Exercise Regions of American Options on several assets

Stephane Villeneuve
Equipe d'Analyse et de Mathématiques appliquées
Université de Marne-la-Vallée
2 rue de la Butte Verte
93166 Noisy-le-Grand Cedex

May 1998

Keywords: Optimal stopping, Free boundary problems, American options.

JEL CLASSIFICATION: *G13*
AMS CLASSIFICATION: *90A09, 93E20*

Abstract

In this paper, we study the nonemptiness and the shape of the exercise region of American options written on several assets. Our contribution is threefold. First, we state an analytic theorem which characterizes the nonemptiness of the exercise region. Second, we study a particular class of payoff functions for which we explicitly identify the shape and the asymptotic behavior near maturity of the associated exercise region. Finally, we present additional results which complement the Broadie and Detemple results concerning the valuation of various types of American options on several assets.

Introduction

The study of American options written on several assets, also called Rainbow options by practitioners, is doubly motivated. On the one hand, many contracts that are traded in financial markets involve such options (index options or exchange options). On the other hand, American Rainbow options contribute to enlarge the derivatives supply on the Over the Counter market. From a theoretical point of view, Bensoussan [3] and Karatzas [10] established the connections between American options and optimal stopping and the variational inequalities techniques of Bensoussan-Lions [2] were applied to American option pricing by Jaillet-Lamberton-Lapeyre [9]. However, we had to wait until the paper of Broadie and Detemple (1996) [4] appeared to realize how important it is to identify the exercise region (i.e the set of coincidence between the option's value and its intrinsic value) in order to have a better understanding of these contracts.
In the last few years, there has been much progress in the study of the exercise region of American options written on a single asset (see Kim [13], Jacka [8], Barles et al. [1] and Myneni [17]). However, our perception of the structure of the exercise region in the multidimensional case remains vague and conjectures based on the knowledge of the one dimensional case turn out to be false (see [4]). The most striking examples are given by the call on the maximum or

the minimum of two assets.
Hence, our goal is to clarify the results concerning the exercise region of American Rainbow options. In the first section, we state a very general analytic theorem which characterizes the nonemptiness of the exercise region thanks to the differential operator associated to the diffusion model. In the second section, we study a particular class of payoff functions that are traded in financial markets. For this class, we define the notion of critical surface for which we can extend some results proved for the free boundary in the one dimensional case. Finally, we present additional results which complement the Broadie and Detemple results concerning the valuation of various types of American options on several assets.
In order to make our results more readable, we list below the various types of options that are treated in the paper with references to the relevant statements:
- American call on the minimum of two assets (see section 2.2)
- American spread option and index option (see section 3)
- American call on the maximum and American put on the minimum of two assets, finite and perpetual case (section 4).

1 American options on several assets

1.1 The model

We consider American options written on n underlying assets. In the multidimensional Black-Scholes setting, the logarithm of the stock prices satisfies the following stochastic differential equation

$$(1) \qquad dX_t^i = (r - \delta_i - \frac{1}{2}\sum_{j=1}^n \sigma_{ij}^2)dt + \sum_{j=1}^n \sigma_{ij} dW_t^j \quad i = 1, \dots, n$$

where, under the so-called risk neutral probability measure which will be denoted by P, $W = \{W_t = (W_t^1, .., W_t^n), \mathcal{F}_t, 0 \le t \le T\}$ is a standard n-dimensional Brownian motion. We denote by $(\mathcal{F}_t)_{0 \le t \le T}$ the augmented filtration of $\mathcal{F}_t^W = \sigma(W_s; s \le t)$. The nonnegative constant r is the interest rate, the nonnegative constant δ_i is the dividend rate of the asset i.
We assume that the matrix $\Sigma = (\sigma_{ij})_{1 \le i,j \le n}$ is invertible.
The value of an American option with date of maturity T, defined by an adapted continuous process $(h(t))_{0 \le t \le T}$ satisfying $E(\sup_{0 \le t \le T} h(t)) < \infty$, where $h(t)$ is the payoff of the option when exercise occurs at time t, is given by:

$$(2) \qquad V_t = e^{rt} \text{ess} \sup_{\tau \in \mathcal{T}_{t,T}} E(e^{-r\tau} h(\tau) \mid \mathcal{F}_t)$$

where $\mathcal{T}_{t,T}$ is the set of all $\mathcal{F}_t$-stopping times with values in the interval $[t, T]$. Recall that the discounted value of an American option $e^{-rt}V_t$ is the Snell envelope of the discounted payoff process, namely the smallest supermartingale which dominates $e^{-rt}h(t)$. We refer to Karatzas [10], [11] and Myneni [17] for the basics of the modern theory of American option. We will restrict our study to payoff processes given by $h(t) = \psi(X_t)$ where ψ is a continuous nonnegative function satisfying the following assumption:

$$\textbf{(H1)} \quad \exists M \ge 0 \quad \forall x \in R^n \quad \psi(x) + \sum_{i=1}^n \mid \frac{\partial \psi}{\partial x_i} \mid \le M e^{M\|x\|}$$

where $\| \, . \, \|$ denotes the Euclidean norm in R^n.
We define for $0 \leq s \leq t \leq T$ and $x = (x_1, \dots, x_n) \in R^n$,

$$(3) \qquad (X_t^{s,x})_i = x_i + (r - \delta_i - \frac{1}{2} \sum_{j=1}^{n} \sigma_{ij}^2)(t-s) + \sum_{j=1}^{n} \sigma_{ij}(W_t^j - W_s^j).$$

Note that $(X_t^{s,x})_i$ is a continuous version of the flow of equation (1). Due to the Markovian properties of the model, it is well-known that the process V_t is given by a function $C(t, X_t)$ where

$$\begin{aligned} C(t,x) &= \sup_{\tau \in \mathcal{T}_{0,T-t}} E(e^{-r\tau} \psi(X_\tau^{0,x})) \\ &= E(e^{-r\tau^*} \psi(X_{\tau^*}^{0,x})) \end{aligned}$$

where $\tau^* = \inf\{u \in [0, T-t] \ | \ C(t+u, X_u^{0,x}) = \psi(X_u^{0,x})\}$.
Recall that τ^* is the smallest optimal stopping time (cf [10]). We deduce easily the following properties of the value function C.

i) $C(t,x) \geq \psi(x)$ on $[0, T[\times R^n$.

ii) $\forall x \in R^n$, $C(T,x) = \psi(x)$.

iii) $\forall x \in R^n$, $C(.,x)$ is nonincreasing.

iv) C is continuous on $[0,T] \times R^n$.

1.2 Exercise region

We introduce the following set:

$$\mathcal{E} = \{(t,x) \in [0, T[\times R^n \ | \ C(t,x) = \psi(x)\}.$$

Clearly, it is never optimal to exercise prior to maturity out of $\mathcal{E}$ where the payoff due to the option's sale is greater than the one due to the exercise. Moreover, the smallest optimal stopping times τ^* satisfies

$$\tau^* = \inf\{u \geq 0 \ | \ (t+u, X_u) \in \mathcal{E}\} \wedge (T-t).$$

Definition 1.1 *The coincidence set $\mathcal{E}$ is called the exercise region of the American option.*

Define the t-sections for every $t \in [0, T[$ by

$$\mathcal{E}_t = \{x \in R^n \ | \ C(t,x) = \psi(x)\}.$$

Clearly, we have $\mathcal{E} = \bigcup_{t<T} \{t\} \times \mathcal{E}_t$.

Proposition 1.1 *Assume ψ is a nonzero function.*

i) $\mathcal{E}$ is closed in $[0, T[\times R^n$.

ii) The family $(\mathcal{E}_t)_{0\leq t\leq T}$ is nondecreasing.

iii) $\forall t \in [0,T[\ \ \mathcal{E}_t \subset \mathcal{O} = \{x \in R^n \mid \psi(x) > 0\}$.

Proof:
i) C and ψ are continuous, we have $\mathcal{E} = (C-\psi)^{-1}(0)$.
ii) Let $s < t$ and $x \in \mathcal{E}_s$, we have

$$\begin{aligned} C(t,x) &\leq C(s,x) \\ &= \psi(x). \end{aligned}$$

Thus, $C(t,x) = \psi(x)$.
iii) Using the definition, we have

$$\forall (t,x) \in [0,T[\times R^n\ C(t,x) \geq E(e^{-r(T-t)}\psi(X^{0,x}_{T-t})).$$

Since the volatility matrix Σ is invertible, the support of the distribution of $X^{0,x}_{T-t}$ is R^n. Therefore, $X^{0,x}_{T-t}$ hits every nonempty open subset with positive probability and in particular $\mathcal{O}$, so that

$$\forall (t,x) \in [0,T[\times R^n, \quad C(t,x) > 0.$$

If $x \in \mathcal{E}_t$ then $C(t,x) = \psi(x) > 0$ thus $x \in \mathcal{O}$. •

1.3 Variational inequalities and American options

We recall in this section the results implicitly shown in Jaillet-Lamberton-Lapeyre [9] because they will be useful hereafter to characterize the exercise region.
Introduce the operator $\mathcal{A}$ defined by

$$\mathcal{A}F = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{n}(r - \delta_i - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2)\frac{\partial F}{\partial x_i} - rF$$

where the matrix $A = (a_{ij})_{1\leq i,j\leq n}$ equals $\Sigma^*\Sigma$.

Proposition 1.2 *Under assumption (H1), the price function C satisfies.*
$\dfrac{\partial C}{\partial t} + \mathcal{A}C \leq 0$ *in the sense of distributions in the open set* $]0,T[\times R^n$.
$\dfrac{\partial C}{\partial t} + \mathcal{A}C = 0$ *in the open set* $\{(t,x) \in]0,T[\times R^n \mid C > \psi\}$.

2 Characterization of the early optimal exercise

2.1 General results

The purpose of this section is to prove a theorem which gives an analytic criterion for the nonemptiness of the exercise region. This criterion is easy to apply to a usual payoff function, in particular we recover the Merton's well-known result which states that the exercise region of the American call on one nondividend-paying asset is empty. Before stating the main theorem, we establish a preliminary result.

Proposition 2.1 *We have the following inequalities in the sense of distributions.*

1) If a t-section $\mathcal{E}_t$ has a nonempty interior then $\mathcal{A}\psi \le 0$ in the open set $\bigcup_{t<T} \overset{\circ}{\mathcal{E}}_t$.

2) $\mathcal{A}\psi \ge 0$ in the open set $\left(\overline{\bigcup_{t<T} \mathcal{E}_t}\right)^c$.

