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Boundary sensitivities for diffusion processes in time dependent domains

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Abstract

We study the sensitivity, with respect to a time dependent domain $\mathcal{D}_s$, of expectations of functionals of a diffusion process stopped at the exit from $\mathcal{D}_s$ or normally reflected at the boundary of $\mathcal{D}_s$. We establish a differentiability result and give an explicit expression for the gradient that allows the gradient to be computed by Monte Carlo methods. Applications to optimal stopping problems and pricing of American options, to singular stochastic control and others are discussed.

KEYWORDS: stopped diffusion, reflected diffusion, time dependent domain, sensitivity analysis, Monte Carlo methods, free boundary
MSC: 49Q12, 60J50, 35R35, 60G40.

1 Introduction

1.1 Presentation of the problem and main results

In this work, we address the problem of the sensitivity of the law of a diffusion process X_s constrained in a time dependent domain $\mathcal{D}_s \subset \Re^d$, with respect to perturbations of the domain. Both situations where the process is stopped at the exit from $\mathcal{D}_s$ and where the process is normally reflected at the boundary are covered. The law of the process is studied by means of the following expectations of functionals:

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1. $\mathbf{E}\left[g(\tau, X_\tau)Z_\tau - \int_t^\tau Z_s f(s, X_s)ds | X_t = x\right]$, where τ is the first exit time of X_s from $\mathcal{D}_s$ and $Z_s = e^{-\int_t^s c(r,X_r)dr}$, when X_s is stopped at the exit from $\mathcal{D}_s$;
2. $\mathbf{E}\left[g(X_T)Z_T - \int_t^T Z_s f(s, X_s)ds - \int_t^T Z_s h(s, X_s)d\Lambda_s | X_t = x\right]$, where Λ_s is the associated increasing process on the boundary and $Z_s = e^{-\int_t^s c(r,X_r)dr + \beta(r,X_r)d\Lambda_r}$, when X_s is normally reflected at the boundary of $\mathcal{D}_s$.

The main result of the paper is the differentiability of the above expectations with respect to perturbations of $\mathcal{D}_s$, with explicit expressions for their gradients (Theorems 2.2 and 3.8). It is worth noticing that these expressions are expectations of other functionals of the form above, and hence allow the gradients to be computed by Monte Carlo methods.

The issue of sensitivity with respect to the domain is classic in the numerical analysis literature, if one thinks of the above expectations as solutions of a Partial Differential Equation (PDE in short) with Cauchy-Dirichlet and Cauchy-Neumann boundary conditions in the time-space domain $\mathcal{D} = \{(s, x) : x \in \mathcal{D}_s, s \in]0, T[\}$: in particular, applications to shape optimization of elastic structures are important (see [All02] and references therein). The first results on this topic date back to Hadamard and have been generalized in [MS76, Sim80, Pir84] among others. These references definitely solve the case of elliptic PDEs. The parabolic case, the one of interest in the present framework, is less studied: in [SZ92], the sensitivity analysis is developed for the Laplacian (corresponding to X_s being Brownian motion) in a fixed domain. Here the analysis is extended to general diffusion processes and time dependent domains: this extension was partly motivated by a new approach to the numerical solution of optimal stopping problems, in particular the valuation of American options, which is presented in Section 4.2 (for optimal stopping and stochastic control problems see, e.g., [Kar81]).

The two cases of a stopped diffusion and a reflected diffusion are exposed in Section 2 and Section 3, respectively. The results of Section 2 have been announced in a slightly stronger form in the note [CEG03]. The leading idea of our proofs is to transfer the pertubation from the domain to the process, which is easier to analyze by stochastic calculus and weak convergence techniques. For related ideas in Malliavin calculus, see [Cat91]. Our techniques are probabilistic and hence different from those employed by the authors mentioned above. The connection with PDEs via Feyman-Kac formulas plays an important role, and is discussed in each case.

Some applications are presented in Section 4. In Subsection 4.1 it is shown how our results can be used to improve the rate of convergence in the simulation of killed diffusion processes. In Subsection 4.2 we discuss the above mentioned application to optimal stopping problems and pricing of American options. The approach we propose is to maximize the expected payoff when the option is exercised at the exit from a continuation region, over all possible regions: the results of this paper provide the main tool to construct a numerical procedure for this optimization problem. In Subsection 4.3 our results are used to establish existence of the density for the joint law of the maximum and the terminal value of a diffusion process.

Finally, applications to singular stochastic control problems are presented in Subsection 4.4. Some technical results are proved in the Appendix.

In the rest of this section, we define the notations used throughout the paper and recall some definitions and results on time-space domains.

1.2 General notation

• *Differentiation.* We adopt the following usual convention on the derivatives: if $\psi : \Re^{d_2} \mapsto \Re^{d_1}$ is a differentiable function, its jacobian $J\psi(x) = (\partial_{x_1}\psi(x), \cdots, \partial_{x_{d_2}}\psi(x))$ takes values in $\Re^{d_1} \otimes \Re^{d_2}$.
For smooth functions $g(t,x)$, we denote by $\partial_x^\beta g(t,x)$ the derivative of g w.r.t. x according to the multi-index β, whereas time derivatives of g are denoted by $\partial_t g(t,x), \partial_t^2 g(t,x)$ and so on. The notation $\nabla g(t,x)$ stands for the usual gradient w.r.t. x (as a row vector) and the Hessian matrix of g (w.r.t. the space variable x) is denoted by $Hg(t,x)$.
The second order linear operator L below stands for the infinitesimal generator of the diffusion process with drift and diffusion coefficients (b,σ):

$$Lu(t,x) = \nabla u(t,x)b(t,x) + \frac{1}{2}\mathrm{Tr}(Hu(t,x)[\sigma\sigma^*](t,x)). \tag{1.1}$$

• *Linear algebra.* The r-th column of a matrix A will be denoted by A_r (or $A_{r,t}$ if A is a time dependent matrix) and the r-th element of a vector a will be denoted by a_r (or $a_{r,t}$ if a is a time dependent vector). A^* stands for the transpose of A. The identity matrix is denoted by $\mathbf{I}$ and the identity function by Id.
• *Metric.* The parabolic distance between two points (t,x) and (s,y) in $\Re \times \Re^d$ is defined by $\mathbf{pd}((t,x),(s,y)) = \max(|s-t|^{1/2}, |x-y|)$, where $|x-y|$ is the usual Euclidean distance. We set $B_{d'}(x,\epsilon)$ for the usual Euclidean d'-dimensional open ball with center x and radius ϵ and $\mathbf{d}(x,C)$ for the Euclidean distance of a point x from a closed set C (and analogously $\mathbf{pd}((t,x),C)$ for the parabolic distance).
• *Functions.* For an open set $\mathcal{D}' \subset \Re \times \Re^d$ and $k \in \mathbf{N}$, $\mathcal{C}^{\lfloor \frac{k}{2} \rfloor, k}(\mathcal{D}')$ (resp. $\mathcal{C}^{\lfloor \frac{k}{2} \rfloor, k}(\overline{\mathcal{D}'})$) is the space of continuous functions f defined on $\mathcal{D}'$ with continuous derivatives $\partial_x^\beta \partial_t^j f$ for $|\beta| + 2j \le k$ (resp. defined in a neighborhood of $\overline{\mathcal{D}'}$). The index b in $\mathcal{C}_b^{\lfloor \frac{k}{2} \rfloor, k}(\mathcal{D}')$ indicates that in addition the functions are bounded as well their derivatives. We may simply write $\mathcal{C}^{\lfloor \frac{k}{2} \rfloor, k}$ and $\mathcal{C}_b^{\lfloor \frac{k}{2} \rfloor, k}$ when $\mathcal{D}' = \Re \times \Re^d$.
$|\cdot|_\infty$ stands for the sup norm.
Denote by $\mathcal{C}([t,T],\Re^d)$ the set of continuous functions from $[t,T]$ into $\Re^d$ and by $\mathcal{I}([t,T],\Re_+)$ the set of continuous and non-decreasing functions from $[t,T]$ into $\Re_+$. With a slight abuse of notation, $d\mathcal{I}$ denotes the measure associated to $\mathcal{I}$ and $d\mathcal{I}_r = d\mathcal{I}(dr)$.
• *Floating constants.* As usual, we keep the same notation K for all finite, non-negative constants which appear in our computations.
• *Miscellaneous.* To be more concise (whenever needed), $g(s,x)$ may be denoted $g_s(x)$.

1.3 Time-space domains

In the sequel $\mathcal{D}$ stands for a bounded time-space domain in $]0,T[\times\Re^d$ (T is a fixed terminal time). The boundary of $\mathcal{D}$ is denoted, as usual, by $\partial\mathcal{D}$. Regularity assumptions on the domain $\mathcal{D}$ will be formulated in terms of Hölder spaces with *time-space* variables (see [Lie96] p.46). Let $\mathcal{D}'$ be an arbitrary time-space domain. If the index of regularity is $a=k+\alpha$ for k a nonnegative integer and $\alpha\in]0,1]$, then $\mathcal{H}_a(\mathcal{D}')$ is the Banach spaces of functions f of class $\mathcal{C}^{\lfloor\frac{k}{2}\rfloor,k}(\mathcal{D}')$ with Hölder continuous k-th derivatives, namely with a finite norm $|f|_{a,\mathcal{D}'}$ where

$$
\begin{aligned}
|f|_{a,\mathcal{D}'} \quad &= \sum_{|\beta|+2j\le k}\sup_{\mathcal{D}'}|\partial_x^\beta\partial_t^j f| + [f]_{a,\mathcal{D}'} + < f >_{a,\mathcal{D}'}\\
\text{with} \quad [f]_{a,\mathcal{D}'} &= \sum_{|\beta|+2j=k}\sup_{(s,y)\in\mathcal{D}'}\sup_{(t,x)\in\mathcal{D}'\setminus\{(s,y)\}}\frac{|\partial_x^\beta\partial_t^j f(t,x)-\partial_x^\beta\partial_t^j f(s,y)|}{[\mathbf{pd}((t,x),(s,y))]^\alpha}\\
\text{and} \quad < f >_{a,\mathcal{D}'} &= \begin{cases}\sum_{|\beta|+2j=k-1}\sup_{(s,x)\in\mathcal{D}'}\sup_{(t,x)\in\mathcal{D}'\setminus\{(s,x)\}}\frac{|\partial_x^\beta\partial_t^j f(t,x)-\partial_x^\beta\partial_t^j f(s,x)|}{|t-s|^{(\alpha+1)/2}} & \text{for } k\ge 1,\\ 0 \quad \text{for } k=0.\end{cases}
\end{aligned}
$$

Whenever convenient, we may denote $(\mathcal{H}_a(\Re\times\Re^d),|\cdot|_{a,\Re\times\Re^d})$ by $(\mathcal{H}_a,|\cdot|_a)$.
The following smoothness definition for the time-space domain $\mathcal{D}$ will be used (cf. [Fri64], page 64).

Definition 1.1 *The domain $\mathcal{D}$ is of class $\mathcal{H}_a$ ($a\ge 1$) ($\mathcal{D}\in\mathcal{H}_a$ in short) if, for every $(t_0,x_0)\in\overline{\partial\mathcal{D}\cap(]0,T[\times\Re^d)}$, there exists a neighborhood $]t_0-\epsilon_0^2,t_0+\epsilon_0^2[\times B_d(x_0,\epsilon_0)$, an index i and a function $\phi_0\in\mathcal{H}_a\left(]t_0-\epsilon_0^2,t_0+\epsilon_0^2[\times B_{d-1}((x_{1,0},\cdots,x_{i-1,0},x_{i+1,0},\cdots,x_{d,0}),\epsilon_0)\right)$ such that*

$$
\begin{gathered}
\overline{\partial\mathcal{D}\cap(]0,T[\times\Re^d)}\cap(]t_0-\epsilon_0^2,t_0+\epsilon_0^2[\times B_d(x_0,\epsilon_0)) =\\
\{(t,x)\in(]t_0-\epsilon_0^2,t_0+\epsilon_0^2[\cap[0,T])\times B_d(x_0,\epsilon_0): x_i=\phi_0(t,x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_d)\}.
\end{gathered}
$$

Let

$$
\begin{aligned}
\mathcal{D}_0 &= \left\{x:\ (0,x)\in\partial\mathcal{D}-\overline{\partial\mathcal{D}\cap(]0,T[\times\Re^d)}\right\},\\
\mathcal{D}_T &= \left\{x:\ (T,x)\in\partial\mathcal{D}-\overline{\partial\mathcal{D}\cap(]0,T[\times\Re^d)}\right\}.
\end{aligned}
$$

$\mathcal{D}_0$ and $\mathcal{D}_T$ are open sets and we assume that they are nonempty domains that coincide with the interior of their closure (cf. [Fri64], Section 3.2). We assume also (cf. again [Fri64], Section 3.2) that the time section of $\mathcal{D}$, $\mathcal{D}_t=\{x:\ (t,x)\in\mathcal{D}\}$, is a domain that coincides with the interior of its closure, for every $t\in]0,T[$. If $\mathcal{D}$ is of class $\mathcal{H}_a$ ($a\ge 1$) the sets $\mathcal{PD}=\partial\mathcal{D}-\{0\}\times\mathcal{D}_0$, $\mathcal{BD}=\{T\}\times\mathcal{D}_T$ and $\mathcal{SD}=\mathcal{PD}-\overline{\mathcal{BD}}$ are the *parabolic boundary*, the *bottom* and the *side* of $\mathcal{D}$ in the sense of [Lie96], pages 7 and 13.

If $\mathcal{D}$ is of class $\mathcal{H}_1$, $\mathcal{D}$ satisfies an exterior tusk condition, which is analogous to the exterior (Wiener's) cone condition for elliptic problems (see [Dur84]). We use this result in our sensitivity analysis and we state it now. The result seems to be standard in the PDE litterature (see [Lie89]), but for the sake of completeness we prove it in Appendix A.

Proposition 1.2 [Tusk condition]. *Assume $\mathcal{D} \in \mathcal{H}_1$. For some $R > 0, \delta > 0$, at any point $(t_0, x_0) \in \mathcal{PD}$, there is a tusk*

$$\mathcal{T} = \{(t,x) : t_0 < t < t_0 + \delta, \left|x - x_0 - \bar{x}_0\sqrt{t - t_0}\right|^2 < R^2(t - t_0)\},$$

for some $\bar{x}_0 \in \Re^d$, such that $\overline{\mathcal{T}}$ intersects $\overline{\mathcal{D}}$ only at (t_0, x_0).

