Weak solutions to the equations of motion for compressible magnetic fluids

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Abstract

We study the differential system governing the flow of a compressible magnetic fluid under the action of a magnetic field. The system is a combination of the compressible Navier–Stokes equations, the angular momentum equation, the magnetization equation and the magnetostatic equations. We prove global-in-time existence of weak solutions with finite energy to the system posed in a bounded domain of $\mathbb{R}^3$ and equipped with initial and boundary conditions.

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1. Introduction and main result

Magnetic fluids (also called ferrofluids) are colloidal suspensions of fine magnetic mono domain nanoparticles in nonconducting liquids. Such fluids have found a wide variety of applications in engineering: magnetic liquid seals, cooling and resonance damping for loudspeaker coils, printing with magnetic inks, rotating shaft seals in vacuum chambers used in the semiconductor industry, see [48] for more details. Magnetic fluids are now days attracting much interest because of their potentiality for many applications such as magnetic separation, drugs or radioisotopes targeted...
by magnetic guidance, hyperthermia treatments, magnetic resonance imaging contrast enhancement, see for instance [21,35].

A number of works show that magnetic fluids can be treated as homogeneous monophase fluids, see [36–38,43] and the references therein. Consider the flow of a compressible and viscous, Newtonian magnetic fluid occupying a domain $D \subset \mathbb{R}^3$, under the action of an applied magnetic field $H_{\text{ext}}$. The field $H_{\text{ext}}$ induces a demagnetizing field $H$ and a magnetic induction $B$ satisfying the law $B = H + \chi(D)M$ where $M$ is the magnetization inside $D$ and $\chi(D)$ denotes the characteristic function of $D$. Let $T > 0$ be a fixed time, $D_T = (0, T) \times D$, $\Gamma_T = (0, T) \times \Gamma$ and let $n$ denote the outward unit normal to $D$. A set of equations describing the flow is proposed by R.E. Rosensweig [38] and it consists of:

- the continuity equation,
  \[ \partial_t \rho + \text{div}(\rho U) = 0 \quad \text{in } D_T, \quad (1) \]
- the linear momentum equation,
  \[ \partial_t (\rho U) + \text{div}(\rho U \otimes U) - \mu \Delta U - (\lambda + \mu) \nabla \text{div} U + \nabla (p(\rho, M)) = R \quad \text{in } D_T, \quad (2) \]
- the angular momentum equation,
  \[ \partial_t (\rho \Omega) + \text{div}(\rho U \otimes \Omega) - \mu' \Delta \Omega - (\lambda' + \mu') \nabla \text{div} \Omega = S \quad \text{in } D_T, \quad (3) \]
- the magnetization equation,
  \[ \partial_t M + \text{div}(U \otimes M) + \frac{1}{\tau}(M - \chi_0 H) = \Omega \times M \quad \text{in } D_T. \quad (4) \]

The right-hand sides of (2) and (3) are given by:

\[ R = \mu_0 M \cdot \nabla H - \zeta \text{curl(curl } U - 2\Omega), \quad (5) \]
\[ S = \mu_0 M \times H + 2\zeta (\text{curl } U - 2\Omega). \quad (6) \]

Here $U$ is the fluid velocity, $p = p(\rho, M)$ is the pressure depending on the density $\rho$ and the magnetization $M$, $\Omega$ is the angular velocity, and the parameters $\lambda$, $\mu$, $\lambda'$, $\mu'$, $\chi_0$, $\mu_0$, $\zeta$ and $\tau$ are positive and their physical meaning can be found in [37,38,42,43] for example. The magnetic field $H$ satisfies the magnetostatic equations:

\[ \text{curl } H = 0, \quad \text{div } B = -\text{div } H_{\text{ext}} \quad \text{in } (0, T) \times \mathbb{R}^3. \quad (7) \]

The equations of motion in the incompressible case have been studied recently. In [47], S. Venkatasubramanian and P. Kaloni consider the differential system introduced by R.E. Rosensweig [37,38] and study the stability and uniqueness of smooth solutions of the system. In [1], we consider the differential system introduced by R.E. Rosensweig [37,38] (see also [36]) and prove existence of global-in-time weak solutions with finite energy to the system posed in a bounded domain of $\mathbb{R}^3$ and supplemented with initial and boundary conditions. In [2], we consider the differential system introduced by M.I. Shliomis [42] and prove existence of global-in-time weak solutions with finite energy to the system posed in a bounded domain of $\mathbb{R}^3$ and supplemented with initial and boundary conditions.