Proof:
1) Assume there exists $t \in [0, T[$ such that $\overset{\circ}{\mathcal{E}}_t$ is nonempty. In the open set $]t, T[\times \overset{\circ}{\mathcal{E}}_t$, we have $C = \psi$ and $\frac{\partial C}{\partial t} + \mathcal{A}C \le 0$ therefore $\mathcal{A}\psi \le 0$ on $\overset{\circ}{\mathcal{E}}_t$.
2) Let $\mathcal{E}_T = \overline{\bigcup_{t<T} \mathcal{E}_t}$ and $\Lambda = \mathcal{E}_T^c$. In the open set $]0, T[\times\Lambda, C > \psi$ and therefore $\frac{\partial C}{\partial t} + \mathcal{A}C = 0$ thanks to proposition 1.2. As the function $C(., x)$ is nonincreasing, the distribution $\frac{\partial C}{\partial t}$ is a negative measure and thus

$$\mathcal{A}C \ge 0 \quad \text{on }]0, T[\times\Lambda.$$

For a fixed time t, we define the distribution $\mathcal{A}C(t, .)$ on Λ by

$$\forall \theta \in \mathcal{C}_c^\infty(\Lambda) \;\; < \mathcal{A}C(t, .), \theta >= \int_\Lambda C(t, x)\mathcal{A}^*\theta(x)\, dx.$$

where the operator $\mathcal{A}^*$ is defined by

$$\mathcal{A}^*F = -\sum_{i=1}^{n}(r - \delta_i - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2)\frac{\partial F}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} - rF.$$

and $\mathcal{C}_c^\infty(\Lambda)$ denotes the set of $\mathcal{C}^\infty$ function with compact support in Λ. The distribution $\mathcal{A}C(t, .)$ is positive on Λ for every $t \in [0, T[$. Since C is continuous, as t tends to T, $< \mathcal{A}C(t, .), \theta >$ tends to $< \mathcal{A}C(T, .), \theta >=< \mathcal{A}\psi, \theta >$ for every $\theta \in \mathcal{C}_c^\infty(\Lambda)$ by the dominated convergence theorem. Hence, $\mathcal{A}\psi \ge 0$.•
We are now in a position to state a theorem which characterizes the nonemptiness of the exercise region. We will need the following lemma:

Lemma 2.1 *If $\mathcal{A}\psi \ge 0$ (resp $= 0$) on R^n then the process $(e^{-rt}\psi(X_t))$ is a submartingale (resp. martingale).*

Proof of the lemma: We only show the case $\mathcal{A}\psi \ge 0$, since the case $\mathcal{A}\psi = 0$ is a straightforward application of Ito's lemma. When, $\mathcal{A}\psi \ge 0$, the only difficulty comes from the possible lack of regularity of ψ.
Introduce a mollifier sequence $(\rho_j)_{j\in N}$:

- $\rho_j \in \mathcal{C}_c^\infty(R^n)$ and $\rho_j \ge 0$ for every j.
- supp$\rho_j \subset B(0, \frac{1}{j})$ where $B(0, \frac{1}{j})$ stands for the ball of radius $\frac{1}{j}$ centred on 0.
- $\int_{R^n} \rho_j(x)\, dx = 1$.

$\psi_j = \rho_j * \psi$ is a $\mathcal{C}^\infty$ function which converges to ψ uniformly on every compact subset of R^n. Moreover, $\mathcal{A}\psi_j = \rho_j * \mathcal{A}\psi$ is a positive function on R^n.
Indeed, for every $\phi \in \mathcal{C}_c^\infty(R^n)$, we have since $\mathcal{A}$ has constant coefficients

$$\begin{aligned} < \mathcal{A}\psi_j, \phi > &= < \mathcal{A}\psi, \check{\rho}_j * \phi > \quad \text{where } \check{\rho}_j(x) = \rho_j(-x) \\ &\geq 0. \end{aligned}$$

Ito's formula yields for $t \geq s$

$$\begin{aligned} E\left(e^{-rt}\psi_j(X_t) \mid \mathcal{F}_s\right) &= e^{-rs}\psi_j(X_s) + E\left(\int_s^t e^{-ru}\mathcal{A}\psi_j(X_u)\,du \mid \mathcal{F}_s\right) \\ &\geq e^{-rs}\psi_j(X_s). \end{aligned}$$

The righthand side tends to $e^{-rs}\psi(X_s)$ when j tends to $+\infty$. It remains to justify the convergence of the lefthand side. We have

$$\begin{aligned} |e^{-rt}\psi_j(X_t)| &\leq \int_{R^n} \rho_j(y)|\psi(X_t - y)|\,dy. \\ &\leq M \int_{B(0,\frac{1}{j})} \rho_j(y) e^{M(|X_t| + \frac{1}{j})}\,dy. \\ &\leq M' e^{M(|X_t|)}. \end{aligned}$$

We apply the dominated convergence theorem to conclude. •
We are now in a position to state the main result.

Theorem 2.1 *The exercise region is empty if and only if $\mathcal{A}\psi$ is a nonzero positive measure on R^n.*

Proof:
a)necessary condition.
If the exercise region is empty, then we have $\mathcal{A}\psi \geq 0$ according to proposition 2.1.
Assume $\mathcal{A}\psi = 0$ on R^n, then for every $x \in R^n$ the process $\left(e^{-rt}\psi(X_t^{0,x})\right)$ is a martingale thanks to lemma 2.1. Therefore, using the definition of the Snell envelope and the optional sampling theorem, we have $C(t,x) = \psi(x)$ for every $(t,x) \in [0,T] \times R^n$, which contradicts the emptyness of $\mathcal{E}$.
b)sufficient condition.
Assume $\mathcal{A}\psi$ is a nonzero positive measure on R^n and $\mathcal{E}_t \neq \emptyset$ for some $t \in [0,T[$.
Let $x \in \mathcal{E}_t$, we have

$$C(t,x) = \psi(x) \geq E\left(e^{-r\tau}\psi(X_\tau^{0,x})\right) \text{ for every } \tau \in \mathcal{T}_{0,T-t}.$$

On the other hand, using lemma 2.1 and the optional sampling theorem we obtain

$$\psi(x) \leq E\left(e^{-r\tau}\psi(X_\tau^{0,x})\right) \text{for every } \tau \in \mathcal{T}_{0,T-t}$$

therefore, the process $e^{-ru}\psi(X_u^{0,x})$ is a martingale. Using the characterization of the Snell envelope as the minimal supermartingale which dominates the reward process, we have for every $u \in [0, T-t]$,

$$e^{-ru}C(u, X_u^{0,x}) = e^{-ru}\psi(X_u^{0,x})\; a.s.$$

Thus, using the continuity of $C(u,.)$ and ψ, we have $C(u,y)=\psi(y)$ on $]0,T-t[\times R^n$. Thanks to proposition 2.1, $\mathcal{A}\psi \leq 0$ on R^n hence $\mathcal{A}\psi = 0$ on R^n, which is a contradiction.•

Remark 2.1 *A referee wonder whether this theorem may be generalized to the case where r is a deterministic function of time. I am afraid not without supplementary assumption because the distribution* $\dfrac{\partial C}{\partial t}$ *may fail to be negative.*

Theorem 2.1 supplies a nice corollary for bounded payoff function.

Corollary 2.1 *Assume* $r>0$.
The exercise region of an American option written on a bounded payoff ψ *is nonempty.*

Proof: According to lemma 2.1, if the exercise region is empty then the process $e^{-rt}\psi(X_t^{0,x})$ is a submartingale for every $x \in R^n$. Hence,

$$\forall\, x \in R^n \quad \psi(x) \leq E e^{-rT}\psi(X_T^{0,x}) \leq \parallel \psi \parallel_\infty e^{-rT}$$

therefore, $\parallel \psi \parallel_\infty \leq \parallel \psi \parallel_\infty e^{-rT}$, which is a contradiction.•

Remark 2.2 *This corollary is of interest if* ψ *does not reach its bounds. Otherwise, we have, for every* x_0 *such that* $\psi(x_0) = \parallel \psi \parallel_\infty$, $\psi(x_0) = C(t,x_0)$.

Remark 2.3 *As a referee pointed out, the American digital option defined by the indicator function of Borel set with nonzero Lebesgue measure, say* $\mathbf{1}_A$, *provides a nice example. By applying the previous remark, it is straightforward to show that the exercise region of the digital option coincides with A.*

Remark 2.4 *We recover the Merton's result on the exercise region of a call option on one nondividend-paying asset. Indeed, an easy computation yields*

$$\mathcal{A}\psi = rK\mathbf{1}_{\{e^x > K\}} + \frac{\sigma^2 K^2}{2}\delta_{\log K}$$

where δ_K *stands for the Dirac measure. It is clear that* $\mathcal{A}\psi$ *is a nonzero positive measure on* R^n.

We characterized the nonemptiness of the exercise region thanks to the distribution $\mathcal{A}\psi$ on R^n. We would like a local characterization (i.e does the distribution $\mathcal{A}\psi$ on some open subset U of R^n inform us on the early exercise on U?). Under some regularity conditions, we state a local version of theorem 2.1.

Proposition 2.2 *Assume that the distribution* $\mathcal{A}\psi$ *is a positive measure on a connected open subset U of* R^n, *that* ψ *is a* $\mathcal{C}^2$ *function on an open subset V of U and* $\mathcal{A}\psi(x) > 0$ *for all* $x \in V$. *Then for every* $(t,x) \in]0,T[\times U, \quad C(t,x) > \psi(x)$.

