of the logarithm change of variables between X and S. For instance, for the Δ, we will compute $\partial_x\mathbf{E}(F)$ instead of $\partial_{S_0}\mathbf{E}(F)$.

Notations and assumptions

In the sequel, we adopt the following usual convention on the gradients. If $\psi : (t,x)\in[0,T]\times\mathbf{R}^d\mapsto\mathbf{R}^q$ is a differentiable function w.r.t. x, its gradient w.r.t. x is denoted $\psi'(t,x)=(\partial_{x_1}\psi(t,x),\cdots,\partial_{x_d}\psi(t,x))$ and it takes values in $\mathbf{R}^q\otimes\mathbf{R}^d$. When the function ψ is evaluated at the point (t,X_t), we may simply note ψ_t instead of

$\psi(t, X_t)$.

Throughout the paper, we will consider the following assumptions on the coefficients defining X.

(H) the functions b and σ are bounded, continuous, continuously differentiable w.r.t. x with uniformly bounded derivatives. The functions $b, b', \sigma, (\sigma'_i)_{1\leq i\leq d}$ satisfy a Hölder continuity property: for some $\eta > 0$, $|b(t,x) - b(s,y)| \leq C(|t-s|^{\eta/2} + |y-x|^{\eta})$ uniformly in s, t, x, y, and analogously for $\sigma, b', (\sigma'_i)_{1\leq i\leq d}$.
Moreover, for some $a_0 > 0$, one has $\xi.[\sigma\sigma^*](t,x)\xi \geq a_0|\xi|^2$ for any $(t, x, \xi) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^d$.

Under the smoothness assumptions above, there is a unique strong solution to (1). Let L be the infinitesimal generator of X, defined by

$$Lu(t,x) = u'(t,x)b(t,x) + \frac{1}{2}\mathrm{Tr}(H_u(t,x)[\sigma\sigma^*](t,x)),$$

where H_u is the Hessian matrix of u w.r.t. the space variable x. Besides, one has for any $c > 0$

$$\mathbf{E}(e^{c\sup_{t\in[0,T]}|X_t|}) < \infty. \tag{2}$$

The payoffs under consideration in the sequel are supposed to satisfy

$$|F| \leq ce^{c\sup_{t\in[0,T]}|X_t|} \tag{3}$$

for a positive constant c. Consequently, owing to (2) they belong to any $\mathbf{L}_p$, which is a simplification in the arguments below but not a restriction in practice.

Finally, we associate to $(X_t)_{t\geq 0}$ its flow, i.e. the Jacobian matrix $Y_t := \partial_x X_t$, and its inverse $Z_t = [Y_t]^{-1}$ (see [Kun84]). They solve the equations

$$Y_t = \mathrm{Id} + \int_0^t b'_s Y_s \, ds + \sum_{i=1}^d \int_0^t \sigma'_{i,s} Y_s \, dW_s^i, \tag{4}$$

$$Z_t = \mathrm{Id} - \int_0^t Z_s(b'_s - \sum_{i=1}^d (b'_s\sigma'_{i,s})^2) \, ds - \sum_{i=1}^d \int_0^t Z_s\sigma'_{i,s} \, dW_s^i. \tag{5}$$

Standard computations show that $\mathbf{E}(\sup_{t\in[0,T]}(|Y_t| + |Z_t|)^p) < \infty$ for any $p \geq 1$.

2. Computations of Delta

For an European option with payoff F satisfying (3), we can write using the predictable representation theorem

$$F = \mathbf{E}(F) + \int_0^T \Delta_s\sigma_s dW_s. \tag{6}$$

The $\mathbf{R}^1 \otimes \mathbf{R}^d$-valued process $(\Delta_t)_{0\le t\le T}$ gives, up to logarithm change of variables, the hedging strategy of the option using the underlying asset S. For an American option with payoff $(F_t)_{0\le t\le T}$, we can proceed analogously to get, under some assumptions,

$$F_{\tau^*} = \mathbf{E}(F_{\tau^*}) + \int_0^{\tau^*} \Delta_s \sigma_s dW_s, \tag{7}$$

τ^* being the optimal exercise time [Kar88].

In this section, we aim at representing Δ_0 for both option styles as an expectation involving explicit terms, which simulations are easy. The key idea consists in using a suitable martingale (see Lemmas 2.1 and 2.2 below), in order to involve in the expected representation only Itô's integrals. This approach has been introduced by Bismut [Bis84] (and intensively used by Thalmaier et al.[TW98], Elworthy et al.[EJL99], Picard [Pic02] and Delarue [Del03] among others) to derive *explicit* ibp formulas.

2.1. *European vanilla options*

Consider first the case $F = f(X_T)$ and define

$$u(t,y) = \mathbf{E}[f(X_T)|X_t = y]. \tag{8}$$

We briefly recall that under **(H)**, u is a $C^{1,2}([0,T[\times\mathbf{R}^d,\mathbf{R})$, satisfying the PDE $\partial_t u + Lu = 0$ in $[0,T[\times\mathbf{R}^d$ with $u(T,\cdot) = f(\cdot)$ as a terminal condition (this directly follows from the existence of a smooth transition density function for the Markov process X, see [Fri75]). Under **(H)**, there is even an additional regularity, namely $(\partial^3_{x_i,x_j,x_k} u)_{i,j,k}$ exist and are locally Hölder continuous (Theorem 10 p.72 in [Fri64]). Remember that in general, derivatives of u explode when $t \to T$. Besides, an application of Itô's formula to $u(T, X_T)$ provides a more explicit form for the predictable representation (6):

$$f(X_T) = u(0,x) + \int_0^T u'(s,X_s)\sigma_s dW_s. \tag{9}$$

Our main tool is the following.