If $\mathcal{D}$ is of class $\mathcal{H}_2$, all domains $\mathcal{D}_t$, for $t \in [0,T]$, satisfy the uniform interior and exterior sphere condition with the same radius r_0. Moreover (see [Lie96], Section X.3), the signed spatial distance F, given by

$$F(t,x) = \begin{cases} -\mathbf{d}(x, \partial\mathcal{D}_t), & \text{for } x \in \mathcal{D}_t^c,\ \mathbf{d}(x,\partial\mathcal{D}_t) < r_0,\ 0 < t < T, \\ \mathbf{d}(x, \partial\mathcal{D}_t), & \text{for } x \in \mathcal{D}_t,\ \mathbf{d}(x,\partial\mathcal{D}_t) < r_0,\ 0 < t < T, \end{cases}$$

is of class $\mathcal{H}_2$ and $\nabla F(t,x)$ is the unit inward normal vector at the nearest point to x in $\partial\mathcal{D}_t$. Then there is a function in $\mathcal{H}_2$ that coincides with F on $\{(t,x) : 0 < t < T,\ \mathbf{d}(x,\partial\mathcal{D}_t) < r_0'\}$, for some $r_0' < r_0$. If $\mathcal{D}$ is of class $\mathcal{H}_{2+\alpha}$, the arguments in [Lie96], Section X.3, show that this function can be taken in $\mathcal{H}_{2+\alpha}$ (hence in $\mathcal{C}_b^{1,2}$).

2 Diffusion processes stopped at the boundary

Here, for $(t,x) \in \bar{\mathcal{D}}$, we consider the $\Re^d$-valued diffusion process $(X^{t,x})$ solution of

$$X_s = x + \int_t^s b(r, X_r)dr + \int_t^s \sigma(r, X_r)dW_r, \tag{2.1}$$

where $(W_t)_{t\geq 0}$ is a q-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions. The assumption **($\mathbf{A}_\alpha$-1)** below ensures the existence of a unique strong solution to (2.1).

($\mathbf{A}_\alpha$) (with $\alpha \in]0,1]$)

1. *Smoothness.* b and σ satisfy $|b(t,x) - b(s,y)| + |\sigma(t,x) - \sigma(s,y)| \leq K(|t-s|^{\alpha/2} + |x-y|)$ uniformly in $(t,x),(s,y) \in [0,T] \times \Re^d$.
2. *Uniform ellipticity.* For some $a_0 > 0$, it holds $\xi.[\sigma\sigma^*](t,x)\xi \geq a_0|\xi|^2$ for any $(t,x,\xi) \in [0,T] \times \Re^d \times \Re^d$.

We mention that the additional smoothness of b and σ w.r.t. the time variable is required for the connection with PDEs. The infinitesimal generator of X is given by (1.1). Now define

$$\tau^{t,x} := \inf\{s > t : (s, X_s^{t,x}) \notin \mathcal{D}\} \tag{2.2}$$

for the first exit time from the domain $\mathcal{D}$ for the time-space process $(s, X_s^{t,x})_{s\in[t,T]}$. Note that $\tau^{t,x}$ is bounded by T. We focus on the expectation of functionals of the process X stopped at the exit from $\mathcal{D}$, of the form

$$u(t,x) = \mathbf{E}(g(\tau^{t,x}, X_{\tau^{t,x}}^{t,x})e^{-\int_t^{\tau^{t,x}} c(r,X_r^{t,x})dr} - \int_t^{\tau^{t,x}} e^{-\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds), \tag{2.3}$$

and on its sensitivity w.r.t. $\mathcal{D}$. The data f, g, c are bounded continuous functions. For (2.3) to be meaningful, it is sufficient that these functions are defined only on $\bar{\mathcal{D}}$; however in the sequel the domain changes, therefore we define them directly on $\Re^{d+1}$ (as for the coefficients b and σ). We are now in a position to state a preliminary result which relates u to the solution of a Cauchy-Dirichlet type PDE in the time-space domain $\mathcal{D}$. This connection is standard but, to our knowledge, it appears in the literature only in the case of cylindrical domains $\mathcal{D} =]0, T[\times D$ with $D \subset \Re^d$ (see Theorem 2.3 p.133 in [Fre85] for instance). The proof is postponed to Appendix B.1.

Proposition 2.1 [Feynman-Kac's formula and a priori estimates on u] *Assume $(\mathbf{A}_\alpha)$, $\mathcal{D} \in \mathcal{H}_1$, $c \in \mathcal{H}_\alpha$, $f \in \mathcal{H}_\alpha$ and $g \in \mathcal{C}^{0,0}$ with $\alpha \in]0, 1[$. Then, u is the unique solution of class $\mathcal{C}^{1,2}(\mathcal{D}) \cap \mathcal{C}^{0,0}(\bar{\mathcal{D}})$ to*

$$\begin{cases} \partial_t u + Lu - cu = f & \text{in } \mathcal{D}, \\ u = g & \text{on } \mathcal{PD}. \end{cases} \tag{2.4}$$

In addition, if $\mathcal{D}$ is of class $\mathcal{H}_{1+\alpha}$ and $g \in \mathcal{H}_{1+\alpha}$, the function u belongs to $\mathcal{H}_{1+\alpha}(\mathcal{D})$ and it holds $|u|_{1+\alpha,\mathcal{D}} \leq K(|f|_{\alpha,\mathcal{D}} + |g|_{1+\alpha,\mathcal{D}})$ (in particular, ∇u is well defined and continuous up to the boundary).

We now turn to one of the main contributions of this paper, namely the sensitivity of $\mathbf{E}(g(\tau^{t,x}, X^{t,x}_{\tau^{t,x}})e^{\int_t^{\tau^{t,x}} c(r,X_r^{t,x})dr} - \int_t^{\tau^{t,x}} e^{\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds)$ w.r.t. spatial perturbations of $\mathcal{D}$. We define a spatial perturbation of the time-space domain $\mathcal{D}$ in the following way:

$$\mathcal{D}^\epsilon = \{(t, x) : (t, x + \epsilon\Theta(t, x)) \in \mathcal{D}\}, \quad \epsilon \in \Re, \tag{2.5}$$

for some map $\Theta : [0, T] \times \Re^d \mapsto \Re^d$. In the sequel

$$\boxed{\Theta \text{ is a function of class } \mathcal{C}_b^{1,2}([0, T] \times \Re^d).}$$

For fixed ω, the exit time from $\mathcal{D}^\epsilon$ of a path $X^{t,x}(\omega)$ is certainly not smooth w.r.t. ϵ. However, the law of related functionals of the form (2.3) is smooth in the sense stated in the theorem below.

Theorem 2.2 *Assume $(\mathbf{A}_\alpha)$, $\mathcal{D} \in \mathcal{H}_{1+\alpha}$, $c \in \mathcal{H}_\alpha$, $f \in \mathcal{H}_\alpha$ and $g \in \mathcal{H}_{1+\alpha}$ with $\alpha \in]0, 1[$. Let (t, x) be in $\mathcal{D} \cup \mathcal{D}_0$ and set*

$$\tau_\epsilon^{t,x} := \inf\{s > t : (s, X_s^{t,x}) \notin \mathcal{D}^\epsilon\}. \tag{2.6}$$

Then, $u^\epsilon(t, x) = \mathbf{E}[g(\tau_\epsilon^{t,x}, X^{t,x}_{\tau_\epsilon^{t,x}})e^{-\int_t^{\tau_\epsilon^{t,x}} c(r,X_r^{t,x})dr} - \int_t^{\tau_\epsilon^{t,x}} e^{-\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds]$ is differentiable w.r.t. ϵ at $\epsilon = 0$ and

$$\partial_\epsilon u^\epsilon(t, x)|_{\epsilon=0} = \mathbf{E}[e^{-\int_t^{\tau^{t,x}} c(r,X_r^{t,x})dr}[(\nabla u - \nabla g)\Theta](\tau^{t,x}, X^{t,x}_{\tau^{t,x}})].$$

Note that ∇u in the above expression is well defined on the boundary since u is of class $\mathcal{H}_{1+\alpha}(\mathcal{D})$. In view of the formula above and because $u = g$ on $\mathcal{PD}$, only normal pertubations of Θ have contributions in the derivative of $u^\epsilon(t,x)$. It is intuitively correct.

Proof. Without loss of generality we can suppose $\epsilon > 0$. Since the initial condition (t,x) is fixed in the proof, we omit the superscripts t,x.
First we prove the result for $c, f \in \mathcal{H}_1$. For convenience, set $Z_s = e^{-\int_t^s c(r,X_r)dr}$. One has to find the limit of $\frac{\Delta_\epsilon}{\epsilon}$ as $\epsilon \to 0$, where

$$\Delta_\epsilon := \mathbf{E}[g(\tau_\epsilon, X_{\tau_\epsilon})Z_{\tau_\epsilon} - \int_t^{\tau_\epsilon} f(s,X_s)Z_s ds] - u(t,x).$$

The idea of the proof is to transform the perturbation of the domain into a perturbation of the process. Namely, we define

$$X_s^\epsilon = X_s + \epsilon\Theta(s,X_s), \quad \tau^\epsilon := \inf\{s > t : (s, X_s^\epsilon) \notin \mathcal{D}\}. \tag{2.7}$$

Then, (2.5) and (2.6) yield the key relation

$$\tau_\epsilon = \tau^\epsilon. \tag{2.8}$$

Since Θ is bounded, the perturbed process X^ϵ converges uniformly on $[t,T]$ to X as ϵ goes to 0. Furthermore, Θ being of class $\mathcal{C}_b^{1,2}$ and $x \mapsto x + \epsilon\Theta(s,x)$ being bijective (for any fixed s) for ϵ small enough, the perturbed process is still a non-homogeneous diffusion process. We denote its infinitesimal generator by L^ϵ. We state now two technical lemmas, which are justified later.

Lemma 2.3 *Assume* $(\mathbf{A}_\alpha)$ *with* $\alpha \in]0,1[$ *and* $\mathcal{D} \in \mathcal{H}_1$. *Then,* $\tau_\epsilon = \tau^\epsilon$ *converges almost surely to* τ *as* ϵ *goes to 0.*

Lemma 2.4 *Assume that* $(\mathbf{A}_\alpha)$, $\mathcal{D} \in \mathcal{H}_{1+\alpha}$, $c \in \mathcal{H}_\alpha$, $f \in \mathcal{H}_\alpha$ *and* $g \in \mathcal{H}_{1+\alpha}$ *with* $\alpha \in]0,1[$. *For any* $p \in [1, \frac{1}{1-\alpha}[$, *one has*

$$\int_t^T \mathbf{E}[\mathbf{1}_\mathcal{D}(s,X_s^\epsilon)(|\partial_s u|^p + |\nabla u|^p + |Hu|^p)(s,X_s^\epsilon)]ds < \infty \tag{2.9}$$

uniformly for ϵ *in a neighborhood of 0.*
Using (2.8) and (2.7), one obtains

$$g(\tau_\epsilon, X_{\tau_\epsilon})Z_{\tau_\epsilon} - \int_t^{\tau_\epsilon} f(s,X_s)Z_s ds = g(\tau^\epsilon, [\mathrm{Id} + \epsilon\Theta(\tau^\epsilon,.)]^{-1}X_{\tau^\epsilon}^\epsilon)Z_{\tau^\epsilon} - \int_t^{\tau^\epsilon} f(s,X_s)Z_s ds.$$

Thus, Δ_ϵ can be decomposed as $\Delta_\epsilon = \Delta_{1,\epsilon} + \Delta_{2,\epsilon} + \Delta_{3,\epsilon} + \Delta_{4,\epsilon}$ with

$$\begin{aligned}
\Delta_{1,\epsilon} &= \mathbf{E}[g(\tau^\epsilon, [\mathrm{Id} + \epsilon\Theta(\tau^\epsilon,.)]^{-1}X_{\tau^\epsilon}^\epsilon)Z_{\tau^\epsilon} - g(\tau^\epsilon, X_{\tau^\epsilon}^\epsilon)Z_{\tau^\epsilon}],\\
\Delta_{2,\epsilon} &= \mathbf{E}[g(\tau^\epsilon, X_{\tau^\epsilon}^\epsilon)Z_{\tau^\epsilon} - u(\tau^\epsilon \wedge \tau, X_{\tau^\epsilon\wedge\tau}^\epsilon)Z_{\tau^\epsilon\wedge\tau} - \int_{\tau^\epsilon\wedge\tau}^{\tau^\epsilon} f(s,X_s)Z_s ds],\\
\Delta_{3,\epsilon} &= \mathbf{E}[u(\tau^\epsilon \wedge \tau, X_{\tau^\epsilon\wedge\tau}^\epsilon)Z_{\tau^\epsilon\wedge\tau} - u(\tau^\epsilon \wedge \tau, X_{\tau^\epsilon\wedge\tau})Z_{\tau^\epsilon\wedge\tau}],\\
\Delta_{4,\epsilon} &= \mathbf{E}[u(\tau^\epsilon \wedge \tau, X_{\tau^\epsilon\wedge\tau})Z_{\tau^\epsilon\wedge\tau} - \int_t^{\tau^\epsilon\wedge\tau} f(s,X_s)Z_s ds] - u(t,x).
\end{aligned}$$

The convergence of $\Delta_{1,\epsilon}$ is straightforward to analyse and we get

$$\lim_{\epsilon\to 0}\frac{\Delta_{1,\epsilon}}{\epsilon} = -\mathbf{E}\left[(\nabla g\ \Theta)(\tau, X_\tau)Z_\tau\right]. \tag{2.10}$$