Proof: Let $\psi_j = \rho_j * \psi$ where $(\rho_j)_{j\in N}$ is a mollifier sequence.
Let $x \in U$ and U_1 an open connected subset of U containing x such that $V \subset U_1 \subset \overline{U_1} \subset U$.
For j large enough, we have $\mathcal{A}\psi_j \geq 0$ on U_1. Indeed, for every $\phi \in \mathcal{C}_c^\infty(U_1)$, we have

$$\begin{aligned} < \mathcal{A}\psi_j, \phi > &= < \rho_j * \mathcal{A}\psi, \phi > \\ &= < \mathcal{A}\psi, \check{\rho}_j * \phi > . \end{aligned}$$

We know that $\text{supp}(\check{\rho}_j * \phi) \subset \text{supp}\check{\rho}_j + \text{supp}\phi \subset \text{supp}\check{\rho}_j + U_1 \subset U$ for j great enough.
We introduce the following stopping time,

$$\tau_1 = \inf\{t \geq 0; X_t^x \notin U_1\}$$

Applying Ito's formula to the process $e^{-rt}\psi_j(X_t^{0,x})$, we have

$$\begin{aligned} E\left[e^{-r(\tau_1\wedge(T-t)}\psi_j(X^{0,x}_{\tau_1\wedge(T-t)})\right] &= \psi_j(x) + E\int_0^{\tau_1\wedge(T-t)} e^{-rs}\mathcal{A}\psi_j(X_s^{0,x})\,ds \\ &\geq \psi_j(x) + E\int_0^{\tau_1\wedge(T-t)} e^{-rs}\mathcal{A}\psi_j(X_s^{0,x})\mathbf{1}_{(X_s^{0,x}\in V)}\,ds. \end{aligned}$$

where the last inequality follows from $\mathcal{A}\psi_j \geq 0$ on U_1.
As j tends to $+\infty$., $\mathcal{A}\psi_j$ converges uniformly to $\mathcal{A}\psi$ on V. Therefore, we have

$$\begin{aligned} C(t,x) &\geq E\left[e^{-r(\tau_1\wedge(T-t)}\psi(X^{0,x}_{\tau_1\wedge(T-t)})\right] \\ &\geq \psi(x) + E\int_0^{\tau_1\wedge(T-t)} e^{-rs}\mathcal{A}\psi(X_s^{0,x})\mathbf{1}_{(X_s^{0,x}\in V)}\,ds \\ &> \psi(x). \end{aligned}$$

It remains to check that $P(\exists s \leq \tau_1 \wedge (T-t); X_s^{0,x} \in V) > 0$. As the matrix Σ is invertible, the support of the distribution of the process $(\Sigma W_t)_{t\leq 0}$ is $\mathcal{C}_0([0,T]; R^n)$ and therefore, applying Girsanov's theorem, the support of the distribution of $X_t^{0,x}$ is the set of all continuous functions from $[0,T]$ with values in R^n starting at x. •

2.2 Application: American call on the minimum of two assets

In this subsection, we highlight the exercise region of American call written on the minimum of two nondividend-paying assets. Therefore, we have $n = 2, \delta_1 = \delta_2 = 0$. Define the payoff by $\psi_m(x) = (\min(e^{x_1}, e^{x_2}) - K)_+$and denote by C_m the value function of this American call. Our knowledge of the one dimensional case made us think that it is not optimal to exercise a call written on nondividend-paying assets prior maturity. Yet, we obtain the following nonintuitive result:

Proposition 2.3 *The exercise region of C_m is nonempty and its t-sections are carried by the line $\{x_2 = x_1\}$.*

Proof: A simple computation yields

$$\mathcal{A}\psi_m = -\frac{1}{2}(a_{11} - 2a_{12} + a_{22})(e^{x_1})^2\sigma + rK1_{x_1 \neq x_2}.$$

where σ is the Lebesgue measure on the line $\{x_1 = x_2\}$. We note that $\mathcal{A}\psi_m$ is not a positive measure on R^2 but satisfies $\mathcal{A}\psi_m > 0$ out of the bisecting line. Theorem 2.1 and proposition 2.2 allow us to conclude. •

We end this application by a more precise description of the t-sections $\mathcal{E}_m(t)$ of the exercise region of the American call on the minimum of two assets.

Proposition 2.4 *There exists a nonincreasing continuous function* $b : [0, T[\rightarrow R^+$ *satisfying* $\lim_{t \to T} b(t) = \log K$ *such that*

$$\mathcal{E}_m(t) = \{(x, x) \in R^2 \mid x \in [b(t), +\infty[\}.$$

Proof: The convexity of the function $x \rightarrow C_m(t, x, x)$ yields immediately that the t-sections are intervals. In order to show that $\mathcal{E}_m(t)$ is of the form stated in the proposition, it suffices to check that $\mathcal{E}_m(0)$ is an unbounded interval of the line $\{x_2 = x_1\}$, since the family $(\mathcal{E}_m(t))_{0 \leq t < T}$ is nondecreasing, which is equivalent by changing the date of maturity to prove that $\mathcal{E}_m(\epsilon)$ is an unbounded interval for every $\epsilon > 0$.

Suppose there exists $\epsilon > 0$ and a date T such that

$$\mathcal{E}_m(\epsilon) = \{(y, y) \in R^2 \mid y \in [m(\epsilon), s(\epsilon)]\}.$$

Choose $x > s(\epsilon)$ and note

$$\tau_x^* = \inf\{t \geq 0 \mid C_m(t, (X_t^{0,x})_1, (X_t^{0,x})_2) = \min(e^{(X_t^{0,x})_1}, e^{(X_t^{0,x})_2}) - K\}.$$

We have,

$$\begin{aligned}
C_m(0, x, x) &= E\left[e^{-r\tau_x^*}(\min(e^{(X_{\tau_x^*}^{0,x})_1}, e^{(X_{\tau_x^*}^{0,x})_2}) - K)_+\right] \\
&\leq e^x E\left(\min(M_{\tau_x^*}^1, M_{\tau_x^*}^2)\right) \text{ where } M_t^i = e^{\sigma_{i1}W_t^1 + \sigma_{i2}W_t^2 - \frac{\sigma_{i1}^2 + \sigma_{i2}^2}{2}t} \\
&= e^x \left(E(M_{\tau_x^*}^1) - E(M_{\tau_x^*}^1 - M_{\tau_x^*}^2)_+\right) \\
&= e^x \left(1 - E(M_{\tau_x^*}^1 - M_{\tau_x^*}^2)_+\right) \\
&= e^x \left(1 - E\left[(M_{\tau_x^*}^1 - M_{\tau_x^*}^2)_+ \mathbf{1}_{\{\tau_x^* = T\}}\right]\right) \text{ since } M_{\tau_x^*}^1 = M_{\tau_x^*}^2 \text{ sur } \{\tau_x^* < T\}.
\end{aligned}$$

We will show that $E\left[(M_{\tau_x^*}^1 - M_{\tau_x^*}^2)_+ \mathbf{1}_{\{\tau_x^* = T\}}\right]$ is uniformly bounded below for x large enough.
We introduce a continuous function $f : [0, T] \rightarrow]0, +\infty[^2$ such that $f(0) = (1, 1)$ and $f^1(t) > f^2(t)$ for every $t \in]0, T]$.
Let $\delta \in]0, \frac{1}{2}[$. By continuity of f, there exists $\eta \in]0, \epsilon[$ such that

$$\forall t \in [0, \eta] \quad \| f(t) - (1, 1) \| < \delta \text{ and } \forall t \in [\eta, T] \; f^1(t) - f^2(t) > \alpha, \text{ with } \alpha > 0.$$

Set $A_\delta = \{ \sup_{t\in[0,T]} \parallel M_t - f(t) \parallel < \delta \wedge \frac{\alpha}{4}\}$. Since the matrix Σ is invertible, we have $P(A_\delta) > 0$ for every $\delta > 0$. On A_δ, we have

$$\forall t \in [\eta, T] \quad M_t^1 - M_t^2 > f^1(t) - f^2(t) - 2 \parallel M_t - f(t) \parallel > \frac{\alpha}{2}.$$

For $t \in [0, \eta]$, we have

$$\begin{aligned} \parallel e^x.M_t - (e^x, e^x) \parallel &\leq e^x (\parallel M_t - f(t) \parallel + \parallel f(t) - (1,1) \parallel) \\ &\leq 2\delta e^x. \end{aligned}$$

Now, we choose x large enough to have $2\delta e^x < e^x - s(\epsilon)$. Then, on $A_\delta, \tau_x^* = T$ and $M^1_{\tau_x^*} > M^2_{\tau_x^*}$. Thus,

$$\begin{aligned} E\left[(M^1_{\tau_x^*} - M^2_{\tau_x^*})_+ 1_{\tau_x^*=T}\right] &\geq E\left[(M^1_{\tau_x^*} - M^2_{\tau_x^*})_+ 1_{A_\delta}\right] \\ &= E\left[(M^1_T - M^2_T)_+ 1_{A_\delta}\right] \\ &= g(\delta) > 0. \end{aligned}$$

Thus, $C_m(0, x, x) \leq e^x(1 - g(\delta)) < e^x - K$ for e^x great enough, which yields a contradiction. At this stage, we have proved that for every t, we have

$$\mathcal{E}_m(t) = \{(x, x) \in R^2 \mid x \in [b(t), +\infty[\}.$$

It remains to check that b is a continuous function satisfying $\lim_{t\to T} b(t) = \log K$. These two assertions are involving the same ideas. We only show the second one.
Clearly, b is a nonincreasing function bounded below by $\log K$ since $\psi(\log K, \log K) = 0$. Hence, b admits a limit as $t \to T$ which will be denoted by $b(T)$. Assume $b(T) > \log K$.
In the open set $]0, T[\times U$ where $U = \{(x, y) \in R^2 \mid \log K < x < b(T)$ and $x - \frac{\log K}{2} < y < x + \frac{\log K}{2}\}$, we have $C_m > \psi_m$ and thus proceeding analogously to the proposition 2.1 $\mathcal{A}\psi_m \geq 0$ on U. But the explicit formula of $\mathcal{A}\psi_m$ given by the proposition 2.3 yields a contradiction.•

3 A particular class of payoffs

Before going further, we review some well-known results for the critical stock-price of an American call written on a single dividend-paying asset. Let $b : t \to b(t)$ be the exercise boundary for such a call. It is well-known that b is continuous, decreasing in time (see Kim [13], Jacka[8]) and even smooth on $[0, T[$ (see Friedman [6]). Moreover, Kim showed that $\lim_{t\to T} b(t) = \max(K, \frac{rK}{\delta})$.
Our goal is to derive some similar results for the immediate exercise boundary of particular American Rainbow options.
Throughout this section, we denote $\log x = (\log x_1, \ldots, \log x_n)$ and for any function defined on R^n, we define $\tilde{\phi}(x) = \phi(\log x)$ for $x \in]0, +\infty[^n$
Now, we focus on the exercise region as subset of $]0, T[\times]0, +\infty[^n$.
Hence, define

$$\tilde{\mathcal{E}} = \{(t, x) \in]0, T[\times]0, \infty[^n; \tilde{C}(x) = \tilde{\psi}(x)\}$$

and its t-sections

$$\tilde{\mathcal{E}}_t = \{x \in]0,\infty[^n; (t,x) \in \tilde{\mathcal{E}}\}$$

Now we introduce the class $\mathcal{B}$ of payoffs ψ defined on R^n by $\psi(x) = \left(\sum_{i=1}^{n} \alpha_i e^{x_i} - K\right)_+$ for $(\alpha_1,\ldots,\alpha_n) \in R^n$ and $K > 0$

Remark 3.1 *1- We suppose that there exists at least an integer i such that $\alpha_i > 0$ in order to have a nonzero payoff. Moreover, we assume that $\alpha_i \neq 0$ for every integer i.*
2-Clearly, the function $x \to \psi(\log x)$ is convex.
3-The class $\mathcal{B}$ provides many examples of contracts that are traded in financial markets like index options, spread options (see the introductory part of [4]).

