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R.I. N^0 559 *November 2004*

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9th November 2004

Abstract

For a multidimensional Itô process $(X_t)_{t\geq 0}$ driven by a Brownian motion, we are interested in approximating the law of $\psi\left((X_s)_{s\in[0,T]}\right)$, $T>0$ deterministic, for a given functional ψ using a discrete sample of the process X. For various functionals (related to the maximum, to the integral of the process, or to the killed/stopped path) we extend to the non Markovian framework of Itô processes the results available in the diffusion case. We thus prove that the order of convergence is more specifically linked to the Brownian driver and not to the Markov property of SDEs.

1 Introduction: statement of the problem

Let $(X_t)_{t\in[0,T]}$ be a d-dimensional Itô process, whose dynamics is given by

$$X_t = x + \int_0^t b_s ds + \int_0^t \sigma_s dW_s \tag{1.1}$$

with a fixed initial data x and a fixed terminal time T. Here, W is a d'-dimensional standard Brownian motion (BM in short) defined on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural completed filtration of W. The progressively measurable coefficients $(b_s)_{s\geq 0}$ and $(\sigma_s)_{s\geq 0}$ are bounded. In this work, we are mainly interested in approximating the law of $\psi\left((X_s)_{s\in[0,T]}\right)$, where ψ is a real valued functional defined on the space of càdlàg functions, using a discrete sample of the process X. For this latter, we use the stepwise constant counterpart of X defined by $(X_{\phi(s)})_{s\in[0,T]}$ where $\phi(s) = t_i$ if $t_i := ih \leq s < t_{i+1}$ ($h = T/N$ being the step size). The main problem consists in controlling the difference

$$\mathsf{Err}(T,h,\psi,x) := \mathbb{E}[\psi\left((X_s)_{s\in[0,T]}\right)] - \mathbb{E}[\psi\left((X_{\phi(s)})_{s\in[0,T]}\right)] \tag{1.2}$$

for a certain class of functionals ψ w.r.t. the time step h. This kind of problem has been widely studied in the Markovian setting (i.e. when X is a solution of a SDE) for a large class of functionals ψ, see the short list and references below. What we want to emphasize in this paper is that the rates of convergence obtained in the Markovian case, through proofs relying on an associated PDE, are still valid in the non Markovian framework of Itô processes. Hence, it is not the Markov property that gives the order of convergence, but actually the Brownian stochastic integral. Here are some controls of $\mathsf{Err}(T,h,\psi,x)$ in the Markovian setting for some specific functionals ψ.

1. Integral of the process.

This case corresponds to $\psi_1(y) := \varphi(\int_0^T y(s)ds)$, where φ is a Lipschitz continuous function from $\mathbb{R}^d$ into $\mathbb{R}$. We know from Temam [Tem01] that $\mathsf{Err}(T,h,\psi_1,x) = O(h)$.

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‡This work has been financially supported by Ecole Polytechnique and Université Pierre et Marie Curie - Paris 6 (during the preparation of the PhD Thesis of the second author).

2. Maximum of the drifted BM when $d = 1$.
This case corresponds to $\psi_2(y) := \max_{s\in[0,T]} y(s)$. For $X_s = x + \mu s + \sigma W_s$, we derive from Lemma 6 in Asmussen et *al.* [AGP95] that there exists a constant $C > 0$ s.t. $0 \le \mathrm{Err}(T, h, \psi_2, x) \le Ch^{1/2}$.
3. Killed/stopped processes.
For the killed case, the functional writes $\psi_3(y) := f(y(T))\mathbb{I}_{\forall s\in[0,T],\ y(s)\in D}$ where f is a measurable function and D a given open set of $\mathbb{R}^d$. In the Markovian setting of uniformly elliptic diffusion processes, the first author showed in [Go 00], Theorem 2.4, that for a smooth domain D and bounded f satisfying a support condition w.r.t. D,
$$\exists C > 0,\ |\mathbb{E}[f(X_T)\mathbb{I}_{\tau^N>T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau>T}]| \le C\sqrt{h}. \tag{1.3}$$
Let us mention that the above result remains valid if we additionally replace the discretely killed diffusion by its discretely killed Euler scheme, see [Go 00] and [GM04] for an extension to a hypoelliptic framework. Anyhow, equation (1.3) emphasizes that, for killed processes, the order $1/2$ is intrinsic to the discrete time killing.

In this work, we show that under suitable assumptions, the previous bounds still hold when X follows the dynamics (1.1).

In terms of financial applications, the above results concerning the discretely sampled integral and maximum, can respectively be seen as preliminary controls to deal with the impact of a time discretization for Asian and look-back options. The estimate associated to the killed path gives an upper bound for the error associated to a discrete time observation for barrier options.

We first detail how standard stochastic analysis arguments provide the necessary tools to control (1.2) in the case of a discretely sampled integral or maximum (cases 1. and 2. of the former list).

Proposition 1.1 *Let X be an Itô process following the dynamics of equation (1.1). Assume the coefficients b and σ are bounded and that φ is a Lipschitz continuous function from $\mathbb{R}^d$ into $\mathbb{R}$. For $p \ge 1$ one has*
$$\varphi\Big(\int_0^T X_s ds\Big) - \varphi\Big(\int_0^T X_{\phi(s)} ds\Big) \underset{\mathrm{L}_p(\mathbb{P})}{=} O(h).$$
Note that a direct use of $\|X_s - X_{\phi(s)}\|_{\mathrm{L}_p} = O(\sqrt{h})$ leads to a sub-optimal rate of convergence.

Proof. Because φ is Lipschitz continuous, it is enough to prove that $\Delta I := \int_0^T X_s ds - \int_0^T X_{\phi(s)} ds \underset{\mathrm{L}_p}{=} O(h)$. Using Fubini's theorem for stochastic integrals, see [RY99] Chapter IV.5, we get
$$\Delta I = \int_0^T \Big(\int_0^T \mathbb{I}_{t\in[\phi(s),s]} dX_t\Big) ds = \int_0^T (\phi(t) + h - t) dX_t.$$
We complete the proof using standard BDG inequalities combined with $|\phi(t) + h - t| \le h$. □

Concerning the discretely sampled maximum we state the following

Proposition 1.2 *Assume $(X_s)_{s\in[0,T]}$ follows the dynamics of equation (1.1), where $(b_u)_{u\ge 0}$ is a bounded progressively measurable coefficient and $\sigma_s = \sigma(X_s)$ where σ is bounded in $C^1(\mathbb{R})$ and s.t. $\exists \sigma_0 > 0$, $\forall y \in \mathbb{R}$, $\sigma(y) \ge \sigma_0$. There exists a constant $C > 0$ s.t.*
$$0 \le \mathrm{Err}(T, h, \psi_2, x) \le C\sqrt{h}.$$

Proof. Define $\Delta M := \psi_2\big((X_s)_{s\in[0,T]}\big) - \psi_2\big((X_{\phi(s)})_{s\in[0,T]}\big) = \max_{s\in[0,T]} X_s - \max_{s\in[0,T]} X_{\phi(s)}$. If X is a BM, as a consequence of Lemma 6 in [AGP95], we have $\mathbb{E}[\Delta M^2]^{1/2} = O(\sqrt{h})$. This estimate is still valid if X is solution of the one dimensional SDE $X_t = x + \int_0^t \frac{1}{2}(\sigma\sigma')(X_s) ds + \int_0^t \sigma(X_s) dW_s$ with the above assumptions on σ. Indeed, introducing the Lamperti transform $(Y_t)_{t\ge0} = (\varphi(X_t))_{t\ge0}$, $\forall y \in \mathbb{R}$, $\varphi(y) = \int_0^y \frac{dz}{\sigma(z)}$, we derive that Y is a standard one dimensional BM with starting point $\varphi(x)$. By construction, the inverse of φ is uniformly Lipschitz continuous. This gives the result. To obtain the statement of the proposition, we finally apply a Girsanov transformation, exploiting that the associated Radon-Nikodym density belongs to any L_p because of the drift's boundedness, and the previous result. $\square$

The limiting factor in our approach is the use of Lamperti's transformation that imposes to have a Markovian diffusion term.