We now prove that $\Delta_{2,\epsilon}/\epsilon$ converges to 0. Observe that $g(\tau^\epsilon, X^\epsilon_{\tau^\epsilon}) = u(\tau^\epsilon, X^\epsilon_{\tau^\epsilon})$. Thus, $\Delta_{2,\epsilon}$ can be decomposed using Itô's formula, which application is justified thanks to Proposition 2.1 and the estimates (2.9) for $p = 1$. We have

$$\begin{aligned}
\Delta_{2,\epsilon} \quad &= \mathbf{E}(\int_{\tau\wedge\tau^\epsilon}^{\tau^\epsilon} [\partial_s u(s, X^\epsilon_s) + L^\epsilon u(s, X^\epsilon_s) - c(s, X_s)u(s, X^\epsilon_s) - f(s, X_s)]Z_s\, ds)\\
&= \mathbf{E}(\int_{\tau\wedge\tau^\epsilon}^{\tau^\epsilon} [[L^\epsilon - L]u(s, X^\epsilon_s) - [c(s, X_s) - c(s, X^\epsilon_s)]u(s, X^\epsilon_s)\\
&\quad -[f(s, X_s) - f(s, X^\epsilon_s)]]Z_s\, ds)
\end{aligned}$$

where in the last equality we used the PDE solved by u (see (2.4), noting that for $s < \tau^\epsilon$, $(s, X^\epsilon_s) \in \mathcal{D}$). Clearly, the difference $[L^\epsilon - L]u$ is bounded by $K\epsilon(|\nabla u| + |Hu|)$ and since $c, f \in \mathcal{H}_1$, one also has $|c(s, X_s) - c(s, X^\epsilon_s)| + |f(s, X_s) - f(s, X^\epsilon_s)| \le K\epsilon$ for some constant K. Since Z_s and $u(s, X^\epsilon_s)$ are bounded, we obtain

$$\begin{aligned}
|\frac{\Delta_{2,\epsilon}}{\epsilon}| \;&\le\; K\mathbf{E}(\int_t^T \mathbf{1}_{[\tau\wedge\tau^\epsilon,\tau^\epsilon[}(s)[|\nabla u| + |Hu| + 1](s, X^\epsilon_s)ds)\\
&\le\; K[\mathbf{E}(\tau^\epsilon - \tau\wedge\tau^\epsilon)]^{1-1/p}[\mathbf{E}\int_t^T \mathbf{1}_{\mathcal{D}}(s, X^\epsilon_s)\ [|\nabla u| + |Hu| + 1]^p(s, X^\epsilon_s)ds]^{1/p}
\end{aligned}$$

applying the Hölder inequality to the measure $d\mathbf{P}\otimes dt$ (with $p \in]1, \frac{1}{1-\alpha}[$). The convergence of Lemma 2.3 clearly holds also in $\mathbf{L}_1$, proving that the first factor converges to 0, while the second one is uniformly bounded using estimates (2.9). This proves that $\lim_{\epsilon\to 0}\Delta_{2,\epsilon}/\epsilon = 0$. For the term $\Delta_{3,\epsilon}$, recalling that $u \in \mathcal{H}_{1+\alpha}(\mathcal{D})$ (Proposition 2.1), we readily obtain

$$\lim_{\epsilon\to 0}\frac{\Delta_{3,\epsilon}}{\epsilon} = \mathbf{E}[(\nabla u\ \Theta)(\tau, X_\tau)Z_\tau]. \tag{2.11}$$

Finally, as far as $\Delta_{4,\epsilon}$ is concerned, it is enough to observe that $(u(s\wedge\tau, X_{s\wedge\tau})Z_{s\wedge\tau} - \int_t^{s\wedge\tau} f(s, X_s)Z_s ds)_{t\le s\le T}$ is a (bounded) martingale, and the result follows. Combining different limits for $\Delta_{1,\epsilon}/\epsilon, \cdots, \Delta_{4,\epsilon}/\epsilon$ completes the theorem, when $c, f \in \mathcal{H}_1$.

Now consider the case where $c, f \in \mathcal{H}_\alpha$ only. There is a sequence of $(c_m, f_m)_m \in \mathcal{H}_1$ which is bounded in $\mathcal{H}_\alpha$ and convergent to (c, f) in $\mathcal{H}_{\alpha'}$-norm (for any $\alpha' \in]0, \alpha[$). Denote u^ϵ_m the associated PDE in the domain $\mathcal{D}^\epsilon$. Clearly $u^\epsilon_m(t, x)$ converges uniformly in ϵ to $u^\epsilon(t, x)$ (t and x are fixed). Moreover, $\mathcal{D}^\epsilon$ is of class $\mathcal{H}_{1+\alpha'}$ (for ϵ small enough, say $\epsilon \le \epsilon_0$) and the previous analysis leads to $\partial_\epsilon u^\epsilon_m(t, x) = \mathbf{E}[e^{-\int_t^{\tau_\epsilon} c_m(r,X_r)dr}[(\nabla u^\epsilon_m - \nabla g)\Theta](\tau_\epsilon, X_{\tau_\epsilon})]$. What remains to be proved to complete our theorem in the general case is the convergence of $\partial_\epsilon u^\epsilon_m(t, x)$ to $\mathbf{E}[e^{-\int_t^{\tau_\epsilon} c(r,X_r)dr}[(\nabla u^\epsilon - \nabla g)\Theta](\tau_\epsilon, X_{\tau_\epsilon})]$ uniformly in ϵ ($\epsilon \le \epsilon_0$). This easily reduces to the uniform convergence of $|u^\epsilon_m - u^\epsilon|_{1+\alpha',\mathcal{D}^\epsilon}$ to 0 as $m\to\infty$. Observe that $\bar u = u^\epsilon_m - u^\epsilon$ is solution of (2.4) in $\mathcal{D}^\epsilon$ with data $\bar c = c_m$, $\bar f = f_m - f + (c_m - c)u^\epsilon$ and $\bar g = 0$. By the $|\cdot|_{1+\alpha',\mathcal{D}^\epsilon}$-estimates from Proposition 2.1 (with a constant K locally uniform in ϵ and depending on $\bar{c}$ only by $|\bar{c}|_{\alpha',\mathcal{D}^\epsilon}$), we obtain $|u_m^\epsilon - u^\epsilon|_{1+\alpha',\mathcal{D}^\epsilon} \leq K(|c_m - c|_{\alpha'} + |f_m - f|_{\alpha'})$ uniformly in ϵ. The proof is finished. □

Proof of Lemma 2.3. To prove $\tau_\epsilon \to \tau$ *a.s.*, we sandwich $\mathcal{D}^\epsilon$ between two domains, $\mathcal{D}^{-\epsilon} = \{(t,x) \in \mathcal{D} : \mathbf{pd}((t,x),\mathcal{SD}) \geq c\epsilon\}$ and $\mathcal{D}^{+\epsilon} = \{(t,x) : \mathbf{pd}((t,x),\mathcal{SD}) < c\epsilon\}$, where the constant c is large enough to ensure $\mathcal{D}^{-\epsilon} \subset \mathcal{D}^\epsilon \subset \mathcal{D}^{+\epsilon}$ and ϵ small enough to have $(t,x) \in \mathcal{D}^{-\epsilon}$. We denote $\tau_{-\epsilon}$ and $\tau_{+\epsilon}$ the relative exit times for X. Since $\tau_{-\epsilon} \leq \tau_\epsilon \leq \tau_{+\epsilon}$, we are reduced to prove that $\tau_{-\epsilon}$ and $\tau_{+\epsilon}$ converge a.s. to τ as $\epsilon \downarrow 0$. Firstly, $\tau_{-\epsilon}$ is an increasing sequence bounded by τ: we write τ_- for its limit. Since $\mathbf{pd}((\tau_{-\epsilon}, X_{\tau_{-\epsilon}}),\mathcal{D}) \leq c\epsilon$, taking the limit gives $\tau_- \geq \tau$, and thus $\tau_- = \tau$. Secondly and analogously, $\tau \leq \tau_+ = \lim_{\epsilon\downarrow 0} \tau_{+\epsilon}$. In view of the estimate (2.12) below, the event $\{\tau < \tau_+\}$ has zero probability. Hence, we get $\tau_+ = \tau$. It remains to prove

$$\forall (s,y) \in \mathcal{PD}, \forall \Delta > 0 \text{ small enough} : p_{s,y,\Delta} = \mathbf{P}(\exists\, t \in]s, s+\Delta] : (t, X_t^{s,y}) \notin \bar{\mathcal{D}}) = 1. \quad (2.12)$$

By the Blumenthal Zero-One law, it suffices to show $p_{s,y,\Delta} > 0$. For this, we combine the tusk condition of Proposition 1.2 and the Aronson's lower bound [Aro67] for the density $p_{(s,y)}(s+\Delta,\cdot)$ (w.r.t. the Lebesgue measure) of the law of $X_{s+\Delta}^{s,y}$, i.e. $p_{(s,y)}(s+\Delta, y') \geq \frac{1}{K\,\Delta^{d/2}} \exp\left(-K\frac{|y-y'|^2}{\Delta}\right)$. Let $\mathcal{T}$ be the tusk of Proposition 1.2 at point (s,y) and take $\Delta < \delta$. We have

$$\begin{aligned} p_{s,y,\Delta} \geq \mathbf{P}((s+\Delta, X_{s+\Delta}^{s,y}) \in \mathcal{T}) \;&\geq\; \int \frac{1}{K\,\Delta^{d/2}} \exp\big(-K\frac{|y-y'|^2}{\Delta}\big)\mathbf{1}_{|[y'-y]-\bar{y}\sqrt{\Delta}|^2 \leq R^2\Delta}\, dy' \\ &=\; \int \frac{1}{K} \exp\big(-K|z|^2\big)\mathbf{1}_{|z-\bar{y}|^2\leq R^2}\, dz > 0. \end{aligned}$$

The proof of (2.12) is complete. □

Proof of Lemma 2.4. Take $p \in [1, \frac{1}{1-\alpha}[$. It is enough to consider the integrability of Hu alone. Indeed, we already know by Proposition 2.1 that ∇u is uniformly bounded, and the control of $\partial_t u$ follows from the other estimates by (2.4). Under our standing assumptions, the second spatial derivatives of u may blow up at the boundary $\mathcal{PD}$ at some rate. Namely, in view of the estimate (4.64) p.79 in [Lie96], we have $|Hu(s,y)| \leq K \inf_{(r,z)\in\mathcal{PD}, r\geq s} [\mathbf{pd}[(s,y),(r,z)]]^{\alpha-1}$. Thus, the assertion of the lemma follows if

$$\int_t^T \mathbf{E}[\mathbf{1}_{(s,X_s^\epsilon)\in\mathcal{D}} \inf_{(r,z)\in\mathcal{PD}, r\geq s} (\mathbf{pd}[(s,X_s^\epsilon),(r,z)])^{p(\alpha-1)}]ds < +\infty, \quad (2.13)$$

with $p(\alpha-1) \in]-1, -1+\alpha]$. This quantity is partly evaluated using an Aronson's upper bound [Aro67] for the density $p_{t,X_t^\epsilon}^\epsilon(s,\cdot)$ of the law of X_s^ϵ conditionnally on X_t^ϵ. We note that for ϵ small enough, the coefficients of the dynamics of the non-homogenuous SDE X^ϵ also satisfy $(\mathbf{A}_\alpha)$, with uniform (w.r.t. ϵ) Lipschitz and ellipticity constants. Thus, one has $p_{t,X_t^\epsilon}^\epsilon(s,y) \leq \frac{K}{(s-t)^{d/2}} \exp\big(-\frac{|X_t^\epsilon - y|^2}{K\,(s-t)}\big)$ with a constant K uniform w.r.t. ϵ. We analyse the quantity (2.13) according to the event $\mathcal{A} = \{\mathbf{pd}((s, X^\epsilon_s), \overline{\mathcal{D}}_T) \le \mathbf{pd}((s, X^\epsilon_s), \mathcal{SD} \cap \{r \ge s\})\}$ and its complementary.

On $\mathcal{A}$, $\inf_{(r,z)\in\mathcal{PD}, r\ge s} \mathbf{pd}[(s, X^\epsilon_s), (r, z)] = \sqrt{T-s}$ which gives a integrable contribution (since $\int_t^T (T-s)^{p(\alpha-1)/2} ds < +\infty$).

On $\mathcal{A}^c$, $\inf_{(r,z)\in\mathcal{PD}, r\ge s} \mathbf{pd}[(s, X^\epsilon_s), (r, z)] = \mathbf{pd}((s, X^\epsilon_s), \mathcal{SD} \cap \{r \ge s\})$. To prove that $\int_t^T \mathbf{E}[\mathbf{1}_{(s,X^\epsilon_s)\in\mathcal{D}} (\mathbf{pd}[(s, X^\epsilon_s), \mathcal{SD} \cap \{r \ge s\}])^{p(\alpha-1)}] ds$ is finite, we can restrict to points (s, X^ϵ_s) in a neighborhood of $\mathcal{SD}$. This set can be covered by a finite number of open balls $(\mathcal{B}_j =]t(j) - \epsilon_0^2, t(j) + \epsilon_0^2[\times B_d(x(j), \epsilon_0))_{1\le j\le J}$ (with $(t(j), x(j)) \in \mathcal{SD}$), on which the local parameterization of $\mathcal{D}$ is available, i.e. $\mathcal{D} \cap (]t(j) - \epsilon_0^2, t(j) + \epsilon_0^2[\times B_d(x(j), \epsilon_0)) = \{(s, z) : s \in](t(j) - \epsilon_0^2)_+, (t(j) + \epsilon_0^2) \wedge T[, z \in B_d(x(j), \epsilon_0), z_i > \phi(s, z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_d)\}$ (see Definition 1.1). Furthermore when $\mathcal{D} \in \mathcal{H}_1$, it is an easy exercice to check that $|z_i - \phi(s, z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_d)| \le K\mathbf{pd}[(s, z), \mathcal{SD} \cap \{r \ge s\}]$. Combining these arguments with the Aronson estimate, we obtain

$$\begin{aligned}
&\int_t^T \mathbf{E}[\mathbf{1}_{(s,X^\epsilon_s)\in\mathcal{D}\cap\mathcal{B}_j} (\mathbf{pd}[(s, X^\epsilon_s), \mathcal{SD} \cap \{r \ge s\}])^{p(\alpha-1)}] ds \\
&\le \int_{(t(j)-\epsilon_0^2)_+\vee t}^{(t(j)+\epsilon_0^2)\wedge T} ds \int_{B_d(x(j),\epsilon_0)} \frac{K}{(s-t)^{d/2}} \exp\left(-\frac{|X^\epsilon_t - z|^2}{K\,(s-t)}\right) \\
&\qquad\qquad \frac{K^{p(1-\alpha)} \mathbf{1}_{z_i > \phi(s,z_1,\cdots,z_{i-1},z_{i+1},\cdots,z_d)}}{|z_i - \phi(s, z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_d)|^{p(1-\alpha)}} dz_1 \cdots dz_d \\
&\le K \int_{(t(j)-\epsilon_0^2)_+\vee t}^{(t(j)+\epsilon_0^2)\wedge T} \frac{ds}{(s-t)^{1/2}} < +\infty,
\end{aligned}$$

where the space integral is easily evaluated by integrating w.r.t. z_i first. □

3 Domain sensitivity for reflecting diffusions

In this section we deal with domain sensitivity of functionals of a normally reflected diffusion process $X^{t,x}$ in a time varying domain $\mathcal{D}$. We consider a general functional of the form

$$u(t, x) = \mathbf{E}[g(X^{t,x}_T) Z_T - \int_t^T Z^{t,x}_s f(s, X^{t,x}_s) ds - \int_t^T Z^{t,x}_s h(s, X^{t,x}_s) d\Lambda^{t,x}_s], \tag{3.1}$$

where $\Lambda^{t,x}$ is the associated increasing process on the boundary and

$$Z^{t,x}_s = e^{-\int_t^s c(r, X^{t,x}_r) dr + \beta(r, X^{t,x}_r) d\Lambda^{t,x}_r}, \tag{3.2}$$

and space perturbations of the domain $\mathcal{D}$ of the form (2.5).