3.1 Nonemptiness of the exercise region and first properties

We introduce the operator $\tilde{\mathcal{A}}$ in the following way:

$$\tilde{\mathcal{A}}\tilde{\psi}(x) = (\mathcal{A}\psi)(\log x).$$

Note that $\tilde{\mathcal{A}}F = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{n}(r-\delta_i)x_i\frac{\partial F}{\partial x_i} - rF$
Theorem 2.1 and proposition 2.2 have a new formulation for the operator $\tilde{\mathcal{A}}$. We have
1) $\tilde{\mathcal{E}} = \emptyset$ if and only if $\tilde{\mathcal{A}}\tilde{\psi}(x)$ is a nonzero positive measure on $]0,+\infty[^n$.
2) If there is a connected open subset U of $]0,\infty[^n$ such that $\tilde{\mathcal{A}}\tilde{\psi}(x) \geq 0$ on U and if $\tilde{\psi}(x)$ is a $\mathcal{C}^2$ function on an open subset V of U with $\tilde{\mathcal{A}}\tilde{\psi}(x) > 0$ on V then

$$\forall (t,x) \in [0,T[\times \mathrm{U}, \quad C(t,\log x) > \psi(\log x).$$

For ψ belonging to $\mathcal{B}$, we remark that $\tilde{\psi}(x)$ is a $\mathcal{B}^2$ function on the open set

$$\tilde{\mathcal{O}} = \{x \in]0,\infty[^n; \sum_{i=1}^{n} \alpha_i x_i > K\}.$$

Moreover, $\tilde{\mathcal{A}}\tilde{\psi}(x) = rK - \sum_{i=1}^{n}\delta_i\alpha_i x_i$ for $x \in \tilde{\mathcal{O}}$. Then, we have the following characterization theorem:

Theorem 3.1 *The exercise region of an American option written on payoff belonging to $\mathcal{B}$ is empty if and only if for every integer i, $\delta_i\alpha_i \leq 0$.*

Proof: Assume $\tilde{\mathcal{E}} = \emptyset$. According to the theorem 2.1, $\tilde{\mathcal{A}}\tilde{\psi}(x)$ is a positive measure on $]0,+\infty[^n$ and in particular on $\tilde{\mathcal{O}}$. Now, $\tilde{\mathcal{A}}\tilde{\psi}(x) = rK - \sum_{i=1}^{n}\delta_i\alpha_i x_i$ on $\tilde{\mathcal{O}}$, which implies $\delta_i\alpha_i \leq 0$ for every i.
On the other hand, the condition $\delta_i\alpha_i \leq 0$ for every i implies $\tilde{\mathcal{A}}\tilde{\psi}(x) \geq 0$ on $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{A}}\tilde{\psi}(x) > 0$ on a subset of $\tilde{\mathcal{O}}$. According to proposition 2.2, $\tilde{\mathcal{O}} \subset \mathcal{E}_t^c$ for every t. Proposition 1.1 allows us to conclude.•

Remark 3.2 *This theorem supplies a very simple characterization of the nonemptiness of the exercise region and allows us to give some examples of options for which early exercise occurs.*

Examples

1) American spread call is defined by the payoff function

$$\tilde{\psi}(x_1, x_2) = (x_2 - x_1 - K)_+.$$

According to the previous results, if the asset two does not pay dividends, early exercise is never optimal.

2) American option on the Standard and Poor 100 index are options on the arithmetic average of the values of 100 assets. It may be modeled by

$$\tilde{\psi}(x) = \left(\sum_{i=1}^{n} \frac{p_i}{p} x_i - K\right)_+ \text{ with } p = \sum_{i=1}^{n} p_i$$

where p_i is the number of shares of asset i. Exercise prior to maturity is optimal if and only if at least one of the assets making up the index pays dividends.

Remark 3.3 *1- We focus on exercise regions of American calls because the exercise region of an American put is never empty (see corollary 2.1).*

2- However, the next properties of exercise region of American options written on payoffs belonging to $\mathcal{B}$ may be proved in the same way for American puts written on payoffs ψ such that $\psi(x) = \left(K - \sum_{i=1}^{n} \alpha_i e^{x_i}\right)_+$.

3.2 Topological features of the t-sections $\tilde{\mathcal{E}}_t$

In this section, we assume that $\psi \in \mathcal{B}$ and there is one integer i such that $\delta_i \alpha_i > 0$.

Proposition 3.1 *The t-sections $\tilde{\mathcal{E}}_t$ satisfy the following assertions.*

i) $\forall t \in [0, T[$ $\tilde{\mathcal{E}}_t$ is a closed convex subset of $]0, +\infty[^n$.

ii) $\forall t \in [0, T[$ $\tilde{\mathcal{E}}_t$ has a nonempty interior .

Proof: Assertion i) follows from Broadie-Detemple [4]. For assertion ii), we first show that $\tilde{\mathcal{E}}_0$ is nonempty. It suffices to check that $\tilde{\mathcal{E}}_\eta \neq \emptyset$ for every $\eta > 0$. Indeed, the section $\tilde{\mathcal{E}}_0$ for an option with maturity T coincides with the section $\tilde{\mathcal{E}}_\eta$ of an option with maturity $T + \eta$.

Suppose there exists some $\eta > 0$ such that $\mathcal{E}_\eta$ is empty. Introduce the stopping time:

$$\tau_x^* = \inf\{t \geq 0; C(t, X_t^{\log x}) = \psi(X_t^{\log x})\}$$

The assumption $\mathcal{E}_\eta = \emptyset$ implies $\tau_x^* \geq \eta$ a.s. But, according to the optimal stopping theory, we have,

$$C(0, \log x) \quad = \quad E\left[e^{-r\tau_x^*}\left(\sum_{i=1}^{n} \alpha_i x_i e^{(r-\delta_i-\frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2)\tau_x^* + \sum_{j=1}^{n}\sigma_{ij}W_{\tau_x^*}^j)} - K\right)_+\right]$$

$$\leq \quad E\left[\sum_{i\in I}\alpha_i x_i e^{(-\delta_i-\frac{1}{2}\sum_{j=1}^n\sigma_{ij}^2)\tau_x^*+\sum_{j=1}^n\sigma_{ij}W^j_{\tau_x^*})}\right]$$
$$\leq \quad \sum_{i\in I}\alpha_i x_i e^{-\delta_i\eta}.$$

where $I=\{i\in(1,\dots,n);\alpha_i>0\}$.
As there exists $i_0\in I$ such that $\delta_{i_0}>0$, it suffices to fix $(x_1,\dots,x_{i_0-1},x_{i_0+1},\dots,x_n)$ and let x_{i_0} tend to $+\infty$ to obtain

$$C(0,\log x)<\left(\sum_{i=1}^n\alpha_i x_i-K\right)_+$$

which is a contradiction.
Now, we prove that the t-sections have nonempty interior. Introduce the following notation

$$x^{\lambda,i}=(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_n)$$

Let $x\in\tilde{\mathcal{E}}_t$. We have, see [4] for details,

i) $x^{\lambda,i}\in\tilde{\mathcal{E}}_t$ for $i\in I,$ for $\lambda>1$

ii) $x^{\lambda,i}\in\tilde{\mathcal{E}}_t$ for $i\notin I,$ for $\lambda\in]0,1[$

Hence, $\tilde{\mathcal{E}}_t$ contains the open set $\prod_{i\in I}]x_i,+\infty[\times\prod_{i\notin I}]0,x_i[$.

3.3 Regularity of the t-sections

In this section, we define the t-sections by

$$\tilde{\mathcal{E}}_t=\{x\in[0,\infty[^n\mid\tilde{C}(t,x)=\tilde{\psi}(x)\}.$$

Now, the t-sections are closed set of $[0,\infty[^n$.
The family $(\tilde{\mathcal{E}}_t)_{0\leq t\leq T}$ is nondecreasing according to proposition 1.1. For payoff functions belonging to $\mathcal{B}$, we shall prove the following regularity property of the family $\left(\tilde{\mathcal{E}}_t\right)$, which can be viewed as an analogue of the continuity in time of the free boundary in one dimension.

Proposition 3.2 *The family $(\tilde{\mathcal{E}}_t)_{0\leq t\leq T}$ satisfies*

1) $\tilde{\mathcal{E}}_t=\bigcap_{u>t}\tilde{\mathcal{E}}_u$ *for every* t *.*

2) $\tilde{\mathcal{E}}_t=\overline{\bigcup_{s<t}\tilde{\mathcal{E}}_s}$ *for every* t *.*

Proof: 1) Let $x\in\bigcap_{u>t}\tilde{\mathcal{E}}_u$, we have

$$\begin{aligned}0 &\leq C(t,\log x)-\psi(\log x)\\ &= C(t,\log x)-C(u,\log x)\text{ for every }u>t.\end{aligned}$$

Using the continuity of C, we let u tend to t to conclude that x belongs to $\tilde{\mathcal{E}}_t$.
The converse inclusion is straightforward.
2) We have $\overline{\bigcup_{s<t}\tilde{\mathcal{E}}_s} \subset \tilde{\mathcal{E}}_t$. Assume this inclusion is strict and set

$$\Phi = \left(\overline{\bigcup_{s<t}\tilde{\mathcal{E}}_s}\right)^c$$

we have $\Phi \cap \tilde{\mathcal{E}}_t \neq \emptyset$. Since, $\tilde{\mathcal{E}}_t$ is a closed convex set with nonempty interior for every t, we have $\tilde{\mathcal{E}}_t = \overline{\overset{\circ}{\tilde{\mathcal{E}}}_t}$. Then, we deduce that $\Phi \cap \overset{\circ}{\tilde{\mathcal{E}}}_t$ is nonempty. Indeed, if $x \in \Phi \cap \tilde{\mathcal{E}}_t$ then there exists some $\rho > 0$ such that $B(x,\rho) \subset \Phi$ and $B(x,\rho) \cap \overset{\circ}{\tilde{\mathcal{E}}}_t \neq \emptyset$.
Hence, in the open set $]0,t[\times(\Phi \cap \overset{\circ}{\tilde{\mathcal{E}}}_t)$, $\tilde{C}(s,x) > \tilde{\psi}(x)$ and thus proceeding analogously as in the proof of proposition 1.2, we deduce

$$\tilde{\mathcal{A}}\tilde{C}(u,.) \geq 0 \text{ on } \Phi \cap \overset{\circ}{\tilde{\mathcal{E}}}_u \, .$$

But, $\tilde{\mathcal{A}}\tilde{C}(u,.) = \tilde{\mathcal{A}}\tilde{\psi} \leq 0$ in this open set. Therefore, $\tilde{\mathcal{A}}\tilde{\psi} = 0$ on $\Phi \cap \overset{\circ}{\tilde{\mathcal{E}}}_u$. But for $\psi \in \mathcal{B}$, we have $\tilde{\mathcal{A}}\tilde{\psi}(x) = rK - \sum_{i=1}^{n} \delta_i \alpha_i x_i$ on $\tilde{\mathcal{O}}$ and thus cannot be identically zero in a nonempty open set. •

Remark 3.4 *The assumption $\psi \in \mathcal{B}$ was used in the proof of the second assertion only.*

We end this section by the study of the asymptotic behavior of the exercise region near the date of maturity. Before stating the main theorem, we show preliminary results concerning the optimal exercise of American options written on payoffs belonging to $\mathcal{B}$.