Propositions 1.1 and 1.2 extend the results stated for ψ_1 and ψ_2 in our initial list to a wider non-Markovian framework without major difficulties. Hence, in the sequel we consider the more difficult cases of discretely killed or stopped processes for which the corresponding functionals are not Lipschitz continuous anymore. We denote the discretization error associated to the killed case by

$$\mathrm{Err}(T,h,f,x) = \mathbb{E}\left[\psi_3\left((X_s)_{s\in[0,T]}\right)\right] - \mathbb{E}\left[\psi_3\left((X_{\phi(s)})_{s\in[0,T]}\right)\right] = \mathbb{E}[f(X_T)\mathbb{I}_{\tau>T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau^N>T}] \qquad (1.4)$$

where, from now on, $\tau := \inf\{t \geq 0 : X_t \notin D\}$, $\tau^N := \inf\{t_i \geq 0 : X_{t_i} \notin D\}$. For the stopped case, and a smooth domain D, for a given real valued bounded function g defined on $[0,T) \times \partial D \cup \{T\} \times \bar{D}$, we introduce

$$\mathrm{Err}(T,h,g,x) := \mathbb{E}[g(T\wedge\tau^N, \pi_{\bar{D}}(X_{T\wedge\tau^N}))] - \mathbb{E}[g(T\wedge\tau, X_{T\wedge\tau})]. \qquad (1.5)$$

The careful reader can object that without further assumptions on the domain (like convexity for instance) the projection on $\bar{D}$ is only locally uniquely defined. By convention, for $y \in \mathbb{R}^d$ s.t. $\pi_{\bar{D}}(y)$ is not unique, we arbitrarily set $\pi_{\bar{D}}(y) = x_0 \in \partial D$. This can seem awkward. Anyhow, we should always keep in mind that, because of the boundedness of the coefficients in (1.1), for h small enough, the events for which the process exits the domain where $\pi_{\bar{D}}$ is uniquely defined, before being discretely stopped are of exponentially small probability. For such events, we derive from the boundedness of g that the definition of the projection has no relevant impact on the convergence analysis. We refer to Section 3.2 for details.

In this work, we extend the result of Theorem 2.4 in [Gob00] to a possibly degenerate non-Markovian framework and to a more general class of functions. For the reader familiar with error decomposition techniques, we guess it is interesting to present below an analogy between standard PDE methods employed in the Markovian setting [TL90] and ours.

Note first that the killed case can be seen as a special case of the stopped one with $\forall t \in [0,T]$, $g(t,.)|_{\partial D} = 0$, $g(T,.)|_D = f(.)|_D$. Introducing $\forall t \in [0,T]$, $V_t := \mathbb{E}[g(T\wedge\tau_t, \pi_{\bar{D}}(X_{T\wedge\tau_t}))|\mathcal{F}_t] := \mathbb{E}[\tilde{g}(T\wedge\tau_t, X_{T\wedge\tau_t})|\mathcal{F}_t]$ where $\tau_t := \inf\{s \geq t : X_s \notin D\}$, the error writes

$$\mathrm{Err}(T,h,g,x) = \mathbb{E}[V_{T\wedge\tau^N}] - V_0. \qquad (1.6)$$

In a Markovian framework, for all $t \leq T\wedge\tau$, $V_t = v(t,X_t)$ where, under suitable assumptions, v is a smooth function satisfying the mixed Cauchy-Dirichlet problem

$$\begin{cases} (\partial_t + L)v(t,x) = 0, \ (t,x) \in [0,T) \times D, \\ v(t,x) = g(t,x), \forall (t,x) \in [0,T) \times \partial D \bigcup \{T\} \times \bar{D}, \end{cases} \qquad (1.7)$$

L being the infinitesimal generator of the diffusion X. The process $(V_{t\wedge\tau})_{t\in[0,T]}$ is associated to the standard Feynman-Kac representation of the solution of (1.7). In our case, we can not rely on a PDE, but on a martingale property that is one of the main ingredients needed for the proof. Namely, one has the following

Proposition 1.3 *Let X be an Itô process that follows the dynamics of equation (1.1). Assume the function g of (1.7) is bounded. Then, $\forall t \in [0,T), (V_{s\wedge\tau_t})_{s\in[t,T]}$ is a martingale.*

Observe that in the Markovian case, one can derive this martingale property from the PDE (1.7) using Itô's formula.

Proof. Note that $\forall s \in [t,T]$, on $\{s < \tau_t\}$, $V_{s\wedge\tau_t} = V_s = \mathbb{E}[\tilde{g}(T\wedge\tau_s, X_{T\wedge\tau_s})|\mathcal{F}_s]$, and on $\{s \geq \tau_t\}$, $V_{s\wedge\tau_t} = V_{\tau_t} = \tilde{g}(\tau_t, X_{\tau_t})$. Turning to the former definition of V it comes

$$\begin{aligned} \mathbb{E}[V_{s\wedge\tau_t} - V_t|\mathcal{F}_t] &= \mathbb{E}[\tilde{g}(T\wedge\tau_{s\wedge\tau_t}, X_{T\wedge\tau_{s\wedge\tau_t}}) - \tilde{g}(T\wedge\tau_t, X_{T\wedge\tau_t})|\mathcal{F}_t] \\ &= \mathbb{E}[\mathbb{I}_{s<\tau_t}(\tilde{g}(T\wedge\tau_s, X_{T\wedge\tau_s}) - \tilde{g}(T\wedge\tau_t, X_{T\wedge\tau_t}))|\mathcal{F}_t] + \mathbb{E}[\mathbb{I}_{s\geq\tau_t}(\tilde{g}(\tau_t, X_{\tau_t}) - \tilde{g}(\tau_t, X_{\tau_t}))|\mathcal{F}_t] = 0 \end{aligned}$$

since on the event $\{s < \tau_t\}$ one has $\tau_t = \tau_s$. $\square$

From (1.6), the strategy in the Markovian setting consists in writing Itô like expansions in order to isolate the leading term o the error (see [Go 00]). The a ove martingale property is crucial or our error decomposition. Namely, it replaces the use o Itô's ormula on v in the Markovian case.

Outline of the paper

In section 2 we state our working assumptions as well as our main results. Section 3 is dedicated to the common decomposition o the errors $\mathsf{Err}(T, h, f, x), \mathsf{Err}(T, h, g, x)$. We give in Section 4 the auxiliary results needed to o tain the ound o the error in the killed and stopped case. In Section 5, we show how our previous techniques can e employed to extend the previous control on $\mathsf{Err}(T, h, f, x)$ to the case o an intersection o smooth domains. We conclude in Section 6 giving some possi le extensions and evoking some remaining open pro lems.

2 Assumptions and main results

2.1 About the process

We assume the coefficients $(b_s)_{s\in[0,T]}, (\sigma_s)_{s\in[0,T]}$ o (1.1) are ounded. Some mild smoothness property on σ (some continuity in pro a ility) will e also needed: the condition stated elow is not restrictive at all and is ulfilled or instance as soon as $(\sigma_s)_{0\le s\le T}$ satisfies a Hölder-continuity property in L_p-norm.

(S) For any $\delta > 0$, there is some unction η_δ with $\lim_{h\to 0^+} \eta_\delta(h) = 0$ such that $a.s$, or $s \in]t_i, t_{i+1}[$ with $X_s \in \partial D$, one has $\mathbb{P}(|\int_s^{t_{i+1}}(\sigma_u - \sigma_s)dW_u| \ge \delta\sqrt{t_{i+1} - s}\,|\,\mathcal{F}_s) \le \eta_\delta(h)$.

2.2 About the domain

In this section we assume the domain D satisfies assumption

(D) The domain D is o class C^2 with ounded oundary ∂D, $X_0 = x \in \bar{D}$.

Additional notations and assumptions concerning the intersection o domains satis ying **(D)** are specified in section 5. For $x \in \partial D$, denote y $n(x)$ the unit inward normal vector at x. For $r \ge 0$, set $V_{\partial D}(r) := \{z \in \mathbb{R}^d : d(z, \partial D) \le r\}$ and $D(r) := \{z \in \mathbb{R}^d : d(z, D) \le r\}$. $B(z, r)$ stands or the closed all with center z and radius r. We now recall standard acts on the distance to the oundary and the orthogonal projection on ∂D (see Lemma 1 and its proo rom [GT77] p. 382).

Proposition 2.1 *Assume* **(D)**. *There is a constant $R > 0$ such that:*

i) for any $x \in V_{\partial D}(R)$, there are unique $s = \pi_{\partial D}(x) \in \partial D$ and $F(x) \in \mathbb{R}$ such that $x = \pi_{\partial D}(x) + F(x)n(\pi_{\partial D}(x))$.

ii) The function $x \longmapsto F(x)$ is the signed normal distance of x to ∂D: this is a C^2-function on $V_{\partial D}(R)$, which can be extended to a C^2 function on $\mathbb{R}^d$ with bounded derivatives. This extension satisfies $F(x) \ge d(x, \partial D) \wedge R$ on D, $F(x) \le -[d(x, \partial D) \wedge R]$ on D^c and $F = 0$ on ∂D.

iii) For $x \in V_{\partial D}(R)$, one has $\nabla F(x) = n(\pi_{\partial D}(x))$.