The definition and construction of a diffusion process with normal reflection in a time varying domain requires few modifications with respect to the analogous definition and construction for a fixed domain, but, to our knowledge, does not appear anywhere in the literature, therefore we formulate it in Subsection 3.1 and add a few more details in Appendix C. The same holds for the Feynman-Kac representation that relates the functional (3.1) to a Cauchy-Neumann parabolic problem in $\mathcal{D}$ (Subsection 3.2 and Appendix B.2).

The sensitivity result we are interested in is contained in Subsection 3.3: we prove that the expectation of the functional (3.1) is differentiable with respect to the perturbation and compute its derivative, which turns out to be an expectation along the paths of $(X^{t,x}, \Lambda^{t,x})$. As in Section 2, the idea of the proof is to transfer the perturbation from the domain to the process, by introducing the perturbed process

$$\tilde{X}_s^{t,x^\epsilon,\epsilon} = (\mathrm{Id} + \epsilon\Theta_s)(X_s^{t,x,\epsilon}), \qquad x^\epsilon = (\mathrm{Id} + \epsilon\Theta_t)(x), \tag{3.3}$$

where $X^{t,x,\epsilon}$ is the normally reflecting diffusion in the perturbed domain (2.5). The process $\tilde{X}^{t,x^\epsilon,\epsilon}$ takes values in $\overline{\mathcal{D}}$ but reflects on the boundary along an oblique direction. Therefore we need to prove some compactness and moment estimates for diffusions with oblique reflection in a time varying domain (Subsection 3.1 and Appendix C.1).

3.1 Reflecting diffusions

In the sequel, we consider a time varying domain $\mathcal{D}$ of class at least $\mathcal{H}_2$ (see Subsection 1.3). Recall that, with this degree of regularity, the time sections $\mathcal{D}_s$, $s \in [0,T]$, verify the uniform exterior and interior sphere condition, uniformly for $s \in [0,T]$; let $\mathbf{n}_s(x)$ denote the unit inward normal with respect to $\mathcal{D}_s$ at $x \in \partial\mathcal{D}_s$. Let γ denote a measurable, unit vector field on $\overline{\mathcal{SD}}$ such that

$$\gamma_s(x) \cdot \mathbf{n}_s(x) > 0, \qquad \forall x \in \partial\mathcal{D}_s, \quad s \in [0,T],$$

and let b be a bounded measurable function on $\overline{\mathcal{D}}$ and σ be a continuous function on $\overline{\mathcal{D}}$. In the sequel (t,x) will be a fixed point in $\overline{\mathcal{D}}$.

Definition 3.1 *A (weak) solution of the stochastic differential equation (RSDE) of coefficients b and σ in $\overline{\mathcal{D}}$ with reflection along γ, starting at (t,x), is a stochastic process $(X^{t,x}, \Lambda^{t,x})$ with paths in $\mathcal{C}([t,T],\Re^d) \times \mathcal{I}([t,T],\Re_+)$, $\Lambda_t^{t,x} = 0$, that satisfies, almost surely, for $s \in [t,T]$,*

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r + \int_t^s \gamma_r(X_r^{t,x})d\Lambda_r^{t,x}, \tag{3.4}$$

$$X_s^{t,x} \in \overline{\mathcal{D}_s}, \ s \in [t,T], \qquad d\Lambda^{t,x}\left(\left\{s \in [t,T] : \ X_s^{t,x} \in \mathcal{D}_s\right\}\right) = 0, \tag{3.5}$$

where W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbf{P})$ and $(X^{t,x}, \Lambda^{t,x})$ is $\{\mathcal{F}_s\}$-adapted. $X^{t,x}$ will be called a reflecting diffusion in $\overline{\mathcal{D}}$ with coefficients b and σ and direction of reflection γ, and $\Lambda^{t,x}$ will be called the associated increasing process on the boundary.

The following two theorems can be proved by arguments similar to those used in [Sai87] and [Cost92] (a few more details are given in Appendix C.1).

Theorem 3.2 *Assume b and σ satisfy* **($\mathbf{A}_\alpha$-1)** *(see Section 2). There exists one and only one (weak) solution to the RSDE of coefficients b and σ in $\overline{\mathcal{D}}$ with reflection along* $\mathbf{n}$ *(normal reflection), starting at $(t,x) \in \overline{\mathcal{D}}$.*

Remark 3.3 *Indeed, under the assumptions of Theorem 3.2, there exists one and only one strong solution to the RSDE of coefficients b and σ in $\overline{\mathcal{D}}$ with normal reflection.*

Let

$$l(r) = \sup_{s,t\in[0,T],\ |s-t|\le r}\ \sup_{x\in\overline{\mathcal{D}_t}} \mathbf{d}(x, \overline{\mathcal{D}_s}).$$

Since $\mathcal{D}$ is of class $\mathcal{H}_2$,

$$\lim_{r\to 0^+} l(r) = 0.$$

Theorem 3.4 *Assume γ is continuous and*

$$\gamma_s(x)\cdot \mathbf{n}_s(x) \ge k_0 > \frac{\sqrt{3}}{2}, \quad \forall x\in\partial\mathcal{D}_s,\ s\in]t,T]. \tag{3.6}$$

Let $(X^{t,x},\Lambda^{t,x})$ be a solution of the stochastic differential equation of coefficients b and σ in $\overline{\mathcal{D}}$ with reflection along γ, starting at $(t,x)\in\overline{\mathcal{D}}$. Denote

$$Y_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dW_r, \quad s\in[t,T]. \tag{3.7}$$

Then there exists a function $\kappa : \mathcal{C}([t,T],\Re^d)\to\Re_+$, depending only on $\mathcal{D}$, k_0 and the modulus of continuity of γ, such that, almost surely,

$$\sup_{s_1\le r_1\le r_2\le s_2} |X_{r_2}^{t,x} - X_{r_1}^{t,x}| + \Lambda_{s_2}^{t,x} - \Lambda_{s_1}^{t,x} \le \kappa(Y^{t,x})\left(\sup_{s_1\le r_1\le r_2\le s_2}|Y_{r_2}^{t,x}-Y_{r_1}^{t,x}| + l(s_2-s_1)\right), \quad t\le s_1\le s_2\le T,$$

κ is bounded over compact subsets of $\mathcal{C}([t,T],\Re^d)$.

We will also need control on the moments of the associated increasing process on the boundary: this is provided by the following proposition, which is proved in Appendix C.2.

Proposition 3.5 *Let $\mathcal{D}$, γ and $(X^{t,x},\Lambda^{t,x})$ be as in Theorem 3.4. Then, for any $p\ge 0$,*

$$\mathbf{E}\left[e^{p\Lambda_T^{t,x}}\right] \le K \tag{3.8}$$

where K is a constant depending only on p, $\mathcal{D}$, $|b|_\infty$, $|\sigma|_\infty$ and k_0.

The above results yield the following proposition, which will be one of the main tools in the proof of the sensitivity result.

Proposition 3.6 *Assume b and σ satisfy* **($\mathbf{A}_\alpha$-1)**. *Let $(\tilde{X}^{t,x^\epsilon,\epsilon},\tilde{\Lambda}^{t,x^\epsilon,\epsilon})$ be a solution of the stochastic differential equation of coefficients $\tilde{b}^\epsilon$ and $\tilde{\sigma}^\epsilon$ in $\overline{\mathcal{D}}$ with reflection along $\tilde{\gamma}^\epsilon$, starting at (t,x^ϵ). Suppose $\tilde{\gamma}^\epsilon$, $\tilde{b}^\epsilon$, $\tilde{\sigma}^\epsilon$ and x^ϵ converge uniformly to $\mathbf{n}$, b, σ and x, respectively, as ϵ goes to zero, and $\tilde{\gamma}^\epsilon$ is continuous. Then $(\tilde{X}^{t,x^\epsilon,\epsilon},\tilde{\Lambda}^{t,x^\epsilon,\epsilon})$ converges weakly to the solution* *of the stochastic differential equation of coefficients b and σ in $\overline{\mathcal{D}}$ with normal reflection, starting at (t,x), and, for some ϵ_0,*

$$\sup_{\epsilon\leq\epsilon_0}\mathbf{E}\left[e^{p\tilde{\Lambda}_T^{t,x^\epsilon,\epsilon}}\right]<\infty, \tag{3.9}$$

for any $p\geq 0$.

Proof. We omit the superscripts t,x^ϵ. For ϵ smaller than some ϵ_0, $\tilde{b}^\epsilon$ and $\tilde{\sigma}^\epsilon$ are bounded uniformly in ϵ and the directions of reflection $\tilde{\gamma}^\epsilon$ satisfy (3.6) with the same $k_0>\frac{\sqrt{3}}{2}$. Therefore (3.9) follows from (3.8).

Now let us turn to convergence. $(\tilde{X}^\epsilon,\tilde{\Lambda}^\epsilon)$ satisfies (3.4) for some Brownian motion $\tilde{W}^\epsilon$. Let $\tilde{Y}_s^\epsilon$ be as in (3.7). Let $\{\epsilon_n\}$ be any sequence converging to zero. The directions of reflection $\tilde{\gamma}^{\epsilon_n}$ are equicontinuous and satisfy (3.6) with the same $k_0>\frac{\sqrt{3}}{2}$, therefore the function κ that appears in Theorem 3.4 does not depend upon n. In addition, since $\tilde{b}^{\epsilon_n}$ and $\tilde{\sigma}^{\epsilon_n}$ are bounded uniformly in n, the family of stochastic processes $\{\tilde{Y}^{\epsilon_n}\}$ is relatively compact. Then the family $\{(\tilde{X}^{\epsilon_n},\tilde{\Lambda}^{\epsilon_n})\}$ is relatively compact and hence so is the family $\{(\tilde{X}^{\epsilon_n},\tilde{W}^{\epsilon_n},\tilde{\Lambda}^{\epsilon_n})\}$. Now let $\{\epsilon_n\}$ be a sequence converging to zero such that $\{(\tilde{X}^{\epsilon_n},\tilde{W}^{\epsilon_n},\tilde{\Lambda}^{\epsilon_n})\}$ converges in law to a limit point $(\tilde{X},\tilde{W},\tilde{\Lambda})$, and let $\{(\overline{X}^{\epsilon_n},\overline{W}^{\epsilon_n},\overline{\Lambda}^{\epsilon_n})\}$ and $(\overline{X},\overline{W},\overline{\Lambda})$ be versions of $\{(\tilde{X}^{\epsilon_n},\tilde{W}^{\epsilon_n},\tilde{\Lambda}^{\epsilon_n})\}$ and $(\tilde{X},\tilde{W},\tilde{\Lambda})$, respectively, such that $\{(\overline{X}^{\epsilon_n},\overline{W}^{\epsilon_n},\overline{\Lambda}^{\epsilon_n})\}$ converges almost surely, on a suitable probability space, to $(\overline{X},\overline{W},\overline{\Lambda})$, uniformly on $[0,T]$. Clearly $\{(\tilde{b}^{\epsilon_n}(\cdot,\overline{X}^{\epsilon_n}),\tilde{\sigma}^{\epsilon_n}(\cdot,\overline{X}^{\epsilon_n}),\tilde{\gamma}^{\epsilon_n}(\cdot,\overline{X}^{\epsilon_n}),\overline{W}^{\epsilon_n},\overline{\Lambda}^{\epsilon_n}\}$ converges almost surely to $\{(b(\cdot,\overline{X}),\sigma(\cdot,\overline{X}),$ $\mathbf{n}(\cdot,\overline{X}),\overline{W},\overline{\Lambda}\}$, uniformly on $[t,T]$. Theorem 2.2 in [KP91] ensures that the right hand side of (3.4) converges in probability, uniformly on $[t,T]$, to $x+\int_t^\cdot b(r,\overline{X}_r)dr+\int_t^\cdot\sigma(r,\overline{X}_r)d\overline{W}_r+\int_t^\cdot\mathbf{n}_r(\overline{X}_r)d\overline{\Lambda}_r$ as soon as

$$\sup_n\mathbf{E}\left[\langle\overline{W}^{\epsilon_n}\rangle_T\right]<\infty,\quad \sup_n\mathbf{E}\left[\overline{\Lambda}_T^{\epsilon_n}\right]<\infty. \tag{3.10}$$

The first inequality in (3.10) holds trivially and the second one holds by (3.9). In fact the convergence holds almost surely, because $\overline{X}^{\epsilon_n}$ converges to $\overline{X}$ almost surely.

In order to show that $(\overline{X},\overline{\Lambda})$ satisfies (3.4-3.5) it only remains to show that $d\overline{\Lambda}(\{s\in[t,T]:$ $\overline{X}_s\in\mathcal{D}_s\})=0$ almost surely. To this end, observe that the sequence of measures $\left\{d\overline{\Lambda}^{\epsilon_n}\right\}$ on $[t,T]$ converges, almost surely, $*$weakly to the measure $d\overline{\Lambda}$. For every $\eta>0$,

$$\left\{s\in[t,T]:\ d(\overline{X}_s,\partial\mathcal{D}_s)>\eta\right\}\subseteq\left\{s\in[t,T]:\ d(\overline{X}_s^{\epsilon_n},\partial\mathcal{D}_s)>\eta/2\right\},$$

for all n large enough, almost surely. The set on the right hand side has zero measure under $d\overline{\Lambda}^{\epsilon_n}$ and hence so does the set on the left hand side. In addition the set on the left hand side is open, so that it has zero measure under $d\overline{\Lambda}$ as well, for every $\eta>0$, which implies that $d\overline{\Lambda}\left(\left\{s\in[t,T]:\ \overline{X}_s\in\mathcal{D}_s\right\}\right)=0$.

The assertion of the proposition then follows from uniqueness of the solution to the RSDE of coefficients b and σ with normal reflection in D (Theorem 3.2). □

3.2 Feynman-Kac's representation

As in Section 2, in order to study the sensitivity of the function u defined by (3.1) with respect to perturbations of $\mathcal{D}$ we need to represent u as the solution of a suitable partial differential equation, i.e. to extend the Feynman-Kac formula.