3.4 Behavior of the exercise region near maturity

The purpose of this section is to identify the exercise region near maturity. For payoff functions belonging to $\mathcal{B}$, we state the following theorem which characterizes the shape of the exercise region near maturity and can be viewed as the multidimensional version of Kim's result.

Theorem 3.2 *Assume there is some integer i such that $\delta_i \alpha_i > 0$. We have*

$$\bigcup_{t<T} \tilde{\mathcal{E}}_t = \left\{ x \in]0,+\infty[^n \mid \sum_{i=1}^{n} \alpha_i x_i - K > 0 \text{ and } rK - \sum_{i=1}^{n} \delta_i \alpha_i x_i < 0 \right\}.$$

Before proceeding further in the proof of this theorem, we establish some preliminary results.
In the open set $\tilde{\mathcal{O}} = \{x \in (]0,+\infty[)^n \mid \sum_{i=1}^{n} \alpha_i x_i > K\}$, we point out the vector which belongs to the affine hyperplan $H = \{x \in (]0,+\infty[)^n \mid rK - \sum_{i=1}^{n} \delta_i \alpha_i x_i = 0\}$.
We shall prove that the t-sections do not intersect H. The proof requires the following probabilistic lemma.

Lemma 3.1 *Let $W = (W_t^1, \dots, W_t^n)$ be a standard n-dimensional $(\mathcal{F}_t)$-Brownian motion. Let θ be a positive number, U an open subset of R^n containing the origin and g a $\mathcal{C}^2$ function on $[0, \theta] \times U$ with bounded first and second derivatives satisfying:*

- $g(0; 0) = 0$
- $\overrightarrow{grad}\ g(0; 0) \neq 0$.

Then, there exists a stopping time τ in $\mathcal{T}_{0,\theta}$ satisfying $\tau \leq \tau_U$ where $\tau_U = \inf\{t \geq 0 \mid W_t \notin U\}$ such that $E\int_0^\tau g(s; W_s)\, ds > 0$.

Proof: Let $u_1 = \dfrac{\overrightarrow{\text{grad}}\ g(0; 0)}{\|\overrightarrow{\text{grad}}\ g(0; 0)\ \|}$.
We construct an orthonormal basis $(u_1, \dots, u_n)$ and we note $B_t = (B_t^1, \dots, B_t^n)$ the coordinates of W in this new basis. $(B_t)_{t\geq 0}$ is still a standard n-dimensional $(\mathcal{F}_t)$-Brownian motion (see [12]).
We fix 3 positive real numbers a, b, λ such that $\max(a, b) < \lambda$.
Introduce the following stopping times:

$$\tau_{a,b} = \inf\{s \geq 0 |\ B_s^1 \notin [-a, b]\}$$

$$\tau_\lambda = \inf\{s \geq 0 |\ \left(\sum_{j=2}^n (B_t^j)^2\right)^{\frac{1}{2}} > \lambda\}.$$

Finally, we suppose that λ is such that

$$\{(x_1, \dots, x_n) \mid x_1 \in [-a, b] \text{ and } \sum_{j=2}^n (x_j)^2 \leq \lambda\} \subset U.$$

Applying Taylor's formula,

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} g(s; W_s)\, ds = E\int_0^{\tau_{ab}\wedge\tau_{a,b}\wedge\theta} \left(\overrightarrow{\text{grad}}\ g(0; 0).W_s + \frac{\partial g}{\partial s}(0; 0)s + R(s; W_s)\right)\, ds$$

hence

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} g(s; W_s)\, ds = E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} \left(\|\overrightarrow{\text{grad}}\ g(0; 0)\ \| .B_s^1 + \frac{\partial g}{\partial s}(0; 0)s + R(s; W_s)\right)\, ds.$$

But the assumptions on g imply for $s \leq \tau_U$

$$|\ R(s; W_s)\ | \leq C \sup_{[0,\theta]\times U} \|\ g"(t, x)\ \| \left(s^2 + \|\ W_s\ \|^2\right).$$

For $s \leq \tau_{a,b} \wedge \tau_\lambda \wedge \theta$, we have $\|\ W_s\ \|^2 \leq 2\lambda^2$ and using the fact that τ_λ has the same distribution as $\lambda^2\tau_1$ (scaling property of Brownian motion), we have

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} |\ R(s; W_s)\ |\ \ ds \leq C(\lambda^6 + \lambda^4)E(\tau_1).$$

On the other hand,

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} \mid \frac{\partial g}{\partial s}(0;0)s \mid \; ds \le CE(\tau_\lambda)^2 \le C\lambda^4 E(\tau_1).$$

When λ tends to 0, we have

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} g(s;W_s)\; ds = \| \overrightarrow{\text{grad}} \; g(0;0) \| \; E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} B_s^1 \; ds + O(\lambda^4)$$

but applying the Ito's formula, we have

$$E\int_0^{\tau_{a,b}} B_s^1 \; ds = \frac{2}{3}E(B^1_{\tau_{a,b}})^3 = \frac{2}{3}ab(b-a)$$

therefore

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} B_s^1 \; ds = \frac{2}{3}ab(b-a) - E\int_{\tau_{a,b}\wedge\tau_\lambda\wedge\theta}^{\tau_{a,b}} B_s^1 \; ds.$$

Then, we choose $a = a_0\lambda^{1+\epsilon}$ and $b = b_0\lambda^{1+\epsilon}$ with $\epsilon > 0$

$$\begin{aligned} \left| E\int_{\tau_{a,b}\wedge\tau_\lambda\wedge\theta}^{\tau_{a,b}} B_s^1 \; ds \right| &\le \lambda^{1+\epsilon}\max(a_0,b_0)E(\tau_{a,b}1_{\{\tau_{a,b}\ge\tau_\lambda\wedge\theta\}}) \\ &\le \lambda^{3+3\epsilon}\max(a_0,b_0)E(\tau_{a_0b_0}1_{\{\lambda^{2+2\epsilon}\tau_{a_0b_0}\ge(\lambda^2\tau_1)\wedge\theta\}}) \end{aligned}$$

where, the second inequality follows from the scaling property of Brownian motion. Since $\tau_{a_0b_0} = \inf\{t \ge 0 |\; B_t^1 \notin [-a_0, b_0]\}$ is integrable, $E(\tau_{a_0b_0}1_{\{\lambda^{2+2\epsilon}\tau_{a_0b_0}\ge(\lambda^2\tau_1)\wedge\theta\}})$ converges to 0 when λ tends to 0 by dominated convergence.
Hence,

$$E\int_0^{\tau_{a,b}\wedge\tau_\lambda\wedge\theta} g(s;W_s)\; ds = \| \overrightarrow{\text{grad}} \; g(0;0) \| \; \lambda^{3+3\epsilon}a_0b_0(b_0-a_0) + o(\lambda^{3+3\epsilon}).$$

It suffices to choose $b_0 > a_0$ and λ small enough to conclude that $\tau = \tau_{a,b}\wedge\tau_\lambda\wedge\theta$ satisfies the desired property.•

Proposition 3.3 *For (t,x) in the open set $[0,T[\times(H\cap\tilde{\mathcal{O}})$, we have*

$$\tilde{\mathcal{C}}(t,x) > \tilde{\psi}(x).$$

Proof: We set $S_t = e^{X_t}$ for every t
Let $\tau_o = \inf\{t \ge 0; S_t^{0,x} \notin \tilde{\mathcal{O}}\}$.
For $x \in H\cap\tilde{\mathcal{O}}$, we have for every $\tau \le \tau_o \wedge (T-t)$,

$$\begin{aligned} \tilde{\mathcal{C}}(t,x) &\ge E\left(e^{-r\tau}\tilde{\psi}(S_\tau^{0,x})\right) \\ &= \tilde{\psi}(x) + E\int_0^\tau e^{-rs}(rK - \sum_{i=1}^n \delta_i\alpha_i x_i e^{(r-\delta_i-\frac{1}{2}\sum_{j=1}^n\sigma_{ij}^2)s+\sum_{j=1}^n\sigma_{ij}W_s^j)}\; ds \\ &= \tilde{\psi}(x) + E\int_0^\tau g(s;W_s)\; ds. \end{aligned}$$

where $g(s;y) = e^{-rs}(rK - \sum_{i=1}^n \delta_i\alpha_i x_i e^{(r-\delta_i-\frac{1}{2}\sum_{j=1}^n \sigma_{ij}^2)s+\sum_{j=1}^n \sigma_{ij}y_j)})$.
$g(0;0) = 0$ since $x \in H \cap \tilde{\mathcal{O}}$ and $\dfrac{\partial g}{\partial x_i}(0;0) = \sum_{j=1}^n \sigma_{ij}\delta_j\alpha_j x_j \neq 0$ for at least one integer i. Otherwise, as the matrix Σ is invertible, the vector with components $\delta_j\alpha_j x_j$ would equal zero, which yields a contradiction since $x \in H$.
Applying lemma 3.1, there exists a stopping time $\tau^* \in \mathcal{T}_{0,T-t} \wedge \tau_o$ such that

$$E\int_0^{\tau^*} g(s;W_s)\, ds > 0$$

which proves the desired inequality. •
Now, we are in a position to prove theorem 3.2.
Proof of theorem 3.2: Set

$$\Delta = \left\{ x \in]0,+\infty[^n \mid \sum_{i=1}^n \alpha_i x_i - K > 0 \text{ and } rK - \sum_{i=1}^n \delta_i\alpha_i x_i < 0 \right\}.$$

We first show the inclusion $\bigcup_{t<T} \tilde{\mathcal{E}}_t \subset \Delta$.
Let $x \in \bigcup_{t<T} \tilde{\mathcal{E}}_t$.
According to the proposition 1.1 iii), $\psi(\log x) = \sum_{i=1}^n \alpha_i x_i - K > 0$.
In the open set $\tilde{\mathcal{O}}$ we have $\tilde{\mathcal{A}}\tilde{\psi}(x) = rK - \sum_{i=1}^n \delta_i\alpha_i x_i$.
It follows from proposition 2.2 that $\tilde{\mathcal{A}}\tilde{\psi}(x) \leq 0$. Moreover, if $\tilde{\mathcal{A}}\tilde{\psi}(x) = 0$, $x \in \tilde{\mathcal{O}} \cap H$ and then $x \notin \bigcup_{t<T} \tilde{\mathcal{E}}_t$ according to proposition 3.3. Hence, $x \in \Delta$. To show the converse inclusion, we use the following elementary result of convex analysis (see Webster [21]).

