

Efficient schemes for the weak approximation of reflected diffusions

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In this paper, we present two new discretization schemes for reflected stochastic differential equa-

(R1) the domain D is bounded and infinitely differentiable: we denote $n(x)$ the unit inward normal vector at $x \in \partial D$.

(R2) the functions B, σ, γ are of class C^∞ on \overline{D} .

(R3) the direction γ is uniformly non tangent to ∂D , i.e.

$$\forall x \in \partial D \quad \gamma(x) \cdot n(x) \geq \rho_0 > 0.$$

Under these assumptions¹, there is an unique strong solution to (1) (see Lions and Sznitman [11], Saisho [13] and references therein). Let us denote by \mathcal{L} the second order operator defined on C^2 functions by

$$\mathcal{L}g(x) = \sum_{i=1}^d B_i(x) \partial_{x_i} g(x) + \frac{1}{2} \sum_{i,j=1}^d [\sigma(x) \sigma^*(x)]_{i,j} \partial_{x_i, x_j}^2 g(x).$$

In the following, the regularity of the law of X_t is going to be involved; to ensure it, we assume an uniform strict ellipticity condition (see Cattiaux [3] for hypoelliptic hypotheses):

Assumption (E): for all $x \in D$, one has $\sigma(x) \sigma(x)^* \geq \epsilon I_d$ where $\epsilon > 0$.

Our main objective is to discuss how to accurately evaluate $\mathbb{E}_x[f(X_T)]$, or $\mathbb{E}_x[\int_0^\infty f(X_s) ds]$ when it is well defined, using Monte Carlo simulations.

Let us now stress our attention on the evaluation of $\mathbb{E}_x[f(X_T)]$ for bounded measurable functions f . Generally speaking, the law of X_t is not explicit, hence only numerical procedures are available: we focus on an approach using discretization schemes for the equation (1), which enable to evaluate the expectation of interest using Monte Carlo simulations.

For SDEs without reflection solving simply $X_t = x + \int_0^t B(X_s) ds + \int_0^t \sigma(X_s) dW_s$, we may use an Euler scheme X^h defined, if we consider N discretization times $t_i = i h$ with $h := T/N$, by

$$(2) \quad \begin{cases} X_0^h & = x \\ X_{t_{i+1}}^h & = X_{t_i}^h + B(X_{t_i}^h) (t_{i+1} - t_i) + \sigma(X_{t_i}^h) (W_{t_{i+1}} - W_{t_i}). \end{cases}$$

This numerical procedure is easy to implement since at each step, it requires only the additional simulations of d independent Gaussian variables for the Brownian increments. This provides an order 1 scheme for the weak approximation, in the sense that the weak error converges to 0 at the rate h :

$$\mathbb{E}_x \left(f(X_T^h) \right) - \mathbb{E}_x \left(f(X_T) \right) = O(h).$$

Actually, one even has an asymptotic expansion in power of h at any order, under some conditions (for smooth functions f , see Talay *et al.* [15]; for measurable functions f and non degenerate diffusion coefficient σ , see Bally *et al.* [1]).

But, when we consider SDEs with boundary conditions, the derivation of 1-order tractable schemes is not as easy as before.

1. *SDE with killing boundary.* If we are interested in the computation of $\mathbb{E}_x[\mathbf{1}_{T < \tau} f(X_T)]$ where $\tau = \inf\{t : X_t \notin D\}$ is the first exit time from D for X , an Euler scheme X^h defined by (2) with the rough exit time $\tau^h = \inf\{t_i : X_{t_i}^h \notin D\}$ yields only an 1/2-order scheme (for the weak error) (see Gobet [8]) and additional simulations involving Brownian bridge laws are necessary to obtain an 1-order scheme (see Gobet [8] []).

¹the C^∞ condition is too strong for this result, but additional regularity will be needed later in the paper.

2. *SDE with reflection in a half-space.* For RSDE in a half-space with a constant reflecting direction $\gamma(x) = \gamma$, Lépingle [10] suggests to use a reflected Euler scheme, defined by

$$(3) \quad \begin{cases} X_0^h &= x \\ X_{t_{i+1}}^h &= X_{t_i}^h + B(X_{t_i}^h) (t_{i+1} - t_i) + \sigma(X_{t_i}^h) (k_{t_{i+1}}^h - k_{t_i}^h). \end{cases}$$

The key fact is that once obtained $X_{t_i}^h$, the simulation of $X_{t_{i+1}}^h$ is easy, using d Gaussian variables and an exponential one, all being independently drawn: the precise formulation is stated in Proposition 2.1 below. This scheme is of order 1/2 for the strong error.