Assume D satisfies **(D)**. Following the notations o Proposition 2.1, we now introduce the non characteristic oundary condition

(C) $\exists a_0 > 0$ such that $a.s.$ $\left(X_s \in V_{\partial D}(R), s \in [0, T] \Longrightarrow \alpha_s := \nabla F(X_s).\sigma_s\sigma_s^*\nabla F(X_s) \ge a_0\right)$

which en orces the process to exit the domain in a non-tangential manner.

2.3 Main results

We are now in a position to state our main results for killed and stopped processes in the case of smooth domains.

Theorem 2.2 Upper bound in the smooth domain case for a killed process.
Assume **(C)**, **(D)**, **(S)** *and suppose f is a borelian and bounded function s.t.* $\exists \varepsilon > 0,\ d(\mathrm{supp}(f), \partial D) \geq 2\varepsilon$. *For some constant C, one has*

$$|\mathrm{Err}(T, h, f, x)| = |\mathbb{E}[f(X_T)\mathbb{I}_{\tau^N > T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau > T}]| \leq C\frac{\|f\|_\infty}{1 \wedge \varepsilon}\sqrt{h}.$$

Remark 2.1 *Note that if f is non-negative one also has $\mathrm{Err}(T, h, f, x) \geq 0$. This readily derives from the inequality $\tau^N \geq \tau$ a.s.*

Theorem 2.3 Upper bound in the smooth domain case for a stopped process.
Assume **(C)**, **(D)**, **(S)** *and suppose g is bounded in $C^{1,2}([0, T] \times \mathbb{R}^d)$. For some constant C, one has*

$$|\mathrm{Err}(T, h, g, x)| = |\mathbb{E}[g(T \wedge \tau^N, \pi_{\bar{D}}(X_{T\wedge\tau^N})) - g(T \wedge \tau, X_{T\wedge\tau})]| \leq C\sqrt{h}.$$

Remark 2.2 *Let us first mention that we can not improve the above rate in our framework, since in the Brownian case, one has an expansion w.r.t. $\sqrt{h}$ (cf. Siegmund and Yuh [SY82] and [Men04]).*

Remark 2.3 *To study the impact of the time discretization, few asssumptions are needed to get, as indicated in the previous remark, the expected rate of convergence. To obtain the same upper bound with the discretely killed Euler scheme of a diffusion process, an additional hypoellepticity condition is necessary (see [GM04]).*

Note also that Assumptions **(D)** *and* **(S)** *could possibly be weakened. On the other hand, Assumption* **(C)** *is somehow a minimal condition to ensure a convergent approximation. Indeed, it easy to imagine a deterministic path which hits ∂D only at time $\tau = \chi T$ where χ is an irrational number in $]0, 1[$: for this, $\tau^N > T$ for any $N \geq 1$ and $\mathrm{Err}(T, h, f, x) = f(X_T)$ is constant.*

Remark 2.4 *Recall also that the results of Theorems 2.2 and 2.3 concern respectively the impact of a discretization time in the quantities $\mathbb{E}[f(X_T)\mathbb{I}_{\tau>T}]$ and $\mathbb{E}[g(T\wedge\tau, X_{T\wedge\tau})]$. They can therefore not be directly compared to the results of Theorem 2.3 in [Gob00] or Section 6.4 Chapter I in [Men04] except in the special case of Brownian motion. Note anyhow that in that case we obtain the upper bound of the weak error with a much simpler proof. The next natural question, in the killed case and when $f \geq 0$, concerns a possible lower bound of the same order for $\mathrm{Err}(T, h, f, x)$ as stated in Theorem 5 in [GM04] in a Markovian framework. We give a counter example that illustrates this property can fail under the sole assumption* **(C)**. *Define for all $t \geq 0$, the one dimensional diffusion process $X_t = \pi/2 + \int_0^t \cos(X_s)ds + \int_0^t \sin(X_s)dW_s$ and put $D :=]-\pi/2, 3\pi/2[$.* **(C)** *is readily satisfied and by construction one has $X_s \in [0, \pi]$ a.s. Hence, $\mathbb{I}_{\tau^N>T} = \mathbb{I}_{\tau>T} = 1$ and $\mathrm{Err}(T, h, f, x) = 0$. A minimal necessary condition to have a lower bound of order $1/2$ w.r.t h is to reach the boundary on the interval $[0, T]$ with positive probability.*

3 Common decomposition of the error

In this section we assume **(D)** is in force. The constant R is the one of Proposition 2.1. In particular, on $D(R)$ the projection on $\bar{D}$ is uniquely defined.

3.1 Miscellaneous

We will keep the same notation C (or C') for all finite, non-negative constants which will appear in our computations: they may depend on D, T, b, σ, f or g, but they will not depend on the number of time steps N and the initial value x. We reserve the notation c and c' for constants also independent of x, T, f or g.

3.2 Localization of X in $D(R)$

In this subsection we justify that for studying $\mathrm{Err}(T,h,g,x)$, we can assume w.l.o.g. that $\forall t\in[0,T]$, $X_t\in D(R)$ *a.s.* Indeed, if it is not the case, we introduce $\tau_R := \inf\{s\geq 0 : X_s\notin D(R)\}$, $\bar X_t = X_{t\wedge\tau_R}$, $\bar\tau^N := \inf\{t_i\geq 0 : \bar X_{t_i}\notin D\}$, $\bar\tau := \inf\{t\geq 0 : \bar X_t\notin D\} = \tau$. Note that

$$\begin{aligned}
&\left|\mathrm{Err}(T,h,g,x) - \left(\mathbb{E}[g(T\wedge\bar\tau^N,\pi_{\bar D}(\bar X_{T\wedge\bar\tau^N}))] - \mathbb{E}[g(T\wedge\bar\tau,\bar X_{T\wedge\bar\tau})]\right)\right|\\
&:= |\mathrm{Err}(T,h,g,x) - \mathrm{Err}_2(T,h,g,x)| \leq 2|g|_\infty\mathbb{P}[\tau_R<\tau^N].
\end{aligned}$$

The process $\bar X$ satisfies **(C)**, **(S)** and is $D(R)$ valued. Hence, from Assumption **(D)**, the projection on $\bar D$ is uniquely defined in the term $\mathrm{Err}_2(T,h,g,x)$. It therefore remains to control the probability $\mathbb{P}[\tau_R<\tau^N]$. To this end, a key tool is the following

Lemma 3.1 (Bernstein's type inequality) *Consider two stopping times S, S' upper bounded by T with $0\leq S'-S\leq\Delta\leq T$. Then for any $p\geq 1$ and $c'>0$, there are some constants $c>0$ and C, such that for any $\eta\geq 0$, one has a.s:*

$$\begin{aligned}
\mathbb{P}[\sup_{t\in[S,S']}\|X_t-X_S\|\geq\eta\,|\,\mathcal{F}_S] &\leq C\exp\left(-c\frac{\eta^2}{\Delta}\right),\\
\mathbb{E}[\sup_{t\in[S,S']}\|X_t-X_S\|^p\,|\,\mathcal{F}_S] &\leq C\Delta^{p/2}.
\end{aligned}$$

Proof. We omit the proof of the first inequality which is standard and refer the reader to Lemma 4.1 in [Gob00] for instance. The other one easily follows from the first one. □

Lemma 3.1 readily gives $\mathbb{P}[\tau_R<\tau^N]\leq C\exp\left(-c\frac{R^2}{h}\right)$. Thus, taking $(\bar X,\bar\tau^N)$ instead of (X,τ^N) has no significant impact. This has however the advantage to keep the projection on $\bar D$ well defined. Hence, in the following we assume

$$(X_t)_{t\in[0,T]}\in D(R)\ a.s.$$

3.3 Error decomposition and proof of the main results

The error decomposition is common to both the killed and stopped cases. Put $\forall(t,z)\in[0,T]\times D(R)$,

$$\tilde g(t,z) := \begin{cases}\mathbb{I}_{t<T}f(z) \text{ in the killed case,}\\ g(t,\pi_{\bar D}(z)) \text{ in the stopped case.}\end{cases}$$

We denote by $\mathrm{Err}(T,h,\tilde g,x)$ the error corresponding to $\mathrm{Err}(T,h,f,x)$ in the killed case (resp. $\mathrm{Err}(T,h,g,x)$ in the stopped case). It comes

$$\begin{aligned}
\mathrm{Err}(T,h,\tilde g,x) &= \mathbb{E}[\tilde g(T\wedge\tau^N,X_{T\wedge\tau^N}) - \tilde g(T\wedge\tau,X_{T\wedge\tau})]\\
&= \mathbb{E}[\mathbb{I}_{\tau<T}\mathbb{E}[\tilde g(T\wedge\tau^N,X_{T\wedge\tau^N}) - \tilde g(\tau,X_\tau)|\mathcal{F}_\tau]].
\end{aligned}$$