Lemma 3.2 *Let $\mathcal{K}$ be a convex subset of R^n with nonempty interior. We have $\overset{\circ}{\overline{\mathcal{K}}} \subset \mathcal{K}$.*

Let $x \in \Delta$. As Δ is open, there is a ball $B(x,\rho)$ within Δ and thus $rK - \sum_{i=1}^n \delta_i\alpha_i x_i < 0$ on this ball. According to proposition 2.1, $B(x,\rho) \subset \overline{\bigcup_{t<T} \tilde{\mathcal{E}}_t}$. Therefore, x belongs to the interior of $\overline{\bigcup_{t<T} \tilde{\mathcal{E}}_t}$. But, $\mathcal{K} = \bigcup_{t<T} \tilde{\mathcal{E}}_t$ is convex as an increasing union of convex sets. We conclude thanks to lemma 3.2. •

3.5 Application: American spread option

We consider an American spread option which is a contingent claim on two underlying assets (S_t^1, S_t^2) that has the payoff upon exercise $\tilde{\psi}(S_t^1, S_t^2) = (S_t^1 - S_t^2 - K)_+$.
Let $C_s(t, S_t^1, S_t^2)$ denote the value of the spread option at time t with

$$C_s(t,x_1,x_2) = \sup_{\tau \in \mathcal{T}_{0,T-t}} E\left[e^{-r\tau}(S_\tau^{2,x_2} - S_\tau^{1,x_1} - K)_+ \right]$$

where $S_t^{i,x} = xe^{(r-\delta_i-\frac{1}{2}\sum_{j=1}^2 \sigma_{ij}^2)t+\sum_{j=1}^2 \sigma_{ij}W_t^j}$. We note that
i) $C_s(.,x_1,x_2)$ is nonincreasing in time for every $(x_1,x_2) \in]0,\infty[^2$.

ii) $C_s(t,.)$ is nonincreasing (resp nondecreasing) in x_1 (resp in x_2) for every t.
In the open set $\{(x_1,x_2)\in]0,\infty[^2 \mid x_2-x_1-K>0\}$ $\tilde{\psi}$ is a $\mathcal{C}^2$ function and $\tilde{\mathcal{A}}\tilde{\psi}(x_1,x_2) = \delta_1 x_1 - \delta_2 x_2 + rK$.
The condition $\delta_2 > 0$ is necessary and sufficient to ensure the nonemptiness of the exercise region, according to theorem 3.1.
Assume this condition holds. We then define the critical surface of the American option on spread by:

$$b_2^*(t,x_1) = \inf\{x_2 > x_1 + K \mid C_s(t,x_1,x_2) = x_2 - x_1 - K\}.$$

At a fixed time t , we have

$$\tilde{\mathcal{E}}_t = \bigcup_{x_1>0} [b_2^*(t,x_1), +\infty[.$$

We deduce from the previous section the following proposition:

Proposition 3.4 *The critical surface satisfies*

1) $b_2^*(.,x_1)$ *is continuous on* $[0,T[$ *for every* x_1*.*

2) $b_2^*(t,.)$ *is convex on* $]0,+\infty[$*.*

3) $\lim_{t\to T} b_2^*(t,x_1) = \max(x_1+K, \frac{\delta_1}{\delta_2}x_1 + \frac{rK}{\delta_2})$.

Proof: 1) The regularity of the t-sections implies the continuity of the critical surface $b_2^*(.,x_1)$ on $[0,T[$ for every x_1.
2)The convexity of $\tilde{\mathcal{E}}_t$ implies that of $b_2^*(t,.)$.
3) $\lim_{t\to T} b_2^*(t,x_1) = \max(x_1+K, \frac{\delta_1}{\delta_2}x_1 + \frac{rK}{\delta_2})$ follows from the asymptotic behavior of the t-sections near maturity. •

4 American options on the maximum, minimum of two assets

We consider now American options written on convex payoff functions which do not belong to $\mathcal{B}$ but that are of interest in practice: the American call on the maximum of two assets and the American put on the minimum of two assets.
Hereafter, we denote $\mu_i = r - \delta_i - \frac{\sigma_{i1}^2}{2} - \frac{\sigma_{i2}^2}{2}$.
Let $\psi_M(x_1,x_2) = (\max(x_1,x_2)-K)_+$ and $\psi_m(x_1,x_2) = (K-\min(x_1,x_2))_+$. The value function of the associated American option is given by

$$C_M(t,x_1,x_2) = \sup_{\tau\in\mathcal{T}_{0,T-t}} Ee^{-r\tau}(\max(S_\tau^{1,x_1}, S_\tau^{2,x_2}) - K)_+$$

and

$$P_m(t,x_1,x_2) = \sup_{\tau\in\mathcal{T}_{0,T-t}} E\left(e^{-r\tau}(K - \min(S_\tau^{1,x_1}, S_\tau^{2,x_2}))_+\right).$$

4.1 Nonconvexity of the exercise region

This two options offer a nice example of options written on a convex payoff function with t-sections that are not convex set (see Broadie and Detemple [4]).
The next proposition will complement a result of Broadie and Detemple. Before going further, we recall a useful elementary lemma.

Lemma 4.1 *Let g, h be two integrable random variable such that*
1- $E(g) = E(h) = 0$
2- $P(g \neq h) > 0$
Then $E(\max(g, h)) > 0$.

Proposition 4.1 *For every fixed t, there is an open cone Γ containing the bissecting line with aperture depending on t such that for every (x_1, x_2) belonging to Γ we have:*

$$C_M(t, x_1, x_2) > \max(x_1, x_2) - K$$

and

$$P_m(t, x_1, x_2) > K - \min(x_1, x_2).$$

Proof: We only show the proposition for the call at $t = 0$.

$$\forall t \geq 0 \; C_M(0, x, x) \geq Ee^{-rt}(\max\left(xe^{\mu_1 t + \sigma_{11} W_t^1 + \sigma_{12} W_t^2}, xe^{\mu_2 t + \sigma_{2,1} W_t^1 + \sigma_{2,2} W_t^2}\right) - K)_+.$$

For t close enough to 0, we have

$$C_M(0, x, x) \geq xE\left(1 + \max(\sigma_{1,1} W_t^1 + \sigma_{1,2} W_t^2, \sigma_{21} W_t^1 + \sigma_{2,2} W_t^2) + f(t, W_t^1, W_t^2)\right) - K$$

with

$$\begin{aligned} f(t, y_1, y_2) &= \max\left(e^{\mu_1 t + \sigma_{11} y_1 + \sigma_{12} y_2}, e^{\mu_2 t + \sigma_{2,1} y_1 + \sigma_{2,2} y_2}\right) \\ &- 1 - \max(\sigma_{1,1} y_1 + \sigma_{1,2} y_2, \sigma_{2,1} y_1 + \sigma_{2,2} y_2). \end{aligned}$$

But, W_t^i has the same distribution as $\sqrt{t} W_1^i$ hence

$$C_M(0, x, x) \geq xE(1 + \sqrt{t} \max(\sigma_{1,1} g_1 + \sigma_{1,2} g_2, \sigma_{21} g_1 + \sigma_{22} g_2) + f(t, W_t^1, W_t^2)) - K.$$

where g_1, g_2 are standard normal random variables.
Using, $\mid e^y - 1 - y \mid \leq \frac{y^2}{2} e^{|y|}$
with $y = \max(\mu_1 t + \sigma_{1,1} W_t^1 + \sigma_{1,2} W_t^2, \mu_2 t + \sigma_{21} W_t^1 + \sigma_{2,2} W_t^2)$, we obtain

$$\begin{aligned} |f(t, W_t^1, W_t^2)| &\leq \frac{y^2}{2} e^{|y|} + |y - \max(\sigma_{1,1} W_t^1 + \sigma_{1,2} W_t^2, \sigma_{2,1} W_t^1 + \sigma_{2,2} W_t^2)| \\ &\leq \frac{y^2}{2} e^{|y|} + rt. \end{aligned}$$

Hence,

$$E|f(t, W_t^1, W_t^2)| \leq Ct$$

Set for i=1,2 $h_i = \sigma_{i,1}g_1 + \sigma_{i,2}g_2$. We have

$$C_M(0,x,x) \geq x - K + x\sqrt{t}\,(E(\max(h_1,h_2)) + o(1))$$

Using lemma 4.1, we may choose t such that $o(1) > -\frac{1}{2}E(\max(h_1,h_2))$.
Hence,

$$C_M(0,x,x) > x - K + \frac{x\sqrt{t}}{2}E(\max(h_1,h_2)).$$

Set $\alpha(t) = \frac{\sqrt{t}}{2}E(\max(h_1,h_2))$.
for every $x > 0$ and $h \in]0,\alpha(t)[$
$C_M(0,(1+h)x,x) > (1+h)x - K$ et $C_M(0,x,(1+h)x) > (1+h)x - K$. •
Using the notation of Broadie and Detemple [4], we define for a fixed time t in $[0,T[$ the following subsets:

$$\mathcal{E}_t^{i,M} = \mathcal{E}_t^M \cap \{x_i = \max(x_1,x_2)\}$$

where $\mathcal{E}_t^M = \{(x_1,x_2) \mid C_M(t,x_1,x_2) = \max(x_1,x_2) - K\}$ According to proposition 4.1, we have $\mathcal{E}_t^M = \mathcal{E}_t^{1,M} \cup \mathcal{E}_t^{2,M}$. In the open set $\{(x_1,x_2) \in (]0,+\infty[)^2 / x_1 > \max(x_2,K)\}$, we have

$$\tilde{A}\psi_M(x_1,x_2) = rK - \delta_1 x_1.$$

According to theorem 3.1, we have $\mathcal{E}_t^{i,M} = \emptyset$ if and only if $\delta_i = 0$. Moreover, Broadie and Detemple proved that $\mathcal{E}_t^{i,M}$ $(i=1,2)$ are closed convex subsets with nonempty interiors, hence we may state the next proposition which identify the shape of $\mathcal{E}_t^i$ near maturity.