3. *SDE with normal reflection in general domain.* In this setting, Constantini *et al.* [4] study an Euler scheme with projection on the boundary. If we denote by $\pi_D(x)$ the orthogonal projection of x on \overline{D} , the approximation process is defined by

$$(4) \quad \begin{cases} X_0^h &= x \\ X_{t_{i+1}}^h &= \pi_D [X_{t_i}^h + B(X_{t_i}^h) (t_{i+1} - t_i) + \sigma(X_{t_i}^h) (k_{t_{i+1}}^h - k_{t_i}^h)]. \end{cases}$$

They prove that this scheme achieves the 1/2 order convergence for the computation of $\mathbb{E}_x[f(X_T)]$ for smooth functions f with vanishing conditions on ∂D (the approximation of $\mathbb{E}_x[f(X_T) \exp(-\int_0^T c(X_s) ds - \int_0^T \lambda(X_s) dk_s) - \int_0^T h(X_t) \exp(-\int_0^t c(X_s) ds - \int_0^t \lambda(X_s) dk_s) dk_t]$ is also studied under some conditions).

Actually, $\pi_D(x)$ is uniquely defined only in a neighborhood of D and it may happen that $X_{t_i}^h + B(X_{t_i}^h) (t_{i+1} - t_i) + \sigma(X_{t_i}^h) (k_{t_{i+1}}^h - k_{t_i}^h)$ does not belong to this neighborhood. In that case, the choice of the projected point can be arbitrary made, since this event has a probability of occurrence which decreases to 0 as $\exp(-c/h)$ for some $c > 0$, and hence has no incidence on the order of convergence.

Hence, for general RSDEs, an 1/2 order scheme is available, which provides a quite slow convergence for numerical algorithms.

OBJECTIVE. Our aim is to **construct new approximation processes** X^h based on $N = T/h$ regularly spaced discretization times $(t_i)_i$, **for which the weak convergence is of order 1:**

$$\mathbb{E}_x \left(f(X_T^h) \right) - \mathbb{E}_x (f(X_T)) = O(h).$$

It is worth saying that these schemes should be easy to simulate. In this paper, we give two answers to this problem:

1. first, we suggest an Euler scheme with symmetry w.r.t. ∂D (see section 1).
2. second, we propose an Euler scheme with oblique reflection in a half-space, which locally approximates \overline{D} (see section 2).

The first procedure is a bit easier to implement, but it enables to obtain good evaluations only on quantities of the form $\mathbb{E}_x [f(X_T) \exp(-\int_0^T c(X_s) ds)]$ under some conditions. The second scheme turns to be more sophisticated and yields good approximations on more general quantities involving in particular the local time $(k_t)_{0 \leq t \leq T}$.

For sake of simplicity, we restrict in section 1 and 2 to the problem of computation of $\mathbb{E}_x[f(X_T)]$, when f is smooth: the case of measurable functions is handled in section 3, whereas the stationary problem (i.e. the evaluation of $\mathbb{E}_x [\int_0^\infty f(X_s) ds]$) is discussed in section 4. Detailed proofs of the theorems presented in this paper and additional results are given in Bossy *et al.* [2] and Gobet [].

1 Euler scheme with symmetry w.r.t. ∂D

We now recall few basic facts from differential geometry involving the functions *projection on \overline{D} parallel to γ* or *symmetry w.r.t. ∂D parallel to γ* . We follow the notation from Gobet [8].

Property 1.1. *Under **(R)**, there is a constant $R = R(D, \gamma) > 0$ (which depends only on the geometry of the domain D and on the constant ρ_0 in (R3) related to γ) such that the following properties hold.*

1. *For any $x \in \{x : d(x, \partial D) \leq R\}$, there is an unique $s \in \partial D$ and $z_1 \in \mathbb{R}$, s.t.*

$$x = s + z_1 \gamma(s).$$

The point $s := \pi_{\partial D}^\gamma(x)$ is the projection of x on ∂D parallel to γ .

The real $z_1 := F^\gamma(x)$ is the algebraic distance (related to the vector fields γ) between x and ∂D .