Hence, to show Theorems 2.2 and 2.3, it is enough to derive

$$|\mathcal{E}| := |\mathbb{E}[\tilde g(T'\wedge\tau^{N'},X_{T'\wedge\tau^{N'}}) - \tilde g(t,x)]| \leq C\sqrt{h}, \tag{3.1}$$

for an initial point $x\in\partial D$, $t\in[0,T)$, for a shifted time mesh defined by $\{t_i : 0\leq i\leq N'\}$ with $t_0=0$, $0<t_1\leq h$, $t_{i+1}=t_i+h$ $(i\geq 1)$, for a new terminal time $T'=t_{N'}$ and a modified exit time $\tau^{N'}=\inf\{t_i\geq t_1 : X_{t_i}\notin D\}$. The constant C in (3.1) has to be uniform in T' in a compact set, in N', in x and in t. For the sake of simplicity, we still write N for N', T for T' and take $t=0$. Introduce now for all $s\in[0,T]$, $V_s := \mathbb{E}[\tilde g(T\wedge\tau_s,X_{T\wedge\tau_s})|\mathcal{F}_s]$ where $\tau_s := \inf\{u\geq s : X_u\notin D\}$ and recall from Proposition 1.3 that $(V_{u\wedge\tau_s})_{u\in[s,T]}$ is a martingale. For $x\in\partial D$, $\tau_0=0$ so $V_0=g(0,x)$. On the other hand $V_{T\wedge\tau^N}=\tilde g(T\wedge\tau^N,X_{T\wedge\tau^N})$. Thus,

$$\begin{aligned}
\mathcal{E} &= \mathbb{E}[V_{T\wedge\tau^N}] - V_0 = \sum_{i=0}^{N-1}\mathbb{E}[V_{t_{i+1}\wedge\tau^N} - V_{t_i\wedge\tau^N}] = \sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^N>t_i}\left(V_{t_{i+1}} - V_{t_i}\right)]\\
&= \sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^N>t_i}\left(V_{t_{i+1}} - V_{t_{i+1}\wedge\tau_{t_i}}\right)] + \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\left(V_{t_{i+1}\wedge\tau_{t_i}} - V_{t_i}\right)].
\end{aligned}$$

It readily follows from the martingale property of $(V_{u\wedge\tau_{t_i}})_{u\in[t_i,T]}$ (see Proposition 1.3) that $\mathbb{E}[\mathbb{I}_{\tau^N>t_i}(V_{t_{i+1}\wedge\tau_{t_i}} - V_{t_i})] = 0$. Therefore we have

$$\mathcal{E} = \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\mathbb{I}_{\tau_{t_i}<t_{i+1}}(V_{t_{i+1}} - V_{\tau_{t_i}})]. \tag{3.2}$$

Remark 3.1 *Note that to obtain (3.2) we did not use any smoothness properties of $\tilde{g}$.*

To control $\mathcal{E}$ we state two auxiliary Lemmas whose proofs are postponed to section 4.

Lemma 3.2 *Assume* **(C)**, **(D)**, **(S)** *and that in the killed case f satisfies the assumptions of Theorem 2.2 (resp. in the stopped case g satisfies the assumptions of Theorem 2.3). For all $i \in [\![0, N-1]\!]$, on the set $\{\tau^N > t_i, \tau_{t_i} < t_{i+1}\}$ one has*

$$|\mathbb{E}[V_{t_{i+1}} - V_{\tau_{t_i}}|\mathcal{F}_{\tau_{t_i}}]| \le C\sqrt{h}.$$

Lemma 3.3 *Assume* **(C)**, **(D)** *and* **(S)**. *There are some positive constants C and N_0 such that for $N \ge N_0$, for any $i \in [\![0, N-1]\!]$, one has for $X_{t_i} \in D$*

$$\mathbb{P}[\exists t \in [t_i, t_{i+1}] : X_t \notin D \mid \mathcal{F}_{t_i}] \le C\, \mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{t_i}].$$

Plugging the control of Lemma 3.2 into (3.2) we obtain

$$|\mathcal{E}| \le C\sqrt{h}\sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\mathbb{I}_{\tau_{t_i}<t_{i+1}}].$$

Using now Lemma 3.3 it comes

$$|\mathcal{E}| \le C\sqrt{h}\sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\mathbb{I}_{X_{t_{i+1}}\notin D}] = C\sqrt{h}\sum_{i=0}^{N-1} \mathbb{P}[\tau^N = t_{i+1}] \le C\sqrt{h}$$

which completes the proof of Theorems 2.2 and 2.3. □

4 Proof of the technical Lemmas

This section is devoted to the proof of Lemmas 3.2 and 3.3. For smooth functions $g(t,x)$, we denote by $\partial_t g(t,x)$ its time derivative, by $\nabla g(t,x)$ its gradient w.r.t. x and by $H_g(t,x)$ its Hessian matrix w.r.t. x. The notation $\frac{\partial g}{\partial n}(t,x) = \nabla g(t,x).n(x)$ stands for the normal derivative on the boundary.

Using the results of Proposition 2.1 and Lemma 3.1, we prove the following Lemma that will be repeatedly used.

Lemma 4.1 *Assume* **(D)**. *For all $i \in [\![0, N-1]\!]$, on the set $\{\tau_{t_i} \le t_{i+1}\}$, one has*

$$\mathbb{E}\Big[\Big(F(X_{t_{i+1}})\Big) |\mathcal{F}_{\tau_{t_i}}] = \mathbb{E}\Big[\Big(F(X_{t_{i+1}}) - F(X_{\tau_{t_i}})\Big) |\mathcal{F}_{\tau_{t_i}}] \le C\sqrt{h}.$$

4.1 Proof of Lemma 3.2

For this proof we distinguish the killed and stopped cases.

4.1.1 Proof in the killed case

In that case Lemma 3.2 is a direct consequence of the following

Lemma 4.2 *Assume* **(C)**, **(D)**, **(S)** *and let the function f be as in Theorem 2.2. There is some constant C such that for any $t \in [0,T]$, one has a.s*

$$|V_t| \le C\frac{\|f\|_\infty}{1\wedge\varepsilon}[F(X_t)]_+.$$

Indeed, we deduce from Lemma 4.1 that $\forall i \in [\![0, N-1]\!]$, on $\{\tau^N > t_i, \tau_{t_i} \le t_{i+1}\}$ one has

$$|\mathbb{E}[V_{t_{i+1}}|\mathcal{F}_{\tau_{t_i}}]| \quad \le \quad C\frac{\|f\|_\infty}{1\wedge\varepsilon}\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}] \le C\sqrt{h}\frac{\|f\|_\infty}{1\wedge\varepsilon}.$$

Proof of Lemma 4.2

W.l.o.g. we assume $f \ge 0$. Since $V_t = 0$ for $X_t \notin D$, it is enough to prove the estimate for $X_t \in D\cap V_{\partial D}(R\wedge\varepsilon/2)$ for which $0 < F(X_t) \le R\wedge\varepsilon/2$. Denote $\tau_t^R = \inf\{s \ge t : F(X_s) \ge R\}$ and split V into two parts $V_t = V_t^1 + V_t^2$ with $V_t^1 = \mathbb{E}\left(\mathbb{I}_{T<\tau_t}\mathbb{I}_{T<\tau_t^R}f(X_T) \mid \mathcal{F}_t\right)$ and $V_t^2 = \mathbb{E}\left(\mathbb{I}_{T<\tau_t}\mathbb{I}_{T\ge\tau_t^R}f(X_T) \mid \mathcal{F}_t\right)$.