The study of magnetic fluids differs from magnetohydrodynamics (MHD) that concerns itself with nonmagnetizable but electrically conducting fluids. The set of equations which describe MHD is a combination of the Navier–Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism; see the papers by R. Duvaut and J.-L. Lions [10], M. Sermange and R. Temam [39], J.-F. Gerbeau and C. Le Bris [19,20] and H. Inoue [25] for some results of existence of solutions. Let us also mention some works on equations arising in the theory of micropolar fluids introduced by A.C. Eringen [11] which focuses on the fluids consisting of randomly oriented particles suspended in a viscous medium, when the deformation of fluid particles is ignored. We refer to the papers by G.P. Galdi and S. Rionero [16], G. Lukaszewicz [30], E.E. Ortega-Torres and M.A. Rojas-Medar [34], and the references therein, for some results of existence of solutions.

In this paper we pursue our analysis on magnetic fluid flows and discuss a compressible model. As first step in this study we consider a regularized system where the magnetization equation (4), which is a Bloch type equation, is replaced by the equation:

\[ \partial_t M + \text{div}(U \otimes M) - \sigma \Delta M + \frac{1}{\tau}(M - \chi_0 H) = \Omega \times M \quad \text{in } D_T. \]
which is of Bloch–Torrey type, $\sigma > 0$ being a diffusion coefficient that carry spins. The Bloch–Torrey equations were proposed by H.C. Torrey [45] as a generalization of the Bloch equations to describe situations when the diffusion of the spin magnetic moment is not negligible; see also [18] for the derivation of the Bloch–Torrey equations. We consider, instead of (1)–(7), the system:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho U) &= 0 \quad \text{in } D_T, \\
\partial_t (\rho U) + \text{div}(\rho U \otimes U) - \mu \Delta U - (\lambda + \mu) \nabla(\text{div} U) + \nabla(p(\rho, M)) &= R \quad \text{in } D_T, \\
\partial_t (\rho \Omega) + \text{div}(\rho U \otimes \Omega) - \mu' \Delta \Omega - (\lambda' + \mu') \nabla(\text{div} \Omega) &= S \quad \text{in } D_T, \\
\partial_t M + \text{div}(U \otimes M) - \sigma \Delta M + \frac{1}{\tau}(M - \chi_0 H) &= \Omega \times M \quad \text{in } D_T,
\end{align*}
\]

where the forces $R$ and $S$ are defined by,

\[
\begin{align*}
R &= \mu_0 M \cdot \nabla H - \zeta \text{curl(curl } U - 2\Omega), \\
S &= \mu_0 M \times H + 2\zeta(\text{curl } U - 2\Omega).
\end{align*}
\]

We also assume that the magnetic field $H$ satisfies, instead of (7), the magnetostatic equations:

\[
\begin{align*}
H &= \nabla \psi, \\
\text{div}(H + M) &= F \quad \text{in } D_T,
\end{align*}
\]

where $F$ is a given function in $D_T$ such that $\int_D F \, dx = 0$, for all $t \in [0, T]$.

We assume that the flow obeys the state law (see [38]):

\[
p = p(\rho, M) = p_e(\rho) + p_m(M),
\]

where $p_e$ is the isentropic pressure given by $p_e(\rho) = \rho \gamma$, where $\alpha > 0$ and $\gamma > \frac{3}{2}$ are constants ($\gamma$ is the adiabatic constant) and $p_m$ is the magnetic pressure given by $p_m(M) = \frac{\mu_0}{2} |M|^2$.