Proposition 4.2 *We have*

$$\bigcup_{t<T} \mathcal{E}_t^{1,M} = \{(x_1,x_2) \in]0,+\infty[^2 \mid x_1 > \max(x_2,K) \text{ and } rK - \delta_1 x_1 < 0\}$$

Proof: It suffices to proceed analogously from theorem 3.2.•
Similar results can be derived for the exercise region of the American put on the minimum of two assets. Define $\mathcal{E}_t^m = \{(x_1,x_2) \mid P_m(t,x_1,x_2) = K - \min(x_1,x_2)\}$. According to corollary 2.1 and proposition 4.1, we know that $\mathcal{E}_t^m$ is nonempty for every t and $\mathcal{E}_t^m = \mathcal{E}_t^{1,m} \cup \mathcal{E}_t^{2,m}$ where $\mathcal{E}_t^{i,m} = \mathcal{E}_t^m \cap \{x_i = \min(x_1,x_2)\}$. We may identify the shape of $\mathcal{E}_t^{i,m}$ near maturity.

Proposition 4.3 *We have*

$$\bigcup_{t<T} \mathcal{E}_t^{1,m} = \{(x_1,x_2) \in]0,+\infty[^2 \mid x_1 < \min(x_2,K) \text{ and } \delta_1 x_1 - rK < 0\}.$$

4.2 Perpetual American call on the maximum of two assets

This section is devoted to the study of the asymptotic behavior of the t-sections for large values of underlying assets. We are now interested in the perpetual case because the results that we present carry over the case of option with finite maturity. Let

$$C_M^\infty(x_1,x_2) = \sup_{\tau \in \mathcal{T}_{0,\infty}} Ee^{-r\tau}(\max(S_\tau^{1,x_1}, S_\tau^{2,x_2}) - K)_+$$

be the value function of the perpetual American call on the maximum of two assets.
As usual, define the exercise region by

$$\mathcal{E}_M = \{(x_1, x_2) \in]0, \infty[^2 / C_M^\infty(x_1, x_2) = \max(x_1, x_2) - K\}$$

and

$$\mathcal{E}_M^i = \{(x_1, x_2) \in \mathcal{E}_M \mid x_i = \max(x_1, x_2) - K\}\,.$$

The inequality $C_M(t, x_1, x_2) \le C_M^\infty(x_1, x_2)$ and the proposition 4.2 implies

$$\mathcal{E}_M = \mathcal{E}_M^1 \cup \mathcal{E}_M^2\,.$$

We are looking for a condition that ensures the nonemptiness of $\mathcal{E}_M^i$.

Proposition 4.4 $\mathcal{E}_M^i$ *is nonempty if and only if* $\delta_i > 0$ *.*

Proof: If $\mathcal{E}_M^i \neq \emptyset$ then $\mathcal{E}_t^i$ is nonempty. According to theorem 3.1, we have $\delta_i > 0$. To establish the converse assertion, we assume $\delta_1 > 0$ and introduce the value function of a perpetual American call on the asset 1 and the one of the perpetual exchange option.

$$C^\infty(x_1, K, r, \delta_1) = \sup_{\tau \in \mathcal{T}_{0,\infty}} E\left(e^{-r\tau}(x_1 e^{\mu_1 \tau + \sigma_{11} W_\tau^1 + \sigma_{12} W_\tau^2} - K)_+\right).$$

and

$$C_e^\infty(x_1, x_2, K, r, \delta_1) = \sup_{\tau \in \mathcal{T}_{0,\infty}} E\left(e^{-r\tau}(x_1 e^{\mu_1 \tau + \sigma_{11} W_\tau^1 + \sigma_{12} W_\tau^2} - x_2 e^{\mu_2 \tau + \sigma_{21} W_\tau^1 + \sigma_{22} W_\tau^2}\right).$$

When $\delta_1 > 0$, there is a critical stock-price $s_1(K) > K$ such that

$$\forall x_1 \ge s_1(K) \quad C^\infty(x_1, K, r, \delta_1) = x_1 - K.$$

Moreover, we can define

$$\beta^* = \inf\{\beta \ge 0 \mid \forall x_1 \ge \beta x_2 \quad C_e^\infty(x_1, x_2, K, r, \delta_1) = x_1 - x_2\}.$$

Remark 4.1 *Gerber and Shiu have shown that* β^* *is finite when* $\delta_1 > 0$*.*

Note the inequality,

$$\begin{aligned}(\max(x, y) - K)_+ &= ((y - x)_+ + x - K)_+ \\ &\le \left(\frac{1}{2}(y - 2x)_+ + \frac{1}{2}(y - 2K) + x\right)_+ \\ &\le \frac{1}{2}(y - 2x)_+ + \frac{1}{2}(y - 2K)_+ + x.\end{aligned}$$

Hence,

$$C_M^\infty(x_1, x_2) \le \frac{1}{2} C_e^\infty(x_1, 2x_2, K, r, \delta_1, \delta_2) + \frac{1}{2} C^\infty(x_1, 2K, r, \delta_1) + x_2$$

We choose $x_2 \ge s_1(2K)$ and $x_1 \ge 2\beta^* x_2$ to obtain

$$C_M^\infty(x_1, x_2) = \max(x_1, x_2) - K.$$

•

Remark 4.2 *We refer to Gerber and Shiu [7] for a detailed study of the perpetual exchange option.*

Proceeding analogously from the proposition 3.1, we can define the perpetual critical surface

$$b^*(x_2) = \inf\{x_1 > x_2 \mid C_M^\infty(x_1, x_2) = x_1 - K\}\,.$$

Since C_M^∞ is convex, we deduce immediately the convexity of $b^*(.)$. This convexity implies the existence of $\lim_{x_2\to\infty} \frac{b^*(x_2)}{x_2}$ in $]0, \infty]$. Propositions 4.1 and 4.4 supply the next corollary which states that this limit is finite.

Corollary 4.1

$$\lim_{x_2\to\infty} \frac{b^*(x_2)}{x_2} \in]1, 2\beta^*]$$

where β^* is given in the proof of the proposition 4.4 .

Remark 4.3 *If we suppose in addition that $\delta_2 > 0$ then we can improve the previous corollary as it is shown in the next proposition.*

Proposition 4.5 *Assume* $\min(\delta_1, \delta_2) > 0$. *Then,*

$$\lim_{x_2\to\infty} \frac{b^*(x_2)}{x_2} \in]1, \beta^*]\,.$$

Proof: Using the inequality,

$$(\max(x, y) - K)_+ \le (y - x)_+ + (x - K)_+$$

we obtain

$$C_M^\infty(x_1, x_2) \le C_e^\infty(x_1, x_2, K, r, \delta_1, \delta_2) + C^\infty(x_2, K, r, \delta_2).$$

As $\delta_2 > 0$, there is s_2 such that $C^\infty(x_2, K, r, \delta_2) = x_2 - K$ for every $x_2 \ge s_2$. It suffices to choose $x_2 \ge s_2$ and $x_1 \ge \beta^* x_2$ to obtain

$$C_M^\infty(x_1, x_2) = x_1 - K\,.$$

This implies $b^*(x_2) \le \beta^* x_2$ for every $x_2 \ge s_2$. •

4.3 Perpetual American put on the minimum of two assets

We end this section by a more precise description of the structure of the exercise region of the perpetual American put on the minimum of two assets. The value function of such an option is given by

$$P_m^\infty(x_1, x_2) = \sup_{\tau\in\mathcal{T}_{0,\infty}} E\left(e^{-r\tau}(K - \min(S_\tau^{1,x_1}, S_\tau^{2,x_2}))_+\right).$$

Note that P_m^∞ is nonincreasing in x_1 and x_2. Define

$$\mathcal{E}_m = \{(x_1, x_2) \in (]0, +\infty[)^2 \mid P_m^\infty(x_1, x_2) = K - \min(x_1, x_2)\}.$$

We deduce from in proposition 4.1

$$\mathcal{E}_m = \mathcal{E}_m^1 \cup \mathcal{E}_m^2 \text{ where } \mathcal{E}_m^i = \mathcal{E}_m \cap \{x_i = \min(x_1, x_2)\}.$$

According to corollary 2.1, we know that $\mathcal{E}_m$ is nonempty. The next proposition informs us on some topological features of $\Pi\mathcal{E}_m^1$.

Proposition 4.6

$$i)\ (x_1, x_2) \in \mathcal{E}_m^1 \quad Rightarrow \quad (x_1, \lambda x_2) \in \mathcal{E}_m^1 \text{ for every } \lambda \geq 1$$
$$ii)\ (x_1, x_2) \in \mathcal{E}_m^1 \quad Rightarrow \quad (\lambda x_1, x_2) \in \mathcal{E}_m^1 \text{ for every } \lambda \in]0, 1[.$$

Similar results hold for $\mathcal{E}_m^2$.

Proof: We refer to proposition 2.3 of [4] for a detailed proof. •
Define the critical surface of $\mathcal{E}_m^1$ by

$$\hat{b}_1(x_2) = \sup\{x_1 \in]0, x_2[; (x_1, x_2) \in \mathcal{E}_m\}.$$

with the convention $\sup \emptyset = 0$. According to the previous proposition, the function $\hat{b}_1$ is nondecreasing. Moreover, the convexity of P_m^∞ implies the concavity of $\hat{b}_1$. We wonder whether $\hat{b}_1$ is positive for every x_2 ? The next proposition gives an answer.

Proposition 4.7 *If* $\delta_2 > 0$ *then* $\hat{b}_1(x) > 0$ *for every* $x > 0$.

Proof: Let P_i^∞ be the value function of the perpetual put on the asset i.
We have

$$K - \min(S_\tau^{1,x_1}, S_\tau^{2,x_2}) \leq K - S_\tau^{2,x_2} + (S_\tau^{2,x_2} - S_\tau^{1,x_1})_+ \text{ for every stopping times } \tau$$

Hence,

$$P_m^\infty(x_1, x_2) \leq P_2^\infty + C_e^\infty(x_1, x_2).$$

There is a real number x_2^* such that for every $x_2 \leq x_2^*$, we have $P_m^2 = K - x_2$.
Choose $x_1 \leq \dfrac{x_2}{\beta^*} \leq \dfrac{x_2^*}{\beta^*}$ where β^* has been previously defined
$P_m^\infty(x_1, x_2) = K - x_1$.
thus, $\hat{b}_1(x_2) > 0$ for every $x_2 \leq x_2^*$. Proposition 4.6 i) allows us to conclude.•
We are interested in the asymptotic behavior of the critical surface.
Define the critical stock-price of the perpetual put on the asset i by

$$x_i^* = \sup\{x_i > 0, P_i^\infty(x_i) = K - x_i\}.$$

We have the following proposition which gives the asymptotic behavior of the critical surface.