2. *The functions $\pi_{\partial D}^\gamma(x)$ and $F^\gamma(x)$ are of class C^∞ on the compact set $\{x : d(x, \partial D) \leq R\}$. We arbitrarily extend them in C_b^∞ on the whole space, with the conditions $F^\gamma(\cdot) > 0$ on D and $F^\gamma(\cdot) < 0$ on \overline{D}^c : hence, one has $\partial D = \{x \in \mathbb{R}^d : F^\gamma(x) = 0\}$.*

3. *The projection of x on \overline{D} parallel to γ is defined on $\{x : d(x, D) \leq R\}$ by:*

$$\pi_D^\gamma(x) = x - (F^\gamma(x))^- \gamma(\pi_{\partial D}^\gamma(x)).$$

4. *The symmetric of x w.r.t. ∂D parallel to γ is defined on $\{x : d(x, \partial D) \leq R\}$ by:*

$$\text{Sym}_{\partial D}^\gamma(x) = \pi_{\partial D}^\gamma(x) - F^\gamma(x) \gamma(\pi_{\partial D}^\gamma(x)) = x - 2F^\gamma(x) \gamma(\pi_{\partial D}^\gamma(x)).$$

5. *We denote by $\text{Sym}_D^\gamma(x)$ the function which is equal to x if $x \in D$, and $\text{Sym}_{\partial D}^\gamma(x)$ if $x \notin D$ and $d(x, \partial D) \leq R$, i.e.*

$$\text{Sym}_D^\gamma(x) = x - 2(F^\gamma(x))^- \gamma(\pi_{\partial D}^\gamma(x)).$$

Definition 1.1. **Euler scheme \overline{X}^h with symmetry w.r.t. ∂D .**

It is defined by:

$$(5) \quad \begin{cases} \overline{X}_0^h &= x, \\ \overline{X}_{t_{i+1}}^h &= \text{Sym}_D^\gamma \left[\overline{X}_{t_i}^h + B(\overline{X}_{t_i}^h) (t_{i+1} - t_i) + \sigma(\overline{X}_{t_i}^h) (\quad_{t_{i+1}} - \quad_{t_i}) \right]. \end{cases}$$

Its simulation is straightforward since it only requires realizations of the Brownian increments.

Actually, $\overline{X}_{t_i}^h + B(\overline{X}_{t_i}^h) (t_{i+1} - t_i) + \sigma(\overline{X}_{t_i}^h) (\quad_{t_{i+1}} - \quad_{t_i})$ may not belong to $\{x : d(x, D) \leq R\}$, the set of definition of Sym_D^γ : in that case, we shall take $\overline{X}_{t_{i+1}}^h = \overline{X}_{t_i}^h$ e.g., this choice having anyhow no incidence on theoretical and numerical convergences. Note that this scheme enables to approximately simulate quantities such as $f(X_T)$ (or $f(X_T) \exp(-\int_0^T c(X_t) dt)$) but not $\int_0^T h(X_t) dk_t$.

We now state the main result concerning the analysis of the weak error for \overline{X}^h .

Theorem 1.1. *Assume **(R)**, **(E)** and that f is a $C_b^{4+\alpha}(\overline{D}, \mathbb{R})$ function (for some $\alpha \in (0, 1)$) satisfying $\gamma \cdot \nabla f|_{\partial D} = \gamma \cdot \nabla(\mathcal{L}f)|_{\partial D}$.*

If any case, the convergence is at least of order 1/2 for \overline{X}^h :

$$(6) \quad \mathbb{E}_x \left(f(\overline{X}_T^h) \right) - \mathbb{E}_x (f(X_T)) = O \left(h^{1/2} \right).$$

But if γ is the co-normal vector (i.e. $\gamma(s) \parallel [\sigma\sigma^*](s)n(s)$ for all $s \in \partial D$), \overline{X}^h achieves the 1-order convergence:

$$() \quad \mathbb{E}_x \left(f(\overline{X}_T^h) \right) - \mathbb{E}_x (f(X_T)) = O(h).$$

We shall briefly comments the above result.

- 1) First, the fact that the co-normal vector plays a key role in this setting is not surprising, since in the PDE's theory, this is also a privileged vector for the reflecting direction. Indeed, if we consider the function $u(t, x) = \mathbb{E}_x (f(X_{T-t}))$ as the solution of the second-order parabolic PDE with a Neumann condition

$$\begin{cases} \partial_t u + \mathcal{L} u = 0 & (t, x) \in [0, T) \times D \\ u(T, x) = f(x) & x \in D \\ \gamma(x) \cdot \nabla u(t, x) |_{\partial D} = 0, \end{cases}$$

it is well known that the analysis of the above PDE is much easier when the co-normal vector coincides with the reflecting direction (see Freidlin [6] e.g.).