Proposition 3.7 *Assume* $(\mathbf{A}_\alpha)$*,* $\mathcal{D} \in \mathcal{H}_{2+\alpha}$*,* $\beta \in \mathcal{H}_{1+\alpha}$*,* $c \in \mathcal{H}_\alpha$*,* $f \in \mathcal{H}_\alpha$ *and* $h \in \mathcal{H}_{1+\alpha}$ *with* $\alpha \in]0,1[$*. Let* g *be a bounded, continuous function on* $\Re^d$*. Then there exists a unique solution of class* $\mathcal{C}^{1,2}(\mathcal{D}) \cap \mathcal{C}^{0,0}(\bar{\mathcal{D}})$ *to the parabolic problem*

$$\begin{cases} \partial_t u + Lu - cu = f & \text{in } \mathcal{D}, \\ \nabla u \, \mathbf{n} - \beta u = h & \text{on } \mathcal{SD}, \\ u(T,\cdot) = g & \text{on } \mathcal{D}_T. \end{cases} \tag{3.11}$$

The solution is given by (3.1). If, in addition, g *is twice continuously differentiable with bounded derivatives, the second order derivatives of* g *are uniformly Hölder continuous of order* α *and* $\nabla g \, \mathbf{n}(T,\cdot) - \beta(T,\cdot) g = h(T,\cdot)$ *on* $\partial\mathcal{D}_T$*, then* u *belongs to* $\mathcal{H}_{2+\alpha}(\mathcal{D})$*.*
The proof is given in Appendix B.2.

3.3 Boundary sensitivity

Let $X^{t,x}$ be the reflecting diffusion with coefficients b and σ and normal reflection in the time space domain $\overline{\mathcal{D}}$, starting at (t,x), and let $\Lambda^{t,x}$ be the associated increasing process on the boundary. Our goal is to study the sensitivity of the functional (3.1) with respect to space perturbations of $\mathcal{D}$ of the form (2.5), with $\Theta \in \mathcal{C}_b^{1,2}([0,T] \times \Re^d)$. We will suppose that $|\epsilon| \leq \epsilon_0$, where ϵ_0 is chosen suitably (in particular $\epsilon_0 \leq \frac{1}{2|J\Theta|_\infty} \wedge 1$, so that the map $\mathrm{Id} + \epsilon\Theta(t,\cdot)$ is invertible for $|\epsilon| \leq \epsilon_0$). A straightforward computation shows that the unit inward normal vector on $\partial\mathcal{D}_s^\epsilon$ is given by

$$\mathbf{n}_s^\epsilon(x) = \frac{(\mathbf{I} + \epsilon J\Theta_s^*)(x) \mathbf{n}_s \circ (\mathrm{Id} + \epsilon\Theta_s)(x)}{|(\mathbf{I} + \epsilon J\Theta_s^*)(x) \mathbf{n}_s \circ (\mathrm{Id} + \epsilon\Theta_s)(x)|}, \quad \forall x \in \partial\mathcal{D}_s^\epsilon. \tag{3.12}$$

Let $X^{\epsilon,t,x}$ be the reflecting diffusion with coefficients b and σ and normal reflection in $\mathcal{D}^\epsilon$, starting at (t,x), and let $\Lambda^{\epsilon,t,x}$ be the associated increasing process on the boundary. Denote by u^ϵ the corresponding functional (3.1). The following theorem contains the sensitivity result we are interested in.

Theorem 3.8 *Assume, for some* $\alpha \in]0,1[$*,* $(\mathbf{A}_\alpha)$*,* $\mathcal{D} \in \mathcal{H}_{2+\alpha}$*,* $\beta \in \mathcal{H}_{1+\alpha}$*,* $c \in \mathcal{H}_\alpha$*,* $f \in \mathcal{H}_\alpha$ *and* $h \in \mathcal{H}_{1+\alpha}$*. Suppose* g *is a twice continuously differentiable function on* $\Re^d$ *with bounded derivatives, the second order derivatives of* g *are uniformly Hölder continuous of order* α *and* $\nabla g \, \mathbf{n}(T,\cdot) - \beta(T,\cdot) g = h(T,\cdot)$ *on* $\partial\mathcal{D}_T$*. Then, for every fixed* $(t,x) \in \mathcal{D} \cup \mathcal{D}_0$*,* $u^\epsilon(t,x)$ *is differentiable with respect to* ϵ *at* $\epsilon = 0$ *and*

$$\frac{du^\epsilon(t,x)}{d\epsilon}|_{\epsilon=0} = \quad \mathbf{E}\Big[-\int_t^T Z_s^{t,x} (\beta u + h)(s, X_s^{t,x}) \mathbf{n}^*(s, X_s^{t,x}) J\Theta(s, X_s^{t,x}) \mathbf{n}(s, X_s^{t,x}) d\Lambda_s^{t,x}$$

$$
\begin{aligned}
&- \int_t^T Z_s^{t,x} \Theta^*(s, X_s^{t,x}) H u(s, X_s^{t,x}) \mathbf{n}(s, X_s^{t,x}) d\Lambda_s^{t,x} \\
&+ \int_t^T Z_s^{t,x} \nabla u(s, X_s^{t,x}) J \Theta^*(s, X_s^{t,x}) \mathbf{n}(s, X_s^{t,x}) d\Lambda_s^{t,x} \\
&+ \int_t^T Z_s^{t,x} \nabla(\beta u + h)(s, X_s^{t,x}) \Theta(s, X_s^{t,x}) d\Lambda_s^{t,x} \Big]
\end{aligned} \tag{3.13}
$$

where $Z^{t,x}$ is given by (3.2).

Proof. In the rest of this section we omit the superscripts t, x and denote

$$
Z_s^\epsilon = e^{-\int_t^s c(r, X_r^\epsilon) dr + \beta(r, X_r^\epsilon) d\Lambda_r^\epsilon}.
$$

Without loss of generality we can suppose $\epsilon > 0$. Since $\mathcal{D}$ is of class $\mathcal{H}_{2+\alpha}$ and $u \in \mathcal{H}_{2+\alpha}(\mathcal{D})$, we can extend u to a function in $\mathcal{H}_{2+\alpha}(]0, T[\times \Re^d)$, which we will still denote by u. Then, for ϵ small enough that $(t, x^\epsilon) \in \mathcal{D}^\epsilon \cup \mathcal{D}_0^\epsilon$,

$$
\begin{aligned}
u^\epsilon(t,x) - u(t,x) = \quad & \mathbf{E}[Z_T^\epsilon g(X_T^\epsilon) - \int_t^T Z_s^\epsilon f(s, X_s^\epsilon) ds - \int_t^T Z_s^\epsilon h(s, X_s^\epsilon) d\Lambda_s^\epsilon] - u(t,x) \\
= \quad & \mathbf{E}[Z_T^\epsilon (g(X_T^\epsilon) - u(T, X_T^\epsilon))] \\
& + \mathbf{E}[Z_T^\epsilon u(T, X_T^\epsilon) - u(t,x) - \int_t^T Z_s^\epsilon f(s, X_s^\epsilon) ds - \int_t^T Z_s^\epsilon h(s, X_s^\epsilon) d\Lambda_s^\epsilon] \\
= \quad & \mathbf{E}[Z_T^\epsilon (g(X_T^\epsilon) - u(T, X_T^\epsilon))] \\
& + \mathbf{E}[\int_t^T Z_s^\epsilon (\partial_s u + Lu - cu - f)(s, X_s^\epsilon) ds] \\
& + \mathbf{E}[\int_t^T Z_s^\epsilon (\nabla u \, \mathbf{n}^\epsilon - \beta u - h)(s, X_s^\epsilon) d\Lambda_s^\epsilon] \\
= \quad & \Delta_{1,\epsilon} + \Delta_{2,\epsilon} + \Delta_{3,\epsilon}.
\end{aligned} \tag{3.14}
$$

As in the proof of Theorem 2.2, the idea is to transfer the perturbation from the domain to the process, by introducing the perturbed process $\tilde{X}_s^\epsilon = \tilde{X}_s^{t,x^\epsilon,\epsilon}$ given by (3.3). The following lemma is proved after this theorem.

Lemma 3.9 *$\tilde{X}^\epsilon$ is a reflecting diffusion in the domain $\mathcal{D}$ with coefficients*

$$
\begin{aligned}
\tilde{b}^\epsilon(s,x) = \quad & (b_s + \epsilon (\partial_s + L) \Theta_s) \circ (\mathrm{Id} + \epsilon \Theta_s)^{-1}(x), \\
\tilde{\sigma}^\epsilon(s,x) = \quad & (\sigma_s + \epsilon J \Theta_s \sigma_s) \circ (\mathrm{Id} + \epsilon \Theta_s)^{-1}(x),
\end{aligned} \tag{3.15}
$$

and direction of reflection

$$
\tilde{\gamma}^\epsilon(s,x) = \frac{\left(\mathbf{I} + \epsilon J \Theta_s \left((\mathrm{Id} + \epsilon \Theta_s)^{-1}(x)\right)\right) \left(\mathbf{I} + \epsilon J \Theta_s^* \left((\mathrm{Id} + \epsilon \Theta_s)^{-1}(x)\right)\right) \mathbf{n}_s(x)}{\left|\left(\mathbf{I} + \epsilon J \Theta_s \left((\mathrm{Id} + \epsilon \Theta_s)^{-1}(x)\right)\right) \left(\mathbf{I} + \epsilon J \Theta_s^* \left((\mathrm{Id} + \epsilon \Theta_s)^{-1}(x)\right)\right) \mathbf{n}_s(x)\right|}. \tag{3.16}
$$

The associated increasing process is

$$
\tilde{\Lambda}_s^\epsilon = \int_t^s \frac{\left|(\mathbf{I} + \epsilon J \Theta_r(X_r^\epsilon)) (\mathbf{I} + \epsilon J \Theta_r^*(X_r^\epsilon)) \mathbf{n}_r(\tilde{X}_r^\epsilon)\right|}{\left|(\mathbf{I} + \epsilon J \Theta_r^*(X_r^\epsilon)) \mathbf{n}_r(\tilde{X}_r^\epsilon)\right|} d\Lambda_r^\epsilon. \tag{3.17}
$$

By Proposition 3.6, $(\tilde{X}^\epsilon, \tilde{\Lambda}^\epsilon)$ converges weakly to (X, Λ), as ϵ goes to 0, and (3.9) holds. Since $|X_s^\epsilon - \tilde{X}_s^\epsilon| \leq \epsilon\, |\Theta|_\infty$ for every $t \leq s \leq T$, X^ϵ too converges weakly to X. As far as Λ^ϵ is concerned, we have

$$\Lambda_s^\epsilon = \int_t^s \frac{\left|(\mathbf{I} + \epsilon J \Theta_r^*(X_r^\epsilon))\, \mathbf{n}_r(\tilde{X}_r^\epsilon)\right|}{\left|(\mathbf{I} + \epsilon J \Theta_r(X_r^\epsilon))\, (\mathbf{I} + \epsilon J \Theta_r^*(X_r^\epsilon))\, \mathbf{n}_r(\tilde{X}_r^\epsilon)\right|} d\tilde{\Lambda}_r^\epsilon, \tag{3.18}$$

so that Λ^ϵ converges weakly to Λ by (3.9) and Theorem 2.2 in [KP91]. In addition, since the integrand in (3.18) is bounded from above by a positive constant, uniformly for $\epsilon \leq \epsilon_0$, (3.9) holds for Λ^ϵ as well. Therefore Z^ϵ converges weakly to Z and

$$\sup_{\epsilon \leq \epsilon_0} \mathbf{E}\left[\sup_{t \leq s \leq T} |Z_s^\epsilon|^p\right] < \infty. \tag{3.19}$$

We are now ready to analyze the right hand side of (3.14). Since g and $u(T, \cdot)$ coincide on $\overline{\mathcal{D}_T}$ and their second order derivatives derivatives are Hölder continuous of order α, one has

$$\Delta_{1,\epsilon} = O(\epsilon^{2+\alpha}).$$

Next consider $\Delta_{2,\epsilon}$ and let

$$A(s, x) = (\partial_s u + Lu - cu - f)(s, x).$$

Notice that A is of class $\mathcal{H}_\alpha$. By (3.19) we have, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$|\Delta_{2,\epsilon}| \leq \quad \mathbf{E}[\int_t^T |Z_s^\epsilon|^p ds]^{1/p} \mathbf{E}[\int_t^T |A(s, X_s^\epsilon)|^q ds]^{1/q} \leq K\, \mathbf{E}[\int_t^T |A(s, X_s^\epsilon)|^q ds]^{1/q}.$$

Now observe that, as u satisfies (3.11), $A(s, X_s^\epsilon) \neq 0$ implies

$$\mathbf{d}(X_s^\epsilon, \partial\mathcal{D}_s) \leq |\Theta|_\infty \epsilon.$$

Therefore we have

$$|\Delta_{2,\epsilon}| \leq K\ \mathbf{E}[\int_t^T \mathbf{1}_{\mathbf{d}(X_s^\epsilon, \partial\mathcal{D}_s) \leq |\Theta|_\infty \epsilon} |A(s, X_s^\epsilon) - A(s, \tilde{X}_s^\epsilon)|^q ds]^{1/q}, \tag{3.20}$$

and, by the smoothness of A,

$$|\Delta_{2,\epsilon}| \leq K\ \epsilon^\alpha \mathbf{E}\left[\int_t^T \mathbf{1}_{\mathbf{d}(X_s^\epsilon, \partial\mathcal{D}_s) \leq |\Theta|_\infty \epsilon} ds\right]^{1/q}.$$

The following lemma is proved after this theorem.