Lemma 2.1. *Assume **(H)** and define $M_t = u'(t,X_t)Y_t$ for $t < T$. Then $M = (M_t)_{0\le t<T}$ is a $\mathbf{R}^1 \otimes \mathbf{R}^d$-valued martingale.*

Proof. We give two different arguments.
Proof 1. From the Markov property, note that $[u(t,X_t) = \mathbf{E}(F(X_T)|\mathcal{F}_t)]_{0\le t<T}$ is a martingale for any $X_0 \in \mathbf{R}^d$: differentiating w.r.t. X_0 provides also a martingale. This is our statement. For full details, see Lemma 2.10 in [GM02].
Proof 2. This second proof is easily adaptable to other options as we will see later on. First, M is a local martingale. Indeed, apply Ito's formula to $u'(t,X_t)Y_t$: the key point is to note that the dt-terms cancel because of the PDE solved by u. It

remains to prove the uniform integrability of $M = (M_t)_{0\leq t\leq T-\epsilon}$ for any $\epsilon > 0$ (the presence of ϵ avoids t being too close to T). But under **(H)**, it is easy to prove that $\mathbf{E}[\sup_{0\leq t\leq T-\epsilon}|M_t|] < \infty$ and the uniform integrability follows. □

We now are in a position to prove

Proposition 2.1. *Assume **(H)**. Then, one has*

$$\Delta_0 = \mathbf{E}\big(f(X_T)\frac{1}{T}\big[\int_0^T [\sigma_s^{-1}Y_s]^* dW_s\big]^*\big).$$

Proof. The uniform ellipticity condition yields $\mathbf{E}(\int_0^T |u'(t,X_t)|^2 dt) \leq C\mathbf{E}(\int_0^T |u'(t,X_t)\sigma_t|^2 dt) = C\mathbf{E}(f^2(X_T))$ using the equation (9). Consequently, $\mathbf{E}(\int_0^T |M_t| dt) \leq [\mathbf{E}(\int_0^T |u'(X_t)|^2 dt)]^{1/2}[\mathbf{E}(\int_0^T |Y_t|^2 dt)]^{1/2} < \infty$. Hence, it allows to write, using the martingale property of M and the isometry of Itô's integral, that Δ_0 equals

$$\begin{aligned} u'(0,x) &= \mathbf{E}\big(\frac{1}{T}\int_0^T M_s\, ds\big) = \mathbf{E}\big(\frac{1}{T}\big[\int_0^T u'(s,X_s)\sigma_s dW_s\big]\big[\int_0^T [\sigma_s^{-1}Y_s]^* dW_s\big]^*\big) \\ &= \mathbf{E}\big(\frac{f(X_T)-u(0,x)}{T}\big[\int_0^T [\sigma_s^{-1}Y_s]^* dW_s\big]^*\big) = \mathbf{E}\big(\frac{f(X_T)}{T}\big[\int_0^T [\sigma_s^{-1}Y_s]^* dW_s\big]^*\big). \end{aligned}$$ □

2.2. *Some European path-dependent options*

The analysis above can be immediately extended to the case of payoff of the form

$$F = f(X_s : t_1 \leq s \leq T)$$

for a fixed time $t_1 > 0$. It covers the situations considered in [FLLLT99], where $F = f(X_{t_1}, \cdots, X_{t_i}, \cdots, X_{t_N})$ with $0 < t_1 < \cdots < t_i < \cdots < t_N = T$, which allows to deal with discrete monitored barrier and lookback options, discrete Asian options... But it also enables us to consider more generally any option, which payoff does not depend on the underlying assets on a short time period: here, we address the case of forward start options [Rub91]. Some examples may be:

- Forward start put on maximum: $F = (\max_{t\in[t_1,T]} e^{X_t} - e^{X_T})$;
- Forward start Asian call: $F = (e^{X_T} - \frac{1}{T-t_1}\int_{t_1}^T e^{X_t} dt)_+$.