Proposition 4.8 *We have* $\lim\limits_{x_2 \to +\infty} \hat{b}_1(x_2) = x_1^*$.

The inequality $\lim_{x_2\to+\infty} \hat{b}_1(x_2) \leq x_1^*$ is immediate thanks to $P_1^\infty \leq P_m^\infty(x_1, x_2)$.
Indeed, take $x_2 > x_1 > x_1^*$, we have

$$\begin{aligned} P_m^\infty(x_1, x_2) &> P_1^\infty(x_1) \\ &> K - x_1 \\ &= K - \min(x_1, x_2). \end{aligned}$$

Thus, $\hat{b}_1(x_2) < x_1$ for every $x_1 > x_1^*$
Hence,

$$\mathcal{E}_m^1 \subset \{(x_1, x_2) \in]0, +\infty[^2 \mid x_1 \leq x_1^*\}.$$

In the same manner, we can prove that

$$\mathcal{E}_m^2 \subset \{(x_1, x_2) \in]0, +\infty[^2 \mid x_2 \leq x_2^*\}.$$

To show the converse inequality, we need two lemmas.
Let $(x_2^{(n)})_{n\geq 0}$ a sequence satisfying $\lim_{n\to\infty} x_2^{(n)} = +\infty$. We define the sequence $(\tau_n)_{n\in N}$ of stopping times by setting

$$\tau_n = \inf\{t \geq 0; (S_t^{1,x_1^*}, S_t^{2,x_2^{(n)}}) \in \mathcal{E}_m\}. \tag{4}$$

By the optimal stopping theory,

$$P_m^\infty(x_1^*, x_2^{(n)}) = E\left(e^{-r\tau_n}(K - \min(S_{\tau_n}^{1,x_1^*}, S_{\tau_n}^{2,x_2^{(n)}}))_+\right).$$

Lemma 4.2

$$\lim_{n\to+\infty} \left(P_m^\infty(x_1^*, x_2^{(n)}) - E\left(e^{-r\tau_n}(K - S_{\tau_n}^{1,x_1^*})_+\right)\right) = 0$$

Proof of the lemma: for every $n \in N$, we have

$$\begin{aligned} \left|P_m^\infty(x_1^*, x_2^{(n)}) - E\left(e^{-r\tau_n}(K - S_{\tau_n}^{1,x_1^*})_+\right)\right| &\leq E\left(e^{-r\tau_n}(S_{\tau_n}^{1,x_1^*} - S_{\tau_n}^{2,x_2^{(n)}}))_+\right) \\ &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}} E\left(e^{-r\tau}(S_\tau^{1,x_1^*} - S_\tau^{2,x_2^{(n)}}))_+\right) \\ &= \Theta(x_1^*, x_2^{(n)}). \end{aligned}$$

Gerber and Shiu [7] have explicitly computed the function Θ and it can be checked using their formula that $\lim_{x_2\to+\infty} \Theta(x_1, x_2) = 0$ for every x_1. •

Remark 4.4 *Proceeding analogously from the previous proposition, we can prove for every* x_1

$$\lim_{x_2\to+\infty} P_m^\infty(x_1, x_2) = P_1^\infty(x_1).$$

In particular, we have for $x_1 \leq x_1^*$

$$\lim_{x_2\to+\infty} P_m^\infty(x_1, x_2) = K - x_1.$$

Lemma 4.3 *The sequence $(\tau_n)_{n\in N}$ converges to 0 in probability .*

Proof of the lemma: Thanks to lemma 4.2, we deduce from remark 4.4 that

$$\lim_{n\to+\infty} E\left(e^{-r\tau_n}(K-S^{1,x_1^*}_{\tau_n})_+\right) = K - x_1^*.$$

But,

$$\begin{aligned} E\left(e^{-r\tau_n}(K-S^{1,x_1^*}_{\tau_n})_+\right) &\leq E\left(e^{-r\tau_n}P_1^\infty(S^{1,x_1^*}_{\tau_n})\right) \\ &= K - x_1^* + E\int_0^{\tau_n} e^{-ru}(\delta_1 S_u^{1,x_1^*} - rK)1_{\{S_u^{1,x_1^*}\leq x_1^*\}}\,du. \end{aligned}$$

where for the second equality, we used the generalized Ito's formula (see Krylov [14]). But, according to theorem 3.2 we have $x_1^* < \dfrac{rK}{\delta_1}$ and thus

$$E\int_0^{\tau_n} e^{-ru}(\delta_1 S_u^{1,x_1^*} - rK)1_{\{S_u^{1,x_1^*}\leq x_1^*\}}\,du \leq (\delta_1 x_1^* - rK)E\int_0^{\tau_n} 1_{\{B_s+\mu s\leq 0\}}\,ds < 0$$

where $B_s = \sigma_{11}W_s^1 + \sigma_{12}W_s^2$ is a nonstandard one-dimensional Brownian motion and $\mu = r - \frac{\sigma_{11}^2}{2} - \frac{\sigma_{12}^2}{2}$.
We obtain,

$$\lim_{n\to+\infty} E\int_0^{\tau_n} 1_{\{B_s+\mu s\leq 0\}}\,ds = 0.$$

We end the proof thanks to the following deterministic result, the proof of which is left to the reader.

Lemma 4.4 *Let $f: R^+ \to R$ be a continuous function satisfying the condition :*
For every $\epsilon > 0$ there is $t \in]0,\epsilon[$ such that $f(t) > 0$.
Then every sequence $(t_n)_{n\in N}$ such that $\int_0^{t_n} 1_{\{f(s)>0\}}\,ds Rightarrow 0$ converges to 0.

We are in a position to prove proposition 4.8.
Proof of proposition 4.8:
We know that the limit exists and is smaller than x_1^* .
Assume $\lim_{x_2\to+\infty} \hat{b}_1(x_2) < x_1^*$ and let $\epsilon < x_1^* - \lim_{x_2\to+\infty}\hat{b}_1(x_2)$. Fix $M > x_2^*$ and let $(x_2^{(n)})_{n\geq 0}$ a sequence satisfying $\lim_{n\to\infty} x_2^{(n)} = +\infty$ and $(\tau_n)_{n\geq 0}$ the sequence of stopping times defined by 4.
Choose n large enough so that $x_2^{(n)} > M$ and introduce the following stopping times,

$$\begin{aligned} \tau_M^{(2),n} &= \inf\{t\geq 0; S_t^{2,x_2^{(n)}} \leq M\},\ \text{for } M > x_2^* \\ \text{and } \tau_\epsilon^{(1)} &= \inf\{t\geq 0; S_t^{1,x_1^*} \leq x_1^* - \epsilon\}. \end{aligned}$$

Using the inclusion

$$\mathcal{E}_m^1 \subset \{(x_1,x_2)\in]0,+\infty[^2 \mid x_1 \leq x_1^* - \epsilon \text{ or } x_2 \leq x_2^*\}$$

we have

$$\tau_n > \tau_M^{(2),n} \wedge \tau_\epsilon^{(1)} \text{ for every } n.$$

Obviously, $\lim_{n\to+\infty} \tau_M^{(2),n} = +\infty$ so that $\lim_{n\to+\infty} \tau_M^{(2),n} \wedge \tau_\epsilon^{(1)} = \tau_\epsilon^{(1)}$ and thus $\lim_{n\to+\infty} \tau_n \geq \tau_\epsilon^{(1)}$ which contradicts the lemma 4.3.

Acknowledgment

I would like specially to thank my supervisor Professor Damien Lamberton for his constant guidance and encouragement.

References

[1] Barles G., Burdeau J., Romano M. et Sansoen N. : Critical Stock Price near expiration, *Mathematical finance* **5**, 77-95 (1995).

[2] Bensoussan A, Lions J.L.: *Applications of Variational Inequalities in Stochastic Control*, North-Holland 1982.

[3] Bensoussan A.: On the theory of option pricing, *Acta Applicandae Mathematicae* **2**, 139-158 (1984).

[4] Broadie M, Detemple J.: The Valuation of American Options on Multiple Assets, *Mathematical Finance*, Vol 7, **3**, 241-286 (1996).

[5] Carr P, Jarrow R, Myneni R.: Alternative Characterisation of American Put Options, *Mathematical Finance* **2**, 87-106 (1992).

[6] Friedman A.: Parabolic variational inequalities in one space dimension and smoothness of the free boundary, *Journal of functional analysis* **18**, 151-176 (1975).

[7] Gerber H, Shiu E.: Martingale approach to pricing perpetual American options on two stocks, *Mathematical Finance* **3**, 303-322 (1996).

[8] Jacka S.D.: Optimal stopping and the American put, *Mathematical finance* **1**, 1-14 (1991).

[9] Jaillet P, Lamberton D, Lapeyre B.: Variational inequalities and the pricing of American options, *Acta Appl.Math.* **21**, 263-289 (1990).

[10] Karatzas I.: On the pricing of American options, *Applied Math. Optimization* **17**, 37-60 (1988).

[11] Karatzas I.: Optimization problems in the theory of continuous trading, *SIAM J. Control Optimization.* **27**, 1221-1259 (1989).

[12] Karatzas I, Shreve S.E.: *Brownian Motion and Stochastic Calculus*, Springer 1988.

[13] Kim I.J.: The analytic valuation of American options , *Review of Financial Studies* **3**, 547-572 (1990).

[14] Krylov N.V.: *Controlled Diffusion Process*, Springer Verlag 1980.

[15] Lamberton D, Lapeyre B.: *Introduction to Stochastic Calculus applied to finance*, Chapman and Hall 1996.

[16] Merton R.C.: Theory of rational option pricing , *Bell Journal of Economics and Management Science* **1**, 141-183 (1973).

[17] Myneni R.: The pricing of the American option, *Annals of applied probability* **2**, 1-23 (1992).

[18] Rudin W.: *Functional Analysis*, McGraw-Hill, Inc 1991.

[19] Tan K, Vetzal K: Early exercice region for exotic options, *The Journal of Derivatives* **3**, 42-55 (1994).

[20] Villeneuve S : Current PhD thesis, Université de Marne-la-Vallée.

[21] Webster R.: *Convexity*, Oxford University Press 1994.