Lemma 3.10 *There is a constant K depending only on $\mathcal{D}$ and the coefficients of L, such that, for some $\epsilon_0, \eta_0 > 0$,*

$$\sup_{\epsilon \leq \epsilon_0} \mathbf{E}\left[\int_t^T \mathbf{1}_{\mathbf{d}(X_s^\epsilon, \partial\mathcal{D}_s) \leq \eta} ds\right] \leq K\eta,$$

for every $0 < \eta < \eta_0$.
By Lemma 3.10, $|\Delta_{2,\epsilon}| \leq K\epsilon^{\alpha+\frac{1}{q}}$, and, by choosing $\frac{1}{q} > 1-\alpha$, we obtain

$$\Delta_{2,\epsilon} = o(\epsilon).$$

Now consider $\Delta_{3,\epsilon}$. Since the unit inward normal to D_s, $\mathbf{n}^\epsilon_s$, is given by (3.12), we have

$$\begin{aligned}
\Delta_{3,\epsilon} = \quad & \mathbf{E}\left[\int_t^T Z^\epsilon_s\, (\nabla u\, \mathbf{n}^\epsilon - \beta u - h)\,(s, X^\epsilon_s) d\Lambda^\epsilon_s\right] \\
= \quad & \mathbf{E}\left[\int_t^T Z^\epsilon_s \left(\nabla u(s, X^\epsilon_s)\, \frac{\mathbf{n}_s(\tilde{X}^\epsilon_s)}{|(\mathbf{I} + \epsilon J\Theta^*_s)(X^\epsilon_s)\, \mathbf{n}_s(\tilde{X}^\epsilon_s))|} - (\beta u + h)(s, X^\epsilon_s)\right) d\Lambda^\epsilon_s\right] \\
& + \epsilon\, \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \nabla u(s, X^\epsilon_s)\, (J\Theta^*)(s, X^\epsilon_s)\ \ \frac{\mathbf{n}_s(\tilde{X}^\epsilon_s)}{|(\mathbf{I} + \epsilon J\Theta^*_s)(X^\epsilon_s)\, \mathbf{n}_s(\tilde{X}^\epsilon_s)|} d\Lambda^\epsilon_s\right].
\end{aligned}$$

Taking into account the boundary condition (3.11) satisfied by u, and the fact that $d\Lambda^\epsilon_s$ increases if and only if $X^\epsilon_s \in \partial\mathcal{D}^\epsilon_s$, that is if and only if $\tilde{X}^\epsilon_s \in \partial\mathcal{D}_s$, we obtain

$$\begin{aligned}
\Delta_{3,\epsilon} = \quad & \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \nabla u(s, X^\epsilon_s) \mathbf{n}_s(\tilde{X}^\epsilon_s) \left(\frac{1}{|(\mathbf{I} + \epsilon J\Theta^*_s)(X^\epsilon_s)\, \mathbf{n}_s(\tilde{X}^\epsilon_s)|} - 1\right) d\Lambda^\epsilon_s\right] \\
& + \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \left(\nabla u(s, X^\epsilon_s) - \nabla u(s, \tilde{X}^\epsilon_s)\right) \mathbf{n}_s(\tilde{X}^\epsilon_s) d\Lambda^\epsilon_s\right] \\
& + \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \left((\beta u + h)(s, \tilde{X}^\epsilon_s) - (\beta u + h)(s, X^\epsilon_s)\right) d\Lambda^\epsilon_s\right] \\
& + \epsilon\, \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \nabla u(s, X^\epsilon_s)\, (J\Theta^*)(s, X^\epsilon_s)\ \ \frac{\mathbf{n}_s(\tilde{X}^\epsilon_s)}{|(\mathbf{I} + \epsilon J\Theta^*_s)(X^\epsilon_s)\, \mathbf{n}_s(\tilde{X}^\epsilon_s)|} d\Lambda^\epsilon_s\right]
\end{aligned}$$

and, by exploiting the identity $\frac{1}{|v|} - 1 = \frac{1-|v|^2}{|v|\,(|v|+1)}$, $v \in \Re^d$,

$$\begin{aligned}
\Delta_{3,\epsilon} = \quad & \epsilon\, \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \nabla u(s, X^\epsilon_s) \mathbf{n}_s(\tilde{X}^\epsilon_s)\right. \\
& \left. \frac{-2\mathbf{n}^*_s(\tilde{X}^\epsilon_s)\, J\Theta_s(X^\epsilon_s) \mathbf{n}_s(\tilde{X}^\epsilon_s)}{|(\mathbf{I} + \epsilon J\Theta^*_s)(X^\epsilon_s)\, \mathbf{n}_s(\tilde{X}^\epsilon_s)|(|(\mathbf{I} + \epsilon J\Theta^*_s)(X^\epsilon_s)\, \mathbf{n}_s(\tilde{X}^\epsilon_s)| + 1)} d\Lambda^\epsilon_s\right] \\
& + \epsilon\, \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \left(-\Theta^*_s(X^\epsilon_s) H u(s, X^\epsilon_s) \mathbf{n}_s(\tilde{X}^\epsilon_s) + \nabla(\beta u + h)(s, X^\epsilon_s) \Theta_s(X^\epsilon_s)\right) d\Lambda^\epsilon_s\right] \\
& + \epsilon\, \mathbf{E}\left[\int_t^T Z^\epsilon_s\, \nabla u(s, X^\epsilon_s)\, (J\Theta^*)(s, X^\epsilon_s)\ \ \mathbf{n}_s(\tilde{X}^\epsilon_s) d\Lambda^\epsilon_s\right] \\
& + \epsilon^{1+\alpha} \mathbf{E}\left[\int_t^T Z^\epsilon_s\, R^\epsilon_s(X^\epsilon_s, \tilde{X}^\epsilon_s) d\Lambda^\epsilon_s\right],
\end{aligned} \tag{3.21}$$

where $|R^\epsilon_s(x, \tilde{x})|$ is uniformly bounded by $(3|J\Theta|^2_\infty + |\Theta|^{1+\alpha}_\infty)(|u|_{2+\alpha} + |\beta u + h|_{1+\alpha})$. The last summand in the right hand side of (3.21) is $o(\epsilon)$ because $\{\Lambda^\epsilon\}$ satisfies (3.9) and $\{Z^\epsilon\}$ satisfies (3.19). Moreover all other integrals on the right hand side of (3.21) converge in law to the integrals on the right hand side of (3.13) because $(\tilde{X}^\epsilon, X^\epsilon, Z^\epsilon, \Lambda^\epsilon)$ converges weakly to (X, X, Z, Λ) and $\{\Lambda^\epsilon\}$ satisfies (3.9) (by Theorem 2.2 in [KP91]). Then the assertion follows from the fact that the sum of the integrals on the right hand side of (3.21) is bounded in absolute value by

$$K(\sup_{s\le T}|Z_s^\epsilon|)(|\nabla u|_\infty|J\Theta|_\infty + |Hu|_\infty|\Theta|_\infty + |\nabla(\beta u + h)|_\infty|\Theta|_\infty)\Lambda_T^\epsilon,$$

and $\{Z^\epsilon\}$, $\{\Lambda^\epsilon\}$ satisfy (3.19) and (3.9), respectively. □

Proof of Lemma 3.9. By Ito's formula for semimartingales, taking into account (3.12), $\tilde{X}^\epsilon$ satisfies (3.4), almost surely, with $\tilde{b}^\epsilon$, $\tilde{\sigma}^\epsilon$, $\tilde{\gamma}^\epsilon$ and $\tilde{\Lambda}^\epsilon$ given by (3.15), (3.16) and (3.17), respectively. $d\tilde{\Lambda}^\epsilon$ is equivalent to $d\Lambda^\epsilon$ because the integrand in (3.17) is uniformly bounded from below and from above by two positive constants, for $\epsilon \le \epsilon_0$. On the other hand $\tilde{X}_s^\epsilon \in \partial\mathcal{D}_s$ if and only if $X_s^\epsilon \in \partial\mathcal{D}_s^\epsilon$, so that $d\tilde{\Lambda}^\epsilon\left(\left\{s \in [t,T] : \ \tilde{X}_s^\epsilon \in \mathcal{D}_s\right\}\right) = 0$. Therefore all conditions in Definition 3.1 are satisfied. □

Proof of Lemma 3.10. Since $\mathcal{D}$ is of class $\mathcal{H}_{2+\alpha}$, there is a function $F \in \mathcal{C}_b^{1,2}$ that coincides with the signed spatial distance of x from $\partial\mathcal{D}_s$ on $\{(t,x) : 0 < t < T, \mathbf{d}(x, \partial\mathcal{D}_t) < r_0'\}$, for some r_0' (see Subsection 1.3). Then $F_s^\epsilon = F(s, X_s^\epsilon)$, $t \le s \le T$, is a semimartingale and, by **($\mathbf{A}_\alpha$-2)**, for $t \le s_1 \le s_2 \le T$,

$$\langle F^\epsilon\rangle_{s_2} - \langle F^\epsilon\rangle_{s_1} = \int_{s_1}^{s_2} |\nabla F\ \sigma|^2 (r, X_r^\epsilon)\, dr \ge a_0 \int_{s_1}^{s_2} \mathbf{1}_{\mathbf{d}(X_r^\epsilon, \partial\mathcal{D}_r) < r_0'} dr. \tag{3.22}$$

Therefore, for $\eta < \eta_0 = r_0'$,

$$\begin{aligned}
\mathbf{E}\left[\int_t^T \mathbf{1}_{\mathbf{d}(X_s^\epsilon,\partial\mathcal{D}_s)\le\eta} ds\right] &= \mathbf{E}\left[\int_t^T \mathbf{1}_{|F(s,X_s^\epsilon)|\le\eta}\mathbf{1}_{\mathbf{d}(X_s^\epsilon,\partial\mathcal{D}_s)<\eta_0} ds\right] \\
&\le \frac{1}{a_0}\mathbf{E}\left[\int_t^T \mathbf{1}_{|F_s^\epsilon|\le\eta} d\langle F^\epsilon\rangle_s\right] = \frac{1}{a_0}\mathbf{E}\left[\int_{-\eta}^{\eta} \mathcal{L}_s^\epsilon(y)dy\right] \\
&\le \frac{2\eta}{a_0} \sup_{|y|\le\eta_0} \mathbf{E}\left[\mathcal{L}_T^\epsilon(y)\right],
\end{aligned}$$

where $\mathcal{L}^\epsilon(y)$ denotes the local time of F^ϵ at y. On the other hand, by the Tanaka formula,

$$\begin{aligned}
\mathbf{E}\left[\mathcal{L}_T^\epsilon(y)\right] &= \mathbf{E}\left[|F_T^\epsilon - y| - |F_t^\epsilon - y| - \int_t^T \mathrm{sign}(F_s^\epsilon - y) dF_s^\epsilon\right] \\
&\le |F|_\infty + |y| + |(\partial_s + L)\, F|_\infty (T - t) + |\nabla F|_\infty \mathbf{E}\left[\Lambda_T^\epsilon\right],
\end{aligned}$$

and the assertion follows by the fact that (3.9) holds for $\{\Lambda^\epsilon\}$. □

4 Applications

The sensitivity results proved in the previous sections can be used for a variety of problems. In this section we give a few relevant examples of applications. Some of them are the object of current or future work by the authors or other researchers and are only sketched here.

4.1 Correcting the bias in the simulation of killed Brownian motion

Let us consider the problem of evaluating $\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O>T}]$, where $X_s^{t,x} = x + \mu(s-t) + \sigma(W_s - W_t) \in \Re^d$, O is a domain in $\Re^d$ and τ_O is the first exit time of $X^{0,x}$ from O. For instance, in finance this is the problem of pricing barrier options written on d risky assets that follow the Black and Scholes model. We can compute $\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O>T}]$ numerically by Monte Carlo methods, i.e. by simulating independent copies of $g(X_T^{0,x})\mathbf{1}_{\tau_O>T}$ and averaging them out. Even though $X_T^{0,x}$ can be simulated exactly, the exit time τ_O cannot; therefore it is usually approximated by $\tau_O^h = \inf\{t_i = ih > 0 : X_{t_i}^{0,x} \notin O\}$ where h is a time discretization step.

Let us consider the simplest possible situation, i.e. $X_s^{t,x} = x + (W_s - W_t) \in \Re$, $O =]-\infty, b[$, and suppose that g vanishes in some neighborhood of b (we could alternatively assume that g is smooth and vanishes in b). Then a special case of a theorem in [GM04] shows that the discretization error can be expanded to the first order w.r.t. $\sqrt{h}$, i.e.

$$\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O^h>T}] - \mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O>T}] = -c_0 u_x(\tau_O, b)\sqrt{h} + o(\sqrt{h}),$$

where $c_0 = 0.5823\ldots$ is a universal constant and u is the solution of (2.4) in $\mathcal{D} =]0,T[\times O$. When $u_x(\tau_O, b) < 0$ (which actually holds as soon as g is non-negative and non-identically zero), this approximation overestimates the exact value. Actually, this is clear since $\tau_O \le \tau_O^h$ systematically. In order to correct this bias, we can restrict the domain O to $O^h =]-\infty, b - C\sqrt{h}[$ (C to be fixed) and approximate τ_O by $\tau_{O^h}^h = \inf\{t_i = ih > 0 : X_{t_i} \notin O^h\}$. Then the error can be split into

$$\left(\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}^h>T}] - \mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}>T}]\right) + \left(\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}>T}] - \mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O>T}]\right).$$

By the result of [GM04] and by the space homogeneity of X, we have

$$\begin{aligned}\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}^h>T}] - \mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}>T}] &= -c_0 u_x(\tau_O^h, b - C\sqrt{h})\sqrt{h} + o(\sqrt{h}) \\ &= -c_0 u_x(\tau_O, b)\sqrt{h} + o(\sqrt{h}).\end{aligned}$$

As far as the second summand is concerned, recalling that g vanishes in some neighborhood of b we have $\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O>T}] = \mathbf{E}[g(X_{\tau_O\wedge T}^{0,x})] = \mathbf{E}[g(X_{\tau_{\mathcal{D}}}^{0,x})]$ and $\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}>T}] = \mathbf{E}[g(X_{\tau_{O^h}\wedge T}^{0,x})] = \mathbf{E}[g(X_{\tau_{\mathcal{D}^h}}^{0,x})]$, with $\mathcal{D} =]0,T[\times O$ and $\mathcal{D}^h =]0,T[\times O^h$. Then we can apply Theorem 2.2, which yields (see [GM04])

$$\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}>T}] - \mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O>T}] = u_x(\tau_O, b)C\sqrt{h} + o(\sqrt{h}).$$

By choosing $C = c_0$, the leading terms in the two expansions above cancel and we get

$$\mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_{O^h}^h > T}] - \mathbf{E}[g(X_T^{0,x})\mathbf{1}_{\tau_O > T}] = o(\sqrt{h}).$$

The case of Brownian motion killed at the exit from $O =]-\infty, b[$ (as well as its extension to multidimensional Brownian motion with drift killed at the exit from a half space) is actually a toy example because in this case one can simulate τ_O exactly by Brownian bridge techniques [Gob01]. Much more interesting situations (such as 2-dimensional wedges) are dealt with in [Men05].