Actually, this path-dependent case can be connected to the vanilla one. Indeed, define $\tilde{f}(X_{t_1}) = \mathbf{E}(F|\mathcal{F}_{t_1})$ using the Markov property for X. Then, take the conditional expectation in (6) to get

$$\tilde{f}(X_{t_1}) = \mathbf{E}(F) + \int_0^{t_1} \Delta_s\sigma_s dW_s. \tag{10}$$

Consequently, the hedging strategy on the time interval $[0, t_1]$ equals the one used for a vanilla option of maturity t_1 and payoff $\tilde{f}(X_{t_1})$. In terms of no-arbitrage,

this is the well established principle that the fair price $(V_t)_{0\leq t\leq T}$ of an option of maturity T coincides on $[0,T']$ with the fair price of an option with maturity $T'<T$ and payoff $V_{T'}$; furthermore, both hedging strategies also coincide.
In particular, Proposition 2.1 gives $\Delta_0 = \mathbf{E}\big(\tilde f(X_{t_1})\frac{1}{t_1}\big[\int_0^{t_1}[\sigma_s^{-1}Y_s]^* dW_s\big]^*\big)$. Using the tower property for conditional expectations, we finally obtain:

Proposition 2.2. *Assume* ***(H)****. Then, one has*

$$\Delta_0 = \mathbf{E}\big(f(X_s : t_1 \leq s \leq T)\frac{1}{t_1}\big[\int_0^{t_1}[\sigma_s^{-1}Y_s]^* dW_s\big]^*\big).$$

2.3. *European barrier options*

Now, we consider continuous time monitored barrier options, with payoff of the form

$$F = f(\tau_0 \wedge T, X_{\tau_0\wedge T}), \qquad \tau_t = \inf\{s \geq t : X_s \notin D\}$$

for some open set $D \subset \mathbf{R}^d$ containing $X_0 = x$. The situation $d = 1$ has been developed by Gobet et al.[GKH03], while a multidimensional extension in the Black-Scholes model (constant coefficients b and σ) has been carried out by Bernis et al.[BGKH03]. Here, we cover general models, but unlike the cited references, we are not able to deal with other Greeks than the Delta. Note also that the form of the payoff allows wider situations, since the dependence through the state $\tau_0 \wedge T$ was not possible in the previous works. In this context, the technique of martingales is borrowed to [TW98] and [Del03].

Analogously to (8), we set

$$v(t,y) = \mathbf{E}[f(\tau_t \wedge T, X_{\tau_t\wedge T})|X_t = y]. \tag{11}$$

Note that we do not assume any regularity property on D, thus we cannot expect v to be a smooth function up to the boundary. In particular, computations involving derivatives on v have to be carefully performed. Our analysis below is only based on the interior regularity, which can be formulated as follows. Take a such that $0 < a < d(x, \partial D)$, set $D' = \{y : |y-x| < a\} \subset D$ and put $\tau'_t = \inf\{s \geq t : X_s \notin D'\}$. The strong Markov property easily yields $v(t,y) = \mathbf{E}(v(\tau'_t \wedge T, X_{\tau'_t\wedge T})|X_t = y)$ for any $(t,y) \in [0,T[\times D'$. It enables us to deduce that the function v is of class $C^{1,2}([0,T[\times D', \mathbf{R})$ and solves $\partial_t v + Lv = 0$ in $[0,T[\times D'$. Indeed, this is classic result which may be obtained using the integral representation (see [Mir70] among others) of v with the Green function, which is smooth in (t,x) because *the ball D' is smooth* and coefficients satisfy **(H)**. Moreover, an application of interior estimates (Theorem 5 p.64 [Fri64]) leads to uniform bounds for the derivatives of v on $[0,T/2]\times \bar{D}'$ (up to taking a smaller) [d]. As for (8), $(\partial^3_{x_i,x_j,x_k} v)_{i,j,k}$ exist and are Hölder continuous.

[d] actually, $T/2$ could be replaced by any time smaller than T.

Now, write $\mathbf{E}(f(\tau_0\wedge T, X_{\tau_0\wedge T})|\mathcal{F}_{\tau_0'\wedge T/2}) = v(\tau_0'\wedge T/2, X_{\tau_0'\wedge T/2})$ using the Markov property and apply Itô's formula. Using the PDE solved by v in $[0,T/2[\times D'$, we obtain

$$\mathbf{E}(f(\tau_0\wedge T, X_{\tau_0\wedge T})|\mathcal{F}_{\tau_0'\wedge T/2}) = v(0,x) + \int_0^{T/2} \mathbf{1}_{s<\tau_0'} v'(s,X_s)\sigma_s dW_s. \tag{12}$$

A comparison with the equation (6) leads to $\Delta_0 = v'(0,x)$ as it was expected. Lemma 2.1 in this new situation becomes

Lemma 2.2. *Assume **(H)** and define $N_t = v'(t,X_t)Y_t$ for $t \leq T/2$ and $t \leq \tau_0'$. Then $N = (N_{\tau_0'\wedge t})_{0\leq t\leq T/2}$ is a $\mathbf{R}^1 \otimes \mathbf{R}^d$-valued martingale.*

Proof. The first argument invoked in the proof of Lemma 2.1 cannot be applied here since it is not clear how to differentiate the martingale $[v(\tau_0'\wedge t, X_{\tau_0'\wedge t}) = \mathbf{E}(f(\tau_0\wedge T, X_{\tau_0,\wedge T})|\mathcal{F}_{\tau_0'\wedge t})]_{0\leq t\leq T/2}$ w.r.t. X_0 (because of τ_0'). However, the second argument is still valid. Indeed, v solves the PDE $\partial_t v + Lv = 0$ in $[0,T/2[\times D'$ and consequently, $[v'(\tau_0'\wedge t, X_{\tau_0'\wedge t})Y_{\tau_0'\wedge t}]_{0\leq t\leq T/2}$ is a local martingale using Itô's formula as before. The uniform integrability is clear because $v'(\cdot,\cdot)$ is bounded on $[0,T/2]\times\bar{D}'$. □