System (8)–(15) is equipped with the boundary conditions:

\[
\begin{align*}
U &= 0, \\
M \cdot n &= 0, \\
\Omega &= 0 \quad \text{on } \Gamma_T, \\
\text{curl } M \times n &= 0, \\
H \cdot n &= 0 \quad \text{on } \Gamma_T,
\end{align*}
\]

and the initial conditions

\[
\rho(0) = \rho_0, \quad (\rho U)(0) = V_0, \quad (\rho \Omega)(0) = Q_0, \quad M(0) = M_0 \quad \text{in } D,
\]

where $V_0$ and $Q_0$ have to be at least such that $V_0(x) = Q_0(x) = 0$ whenever $\rho_0(x) = 0$. Note that the boundary conditions in (17) on the magnetization $M$ are consistent with the vectorial Laplace operator, according to the Green formula:

\[
\int_D (-\Delta M) \cdot q \, dx = \int_D (\text{curl } M) \cdot (\text{curl } q) \, dx + \int_D (\text{div } M) (\text{div } q) \, dx
\]

\[
- \int_{\partial D} (\text{curl } M \times n) \cdot q \, dS - \int_{\partial D} (\text{div } M) q \cdot n \, dS.
\]

Note also that the physical boundary conditions on the angular velocity $\Omega$ may be more complicated than that in (16), see [36].

The aim of this paper is to study the weak solvability of problem (8)–(18). We assume that $D$ is an open, bounded and smooth domain of class $C^{2+r}$, $r > 0$. Let $L^p(D)$ and $H^s(D)$ ($1 \leq p \leq \infty$, $s \in \mathbb{R}$) be the usual Lebesgue and Sobolev spaces of scalar-valued functions, respectively. We denote $L^p(D) = (L^p(D))^3$, $H^s(D) = (H^s(D))^3$ and by $\| \cdot \|$ and $(\cdot, \cdot)$ we denote the $L^2$-norm and scalar product, respectively. If $\mathcal{X}$ is a Banach space, by $\| \cdot \|_{L^p(0,T;\mathcal{X})}$ we denote the norm in $L^p(0,T;\mathcal{X})$, and the duality product between $\mathcal{X}'$ (the dual space of $\mathcal{X}$) and $\mathcal{X}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}' \times \mathcal{X}}$ or simply by $(\cdot, \cdot)$ when there is no confusion of notation. We denote by $\mathcal{C}([0,T]; \mathcal{X}')$ the space of functions $v: [0,T] \rightarrow \mathcal{X}'$ which are continuous with respect to the weak topology. We have:

\[
v_n \rightharpoonup v \quad \text{in } C([0,T]; \mathcal{X}') \quad \text{if } \{v_n(t); w\} \rightharpoonup \{v(t); w\}
\]

uniformly with respect to $t \in [0,T]$, for any $w \in \mathcal{X}$.
Denote \( \mathcal{M} = \{q \in \mathbb{L}^2(D); \text{ div } q \in L^2(D), \text{ curl } q \in L^2(D), q \cdot n = 0 \text{ on } \partial D \} \), the Hilbert space equipped with the scalar product:

\[
\langle q_1, q_2 \rangle = \int_D q_1 \cdot q_2 \, dx + \int_D (\text{ div } q_1) (\text{ div } q_2) \, dx + \int_D (\text{ curl } q_1) \cdot (\text{ curl } q_2) \, dx
\]

and the associated norm. We have

\[
\mathcal{M} = \{q \in \mathbb{H}^1(D); q \cdot n = 0 \text{ on } \partial D \},
\]

and \( \| \cdot \|_{\mathcal{M}} \) and \( \| \cdot \|_{\mathbb{H}^1(D)} \) are two equivalent norms on the space \( \mathcal{M} \) (see [7], pp. 244–249). We assume that \( M_0 \in \mathcal{M} \) and \( F \in H^1(0, T; L^2(D)) \) with \( (F(t); 1) = 0 \) in \( (0, T) \) and denote \( H_0 = \nabla \varphi_0 \) where \( \varphi_0 \) is the unique weak solution in \( H^1(D) \) of

\[
\Delta \varphi_0 = -\text{ div } M_0 + F_0 \quad \text{in } D, \quad \frac{\partial \varphi_0}{\partial n} = 0 \quad \text{on } \Gamma_T, \quad (\varphi_0; 1) = 0,
\]

and \( F_0 = F(0) \).