4.2 Pricing of American options

The valuation of American options is still a major issue in asset pricing. The buyer of such a contract is given the right to exercise the option at any time τ between now (t say) and the maturity T. Assume that the vector of the prices of the underlying assets X evolves according to an SDE of type (2.1), that the market is complete and that the instantaneous interest rate is of the form $(c(s, X_s))_{s \le t \le T}$. If the payoff at time τ is $g(\tau, X_\tau)$ (g a continuous function satisfying suitable integrability conditions), the fair price of the option is given by (see [Kar88])

$$P(t,x) = \sup_{\tau \in [t,T]} \mathbf{E}\left[e^{-\int_t^\tau c(s, X_s^{t,x})ds} g(\tau, X_\tau^{t,x})\right],$$

the supremum being taken over all stopping times with values in $[t, T]$. There is no simple numerical method available to evaluate the price of American options. We refer the reader to [FLM$^+$01] for a review of numerical methods to handle this issue. A possible new approach is based on the observation that, since the smallest optimal stopping time is given by the first exit time of $X^{t,x}$ from the (unknown) continuation region $\mathcal{C} = \{(s, y) : P(s, y) > g(s, y)\}$ (see [Kar81]), one has

$$P(t,x) = \sup_{\mathcal{D} \subset]0,T[\times \Re^d} \mathbf{E}\left[e^{-\int_t^{\tau_\mathcal{D}^{t,x}} c(s, X_s^{t,x})ds} g(\tau, X_\tau^{t,x})\right], \tag{4.1}$$

where $\tau_\mathcal{D}^{t,x}$ is the first exit time of $X^{t,x}$ from $\mathcal{D}$, and $\mathcal{D}$ is an open set. The optimization of the r.h.s. of (4.1) can be carried out by a 'gradient' algorithm that uses the sensitivity with respect to the domain $\mathcal{D}$. Theorem 2.2 provides a tractable formula for this sensitivity. In practice, only smooth domains with a suitable parameterization are considered. The algorithm will be fully described in further work. An approximation to the Δ-hedging can also be obtained.

It is worth mentioning that the expression for the sensitivity given in Theorem 2.2 is consistent with the smooth-fit condition ([Fri76], [BØ91]): formally, at the boundary of the continuation region $\mathcal{C}$, we have $\nabla P = \nabla g$. By Theorem 2.2 this gives that at $\mathcal{D} = \mathcal{C}$ the sensitivity of $u = P$ is null.

4.3 Joint density of the maximum and the terminal value of a diffusion process

Let $X^{t,x}$ be the solution of a d-dimensional SDE of the form (2.1), under $(\mathbf{A}_\alpha)$ ($\alpha \in]0,1]$). The aim of this subsection is to prove that the joint law of $(\max_{t\le s\le T} X^{t,x}_{1,s}, X^{t,x}_{1,T}, \cdots, X^{t,x}_{d,T})$, $t<T$, has a density w.r.t. the Lebesgue measure on $\Re^{d+1}$, and to give a representation of this density. What follows extends the one-dimensional result of [CFS87], which is proved by different techniques.

Let $\tau^{t,x}_a = \inf\{s>t : X^{t,x}_{1,s} \ge a\}$. For fixed $(y_1,\cdots,y_d)$,

$$\begin{aligned}&\mathbf{P}\left(\max_{t\le s\le T} X^{t,x}_{1,s} \le a, X^{t,x}_{1,T} \le y_1, \cdots, X^{t,x}_{d,T} \le y_d\right)\\ &= u_a(t,x) = \mathbf{E}\left[\mathbf{1}_{\{T\}}(\tau^{t,x}_a \wedge T)\mathbf{1}_{]-\infty,y_1[\times\cdots]-\infty,y_d[}(X^{t,x}_{\tau^{t,x}_a\wedge T})\right],\end{aligned}$$

is the expectation of a functional of the form (3.1) with $\mathcal{D} =]0,T[\times]-\infty,a[\times\Re^{d-1}$ and $g(t,x) = \mathbf{1}_{\{T\}}(t)\mathbf{1}_{]-\infty,y_1[\times\cdots]-\infty,y_d[}(x)$. Without loss of generality we can take $a>x_1$ and $a>y_1$. We want to apply Theorem 2.2 to compute $\partial_a u_a$. In this example $\mathcal{D}$ is non bounded and g is not of class $\mathcal{H}_{1+\alpha}$. However, denoting by $q_a(t,x,T,\cdot)$ the density of the diffusion process killed at the exit from $]-\infty,a[\times\Re^{d-1}$, $u_a(t,x)$ can be represented as

$$u_a(t,x) = \mathbf{E}\left[\mathbf{1}_{\{\tau^{t,x}_a>T\}}\mathbf{1}_{]-\infty,y_1[\times\cdots]-\infty,y_d[}(X^{t,x}_T)\right] = \int_{z_1<y_1,\cdots,z_d<y_d} q_a(t,x,T,z)dz.$$

Since $q_a(t,x,T,z)$ is a smooth function of (t,x) and its derivatives satisfy exponential integrability conditions in z (see [GM04] for details), u_a is also a smooth function in $\mathcal{D}$ and ∇u_a can be extended continuously and boundedly up to $\mathcal{SD} = \{(t,x) : 0\le t<T,\ x_1 = a\}$. A closer inspection of the proof of Theorem 2.2 shows that in addition, only smoothness of g on the side $\mathcal{SD}$ is needed, and this holds here. Then, by taking $\Theta = (-1,0,\cdots,0)^*$ in Theorem 2.2, we get the differentiability of u_a w.r.t. a, and

$$\begin{aligned}\partial_a u_a(t,x) &= -\mathbf{E}[\mathbf{1}_{\tau^{t,x}_a<T}\partial_{x_1} u_a(\tau^{t,x}_a, X^{t,x}_{\tau^{t,x}_a})]\\ &= -\int_{z_1<y_1,\cdots,z_d<y_d} \mathbf{E}[\mathbf{1}_{\tau^{t,x}_a<T}\partial_{x_1} q_a(\tau^{t,x}_a, X^{t,x}_{\tau^{t,x}_a}, T, z)]dz.\end{aligned}$$

This, together with the observation that $u_a(t,x) = \int_{x_1}^a \partial_{a'} u_{a'}(t,x)da'$, proves that the density of the law of $(\max_{t\le s\le T} X^{t,x}_{1,s}, X^{t,x}_{1,T}, \cdots, X^{t,x}_{d,T})$ exists and is given by

$$r(a,y) = -\mathbf{1}_{a>x_1}\mathbf{1}_{a>y_1}\mathbf{E}[\mathbf{1}_{\tau^{t,x}_a<T}\partial_{x_1} q_a(\tau^{t,x}_a, X^{t,x}_{\tau^{t,x}_a}, T, y)].$$

Note that r is non-negative since $q_a(s,z,T,y)\ge 0$, $q_a(s,z,T,y) = 0$ if $z_1 = a$ and thus $\partial_{z_1} q_a(s,z,T,y)|_{z_1=a} \le 0$. Actually, $\partial_{z_1} q_a(s,z,T,y)|_{z_1=a} < 0$ for $s<T$ and $a>y_1$ (see Lemma 13 in [GM04]), so that $r(a,y)$ is strictly positive on the set $\{a>x, a>y_1\}$. A little extra work would show the continuity of r. Furthermore, we could iterate our arguments to study the differentiability of r: it leads to tedious computations we do not reproduce.

4.4 Singular stochastic control problems

In singular stochastic control problems (see, e.g., [Sh88] or [FS93] for an introduction to these problems) admissible controls (and, in general, optimal controls, when they exist) are not absolutely continuous in time. One considers a family of SDE's in $\Re^d$ of the form

$$X_s^{\phi,t,x} = x + \int_t^s b(r, X_r^{\phi,t,x})dr + \int_t^s \sigma(r, X_r^{\phi,t,x})dW_r + \phi_s \tag{4.2}$$

where the control $\phi_s,\ t \leq s < T,\ \phi_t = 0$, is a process with bounded variation, left continuous paths with right hand limits, such that the direction γ_r defined by

$$\phi_s = \int_{[t,s[} \gamma_r d|\phi|_r, \tag{4.3}$$

satisfies

$$\gamma_r \in \Gamma, \quad d|\phi| - a.e., \tag{4.4}$$

for a given closed cone Γ. The goal is to minimize, over all ϕ's, $J_{0,x}(\phi)$, where

$$J_{t,x}(\phi) = \mathbf{E}\left[g(X_T^{\phi,t,x}) - \int_t^T f(s, X_s^{\phi,t,x})ds - \int_{[t,T[} h(s,\gamma_s)d|\phi|_s\right].$$

When the value function $V(t,x) = \inf_\phi J_{t,x}(\phi)$ is sufficiently smooth (typically under some convexity assumptions), it can be shown that, letting

$$\widehat{\mathcal{D}} = \{(t,x) :\ 0 < t < T,\ H(t, \nabla V(t,x)) < 0\}\,, \tag{4.5}$$

where $H(t,v) = \sup_{|\gamma|=1,\,\gamma\in\Gamma}\{-v^*\gamma - h(t,\gamma)\}$, if the SDE with coefficients b and σ and reflection along Γ in $\widehat{\mathcal{D}}$, starting at $(0,x)$, has a solution $(\widehat{X}^{0,x}, \widehat{\phi}^{0,x})$, then $\widehat{\phi}^{0,x}$ is an optimal control. The definition of solution of a RSDE with reflection along a cone Γ in the closure of a domain $\overline{\mathcal{D}}$ is analogous to Definition 3.1, except that (3.4) has to be replaced by (4.2), (4.3) and (4.4) and accordingly $\Lambda^{t,x}$ has to be replaced by $|\Phi^{t,x}|$. In addition, in the present context (t,x) does not necessarily belong to $\overline{\mathcal{D}}$, therefore, for $(t,x) \notin \overline{\mathcal{D}}$, one has to allow an initial jump $\phi_{t^+}^{t,x} \in \Gamma$ such that $X_{t^+}^{t,x} = x + \phi_{t^+}^{t,x} \in \partial\mathcal{D}$ (as a consequence $(X^{t,x}, \phi^{t,x})$ is required to be continuous on $]t,T]$, with right hand limits in t and (3.5) holds for $s \in]t,T]$). Then the problem can be viewed as that of minimizing, over all domains D such that the RSDE with coefficients b and σ and reflection along Γ in $\overline{\mathcal{D}}$ has a solution, a functional of the form (3.1) (at least if $\widehat{\gamma}_s^{0,x} = \gamma(s, \widehat{X}_s^{0,x})$ for some known function γ, or if h is a function of time alone). If a sensitivity result like Theorem 3.8 can be proved, it can be used to derive necessary conditions for the optimal domain $\widehat{\mathcal{D}}$ and to construct a 'gradient' type stochastic algorithm to approximate $\widehat{\mathcal{D}}$ numerically.

As an example, consider the stochastic control problem analyzed in [SoSh91]:

$$b = 0, \quad \sigma = \sqrt{2}, \quad \Gamma = \{\lambda e,\ \lambda \geq 0\}, \quad e = (0, \cdots, 0, -1)^*, \tag{4.6}$$

$$h(s,\gamma) = h(s).$$

It is shown in [SoSh91], that, under suitable assumptions on f, g and h (in particular f and g convex in the x_d variable, f, g and h smooth and satisfying growth conditions), the domain $\widehat{\mathcal{D}}$ defined by (4.5) is of the form

$$\widehat{\mathcal{D}} = \{(t,x):\ 0 < t < T,\ x_d < \widehat{q}(t, x_1, \cdots, x_{d-1})\}, \tag{4.7}$$

for some (unknown) function $\widehat{q}$, which is locally Lipschitz on $[0,T[\times\Re^{d-1}$. The RSDE with coefficients (4.6) and reflection along e in $\overline{\mathcal{D}}$, starting at (t,x), has a (unique) solution $(X^{t,x}, \Phi^{t,x})$ for any domain $\mathcal{D}$ of the form (4.7). Therefore

$$\inf_{\phi} J_{0,x}(\phi) = u_{\widehat{\mathcal{D}}}(0,x) = \inf_{\mathcal{D}} u_{\mathcal{D}}(0,x),$$

where $u_{\mathcal{D}}$ is defined by (3.1) and the infimum is taken over all domains of the form (4.7) for some function q.

The results of Section 3 can be easily extended to this situation. For a domain $\mathcal{D}$ of the form (4.7), if $q \in \mathcal{H}_{2+\alpha}$, $f \in \mathcal{H}_\alpha$, $h \in \mathcal{H}_{1+\alpha}$ for some α, g is twice continuously differentiable with bounded derivatives, the second order derivatives of g are Hölder continuous of order α and $-\frac{\partial g}{\partial x_d}(x) = h(T)$, for all $x \in \Re^d$, considering perturbations of $\mathcal{D}$ of the form

$$\mathcal{D}^\epsilon = \{(s,x):\ 0 < s < T,\ x_d < (q - \epsilon\Theta')(s, x_1, \cdots, x_{d-1})\}, \quad \Theta' \in C_b^{1,2}([0,T] \times \Re^{d-1}),$$

and a point x such that $x_d < q(0, x_1, \cdots, x_{d-1})$, we have, as in (3.14),

$$\begin{aligned} u_{\mathcal{D}^\epsilon}(0,x) - u_{\mathcal{D}}(0,x) = \quad & \mathbf{E}[(g(X_T^\epsilon) - u_{\mathcal{D}}(T, X_T^\epsilon))] \\ & + \mathbf{E}[\int_0^T (\partial_s u_{\mathcal{D}} + L u_{\mathcal{D}} - f)(s, X_s^\epsilon) ds] \\ & + \mathbf{E}[\int_0^T (\nabla u_{\mathcal{D}}(s, X_s^\epsilon)\, e - h(s)) d|\Phi^\epsilon|_s] \\ = \quad & \Delta_{1,\epsilon} + \Delta_{2,\epsilon} + \Delta_{3,\epsilon}, \end{aligned}$$

where $(X^\epsilon, \Phi^\epsilon)$ is the solution of the RSDE with coefficients (4.6) and reflection along e in $\overline{\mathcal{D}^\epsilon}$, starting at $(0,x)$. Since $u_{\mathcal{D}}$ satisfies (3.11) (with $\mathbf{n}$ replaced by e and $c = \beta = 0$) it can be easily shown that $\Delta_{1,\epsilon} = O(\epsilon^{1+\alpha})$, $\Delta_{2,\epsilon} = o(\epsilon)$ and

$$\Delta_{3,\epsilon} = \mathbf{E}\left[\int_0^T \left(\nabla u_{\mathcal{D}}(s, X_s^\epsilon)\ -\ \nabla u_{\mathcal{D}}(s, X_s^\epsilon - \epsilon\,\Theta'(s, X_s^\epsilon)\, e)\right) e\ d|\Phi^\epsilon|_s\right].$$

Therefore

$$\left.\frac{du_{\mathcal{D}^\epsilon}(0,x)}{d\epsilon}\right|_{\epsilon=0} = \mathbf{E}\left[\int_0^T \Theta'(s, X_s^{0,x}) \frac{\partial^2 u_{\mathcal{D}}}{\partial x_d^2}(s, X_s^{0,x})\, d|\Phi^{0,x}|_s\right]. \tag{4.8}$$

Notice that (4.8) is consistent with the smooth fit condition, which is proved to hold in [SoSh91].