To complete the localization, we introduce the adapted process $h = (h_t)_{0\leq t\leq T}$ defined by

$$h_t = \frac{1}{\lambda}\frac{\mathbf{1}_{t<\tau}}{d^2(X_t,\partial D')(T/2-t)} \tag{13}$$
$$\text{with } \tau = \inf\big\{t \geq 0 : \int_0^t \frac{ds}{d^2(X_s,\partial D')(T/2-s)} = \lambda\big\}.$$

The parameter λ above is a positive real number, that can be chosen arbitrarily. From [Del03], one has $\tau < \tau_0'\wedge T/2$ *a.s.* and $\mathbf{E}(\int_0^T h_t^2 dt)^p < \infty$ for any $p \geq 1$. As a consequence, one gets the crucial properties

$$h_t = 0 \text{ for } t \geq \tau_0'\wedge T/2, \quad \text{and} \quad \int_0^T h_t dt = 1. \tag{14}$$

Now, the representation for Δ_0 can be stated as follows.

Theorem 2.1. *Assume **(H)**. Then, one has*

$$\Delta_0 = \mathbf{E}\big(f(\tau_0\wedge T, X_{\tau_0\wedge T})\big[\int_0^T h_s[\sigma_s^{-1}Y_s]^* dW_s\big]^*\big).$$

Proof. We proceed by verification. Because of the local property of h, one has $\mathbf{E}\big(\mathbf{E}[f(\tau_0\wedge T, X_{\tau_0\wedge T})|\mathcal{F}_{\tau_0'\wedge T/2}][\int_0^{\tau_0'\wedge T/2} h_s[\sigma_s^{-1}Y_s]^* dW_s]^*\big)$. Then, because of (12), it equals

$$\begin{aligned}&\mathbf{E}\big(\int_0^{T/2}\mathbf{1}_{s<\tau_0'}h_s v'(s,X_s)Y_s\,ds\big)\\=&\mathbf{E}\big(\int_0^{T/2}h_s N_s\,ds\big)\\=&\mathbf{E}\big(\int_0^{T/2}h_s N_{\tau_0'\wedge T/2}\,ds\big)=\mathbf{E}\big(N_{\tau_0'\wedge T/2}\big)=N_0=\Delta_0\end{aligned}$$

using successively the property on the support of h, the martingale property of N between times s and $\tau_0'\wedge T/2$ on the event $\{h_s\neq 0\}\subset\{s<\tau_0'\wedge T/2\}$, the normalized time integral of h and once again the martingale property of N. □

2.4. *European lookback options*

The payoff of lookback options depends on the extrema $M^i=\sup_{t\in\mathcal{T}^{M^i}}X_t^i$, $m^i=\inf_{t\in\mathcal{T}^{m^i}}X_t^i$ of the underlying assets, computed on some subsets $\mathcal{T}^{M^i}$ and $\mathcal{T}^{m^i}$ of $[0,T]$. These monitoring sets may be different for each maximum and minimum. Put $M=(M^1,\cdots,M^d)^*$, $m=(m^1,\cdots,m^d)^*$: the payoff is of the form

$$F=f(M,m,X_T).$$

Note that even in the case of smooth functions f, the pathwise approach cannot be applied because in general, M and m are not differentiable w.r.t. X_0. It strengthens the necessity to develop a *likelihood-type* method.

We assume in the sequel the following structure of the payoff function f.

(S) There exists $a_0>0$ such that for any $i\in\{1,\cdots,d\}$, the payoff $f(M^1,\cdots,M^d,m^1,\cdots,m^d,X_T^1,\cdots,X_T^d)$ does not depend on M^i (resp. m_i) if $M^i<X_0^i+a_0$ (resp. $m^i>X_0^i-a_0$).

As it is discussed in [BGKH03], this restriction appears to be a necessary condition to allow a representation of Δ_0 as $\mathbf{E}(FH)$, for an appropriate square integrable random variable H. However, in practice many payoffs fulfill **(S)** [BGKH03].

Now, we make the connection between this framework about general lookback options and the previous situation with barrier options, analogously to what we have done in the paragraph 2.2. For this, put $D=\{y:|y-x|<a_0\}$ and set $\tau_0=\inf\{t\geq 0:X_t\notin D\}$. The strong Markov property for X, the definition of τ_0 and the assumption **(S)** enable us to write $\mathbf{E}(F|\mathcal{F}_{\tau_0\wedge T})=\tilde{f}(\tau_0\wedge T,X_{\tau_0\wedge T})$ for some function $\tilde{f}$. With the arguments from paragraph 2.2, it readily follows that Δ_0 equals the Delta for the barrier option with payoff $\tilde{f}(\tau_0\wedge T,X_{\tau_0\wedge T})$. By Theorem 2.1, it writes $\Delta_0=\mathbf{E}\big(\tilde{f}(\tau_0\wedge T,X_{\tau_0\wedge T})\big[\int_0^T h_s[\sigma_s^{-1}Y_s]^*dW_s\big]^*\big)$ where h is defined in (13) with $0<a<a_0$. Because of the properties of h, $\int_0^T h_s[\sigma_s^{-1}Y_s]^*dW_s=\int_0^{\tau_0\wedge T}h_s[\sigma_s^{-1}Y_s]^*dW_s$ and an application of the tower property for conditional expectations completes the proof of the following result.