The existence of solutions of the equations for viscous compressible flows have been the subject of many studies, see for instance the papers by A.V. Kazhikhov and V.V. Shelukhin [26], A. Matsumura and T. Nishida [31], D. Serre [40,41], D. Hoff [22–24], V.A. Vaigant and A.V. Kazhikhov [46], R. Danchin [6]. P.-L. Lions [29] has developed a theory for viscous compressible barotropic fluids; he proved the global-in-time existence of solutions in any space dimension, without any restrictive assumption on the initial data. The adiabatic constant \( \gamma \) is such that \( \gamma \geq 3/2 \) if the dimension space \( d = 2 \), \( \gamma \geq 9/5 \) if \( d = 3 \), and \( \gamma > d/2 \) if \( d \geq 4 \). This result has been extended later by E. Feireisl, A. Novotný and H. Petzeltová [12] (see also [13]) to the case \( \gamma > d/2 \), and recently by E. Feireisl [14] for variational solutions of the full system of the Navier–Stokes equations with viscosity coefficients depending on the temperature. Let us also mention the recent works of D. Bresch and B. Desjardins [3–5] and A. Mellet and A. Vasseur [32] dealing with barotropic compressible Navier–Stokes equations with density dependent viscosity coefficients. In [9], B. Ducomet and E. Feireisl prove existence of global-in-time weak solutions to the equations of MHD, specifically, the Navier–Stokes–Fourier system describing the evolution of a compressible, viscous and heat conductive fluid coupled with the Maxwell equations.

Here we use ideas and techniques in the books of P.-L. Lions [28,29], E. Feireisl [15] and A. Novotný and I. Straškraba [33] to construct a solution of problem (8)–(18). We begin by recalling the following definition introduced by R.J. DiPerna and P.-L. Lions [8]. We will say that \( \rho \) is a renormalized solution of the continuity equation (8) in \( D_T \) if the integral identity,

\[
\int_{D_T} \left( (b(\rho))^2 \varphi + b(\rho)U \cdot \nabla \varphi + (b(\rho) - b'(\rho)\rho) \text{ div } U \varphi \right) \, dx \, dt = 0,
\]

holds for any function \( b \in C^1[0, \infty) \) such that \( b'(\rho) = 0 \) for all \( \rho \) large enough, and for any test function \( \varphi \in C^\infty(\overline{D_T}) \) such that \( \varphi(0) = \varphi(T) = 0 \).

**Definition 1.** We will say that \((\rho, U, \Omega, M, H)\) is a weak solution with finite energy of problem (8)–(18) if the conditions (i)–(vii) below are satisfied:

(i) the density \( \rho \) belongs to \( L^\infty(0, T; L^\gamma(D)) \cap C([0, T]; L^4(D)) \), \( \rho \geq 0 \) a.e. in \( D_T \), the velocity \( U \) belongs to \( L^2(0, T; \mathbb{H}^1(D)) \), \( \sqrt{\rho} U \) belongs to \( L^\infty(0, T; L^2(D)) \), the momentum \( \rho U \) belongs to the space \( C([0, T]; \ell^{2\gamma/\gamma+1}(D)) \), the pressure \( p \) belongs to \( L^{1}(0, T; L^{4}(D)) \);

(ii) the density \( \rho \) is a renormalized solution of the continuity equation (8) in \((0, T) \times \mathbb{R}^3 \) provided \( \rho \) and \( U \) were extended by zero outside \( D \) and the momentum equation (9) holds in \( D'(D_T) \);