- 2) Second, we can easily understand why this procedure with a symmetry may work better than those with the projection from Constantini [4]. Indeed, consider the case of a normally reflected ($\gamma = n$) Brownian Motion in $D = \mathbb{R}^+$ starting from, say, 0: this process is equal to

$$\left(B_t - \inf_{s \in [0, t]} B_s \right)_{t \geq 0} \stackrel{law}{=} (|B_t|)_{t \geq 0} \stackrel{def}{=} (\text{Sym}_D^n(B_t))_{t \geq 0},$$

the equality in law being derived from Lévy's Theorem (see Revuz and Yor [12]).

ELEMENTS OF PROOFS OF THEOREM 1.1 (for a complete proof, see Bossy *et al.* [2]). The assumptions on f ensure that derivatives of u up to some order are uniformly bounded: indeed, one has $u \in C^{2+\alpha/2, 4+\alpha}([0, T], \overline{D})$ (see Ladyzenskaja *et al.* [9]). The weak error can be decomposed by writing

$$\mathbb{E}_x \left(f(\overline{X}_T^h) \right) - \mathbb{E}_x (f(X_T)) = \sum_{i=0}^{N-1} \mathbb{E}_x \left(u(t_{i+1}, \overline{X}_{t_{i+1}}^h) - u(t_i, \overline{X}_{t_i}^h) \right).$$

We analyze each difference using Itô's formula between t_i and t_{i+1} , the key fact being the identification of the semimartingale decomposition of $\text{Sym}_D^\gamma(Y_t)$ where $Y_t = \overline{X}_{t_i}^h + B(\overline{X}_{t_i}^h)(t-t_i) + \sigma(\overline{X}_{t_i}^h)(t-t_i)$. For this, we adapt arguments from Gobet [8] Proposition 3.1, to derive

$$d(\text{Sym}_D^\gamma(Y_t)) = \mathbf{1}_{Y_t \in D} dY_t + \mathbf{1}_{Y_t \notin D} dY_t^{\partial D} + \gamma(Y_t) dL_t^0(F^\gamma(Y)),$$

where $Y_t^{\partial D}$ is an other Itô process and $L_t^0(F^\gamma(Y))$ is the 1-dimensional local time of the continuous semimartingale $F^\gamma(Y)$ at time t and level 0. The estimate (6) now follows quite easily. But the interesting estimate () is much trickier to obtain, we refer to Bossy *et al.* [2] for the details.

2 Euler scheme locally reflected in half-space approximation of D

2.1 Exact simulation in a half-space

We first recall the useful result from Lépingle [10], which enables to exactly simulate RSDEs in a half-space when the coefficients B , σ , γ are constant. To state a precise formulation, we define $D = \left\{ z \in \mathbb{R}^d : (z - y) \cdot n > 0 \right\}$ and consider the solution of $Y_t = x + B t + \sigma \int_t^x + \gamma k_t$. Here, k_t is explicitly given by:

$$k_t = \frac{1}{n \cdot \gamma} \max \left(0, \sup_{0 \leq s \leq t} -(x + B s + \sigma \int_t^s - y) \cdot n \right).$$

We can simulate Y_t owing the

Proposition 2.1. *Set $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$. If $U \stackrel{law}{=} \mathcal{N}(0, t I_d)$ and $V \stackrel{law}{=} \mathcal{E}(1/2t)$ independent of U , one has*

$$\left(\int_t^x, \sup_{0 \leq s \leq t} (a \cdot s + c s) \right) \stackrel{law}{=} \left(U, \frac{1}{2} \left[a \cdot U + ct + \sqrt{|a|^2 V + (a \cdot U + ct)^2} \right] \right).$$

2.2 Construction of $(\tilde{X}^h, \tilde{k}^h)$, using an Euler scheme with oblique reflection in a half-space approximation of D

To describe the general procedure, we need to introduce a new uniformly non tangent vector field γ' , to which we associate the constant R defined in Property 1.1: the appropriate choice of γ' is discussed in Theorem 2.1 below.

Set $\tilde{X}_0^h = x$ and $\tilde{k}_0^h = 0$. We assume that $z := \tilde{X}_{t_i}^h \in \overline{D}$ and $\tilde{k}_{t_i}^h$ are defined and we now construct $\tilde{X}_{t_{i+1}}^h$ and $\tilde{k}_{t_{i+1}}^h$.