Theorem 2.2. *Assume* ***(H)*** *and* ***(S)****. Then, one has*

$$\Delta_0 = \mathbf{E}\big(f(M,m,X_T)\big[\int_0^T h_s[\sigma_s^{-1}Y_s]^* dW_s\big]^*\big),$$

with h *defined as above.*

2.5. *American vanilla options*

In this paragraph, we focus on American contracts with payoff $F_t = f(t, X_t)$. The fair price is given by $P(0,x) = \sup_{\tau\in[0,T] \text{ stopping times}} \mathbf{E}(f(\tau, X_\tau))$. The existence of an optimal exercise time τ^* such that $P(0,x) = \mathbf{E}(f(\tau^*, X_{\tau^*}))$ has been handled for years by El Karoui [Kar81]: the continuity of f and the uniform integrability of $(F_t)_{0\leq t\leq T}$ are sufficient for this result. Furthermore, in our Markovian setting, τ^* is the entrance time of the process $(t, X_t)_{0\leq t\leq T}$ in the so-called exercise region $\mathcal{E}$ (which is unknown). For a recent work on these issues, see [Vil99].

Hence, from this point of view, the computation of Δ_0 in (7) is a particular case of the previous study on barrier options, except that it needs a minor adaptation since $\mathcal{E}$ is not a cylindrical time-space domain. Namely, the radius a and the intermediate time T' (equal to $T/2$ before) have to be such that $[0, T'] \times \{y : |y - x| \leq a\} \subset \mathcal{E}^c$, everything else being unchanged.

Nevertheless, this argumentation assumes that $\mathcal{E}$ is known in a neighborhood of $(0, x)$, which might be unrealistic in practice. Besides, some numerical methods based on Monte Carlo simulations build for each path the optimal exercise time; see for instance [LS01 Gar01 IZ02]. Hence, deriving an estimator of Δ_0 using the same information is especially relevant for numerical procedures. It is the aim of this section and we develop a *pathwise approach*.

To state our result, we assume that the price function

$$P(t,y) = \sup_{\tau\in[t,T] \text{ stopping times}} \mathbf{E}(f(\tau, X_\tau)|X_t = y) \tag{15}$$

satisfies the *smooth pasting* condition, that is $P'(t,y) = f'(t,y)$ for $(t,y) \in \partial\mathcal{E}$. Sufficient conditions for this are given in [Fri76] Chapter 16 and extensions are discussed by Brekke et al.[BØ91]. Here, we do not report a relevant set of hypotheses which makes the smooth pasting condition valid, but we directly assume that it holds. For instance, it is true for the American put [Lam98].

- **(P)** The function $P(\cdot,\cdot)$ is continuously differentiable w.r.t. y in a neighborhood of the continuation region $\mathcal{C} = \mathcal{E}^c$ and it satisfies $\sup_{t\in[0,T]} \mathbf{E}(|P'(\tau^* \wedge t, X_{\tau^*\wedge t})|^{1+\epsilon}) < \infty$ for some $\epsilon > 0$.
  Moreover, in the neighborood of $\partial\mathcal{C}$, $f(\cdot,\cdot)$ is also continuously differentiable w.r.t. y and one has

$$P'(\tau^*, X_{\tau^*}) = f'(\tau^*, X_{\tau^*}) \quad a.s. \tag{16}$$

Analogously to Lemma 2.2, we assert that $[P'(\tau^* \wedge t, X_{\tau^* \wedge t}) Y_{\tau^* \wedge t}]_{0 \leq t \leq T}$ is a martingale. Indeed, in the open set $\mathcal{C}$, the price satisfies $\partial_t P + LP = 0$ (use the interior regularity as we did for $v(\cdot)$). Thus, an application of Itô's formula proves the property of local martingale as before. Under **(P)**, this is a true martingale. To get a representation for Δ_0 as an expectation, write $\Delta_0 = P'(0, x) = \mathbf{E}(P'(\tau^*, X_{\tau^*}) Y_{\tau^*})$ and use the smooth pasting condition to conclude. We obtain

Theorem 2.3. *Assume **(H)** and **(P)**. One has*

$$\Delta_0 = \mathbf{E}(f'(\tau^*, X_{\tau^*}) Y_{\tau^*}).$$

Similar results have been obtained by Piterbarg [Pit02] in the discrete time setting of Bermuda options: his proof is based on the differentiation of the dynamic programming equation.

3. Other greeks

The extension of previous arguments to the computation of Γ (second derivatives of the price w.r.t. x) is possible for European vanilla options. Formulas still involving only Itô's integrals are proved in Theorem 2.12 [GM02]: we do not report them here. For other path-dependent options, we have not been able to generalize this approach.

3.1. *Vega index*

An other relevant Greek is the so-called Vega index, which usually evaluates the sensitivity of the price w.r.t. the volatility. This is an important index when one needs to measure the impact of a model misspecification on the options' prices. We address the problem of its evaluation in the case of European vanilla options. Using Malliavin calculus techniques, see [FLLLT99]. For the Asian case, see [Ben00]. For barrier and lookback options, the problem is still open to our knowledge. Here, we present some results proved in [GM02].