(iii) the angular velocity \( \Omega \) belongs to \( L^2(0, T; \mathbb{H}^1(D)) \), \( \sqrt{\rho} \Omega \) belongs to \( L^\infty(0, T; L^2(D)) \), the angular momentum \( \rho \Omega \) belongs to \( C([0, T]; \ell^{2\gamma/\gamma+1}(D)) \), and the angular momentum equation (10) holds in \( D'(D_T) \);
(iv) the magnetization \( M \) belongs to \( C([0, T]; 1^2_{\text{weak}}(D)) \cap L^2(0, T; \mathcal{M}) \) and the integral identity:
\[
\frac{d}{dt} \int_M M \cdot q \, dx - \int_M (U \otimes M) \cdot \nabla q \, dx + \sigma \int_M (\text{curl} \, M) \cdot (\text{curl} \, q) \, dx + \sigma \int_M (\text{div} \, M)(\text{div} \, q) \, dx
\]
\[
= \int_M (\Omega \times M) \cdot q \, dx - \frac{1}{\tau} \int_M (M - \chi_0 H) \cdot q \, dx \quad \text{in} \ D'([0, T])
\]
\[(21)\]
holds for every \( q \in \mathcal{M} \);

(v) the magnetic field \( H \) is such that \( H = \nabla \varphi \) where \( \varphi \in L^\infty(0, T; H^1(D)) \cap L^2(0, T; H^2(D)) \) and solves the problem,
\[
-\Delta \varphi = \text{div} \, M - F \quad \text{in} \ D_T,
\]
\[(22)\]
\[
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \ \partial D \times (0, T), \quad \int \varphi \, dx = 0 \quad \text{in} \ (0, T);
\]
\[(23)\]

(vi) the functions \( \rho, \rho \, U, \rho \, \Omega \) and \( M \) satisfy, for any \( \psi \in D(D) \) and \( \phi \in (D(D))^3 \),
\[
\lim_{t \to 0} \int_D \rho(t)\psi \, dx = \int_D \rho_0 \psi \, dx, \quad \lim_{t \to 0} \int_D (\rho \, U)(t)\phi \, dx = \int_D V_0 \phi \, dx,
\]
\[
\lim_{t \to 0} \int_D (\rho, \Omega)(t)\phi \, dx = \int_D Q_0 \phi \, dx, \quad \lim_{t \to 0} \int_D M(t)\phi \, dx = \int_D M_0 \phi \, dx;
\]

(vii) the energy inequality,
\[
\mathcal{E}(t) + C_1 \int_0^t \mathcal{E}^d(s) \, ds \leq \mathcal{E}_0 + C_2 \int_0^t (1 + \| F(s) \|^2 + \| \partial_t F(s) \|^2) \, ds,
\]
\[(24)\]
holds for a.e. \( t \in (0, T) \). Here, \( \mathcal{E}(t), \mathcal{E}_0 \) and \( \mathcal{E}^d(t) \) denote the kinetic energy at time \( t \), the initial kinetic energy and the dissipated energy at time \( t \) defined by:
\[
\mathcal{E}(t) = \int_D \rho(t) \left( \frac{1}{2} |U|^2 + \frac{1}{2} |\Omega|^2 + P_e(\rho) \right) \, dx + \frac{\mu_0}{2} \int_D (|H|^2 + |M|^2) \, dx,
\]
\[(25)\]
\[
\mathcal{E}_0 = \int_D \left( \frac{1}{2} \frac{|V_0|^2}{\rho_0} + \frac{1}{2} \frac{|Q_0|^2}{\rho_0} + \rho_0 P_e(\rho_0) \right) \, dx + \frac{\mu_0}{2} \int_D (|H_0|^2 + |M_0|^2) \, dx,
\]
\[(26)\]
and
\[
\mathcal{E}^d(t) = \frac{\mu_0}{\tau} \int_D |M|^2 \, dx + \frac{\mu_0(2\chi_0 + 1)}{\tau} \int_D |H|^2 \, dx + \mu \int_D |\nabla U|^2 \, dx
\]
\[
+ \mu' \int_D |\nabla \Omega|^2 \, dx + (\lambda + \mu) \int_D |\text{div} \, U|^2 \, dx + (\lambda' + \mu') \int_D |\text{div} \, \Omega|^2 \, dx
\]
\[
+ \mu_0 \sigma \left( \int_D |\text{curl} \, M|^2 \, dx + 2 \int_D |\text{div} \, M|^2 \, dx \right) + \xi \int_0^t |\text{curl} \, U - 2\Omega|^2 \, dx,
\]
\[(27)\]
where \( P_e \) is the internal energy function defined by,
\[
P_e(\rho) = \frac{a}{\gamma - 1} (\rho^{\gamma - 1} - 1),
\]
\[(28)\]
and $C_1$ and $C_2$ are positive constants depending only on the domain $D$ and the physical constants $\lambda$, $\mu$, $\lambda'$, $\mu'$, $\mu_0$, $\xi$, $\tau$, $\sigma$ and $\chi_0$.