Before estimating separately each contribution, we set some standard notations related to time-changed Brownian martingales. Define the increasing continuous process $\mathcal{A}_s = \int_t^s \alpha_u du$ (from $[t, +\infty[$ into $\mathbb{R}^+$) and its increasing right-continuous inverse $\mathcal{C}_s = \inf\{u \ge t : \mathcal{A}_u > s\}$ (from $\mathbb{R}^+$ into $[t, +\infty[$) (see section V.1 in Revuz-Yor [RY99]) and put $M_s = \int_t^{\mathcal{C}_s} \nabla F(X_u).\sigma_u dW_u$, $Z_s = F(X_{\mathcal{C}_s})$. From the Dambis-Dubins-Schwarz theorem, M coincides with a standard BM β (defined on a possibly enlarged probability space) for $s < \int_t^\infty \alpha_u du$ and it is easy to check that β is independent of $\mathcal{F}_t$ (see the arguments in the proof of Theorem V.1.7 in [RY99]). Owing to the assumption **(C)**, $\mathcal{A}$ and $\mathcal{C}$ are strictly increasing on $[t, \tau_t^R]$ and $[0, \int_t^{\tau_t^R} \alpha_u du]$. Thus, for $s \in [0, \int_t^{\tau_t^R} \alpha_u du]$, one easily obtains

$$Z_s = F(X_t) + \beta_s + \int_0^s \lambda_v dv$$

where $\lambda_v = \{[\nabla F(X_u).b_u + \frac{1}{2}\mathrm{tr}(H_F(X_u)\sigma_u\sigma_u^*)]|_{u=\mathcal{C}_v}\}\frac{1}{\alpha_{\mathcal{C}_v}}$ is bounded by $\|\lambda\|_\infty$. Define

$$Z'_s = F(X_t) + \beta_s + \|\lambda\|_\infty s \ge Z_s. \tag{4.1}$$

Finally, put $\tau_0^Z = \inf\{s \ge 0 : Z_s \le 0\}$, $\tau_R^Z = \inf\{s \ge 0 : Z_s \ge R\}$ and analogously $\tau_0^{Z'}$, $\tau_R^{Z'}$ for Z'.

<u>*Estimation of V^1*</u>. Let us first prove that for any stopping time $S \in [t, T]$, one has

$$\begin{aligned}\mathbb{E}\left(f(X_T) \mid \mathcal{F}_S\right) \le& \|f\|_\infty \mathbb{P}[F(X_T) \ge 2\varepsilon \mid \mathcal{F}_S]\\ \le& C\|f\|_\infty \exp\left(-c\frac{(2\varepsilon - F(X_S))_+^2}{T-S}\right) \quad a.s.\end{aligned} \tag{4.2}$$

The first inequality simply results from the support of f included in $D\backslash V_{\partial D}(2\varepsilon)$. To justify the second one, note that $\{F(X_T) \ge 2\varepsilon\} \subset \{|F(X_T) - F(X_S)| \ge 2\varepsilon - F(X_S)\} \subset \{|F(X_T) - F(X_S)| \ge (2\varepsilon - F(X_S))_+\}$ and the proof of (4.2) is complete using Lemma 3.1 applied to the Itô process $(F(X_s))_{s\ge 0}$ with bounded coefficients. We now turn to the evaluation of V_t^1. On $\{T < \tau_t^R\}$, using the notation with the time change above, one has $T = \mathcal{C}_{\mathcal{A}_T} \ge \mathcal{C}_{a_0(T-t)}$ and $a_0(T - \mathcal{C}_{a_0(T-t)}) \le \int_{\mathcal{C}_{a_0(T-t)}}^T \alpha_u du = \mathcal{A}_T - \mathcal{A}_{\mathcal{C}_{a_0(T-t)}}$. Hence, $T - \mathcal{C}_{a_0(T-t)} \le \frac{1}{a_0}(\mathcal{A}_T - a_0(T-t)) \le \frac{\|\alpha\|_\infty}{a_0}(T-t)$. Thus, one obtains

$$\begin{aligned}V_t^1 &\le \mathbb{E}\left(\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t}\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\mathbb{I}_{T-\mathcal{C}_{a_0(T-t)}\le\frac{\|\alpha\|_\infty}{a_0}(T-t)}\mathbb{E}\left(f(X_T) \mid \mathcal{F}_{\mathcal{C}_{a_0(T-t)}}\right) \mid \mathcal{F}_t\right)\\ &\le C\|f\|_\infty\mathbb{E}\left(\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t}\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\exp\left(-c'\frac{(2\varepsilon - F(X_{\mathcal{C}_{a_0(T-t)}}))_+^2}{T-t}\right) \mid \mathcal{F}_t\right)\\ &\le C\|f\|_\infty\mathbb{E}\left(\mathbb{I}_{a_0(T-t)<\tau_0^{Z'}}\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\exp\left(-c'\frac{(2\varepsilon - Z'_{a_0(T-t)})_+^2}{T-t}\right) \mid \mathcal{F}_t\right)\end{aligned}$$

where one has applied at the second line the estimate (4.2) with $S = \mathcal{C}_{a_0(T-t)}$ (here $c' = c\frac{a_0}{\|\alpha\|_\infty}$), at the third one $\{\mathcal{C}_{a_0(T-t)} < \tau_t\} = \{\forall s \in [t, \mathcal{C}_{a_0(T-t)}] : F(X_s) > 0\} = \{\forall u \in [0, a_0(T-t)] : Z_u > 0\} = \{a_0(T-t) < \tau_0^Z\} \subset \{a_0(T-t) < \tau_0^{Z'}\}$ and $(2\varepsilon - F(X_{\mathcal{C}_{a_0(T-t)}}))_+ = (2\varepsilon - Z_{a_0(T-t)})_+ \ge (2\varepsilon - Z'_{a_0(T-t)})_+$. Reminding the law of β, one finally gets that $V_t^1 \le C\|f\|_\infty\Phi_1(a_0(T-t), F(X_t))$ with $\Phi_1(r,z) = \mathbb{E}\left(\mathbb{I}_{\forall u\in[0,r]:z+\beta_u+\|\lambda\|_\infty u>0}\exp\left(-a_0c'\frac{(2\varepsilon - z - \beta_r - \|\lambda\|_\infty r)_+^2}{r}\right)\right)$. With clear notations involving the smooth transition density of the killed drifted BM and Gaussian type estimates of its gradient (see [LSU68] Theorem 16.3), one has $\Phi_1(r,z) = \int_0^\infty q_r(z,y)\exp\left(-a_0c'\frac{(2\varepsilon-y)_+^2}{r}\right)dy$ and

$$|\partial_z\Phi_1(r,z)| \le C\int_0^\infty \frac{1}{r}\exp(-c\frac{(z-y)^2}{r})\exp\left(-a_0c'\frac{(2\varepsilon-y)_+^2}{r}\right)dy.$$

We now justify that $|\partial_z \Phi_1(r,z)| \le \frac{C}{1\wedge\varepsilon}$ for $0 \le z \le \epsilon/2$ and for this, we may split the domain of integration into two parts. For $y < \varepsilon$, $(2\varepsilon - y)_+^2 \ge \varepsilon^2$ and the corresponding contribution for the integral is bounded by $\int_0^\infty \frac{1}{\sqrt{r}} \exp(-c\frac{(z-y)^2}{r}) \frac{1}{\sqrt{r}} \exp\left(- a_0 c' \frac{\varepsilon^2}{r}\right) dy \le \frac{C}{1\wedge\varepsilon}$. For $y \ge \varepsilon$ and $0 \le z \le \varepsilon/2$, $(z-y)^2 \ge \varepsilon^2/4$ and the integral is bounded by $\int_0^\infty \frac{1}{\sqrt{r}} \exp(-\frac{c}{2}\frac{(z-y)^2}{r}) \frac{1}{\sqrt{r}} \exp(-\frac{c}{2}\frac{\varepsilon^2}{4r}) dy \le \frac{C}{1\wedge\varepsilon}$.
Since $\Phi_1(r,0) = 0$, one gets $\Phi_1(r,z) \le \frac{C}{1\wedge\varepsilon} z$ for $z \in [0,\varepsilon/2]$ and this proves that $V_t^1 \le C\frac{\|f\|_\infty}{1\wedge\varepsilon} F(X_t)$.
<u>*Estimation of V^2.*</u> Clearly, one has $V_t^2 \le \|f\|_\infty \mathbb{P}\left(\tau_t^R < \tau_t \mid \mathcal{F}_t\right)$. Note that $\{\tau_t^R < \tau_t\} = \{\tau_R^Z < \tau_0^Z\} \subset \{\tau_R^{Z'} < \tau_0^{Z'}\}$ because of (4.1). Hence, one has $V_t^2 \le \|f\|_\infty \Phi_2(F(X_t))$ where $\Phi_2(z) = \mathbb{P}[(z+\beta_u+\|\lambda\|_\infty u)_{u\ge 0}$ hits R before $0]$. It is well-known that $\Phi_2(z) = \frac{1-\exp(-2\|\lambda\|_\infty z)}{1-\exp(-2\|\lambda\|_\infty R)} \le Cz$ (see Section 5.5 in [KS91] e.g.) and this proves that $V_t^2 \le C\|f\|_\infty F(X_t)$. Combining estimates for V^1 and V^2 gives the result of Lemma 4.2. $\square$