We slightly modify the definition of Vega because of the logarithm change of variables. For $\alpha \in \mathbf{R}$, consider the solution of the stochastic differentiable equation

$$X_t^\alpha = x + \int_0^t [b(s, X_s^\alpha) + \alpha \bar{b}(s, X_s^\alpha)] ds + \sum_{i=1}^d \int_0^t [\sigma_i(s, X_s^\alpha) + \alpha \bar{\sigma}_i(s, X_s^\alpha)] dW_s^i, \quad (17)$$

with coefficients $b + \alpha \bar{b}$ and $\sigma + \alpha \bar{\sigma}$ satisfying **(H)** for α small enough. If no reference to α is given, it means that $\alpha = 0$, for instance $X_t = X_t^0$. We aim at computing

$$\text{Vega} = \mathbf{E}[f(X_T^\alpha)]|_{\alpha=0}. \quad (18)$$

Denote $\dot{X}_t = \partial_\alpha X_t|_{\alpha=0}$, which is solution of

$$\dot{X}_t = \int_0^t \left(\bar{b}_s + b'_s\, \dot{X}_s\right) ds + \sum_{i=1}^d \int_0^t \left(\bar{\sigma}_{i,s} + \sigma'_{i,s}\, \dot{X}_s\right) dW_s^i. \quad (19)$$

When f is smooth, the *pathwise approach* leads to the representation

$$\text{Vega} = \mathbf{E}[f'(X_T)\dot{X}_T]. \tag{20}$$

In [GM02], two new approaches are developed. Firstly, the so-called *adjoint method* uses the underlying PDE to write Vega as an expectation. Briefly, the option price $u^\alpha(t,x) = \mathbf{E}(f(X_T^\alpha)|X_t^\alpha = x)$ is solution of $\partial_t u^\alpha + L^\alpha u^\alpha = 0$ and consequently the Vega index $\partial_\alpha u^\alpha$ solves $\partial_t[\partial_\alpha u^\alpha] + L^\alpha[\partial_\alpha u^\alpha] = -[\partial_\alpha L^\alpha]u^\alpha$. Then, after some substantial work, one can represent the solution as an expectation with only explicit quantities.

We prefer here to expose the second approach, which seems to be the most efficient in practice.

Theorem 3.1. *Assume* ***(F)****:* $\displaystyle\int_0^T \frac{\|f(X_T) - f(X_t)\|_{\mathbf{L}^{p_0}}}{T-t}\, dt < +\infty$ *for some* $p_0 > 1$. *Then, under* ***(H)*** *one has*

$$\begin{aligned}\text{Vega} = \mathbf{E}\Big(&\frac{f(X_T)}{T}\int_0^T [\sigma_s^{-1}\dot{X}_s]^* dW_s \\ &+ \int_0^T dr\, \frac{[f(X_T) - f(X_r)]}{(T-r)^2}\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\Big).\end{aligned}$$

Note that the random variable in the expectation above is integrable. Indeed, this is clear for the first term. For the second one, the combination of the generalized Minkowski inequality, the Hölder inequality and standard estimates from the stochastic calculus gives

$$\begin{aligned}&\Big\|\int_0^T dr\, \frac{[f(X_T) - f(X_r)]}{(T-r)^2}\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\Big\|_{\mathbf{L}^1} \\ \le &\int_0^T \frac{dr}{(T-r)^2}\|f(X_T) - f(X_r)\|_{\mathbf{L}^{p_0}}\Big\|\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\Big\|_{\mathbf{L}^{\frac{p_0}{p_0-1}}} \\ \le &C\int_0^T \frac{\|f(X_T) - f(X_r)\|_{\mathbf{L}^{p_0}}}{T-r}\, dr < \infty. \end{aligned} \tag{21}$$

Assumption **(F)** can be interpreted under **(H)** as a weak regularity condition on f and it allows to consider discontinuous functions. To simplify, say that the support of f is included in a bounded set D: then we assert that if f belongs to the Sobolev space $W_{p_0}^\epsilon(D)$ ($\epsilon \in]0,1[$) equipped with the norm $\|f\|_{\mathcal{W}_{p_0}^\epsilon(D)}^{p_0} = \int_D |f(y)|^{p_0} dy + \int_{D\times D} \frac{|f(y)-f(z)|^{p_0}}{|y-z|^{d+p_0\epsilon}} dz dy$ (see [LSU68]), then f satisfies **(F)**. Indeed, using standard Gaussian upper bounds [Fri64] for the transition density function of X, **(F)** is fulfilled if the following quantity is finite:

$$\begin{aligned}&\int_{T/2}^T \frac{dr}{(T-r)}\Big(\int_{D\times D} \frac{e^{-c\frac{|y-z|^2}{T-r}}}{(T-r)^{d/2}}|y-z|^{d+p_0\epsilon}\frac{|f(z)-f(y)|^{p_0}}{|y-z|^{d+p_0\epsilon}} dy dz\Big)^{1/p_0} \\ \le C &\int_{T/2}^T \frac{dr}{(T-r)^{1-\epsilon/2}}\Big(\int_{D\times D} \frac{|f(z)-f(y)|^{p_0}}{|y-z|^{d+p_0\epsilon}} dy dz\Big)^{1/p_0} \le C\frac{T^{\epsilon/2}}{\epsilon/2}\|f\|_{\mathcal{W}_{p_0}^\epsilon(D)}.\end{aligned}$$