Our main result is the following:

**Theorem 1.** Let $M_0 \in \mathcal{M}$ and $F \in H^1(0, T; L^2(\Omega))$ with $(F(t); 1) = 0$ in $(0, T)$. Assume that, for $\gamma > \frac{3}{2}$,

\[
\rho_0 \in L^\gamma(D), \quad \rho_0 \geq 0 \quad \text{a.e. in } D,
\]
\[
V_0 \in L^{2\gamma}(D), \quad \frac{|V_0|^2}{\rho_0} \in L^1(D),
\]
\[
Q_0 \in L^{2\gamma}(D), \quad \frac{|Q_0|^2}{\rho_0} \in L^1(D).
\]

Then there exists a weak solution with finite energy of problem (8)–(18), in the sense of Definition 1.

To obtain the existence of a (global) weak solution with finite energy, we start by deriving a formal energy inequality and a priori estimates satisfied by any smooth solution of problem (8)–(18). Then we follow the ideas and techniques of P.-L. Lions [29], E. Feireisl [15], and A. Novotný and I. Straškraba [33] to construct a solution of problem (8)–(18).

We introduce a regularized problem (problem (43)–(56)) depending on two small positive parameters $\varepsilon$ and $\delta$ and consisting of:

(i) a regularization of the continuity equation by adding the term $\varepsilon \Delta \rho$ in its right-hand side;

(ii) a regularization of the momentum equation by adding to its left-hand side the terms $\nabla(\delta \rho \beta)$ and $\varepsilon \nabla U \cdot \nabla \rho$;

(iii) a regularization of the angular momentum equation by adding to its left-hand side the term $\varepsilon \nabla \Omega \cdot \nabla \rho$;

(iv) a regularization of the initial data.

The magnetization equation and the magnetostatic equations are not modified. The quantity $\nabla(\delta \rho \beta)$ represents an artificial gradient pressure and the terms $\varepsilon \nabla U \cdot \nabla \rho$ and $\varepsilon \nabla \Omega \cdot \nabla \rho$ are introduced in order to obtain an energy inequality close to (24).

To solve the regularized problem (43)–(56) we use a semi-Galerkin approximation. The a priori estimates obtained formally (in Section 2) for exact solutions are compatible with our approximation scheme. We pass to the limit, first as $\varepsilon \to 0$ and then as $\delta \to 0$, on the solutions of the regularized problem (43)–(56), employing compactness arguments developed by P.-L. Lions [29] and E. Feireisl [15] for the equations of compressible barotropic flows. We thus prove the existence of a weak solution with finite energy of problem (8)–(18), in the sense of Definition 1.

In the paper, $C$ indicates a generic constant, depending only on some bounds of the physical data, which can take different values in different occurrences. By $C(\varepsilon)$ we denote a generic constant, depending only on $\varepsilon$ and on some bounds of the physical data; similarly, $C(\delta)$ (respectively $C(\varepsilon, \delta)$) denotes a generic constant, depending only on $\delta$ (respectively on $\varepsilon$ and $\delta$) and on some bounds of the physical data.