4.1.2 Proof in the stopped case

Assume the function g is as in Theorem 2.3. In this case, we use the smoothness of g. Since we also assumed X_t is $D(R)$ valued, the semi-martingale decomposition stated in Proposition 3.1 in [Gob00] remains valid for $(\pi_{\bar D}(X_t))_{t\ge 0}$. Hence, $\forall i \in [\![0, N-1]\!]$, on the set $\{\tau_{t_i} \le t_{i+1}\}$ we write

$$\begin{aligned}
&\tilde g(T\wedge\tau_{t_{i+1}}, X_{T\wedge\tau_{t_{i+1}}}) - \tilde g(\tau_{t_i}, X_{\tau_{t_i}}) \\
&= \int_{\tau_{t_i}}^{T\wedge\tau_{t_{i+1}}} \partial_u g(u, \pi_{\bar D}(X_u))du + \nabla g(u,\pi_{\bar D}(X_u))\cdot d(\pi_{\bar D}(X_u)) + \frac{1}{2}\mathrm{tr}(H_g(u,\pi_{\bar D}(X_u)) d\langle \pi_{\bar D}(X)\rangle_u) \\
&:= (M_{T\wedge\tau_{t_{i+1}}} - M_{\tau_{t_i}}) + (V_{T\wedge\tau_{t_{i+1}}} - V_{\tau_{t_i}}) + \int_{\tau_{t_i}}^{T\wedge\tau_{t_{i+1}}} \frac{\partial g}{\partial n}(u, X_u) dL_u^0(F(X))
\end{aligned}$$

where M is a local martingale and V a finite variation process. From the boundedness of the derivatives of g and of the coefficients b_s, σ_s, we derive that M is a true martingale and that $a.s$ $|V_{T\wedge\tau_{t_{i+1}}} - V_{\tau_{t_i}}| \le C(T\wedge\tau_{t_{i+1}} - \tau_{t_i})$. It comes

$$\begin{aligned}
|\mathbb{E}[\tilde g(T\wedge\tau_{t_{i+1}}, X_{T\wedge\tau_{t_{i+1}}}) - \tilde g(\tau_{t_i}, X_{\tau_{t_i}}) | \mathcal{F}_{\tau_{t_i}}]| &\le C\Big(\mathbb{E}[L^0_{T\wedge\tau_{t_{i+1}}}(F(X)) - L^0_{\tau_{t_i}}(F(X)) | \mathcal{F}_{\tau_{t_i}}] \\
&+ \mathbb{E}[(T\wedge\tau_{t_{i+1}} - \tau_{t_i})|\mathcal{F}_{\tau_{t_i}}]\Big) := C\left(A^1_{\tau_{t_i}} + A^2_{\tau_{t_i}}\right).
\end{aligned}$$

Term $A^1_{\tau_{t_i}}$: control of the local time.
Since the measure $dL_t^0(F(X))$ is $a.s$ carried by the set $\{t : F(X_t) = 0\}$ we write

$$\begin{aligned}
A^1_{\tau_{t_i}} &= \mathbb{E}[L^0_{t_{i+1}}(F(X)) - L^0_{\tau_{t_i}}(F(X)) | \mathcal{F}_{\tau_{t_i}}] \\
&= 2\mathbb{E}[[F(X_{t_{i+1}})]_- - [F(X_{\tau_{t_i}})]_- + \int_{\tau_{t_i}}^{t_{i+1}} \mathbb{I}_{F(X_s)<0} dF(X_s) | \mathcal{F}_{\tau_{t_i}}] \le C\sqrt{h}. \qquad (4.3)
\end{aligned}$$

The last equality follows from Tanaka's formula. The last inequality is a consequence of the boundedness of F and its derivatives, the boundedness of the coefficients of X and Lemma 4.1.

Term $A^2_{\tau_{t_i}}$: time-change techniques.
Write

$$A^2_{\tau_{t_i}} = (T - \tau_{t_i})\mathbb{P}[\tau_{t_{i+1}} > T | \mathcal{F}_{\tau_{t_i}}] + \mathbb{E}[(\tau_{t_{i+1}} - \tau_{t_i})\mathbb{I}_{\tau_{t_{i+1}} \le T} | \mathcal{F}_{\tau_{t_i}}] := A^{21}_{\tau_{t_i}} + A^{22}_{\tau_{t_i}}.$$

The key idea is now, as in the proof of Lemma 4.2, to use time-changes in order to apply well known results for hitting times in a Brownian framework. We rewrite

$$A^{21}_{\tau_{t_i}} = (T - \tau_{t_i})\mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D} \mathbb{E}[\mathbb{I}_{\tau_{t_{i+1}} > T} | \mathcal{F}_{t_{i+1}}] | \mathcal{F}_{\tau_{t_i}}].$$

Put $C_{t_{i+1}} := \mathbb{P}[\tau_{t_{i+1}} > T | \mathcal{F}_{t_{i+1}}]$ and define $\tau_t^R := \inf\{s \ge t : F(X_s) \ge R\}$. We decompose $C_{t_{i+1}} = \mathbb{P}[\tau_{t_{i+1}} > T, \tau^R_{t_{i+1}} \le T | \mathcal{F}_{t_{i+1}}] + \mathbb{P}[\tau_{t_{i+1}} > T, \tau^R_{t_{i+1}} > T | \mathcal{F}_{t_{i+1}}] := C^1_{t_{i+1}} + C^2_{t_{i+1}}$. Since $C^1_{t_{i+1}} \le \mathbb{P}[\tau_{t_{i+1}} > \tau^R_{t_{i+1}} | \mathcal{F}_{t_{i+1}}]$, we can control this term in the same way we did for V^2 in the proof of Lemma 4.2. Namely, we get

$$\mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D} C^1_{t_{i+1}} | \mathcal{F}_{\tau_{t_i}}] \le C\mathbb{E}[[F(X_{t_{i+1}})]_+ | \mathcal{F}_{\tau_{t_i}}]. \qquad (4.4)$$

In the ollowing we use the notation introduced in the proo o Lemma 4.2 or time-changed martingales with $t = t_{i+1}$. For all $i \in [\![0, N-2]\!]$, on the set $\{X_{t_{i+1}} \in D\}$ we write

$$\begin{aligned}
C^2_{t_{i+1}} &= \mathbb{P}[\inf_{s\in[t_{i+1},T]} F(X_{t_{i+1}}) + \beta_{\mathcal{A}_s} + \lambda_{\mathcal{A}_s} > 0, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}]\\
&\le \mathbb{P}[\inf_{s\in[0,\mathcal{A}_T]} F(X_{t_{i+1}}) + \beta_s + \|\lambda\|_\infty s > 0, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}]\\
&\le \mathbb{P}[\inf_{s\in[0,a_0(T-t_{i+1})]} F(X_{t_{i+1}}) + \beta_s + \|\lambda\|_\infty s > 0, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}]\\
&\le \int_{a_0(T-t_{i+1})}^{\infty} dt \frac{F(X_{t_{i+1}})}{(2\pi t^3)^{1/2}} \exp(-\frac{(F(X_{t_{i+1}}) + \|\lambda\|_\infty t)^2}{2t}) \le \frac{CF(X_{t_{i+1}})}{(T-t_{i+1})^{1/2}}
\end{aligned} \tag{4.5}$$

exploiting the explicit density or the hitting times o the dri ted BM, see e.g. [KS91] section 3.5.C, or the last   ut one inequality. From (4.4) and (4.5) we derive that $\forall i \in [\![0, N-2]\!]$

$$A^{21}_{\tau_{t_i}} \le C(T-\tau_{t_i})\mathbb{E}[[F(X_{t_{i+1}})]_+(1 + \frac{1}{(T-t_{i+1})^{1/2}})|\mathcal{F}_{\tau_{t_i}}].$$

O serving that $\forall i \in [\![0, N-2]\!], T - t_{i+1} \ge \frac{T-t_i}{2} \ge \frac{T-\tau_{t_i}}{2}$ we derive rom Lemma 4.1

$$A^{21}_{\tau_{t_i}} \le C\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}] \le C\sqrt{h}. \tag{4.6}$$

Since or $i = N-1$ we also have $A^{21}_{\tau_{t_i}} \le (T-\tau_{t_i}) \le h$ and we finally o tain that equation (4.6) is valid or all $i \in [\![0, N-1]\!]$. We now turn to the control o $A^{22}_{\tau_{t_i}}$ reintroducing the events $\{\tau^R_{t_{i+1}} > \tau_{t_{i+1}}\}, \{\tau^R_{t_{i+1}} < \tau_{t_{i+1}}\}$. It comes

$$\begin{aligned}
A^{22}_{\tau_{t_i}} &= \mathbb{E}[(\tau_{t_{i+1}} - \tau_{t_i})\mathbb{I}_{\tau_{t_{i+1}}\le T}\mathbb{I}_{X_{t_{i+1}}\in D}(\mathbb{I}_{\tau^R_{t_{i+1}}>\tau_{t_{i+1}}} + \mathbb{I}_{\tau^R_{t_{i+1}}<\tau_{t_{i+1}}})|\mathcal{F}_{\tau_{t_i}}] + O(h)\\
&:= A^{221}_{\tau_{t_i}} + A^{222}_{\tau_{t_i}} + O(h).
\end{aligned}$$