This approach is called the *martingale method* since Theorem 3.1 is obtained in [GM02] using cleverly the martingale

$$\partial_\alpha\big[\mathbf{E}(f(X_T^\alpha)|\mathcal{F}_t)\big] = \partial_\alpha\big[u^\alpha(t,X_t^\alpha)\big] = [\partial_\alpha u^\alpha](t,X_t^\alpha) + [u^\alpha]'(t,X_t^\alpha)\partial_\alpha X_t^\alpha.$$

We present here another proof, which extends the scope of the method. For this, we show

Theorem 3.2. *Consider*

(1) *a continuously differentiable function f satisfying **(F)** and $\|f'(X_T)\|_{\mathbf{L}^{p_0}} < \infty$;*
(2) *a $\mathbf{R}^d\otimes\mathbf{R}^q$-valued continuous semimartingale $(U_t)_{0\le t\le T}$ with $\|U_T - U_t\|_{\mathbf{L}^p} \le C\sqrt{T-t}$ for any $p \ge 1$.*

*Then, under **(H)**, one has*

$$\mathbf{E}\big(f'(X_T)U_T\big) = \mathbf{E}\Big(\frac{f(X_T)}{T}\Big[\int_0^T [\sigma_s^{-1}U_s]^* dW_s\Big]^* + \int_0^T dr\,\frac{[f(X_T)-f(X_r)]}{(T-r)^2}\Big[\int_r^T [\sigma_s^{-1}(U_s - Y_sZ_rU_r)]^* dW_s\Big]^*\Big).$$

The result above is an explicit ibp formula in an elliptic framework. We should emphasize the fact that it uses only the first derivative of the coefficients defining X, whereas the usual ibp requires two derivatives. At least for simulations, this is a crucial advantage.

Note that Proposition 2.1 and Theorem 3.1 are a particular case of the result above. Indeed,

a) one has $\partial_x\mathbf{E}(f(X_T)) = \mathbf{E}(f'(X_T)Y_T)$. Now, take $U_T = Y_T$ ($q = d$): the second term in the representation cancels and Proposition 2.1 is proved.
b) Theorem 3.1 immediately follows by considering (20) and taking $U_T = \dot{X}_T$.

Proof. (of Theorem 3.2.) We first justify the following integral representation, that we have not previously found in the literature. If U is a continuous semimartingale as above, then one has

$$U_T = \frac{1}{T}\int_0^T U_t dt + \int_0^T \frac{dr}{(T-r)^2}\int_r^T (U_s - U_r)ds. \tag{22}$$

Actually, this follows from three integration by parts formula w.r.t. time:

$$\begin{aligned}\int_0^T \frac{dr}{(T-r)^2}\int_r^T (U_s-U_r)ds &= \int_0^T \frac{dr}{(T-r)^2}\int_r^T (T-s)dU_s\\ &= -\frac{1}{T}\int_0^T (T-r)dU_r + \int_0^T dU_s = \frac{1}{T}\int_0^T r dU_r\\ &= U_T - \frac{1}{T}\int_0^T U_r dr.\end{aligned}$$

We apply (22) to the semimartingale $Z_T U_T$, to obtain that $\mathbf{E}(f'(X_T)U_T) = \mathbf{E}(f'(X_T)Y_T\ Z_T U_T)$ equals

$$\mathbf{E}\big(\frac{1}{T}\int_0^T f'(X_T)Y_T Z_t U_t dt + \int_0^T \frac{dr}{(T-r)^2}\int_r^T f'(X_T)Y_T(Z_s U_s - Z_r U_r)ds\big). \quad (23)$$

Both terms can be handled in the same way: we only treat the second one, for a fixed r. By Lemma 2.1, $(u'(s,X_s)Y_s)_{0\le t<T}$ is a martingale which is closed by $f'(X_T)Y_T$ when f is smooth: hence, $u'(s,X_s)Y_s = \mathbf{E}(f'(X_T)Y_T|\mathcal{F}_s)$. It follows that $\mathbf{E}\big(\int_r^T f'(X_T)Y_T(Z_sU_s - Z_rU_r)ds\big) = \mathbf{E}\big(\int_r^T u'(s,X_s)Y_s(Z_sU_s - Z_rU_r)ds\big) = \mathbf{E}\big([f(X_T) - u(r,X_r)][\int_r^T [\sigma_s^{-1}(U_s - Y_sZ_rU_r)]^* dW_s]^*\big)$ using (9). In the last term, $u(r,X_r)$ can be replaced by any $\mathcal{F}_r$-measurable random variable because the stochastic integral is centered. The choice of $f(X_r)$ is more convenient. It remains to plug these expressions in (23), Fubini's theorem completing the proof. Note that the estimates on U help in justifying that all quantities that appear are well defined, as we have done for example in (21). □

3.2. *Numerical experiment*

The purpose of this paragraph is to illustrate how representations with Itô's integrals can be numerically efficient compared to the ones resulting from general Malliavin calculus computations. We report here an example borrowed from [GM02].