- a) z is far from ∂D . If $d(z, \partial D) \geq R$, we set $\tilde{X}_{t_{i+1}}^h = z + B(z) (t_{i+1} - t_i) + \sigma(z) \left(\int_{t_i}^{t_{i+1}} - \int_{t_i}^{t_i} \right)$ and $\tilde{k}_{t_{i+1}}^h = \tilde{k}_{t_i}^h$: in other words, we consider that there is no reflection between t_i and t_{i+1} , which is false only with an exponentially small probability w.r.t. $1/h$. If $\tilde{X}_{t_{i+1}}^h \notin D$ (which also occurs with a negligible probability), replace $\tilde{X}_{t_{i+1}}^h$ by its projection on \overline{D} .
- b) z is not far from ∂D , i.e. $d(z, \partial D) < R$.

b1) We set $s = \pi_{\partial D}^{\gamma'}(z)$ for the projection of z on ∂D parallel to γ' .

b2) Let $D_s = \left\{ z \in \mathbb{R}^d : (z - s) \cdot n(s) > 0 \right\}$ the half-space delimited by the tangent plane to ∂D , at s .

b3) Let $(Y_t)_{t_i \leq t \leq t_{i+1}}$ be the RSDE in D_s defined by

$$Y_t = z + B(z) (t - t_i) + \sigma(z) \left(\int_{t_i}^t - \int_{t_i}^{t_i} \right) + \gamma(s) (\tilde{k}_t^h - \tilde{k}_{t_i}^h).$$

Note that $(Y_{t_{i+1}}^h, \tilde{k}_{t_{i+1}}^h - \tilde{k}_{t_i}^h)$ can be simulated using Proposition 2.1.

b4) To obtain $\tilde{X}_{t_{i+1}}^h$, project $Y_{t_{i+1}}^h$ on D parallel to γ : $\tilde{X}_{t_{i+1}}^h = \pi_D^\gamma(Y_{t_{i+1}}^h)$. Actually, most of the time, one has $Y_{t_{i+1}}^h \in D$ and the projection is obvious.

If we are interested (as in Constantini *et al.* [4]) in the approximation of $f(X_T) \exp(-Z_T) - \int_0^T h(X_t) \exp(-Z_t) dk_t$ with $Z_t = \int_0^t c(X_s) ds + \int_0^t \lambda(X_s) dk_s$, we may use standard discretizations of the integral, which we can simulate since one has obtained realizations of $(\tilde{X}_{t_{i+1}}^h, \tilde{k}_{t_{i+1}}^h - \tilde{k}_{t_i}^h)_{0 \leq i \leq N-1}$.

We now give the weak error for the computation of $\mathbb{E}_x(f(X_T))$.

Theorem 2.1. *Assume (R), (E) and that f satisfies the same assumptions as in Theorem 1.1. For any uniformly non tangent vector field γ' , the weak convergence for \tilde{X}^h holds with order at least equal to 1/2:*

$$(8) \quad \mathbb{E} \left(f(\tilde{X}_T^h) \right) - \mathbb{E}(f(X_T)) = O \left(h^{1/2} \right).$$

But if γ is the co-normal vector, the choice $\gamma'(\cdot) = \gamma(\cdot)$ leads to the 1-order convergence:

$$(9) \quad \mathbb{E}_x \left(f(\tilde{X}_T^h) \right) - \mathbb{E}_x (f(X_T)) = O(h).$$

When the reflecting direction is not the co-normal one, we may use for simplicity $\gamma'(\cdot) = n(\cdot)$.

ELEMENTS OF PROOF (for a complete proof, see Gobet []). As before, we used the PDE solved by $u(t, x)$ and this leads, after some tricky computations, to the general estimate (8). Actually, the term of order $h^{1/2}$ can be interpreted as an integral on the boundary, involving the explicit transition density function $p_t(x, y)$ of Y_t defined in b3). To remove this term and achieve the order 1, we prove that in the case of co-normal vector with $\gamma' = \gamma$, the function $p_t(x, y) |_{y \in \partial D}$ has some nice symmetry properties w.r.t. y . For example, if $D = \{y \in \mathbb{R}^d : y_1 > 0\}$ and $Y_t = x + \sigma \cdot t + \gamma k_t$ is reflected in D , we prove that the function $(y_2, \dots, y_d) \rightarrow p_t(x, \pi_{\partial D}^\gamma(x) + (0, y_2, \dots, y_d)^*)$ is an even function in each y_i .