Conditioning w.r.t. $\mathcal{F}_{t_{i+1}}$ and using the same arguments as or $C^1_{t_{i+1}}$ we readily get $A^{222}_{\tau_{t_i}} \le C\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}] \le C\sqrt{h}$. For $A^{221}_{\tau_{t_i}}$ write

$$\begin{aligned}
A^{221}_{\tau_{t_i}} &\le h + \mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D}\mathbb{E}[(\tau_{t_{i+1}} - t_{i+1})\mathbb{I}_{\tau_{t_{i+1}}\le T}\mathbb{I}_{\tau^R_{t_{i+1}}>\tau_{t_{i+1}}}|\mathcal{F}_{t_{i+1}}]|\mathcal{F}_{\tau_{t_i}}]\\
&:= h + \mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D}Q_{t_{i+1}}|\mathcal{F}_{\tau_{t_i}}].
\end{aligned}$$

Regarding $Q_{t_{i+1}}$, one has

$$\begin{aligned}
Q_{t_{i+1}} &\le \int_0^{T-t_{i+1}} ds\mathbb{P}[\tau_{t_{i+1}} - t_{i+1} \ge s, \tau^R_{t_{i+1}} > \tau_{t_{i+1}}|\mathcal{F}_{t_{i+1}}]\\
&\le \int_0^{T-t_{i+1}} ds\mathbb{P}[\inf_{u\in[0,\mathcal{A}_{s+t_{i+1}}]} F(X_{t_{i+1}}) + \beta_u + \|\lambda\|_\infty u > 0, \tau^R_{t_{i+1}} > \tau_{t_{i+1}}|\mathcal{F}_{t_{i+1}}]\\
&\le \int_0^{T-t_{i+1}} ds\mathbb{P}_y[\tau_0^{\tilde\beta} \ge a_0 s]
\end{aligned}$$

where we denote $y = F(X_{t_{i+1}})$, $\tilde\beta_u = y + \beta_u + \|\lambda\|_\infty u$, $\tau_0^{\tilde\beta} := \inf\{s \ge 0 : \tilde\beta_s = 0\}$. Thus, recalling that $y > 0$ on the set $\{X_{t_{i+1}} \in D\}$, it comes

$$\begin{aligned}
Q_{t_{i+1}} &\le a_0^{-1}\int_0^{(T-t_{i+1})a_0} ds\mathbb{P}_y[\tau_0^{\tilde\beta} \ge s] = a_0^{-1}\mathbb{E}_y[\tau_0^{\tilde\beta}\mathbb{I}_{\tau_0^{\tilde\beta}\le a_0(T-t_{i+1})}]\\
&\le a_0^{-1}\int_0^{(T-t_{i+1})a_0} dt\frac{ty}{(2\pi t^3)^{1/2}}\exp(-\frac{(y + \|\lambda\|_\infty t)^2}{2t}) \le Cy.
\end{aligned}$$

From this last estimate and the previous controls we derive

$$A^{221}_{\tau_{t_i}} \le h + C\mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D}F(X_{t_{i+1}})|\mathcal{F}_{\tau_{t_i}}] \le C\sqrt{h}.$$

Hence, or all $i \in [\![0, N-1]\!]$,

$$A^{22}_{\tau_{t_i}} \le C\sqrt{h}. \tag{4.7}$$

We conclude the proo o Lemma 3.2 in the stopped case putting together the controls (4.3), (4.6), (4.7) . □

4.2 Proof of Lemma 3.3

We adapt some ideas from [Go 00]: in the cited paper, a uniform ellipticity condition was assumed, and this enabled to use a Gaussian type lower bound for the conditional density of $X_{t_{i+1}}$ w.r.t. the Lebesgue measure, together with some computations related to a cone exterior to D. Here, under **(C)**, the conditional law of $X_{t_{i+1}}$ may be degenerate and our proof rather exploits the scaling invariance of the cone and of the Brownian increments.
It is enough to prove that $a.s$ on $\{t_i < \tau_{t_i} < t_{i+1}\}$, one has

$$\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \geq \frac{1}{C}. \tag{4.8}$$

Indeed, it follows that $\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{t_i}] = \mathbb{E}[\mathbb{I}_{\tau_{t_i} \leq t_{i+1}} \mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \mid \mathcal{F}_{t_i}] \geq \frac{\mathbb{P}[\tau_{t_i} \leq t_{i+1} \mid \mathcal{F}_{t_i}]}{C}$ and Lemma 3.3 is proved.
To get (4.8), write $X_{t_{i+1}} = X_{\tau_{t_i}} + \sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}}) + R_i$ where $R_i = \int_{\tau_{t_i}}^{t_{i+1}} b_u du + \int_{\tau_{t_i}}^{t_{i+1}} (\sigma_u - \sigma_{\tau_{t_i}}) dW_u$. The domain D is of class C^2, and thus satisfies a uniform exterior sphere condition with radius $R/2$ (R defined in Proposition 2.1): for any $z \in \partial D$, $B(z - \frac{R}{2} n(z), \frac{R}{2}) \subset D^c$. In particular, if we define for $\theta \in]0, \pi/2[$ the cone $\mathcal{K}(\theta, z) := \{y \in \mathbb{R}^d : (y-z).[-n(z)] \geq \|y - z\| \cos(\theta)\}$, then one has $\mathcal{K}(\theta, z) \cap B(z, R(\theta)) \subset B(z - \frac{R}{2} n(z), \frac{R}{2}) \subset D^c$ for some appropriate choice of the *positive* function $R(.)$. Then, it follows that

$$\begin{aligned}
&\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \geq \mathbb{P}[X_{t_{i+1}} \in \mathcal{K}(\theta, X_{\tau_{t_i}}) \cap B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}] \\
&\geq \mathbb{P}[X_{t_{i+1}} \in \mathcal{K}(\theta, X_{\tau_{t_i}}) \mid \mathcal{F}_{\tau_{t_i}}] - \mathbb{P}[X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}] \\
&\geq \mathbb{P}[(X_{t_{i+1}} - X_{\tau_{t_i}}).(-n(X_{\tau_{t_i}})) \geq \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \geq \|X_{t_{i+1}} - X_{\tau_{t_i}}\| \cos(\theta) \mid \mathcal{F}_{\tau_{t_i}}] \\
&\quad - \mathbb{P}[X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}] \geq A_1 - A_2(\theta) - A_3(\theta),
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\text{where } A_1 &= \mathbb{P}[(X_{t_{i+1}} - X_{\tau_{t_i}}).(-n(X_{\tau_{t_i}})) \geq \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}], \\
A_2(\theta) &= \mathbb{P}[\sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} < \|X_{t_{i+1}} - X_{\tau_{t_i}}\| \cos(\theta) \mid \mathcal{F}_{\tau_{t_i}}], \\
A_3(\theta) &= \mathbb{P}[X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}].
\end{aligned}$$

Term A_1. Clearly, one has $A_1 \geq \mathbb{P}[(-n(X_{\tau_{t_i}})).\sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}}) \geq 2\sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}] - \mathbb{P}[|n(X_{\tau_{t_i}}).R_i| \geq \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}] := A_{11} - A_{12}$. The random variable $(-n(X_{\tau_{t_i}})).\sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}})$ is conditionally to $\mathcal{F}_{\tau_{t_i}}$ a centered Gaussian variable with variance $\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})$, and thus $A_{11} = \Phi(-2) > 0$, where Φ denotes the distribution function of the standard normal law. Owing to the condition **(S)** and since $\alpha_{\tau_{t_i}} \geq a_0$ $a.s$, it is easy to see that the contribution A_{12} converges uniformly to 0 when h goes to 0, and thus for $h = T/N$ small enough, one has $A_1 \geq \frac{A_{11}}{2} > 0$.
Term $A_2(\theta)$. From Markov's inequality, $A_2(\theta) \leq \frac{\mathbb{E}[\|X_{t_{i+1}} - X_{\tau_{t_i}}\|^2 \cos^2(\theta) \mid \mathcal{F}_{\tau_{t_i}}]}{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \leq C \cos^2(\theta)$ using **(C)** and estimates of Lemma 3.1. In particular, taking θ close to $\pi/2$ ensures that $A_2(\theta) \leq \frac{A_{11}}{6}$.
Term $A_3(\theta)$. Using Lemma 3.1, one readily gets $A_3(\theta) \leq C \exp\left(-c \frac{R^2(\theta)}{h}\right) \leq \frac{A_{11}}{6}$ for h small enough ($R(\theta) > 0$). Putting together estimates for $A_1, A_2(\theta)$ and $A_3(\theta)$ into (4.9) gives $\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \geq \frac{A_{11}}{6}$. This proves (4.8). $\square$