We consider a vanilla digital option with payoff $e^{-rT}\mathbf{1}_{X_T^1\ge X_T^2}$ and maturity $T = 1$, using the model

$$\begin{cases} dX_t^1 = [r - \sigma^2(X_t^1,\lambda_1)]dt + \sigma(X_t^1,\lambda_1)dW_t^1 \\ dX_t^2 = [r - \sigma^2(X_t^2,\lambda_2)]dt + \sigma(X_t^2,\lambda_2)\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right). \end{cases}$$

The constant interest rate is defined by $r = 0.04$ and the volatility function is given by $\sigma(x,\lambda) = 0.25(1 + \frac{1}{1+e^{-\lambda\exp(x)}})$. We focus on the Vega index, by evaluating the sensitivity of the price w.r.t. $\lambda_1, \lambda_2, \rho$ at the point $\lambda_1 = 2$, $\lambda_2 = 2$, and $\rho = 0.6$. We take $X_0^1 = X_0^2 = 0$. In the table below, different estimators are compared in terms of computational time and variance:

(1) the usual Malliavin calculus estimator [FLLLT99], requiring the simulation of Skorohod integrals.
(2) the pathwise estimator: since the payoff is not smooth, we approximate it linearly using a regularization parameter ε close to 0.
(3) the martingale estimator from Theorem 3.1.

The measurements have been carried out with 1000 sample paths, on a Pentium III, 700Mhz processor. The process X and the stochastic integrals have been discretized using the Euler scheme with time step $h = 0.01$. For the influence of the time step in such procedures, we refer to [GM02], where it is essentially proved that the error is linear w.r.t. h.

On the one hand, the pathwise method provides a very high variance because the payoff is not smooth. On the other hand, the Malliavin calculus estimator leads to a small confidence interval, but it is too costly to evaluate. Finally, the martingale approach appears to be the most competitive on this example.

Variances	Malliavin	Martingale	Pathwise $\varepsilon = 10^{-3}$	" $\varepsilon = 10^{-4}$
λ_1	0.0011	0.0012	0.0378	3.8951
λ_2	0.0048	0.0018	0.0296	4.9427
ρ	1.5788	1.4323	14.923	100.86
CPU time	$20.8s$	$7.31s$	$2.97s$	$2.97s$

Variance of different estimators of the Vega index.

3.3. *Sensitivity w.r.t. the boundary*

We now open the discussion on a different type of sensitivity analysis, by briefly reporting here a recent result by Costantini et al.[CKG03]. We refer to the cited paper for full details.

Consider a time-space domain $D \subset [0,T] \times \mathbf{R}^d$ and the associated exit time τ_D for the process X defined in (1). How to evaluate the sensitivity of $u(t,x) = \mathbf{E}(g(\tau, X_\tau)e^{-\int_t^\tau c(r,X_r)dr} - \int_t^\tau e^{-\int_t^s c(r,X_r)dr} f(s,X_s)ds | X_t = x)$ w.r.t. D?

This issue may be relevant to devise new algorithms for the pricing of American options by optimizing the continuation region [CKG04]. The answer is the following. Define the pertubation of the domain by $D_\epsilon = \{(t,x) : (t, x + \epsilon\Theta(t,x)) \in D\}$, for which the new exit time is given by τ_ϵ. We are concerned by the regularity of the map

$$\epsilon \mapsto J(\epsilon)(t,x) = \mathbf{E}(g(\tau_\epsilon, X_{\tau_\epsilon})e^{-\int_t^{\tau_\epsilon} c(r,X_r)dr} - \int_t^{\tau_\epsilon} e^{-\int_t^s c(r,X_r)dr} f(s,X_s)ds)|X_t = x).$$

First, to ensure a global regularity to u, all the datas $f, g, c, \cdots$ are assumed to be smooth enough, that is essentially a little more than continuously differentiable. Then, the main result is the differentiability of $J(\epsilon)(t,x)$ at $\epsilon = 0$, for any $(t,x) \in D$, with

$$\partial_\epsilon J(\epsilon)(t,x)|_{\epsilon=0} = \mathbf{E}\left[e^{-\int_t^\tau c(r,X_r)dr}[(u' - g')\Theta](\tau, X_\tau)|X_t = x\right].$$

The expression of the gradient shows a nice connection with the smooth pasting condition **(P)** from the paragraph 2.5. If it satisfies, the gradient above vanishes, which is natural for the optimal domain.

4. Conclusion

In this paper, we have tackled the issue of the Greeks' evaluation. Rather than using the gear of Malliavin calculus techniques, we try hard to develop simple arguments in order to get simple representations. Our arguments take strongly advantage of the Markovian structure of the model. Firstly, it is interesting from the pedagogical point of view. Secondly, since only Itô's integrals have to be sampled, the simulation procedures are easy and quick. Finally, we have succeeded to cover new situations. To conclude, here is a list of open problems:

(1) extension of the current approach to the case of Asian options;
(2) computation of the Vega index for barrier and lookback options;
(3) additional variance reduction for the Greeks' evaluation: some answers are given in [KHP02].

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