3 Extension of the results when f is measurable

Theorem 3.1. *The theorems 1.1 and 2.1 are still valid if f is a bounded measurable function, satisfying $d(\text{Supp}(f), \partial D) > 0$.*

ELEMENTS OF PROOF. Formally, the approach using the PDE solved by u remains the same; actually, the main point consists in deriving time uniform control on quantities such as $\mathbb{E}_x(\partial^k u(t_i, X_{t_i}^h))$, even if derivatives of u for t_i closed to T may explode for irregular functions f .

Following Bally *et al.* [1], we used Malliavin calculus techniques to transform the above expectation using an integration by parts formula. It seems to be especially difficult in our setting because we deal with piecewise RSDs: the assumption on the support of f however enables to develop a quite easy approach. Indeed, this support condition ensures that derivatives of u are bounded near ∂D , so that we only need to apply Malliavin Calculus to the law of X^h restricted to the interior of D . But strictly inside D , it behaves like standard Euler scheme without reflection, so that the classical integration by parts formula can apply. This kind of arguments are used in Gobet [8] to deal with killed diffusions.

4 The stationary problems

In this section, we present a numerical procedure for the computation of

$$u(x) = \mathbb{E}_x \left[\int_0^\infty f(X_t) dt \right],$$

for $x \in \overline{D}$, using Monte Carlo simulations, where X is the RSDE defined (1). For the details, we refer to Bossy *et al.* [2].

Under **(R)** and **(E)**, $(X_t)_{t \geq 0}$ is ergodic: we denote by μ its invariant probability measure and we set $\mu(f) = \int_D f(x) \mu(dx)$ whenever this quantity is finite. Since the Doeblin's condition is satisfied, one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \mu(f) \quad \text{m)} \quad \text{ptio i th t qnè M o eo ve , o e f}$$

5 Conclusion

We have proposed two new implementable schemes for the weak approximation of $(X_t)_{t \geq 0}$, a RSDE with oblique reflection. The first one is built using an Euler scheme on which we apply a symmetry procedure at the boundary: it is convenient if we are interested in the simulation of $f(X_T)$ e.g.. The second one consists in locally approximating the domain in a half space, in which a reflected Euler scheme is easy to simulate: this leads to the evaluation of X_T but also of its local time k_t .

Both schemes give an 1-order convergence for the computation of $\mathbb{E}_x(f(X_T))$ when the reflecting direction is the co-normal one. Anyhow, preliminary numerical experiments illustrate that they work better than the usual projected Euler scheme (see Constantini *et al.* [4]). On the figure 5.1, we compare the projected Euler scheme and the Euler scheme with symmetry, in the case of a 2-dimensional Brownian motion ($x = 0$) reflected in the unit sphere : it is clear that the weak convergence is much faster for the Euler scheme with symmetry. Analogous results are available for the scheme using a half-space approximation.

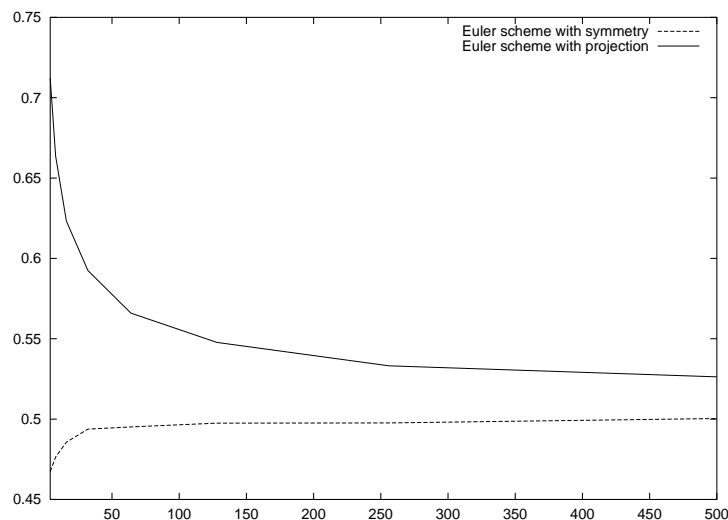


FIG. 5.1: evaluation of $\mathbb{E}_x \|X_1^h\|^2$ w.r.t. the number of discretization times $N = T/h$.

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