4.3 A simple extension in the stopped case

From the previous controls we easily derive the following

Theorem 4.3 *Assume* **(C)**, **(D)**, **(S)** *and that g is bounded, uniformly Hölder continuous with index $\alpha \in (0, 1/2]$ in time and Hölder continuous with index 2α in space. For some constant C, one has*

$$|\mathrm{Err}(T, h, g, x)| \leq C h^{\alpha/2}.$$

Proof. Starting rom (3.2) we write

$$
\begin{aligned}
|\mathcal{E}| &\leq C \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N > t_i} \mathbb{I}_{\tau_{t_i} \leq t_{i+1}} \mathbb{E}[(T \wedge \tau_{t_{i+1}} - \tau_{t_i})^\alpha + \|X_{T \wedge \tau_{t_{i+1}}} - X_{\tau_{t_i}}\|^{2\alpha} | \mathcal{F}_{\tau_{t_i}}]] \\
&\leq C \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N > t_i} \mathbb{I}_{\tau_{t_i} \leq t_{i+1}} \mathbb{E}[(T \wedge \tau_{t_{i+1}} - \tau_{t_i})^\alpha | \mathcal{F}_{\tau_{t_i}}]]
\end{aligned}
$$

using the BDG inequalities or the last inequality. We controlled the term $\mathbb{E}[(T \wedge \tau_{t_{i+1}} - \tau_{t_i}) | \mathcal{F}_{\tau_{t_i}}] := A^2_{\tau_{t_i}} \leq C\sqrt{h}$ in the proo o Lemma 3.2 in the stopped case. Hence, the result is a consequence o Hölder's inequality and Lemma 3.3. □

5 Extension to an intersection of smooth domains

5.1 Additional notations and assumptions

In this section we allow the domain to e singular in the sense o the ollowing Assumption

(D') The domain $D = \bigcap_{j=1}^m D_j$, $m \geq 2$. For all $j \in [\![1, m]\!]$, D_j satisfies **(D)**. We denote its oundary y $\Gamma_j := \partial D_j$.

For $r \geq 0$, we set $\forall j \in [\![1, m]\!]$, $V_{\Gamma_j}(r) := \{z \in \mathbb{R}^d : d(z, \Gamma_j) \leq r\}$, $V_{\partial D}(r) := \{z \in \mathbb{R}^d : d(z, \partial D) \leq r\}$, $D(r) := D \cup V_{\partial D}(r)$. Since the Γ_j are C^2, we recall rom Proposition 2.1 that $\exists R_j > 0$ s.t. on $V_{\Gamma_j}(R_j)$ the projection on Γ_j is uniquely defined. For all $x \in \Gamma_j$, the notation $n_j(x)$ stands or the inner normal unit o D_j. In the ollowing, F_j denotes the signed distance to Γ_j which is C^2 on $V_{\Gamma_j}(R_j)$ and can e extended into a C^2 unction on $\mathbb{R}^d$ with ounded derivatives (see once again Proposition 2.1 or details). Set $R := \wedge_{j=1}^m R_j$. Our non degeneracy assumption on the domain D is stated as ollows:

(C') $\exists a_0 > 0$ such that $a.s.$ $\{X_s \in V_{\Gamma_j}(R) \cap V_{\partial D}(R), s \in [0, T], j \in [\![1, m]\!] \Longrightarrow \nabla F_j(X_s).\sigma_s \sigma_s^* \nabla F_j(X_s) \geq a_0\}$.

This corresponds to a non characteristic oundary condition w.r.t. every hypersur ace in a neigh ourhood o the domain D.

5.2 Main result

We are now in a position to state the main result o the section.

Theorem 5.1 (Upper Bound for an intersection of smooth domains in the killed case) *Assume* **(C')**, **(D')**, **(S)** *and let f be as in Theorem 2.2. For some constant $C := C(m)$, one has*

$$
|\mathrm{Err}(T, h, f, x)| = |\mathbb{E}[f(X_T)\mathbb{I}_{\tau^N > T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau > T}]| \leq C \frac{\|f\|_\infty}{1 \wedge \varepsilon} \sqrt{h}.
$$

We restrict ourselves to the killed case or simplicity ecause we do not need to project X_{τ^N} on the oundary to define our approximation.

Remark 5.1 *The result of Theorem 5.1 is very interesting even in the Markovian setting of Brownian Motion. Indeed, for non smooth domains it is a hard task to use the traditional error analysis techniques that require the smoothness of the derivatives of the solution of the underlying PDE (1.7) up to the boundary, see also [Men04]. We thus provide an alternative technique that points out that the main difficulty to upper-bound the weak error in the Brownian context does not lie in the lack of regularity of the domain.*

5.3 Proof of Theorem 5.1

Without modi ying the rate o convergence, see Section 3.2 or details, we can assume $X_t \in D(R)$ $a.s.$

Using the above definition of $(V_t)_{t\in[0,T]}$, i.e. $\forall t \in [0,T]$, $V_t = \mathbb{E}[f(X_T)\mathbb{I}_{\tau_t>T}|\mathcal{F}_t]$, and for an initial point $x \in \partial D$, we derive in a similar way than for the proof of Theorem 2.2

$$\mathcal{E} := \mathbb{E}[f(X_T)\mathbb{I}_{\tau^N>T}] = \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\mathbb{I}_{\tau_{t_i}<t_{i+1}}V_{t_{i+1}}].$$

Recall that, to prove Theorem 5.1, it is enough to show $|\mathcal{E}| \le C\sqrt{h}$ controlling that C is uniform w.r.t. $x \in \partial D$.

Put $\tau_t^j := \inf\{s > t : X_s \notin D_j\}$ and note that $\tau_t = \wedge_{j=1}^m \tau_t^j$. From **(C')**, we then derive that X satisfies our previous assumption **(C)** w.r.t. $D_j, \forall j \in [\![1,m]\!]$. Hence, as a consequence of Lemma 4.2 it comes

$$\begin{aligned}|V_{t_{i+1}}| &= |\mathbb{E}[f(X_T)\mathbb{I}_{\tau_{t_{i+1}}>T}|\mathcal{F}_{t_{i+1}}]| \le \mathbb{E}[|f(X_T)|\mathbb{I}_{\tau^j_{t_{i+1}}>T}|\mathcal{F}_{t_{i+1}}] \\ &\le \frac{C\|f\|_\infty}{1\wedge\varepsilon}[F_j(X_{t_{i+1}})]_+,\ \forall j \in [\![1,m]\!].\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{E}| &\le \sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^N>t_i,\tau_{t_i}<t_{i+1}}|V_{t_{i+1}}|] = \sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^N>t_i,\cup_{j=1}^m\{\tau^j_{t_i}<t_{i+1}\}}|V_{t_{i+1}}|] \\ &\le \frac{C\|f\|_\infty}{1\wedge\varepsilon}\sum_{j=1}^m\sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^{N,j}>t_i,\tau^j_{t_i}<t_{i+1}}[F_j(X_{t_{i+1}})]_+]\end{aligned}$$

where $\tau^{N,j} := \inf\{s_i \ge 0 : X_{s_i} \notin D_j\}$. Applying Lemma 4.1 we derive that

$$|\mathcal{E}| \le \sqrt{h}\frac{C\|f\|_\infty}{1\wedge\varepsilon}\sum_{j=1}^m\sum_{i=0}^{N-1}\mathbb{P}[\tau^{N,j} > t_i, \tau^j_{ti} < t_{i+1}].$$

We conclude the proof using Lemma 3.3 for all $j \in [\![1,m]\!]$. □

6 Conclusion

In this paper, we first emphasized that, under suitable assumptions, the error $\mathrm{Err}(T,h,\psi,x)$ associated to the discrete sampling of X for a given set of functionals ψ, is not given by the Markov property of SDEs but actually only depends on the Brownian stochastic integral in the dynamics (1.1). For a discretely sampled maximum or integral we used standard arguments to get this result. For killed/stopped processes, we introduced some martingale techniques that allow to go beyond the Markovian framework and also to control $\mathrm{Err}(T,h,f,x)$ at the expected rate for a certain class of non-smooth domains. In the killed/stopped case, as a matter of fact, few technical tools are needed for the error analysis we present. This is promising since even in a Brownian setting, for non-smooth domains the PDE approach for the error analysis is rather tedious or fails. The next natural question concerns the possible extension of our techniques when the stochastic integral in (1.1) is driven by a stable process.

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