A Tusk condition: proof of Proposition 1.2

If $(t_0, x_0) \in \overline{\mathcal{D}}_T$, the result is clear. Now, consider $(t_0, x_0) \in \mathcal{SD}$ and a new (local) coordinate system centered at (t_0, x_0) (see Definition 1.1), such that $x_d > \phi(t, x_1, \cdots, x_{d-1})$ provides a local description of $\mathcal{D}$. In this coordinate system, if we put $\delta = \epsilon_0^2$, $\bar{x}_0 = (0, \cdots, -\lambda)^*$, $0 < R \leq \lambda$ (λ and R are chosen later), the tusk writes $\mathcal{T} = \{(t, x) : 0 < t < \delta, |(x_1, \cdots, x_{d-1})|^2 + (x_d + \lambda\sqrt{t})^2 < R^2 t\}$. Now take a point (t, x) in this neighborhood of $(0, 0)$ and in $\overline{\mathcal{T}} \cap \overline{\mathcal{D}}$: we aim at proving $(t, x) = (0, 0)$. On the one hand, using the tusk and $\lambda \geq R$ we obtain $x_d \leq 0$. Moreover, for $\epsilon > 0$, a Young inequality gives

$$|(x_1, \cdots, x_{d-1})|^2 \leq R^2 t - (x_d + \lambda\sqrt{t})^2 \leq R^2 t - x_d^2 - \lambda^2 t + x_d^2/\epsilon + \lambda^2 t\epsilon.$$

On the other hand, the Hölder continuity property of ϕ when $\mathcal{D} \in \mathcal{H}_1$ writes $x_d \geq -K|(x_1, \cdots, x_{d-1})| - K\sqrt{t}$. Thus,

$$x_d^2 \leq 2K^2(|(x_1, \cdots, x_{d-1})|^2 + t) \leq 2K^2(R^2 t - \lambda^2 t - x_d^2 + x_d^2/\epsilon + \lambda^2 t\epsilon + t).$$

The choice $\epsilon = 2K^2/(1 + 2K^2) < 1$ and $R = \sqrt{\frac{1-\epsilon}{2}}\lambda$ $(\leq \lambda)$ leads to a cancellation of terms with x_d^2 and it reduces to $0 \leq t(1 - \lambda^2 \frac{1-\epsilon}{2})$. For λ large enough, the last inequality can be satisfied only if $t = 0$. Then $x = 0$ easily follows using $(t, x) \in \overline{\mathcal{T}}$.

B Feynman-Kac representation

B.1 Stopped diffusion: proof of Proposition 2.1

Existence and uniqueness of a solution to the PDE (2.4). These are direct consequences of more or less classical results about PDEs in time-space domains. For this, one has to reverse the time when defining functions and domains: namely, set $\tilde{u}(t, x) = u(T - t, x)$ (and analogously for b, σ, c, f and g), denote $\tilde{L}v = \nabla v\ \tilde{b} + \frac{1}{2}\text{Tr}(Hv\tilde{\sigma}\tilde{\sigma}^*)$ and define the time-reversed time-space domain $\tilde{\mathcal{D}} = \{(t, x) : (T - t, x) \in \mathcal{D}\}$. Then, the PDE problem (2.4) is equivalent to $-\partial_t\tilde{u} + \tilde{L}\tilde{u} - \tilde{c}\tilde{u} = \tilde{f}$ in $\tilde{\mathcal{D}}$ with $\tilde{u} = \tilde{g}$ on $\mathcal{P}\tilde{\mathcal{D}}$. An application of Theorems 5.9 and 5.10 p.92 in [Lie96] ensures, under $(\mathbf{A}_\alpha)$, $\mathcal{D} \in \mathcal{H}_1$, $c \in \mathcal{H}_\alpha$, $f \in \mathcal{H}_{\underline{\alpha}}$ and $g \in C^{0,0}$, the existence and uniqueness of a strong solution $\tilde{u}$ of class $\mathcal{C}^{1,2}(\tilde{\mathcal{D}}) \cap \mathcal{C}^{0,0}(\overline{\tilde{\mathcal{D}}})$ to this PDE. Actually, what remains to be justified for an application of these theorems is the existence of local barriers at any point of the parabolic boundary. This property is implied by the tusk condition (see p.43 in [Lie96]), which holds when $\mathcal{D} \in \mathcal{H}_1$ (see Proposition 1.2).
A priori estimates on u. As a strong solution, $\tilde{u}$ is also a weak solution, to which we can apply Theorem 6.45 p.140 in [Lie96] which states that $u \in \mathcal{H}_{1+\alpha}(\mathcal{D})$.
Feynman-Kac's formula. The proof is analogous to the case of cylindrical domains (see [Fre85]): for sake of completeness, we briefly give it. Take $(t, x) \in \mathcal{D} \cup \mathcal{PD}$. If $(t, x) \in \mathcal{PD}$, the verification is clear in view of (2.12). Otherwise, we may apply Ito's formula to $u(s, X_s^{t,x})e^{-\int_t^s c(r, X_r^{t,x})dr}$ but, we have to be careful since derivatives $\partial_t u(t, x)$, $\nabla u(t, x)$ and

$Hu(t,x)$ *a priori* explode for (t,x) close to the boundary (see Theorem 5.9 p.92). However, owing to the same cited estimates, these derivatives are uniformly bounded while the distance from (t,x) to $\partial\mathcal{D}$ remains bounded from below by a positive constant, say η; hence, if we define the bounded stopping time $\tau^{t,x}(\eta) := \inf\{s \geq t : d((s, X_s^{t,x}), \partial\mathcal{D}) \leq \eta\}$ (strictly positive for η small enough), the random variables $\partial_t u(s, X_s^{t,x}), \nabla u(s, X_s^{t,x}), Hu(s, X_s^{t,x})$ are uniformly bounded for $s \leq \tau^{t,x}(\eta)$. It authorizes an application of Itô's formula, which immediately gives $\mathbf{E}(u(\tau^{t,x}(\eta), X^{t,x}_{\tau^{t,x}(\eta)})e^{-\int_t^{\tau^{t,x}(\eta)} c(r,X_r^{t,x})dr}) = u(t,x) + \mathbf{E}(\int_t^{\tau^{t,x}(\eta)} e^{-\int_t^s c(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds)$ taking account the PDE for u. Take the limit as η goes to 0: it is clear that $\tau^{t,x}(\eta)$ converges to $\tau^{t,x}$ almost surely. Then, by continuity of u, $u(\tau^{t,x}(\eta), X^{t,x}_{\tau^{t,x}(\eta)})$ converges to $u(\tau^{t,x}, X^{t,x}_{\tau^{t,x}}) = g(\tau^{t,x}, X^{t,x}_{\tau^{t,x}})$ since $u = g$ on $\mathcal{PD}$. The dominated convergence theorem completes the proof. □

B.2 Reflecting diffusion: proof of Proposition 3.7

The arguments are similar to the one for stopped diffusions. For the PDE results, we apply Theorem 5.18 p.96 in [Lie96]. The Feynman-Kac formula follows from a verification procedure based on Itô's formula. We omit further details. □

C Reflecting diffusions in time dependent domains

C.1 Proofs of Theorems 3.2 and 3.4

Our approach relies on the Skorohod problem.

Definition C.1 *Let $w \in \mathcal{C}([t,T], \Re^d)$, $w_t \in \overline{\mathcal{D}_t}$. A solution to the Skorohod problem for $(\mathcal{D}, \gamma, w)$ is a pair $(x, \lambda) \in \mathcal{C}([t,T], \Re^d) \times ([t,T], \Re_+)$ such that $x_s \in \overline{\mathcal{D}_s}$ for all $s \in [t,T]$ and*

$$x_s = w_s + \int_t^s \gamma_r(x_r)d\lambda_r, \ \forall s \in [t,T], \qquad d\lambda\left\{s \in [t,T] : x_s \in \mathcal{D}_s\right\} = 0.$$

Theorem 3.2 is an immediate consequence of the following.

Theorem C.2 *For every $w \in \mathcal{C}([t,T], \Re^d)$, $w_t \in \overline{\mathcal{D}_t}$, there exists one and only one solution to the Skorohod problem for $(\mathcal{D}, \mathbf{n}, w)$.*

Proof. Let r_0 be the radius of the uniform exterior sphere to $\mathcal{D}_s$, for all $s \in [t,T]$. For n large enough that $\sup_{t \leq s \leq r \leq T, r-s \leq 2^{-n}(T-t)} |w_r - w_s| < \frac{r_0}{2}$ and $l\left(2^{-n}(T-t)\right) < \frac{r_0}{2}$, approximate w by the step functions w^n defined by

$$w_s^n = w_{t+k2^{-n}(T-t)}, \quad \text{for } s \in [t + k2^{-n}(T-t), t + (k+1)2^{-n}(T-t)[, \quad t \leq s \leq T.$$

For $t \leq s \leq T$, define, for $s \in [t, t + 2^{-n}(T-t)[$, $x_s^n = w_t$, and, for $s \in [t + k2^{-n}(T-t), t + (k+1)2^{-n}(T-t)[$, $k \geq 1$,

$$x_s^n = \pi_{t+k2^{-n}(T-t)}\left(x^n_{t+(k-1)2^{-n}(T-t)} + w^n_{t+k2^{-n}(T-t)} - w^n_{t+(k-1)2^{-n}(T-t)}\right),$$

where $\pi_s(x)$ denotes the normal projection of x on $\overline{\mathcal{D}_s}$, and

$$\lambda_s^n = \lambda_{t+(k-1)2^{-n}(T-t)}^n + \left| x_s^n - \left(x_{t+(k-1)2^{-n}(T-t)}^n + w_{t+k2^{-n}(T-t)}^n - w_{t+(k-1)2^{-n}(T-t)}^n \right) \right|.$$

Notice that x_s^n not necessarily belongs to $\overline{\mathcal{D}_s}$ for all $s \in [t,T]$, but

$$\sup_{[t,T]} d(x_s^n, \overline{\mathcal{D}_s}) \leq l\left(2^{-n}(T-t)\right), \quad \forall s \in [t,T].$$

This and the fact that $|y - \pi_r(y)| \leq l(|r-s|)$, for any $y \in \overline{\mathcal{D}_s}$, $s, r \in [t,T]$, yield that

$$\int_{(s_1,s_2]} \left(x_r^n - x_{s_1}^n \right) \cdot \mathbf{n}_r(x_r^n)\, d\lambda_r^n$$
$$\leq \int_{(s_1,s_2]} \left(x_r^n - \pi_r(x_{s_1}^n) \right) \cdot \mathbf{n}_r(x_r^n)\, d\lambda_r^n + \left[l(2^{-n}(T-t)) + l(s_2 - s_1) \right] \left(\lambda_{s_2}^n - \lambda_{s_1}^n \right).$$

Then the same arguments used in [Sai87] and [Cost92] allow one to prove the assertion. □

Theorem 3.4 is an immediate consequence of the following.

Theorem C.3 *Assume γ is continuous and satisfies (3.6). If (x, λ) is a solution of the Skorohod problem for $(\mathcal{D}, \gamma, w)$, $w \in \mathcal{C}([t,T], \Re^d)$, $w_t \in \overline{\mathcal{D}_t}$, then*

$$\sup_{s_1 \leq r_1 \leq r_2 \leq s_2} |x_{r_2} - x_{r_1}| + \lambda_{s_2} - \lambda_{s_1} \leq \kappa(w) \left[\sup_{s_1 \leq r_1 \leq r_2 \leq s_2} |w_{r_2} - w_{r_1}| + l(s_2 - s_1) \right],$$

where κ is a function of w that depends only on $\mathcal{D}$, k_0 and the modulus of continuity of γ and is bounded on compact sets of $\mathcal{C}([t,T], \Re^d)$.

Proof. Since $|y - \pi_r(y)| \leq l(|r-s|)$, for any $y \in \overline{\mathcal{D}_s}$, $r, s \in [t,T]$,

$$\int_{(s_1,s_2]} (x_r - x_{s_1}) \cdot \gamma_r(x_r)\, d\lambda_r \leq \int_{(s_1,s_2]} (x_r - \pi_r(x_{s_1})) \cdot \gamma_r(x_r)\, d\lambda_r + l(s_2 - s_1)\left(\lambda_{s_2} - \lambda_{s_1}\right).$$

Then the proof can be carried out by the same arguments used in [Cost92]. □

C.2 Proof of Proposition 3.5

Since $\mathcal{D}$ is of class $\mathcal{H}_2$ (see Subsection 1.3) there is a function in $\mathcal{H}_2$ that coincides with F on $\{(s,x) : 0 < s < T,\ \mathbf{d}(x, \partial\mathcal{D}_s) < r_0'\}$, for some r_0'; in particular its gradient equals $\mathbf{n}_s(x)$ for $x \in \partial\mathcal{D}_s$. We approximate this function by a sequence $\{F_m\}$ of functions of class $\mathcal{C}_b^{1,2}(\Re^{d+1})$, uniformly bounded in $\mathcal{H}_2$ and convergent in $\mathcal{H}_{3/2}$-norm. Then, in particular, $\sup_m(|F_m|_\infty + |\partial_t F_m|_\infty + |\nabla F_m|_\infty + |HF_m|_\infty) < \infty$, and for m large enough, one has $\nabla F_m(s,x)\,\gamma_s(x) \geq k_0/2$, uniformly for $x \in \partial\mathcal{D}_s$, $s \in]t,T]$. Combining these facts with Ito's formula, we get

$$\Lambda_T^{t,x} \leq \frac{2}{k_0}(2|F_m|_\infty + |(\partial_t + L)F_m|_\infty (T-t) - \int_t^T \nabla F_m(s, X_s^{t,x})\sigma_s(X_s^{t,x})dW_s),$$

and the assertion follows from the properties of the exponential martingale. □

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