2. Formal energy inequality and a priori estimates

We assume in this section that the solutions $(\rho, U, \Omega, M, H)$ to Eqs. (8)–(18) are smooth enough.

2.1. Total mass conservation and $L^\infty$ estimates of the density

The continuity equation (8) integrated with respect to the space variable implies:

\[
\int_D \rho(t, x) \, dx = \int_D \rho_0(x) \, dx, \quad \text{for all } t \in (0, T),
\]

that is the total mass of the fluid is conserved. The continuity equation furnishes also the classical estimates,
\[
\left( \inf_{x \in D} \rho_0(x) \right) \exp \left( - \int_0^t \left\| \text{div} U(s) \right\|_{L^\infty(D)} \, ds \right) \leq \rho(t, x) \\
\leq \left( \sup_{x \in D} \rho_0(x) \right) \exp \left( \int_0^t \left\| \text{div} U(s) \right\|_{L^\infty(D)} \, ds \right)
\]
for all \( t \in [0, T], \ x \in D, \)
from which we deduce, since \( \rho_0 \) is nonnegative, that \( \rho \) is nonnegative.

### 2.2. Energy inequality

Consider first the magnetostatic equations. We have \( H = \nabla \varphi \) and \( \varphi \) is the solution of (22) and (23). Multiplying (22) by \( \varphi \) and integrating by parts yields:

\[
\| H \|_2^2 = - \int_D M \cdot H \, dx - \int_D F \varphi \, dx. \tag{29}
\]

Differentiating (22) with respect to \( t \), multiplying the result by \( \varphi \) and integrating by parts we get:

\[
\frac{d}{dt} \| H \|_2^2 = -2 \int_D \partial_t M \cdot H \, dx - 2 \int_D \partial_t F \varphi \, dx. \tag{30}
\]

Then we multiply the linear momentum equation (9) by \( U \) and integrate over \( D \). After integration by parts we get:

\[
\int_D \partial_t (\rho U) \cdot U \, dx = \int_D (\rho U \otimes U) \cdot \nabla U \, dx - \int_D p(\rho, M) \text{div} U \, dx + \mu \int_D |\nabla U|^2 \, dx + (\lambda + \mu) \int_D |\text{div} U|^2 \, dx.
\]

As usual, using integration by parts and the continuity equation, we rewrite the first two terms of (31) as

\[
\int_D \partial_t (\rho U) \cdot U \, dx = \int_D (\rho U \otimes U) \cdot \nabla U \, dx = \frac{d}{dt} \int_D \frac{1}{2} \rho |U|^2 \, dx. \tag{32}
\]

Using the identity,

\[
p_e(\rho) \text{div} U = - \text{div}(\rho P_e(\rho)U) - \partial_t (\rho P_e(\rho)),
\]
where \( P_e(\rho) \) represents the internal energy defined by (28), we have:

\[
\int_D P(\rho, M) \text{div} U \, dx = - \int_D \partial_t (\rho P_e(\rho)) \, dx + \int_D p_m(M) \text{div} U \, dx. \tag{33}
\]

We transform the first term in the right-hand side of (31) as follows. First we observe that

\[
\int_D M \cdot \nabla H \cdot U \, dx = - \int_D \text{div}(U \otimes M) \cdot H \, dx. \tag{34}
\]

Indeed, using the relation \( \text{curl} H = 0 \), we have,

\[
\int_D M \cdot \nabla H \cdot U \, dx = - \int_D U \cdot \nabla H \cdot M \, dx,
\]
and we deduce (34), using an integration by parts. Then, multiplying the equation of magnetization (11) by \( H \) and integrating over \( D \) yields:

---

\[- \int_D \text{div}(U \otimes M) \cdot H\, dx = \int_D \partial_t M \cdot H\, dx - \int_D (\Omega \times M) \cdot H\, dx + \frac{1}{\tau} \int_{\Omega} (M - \chi_0 H) \cdot H\, dx - \sigma \int_D \Delta M \cdot H\, dx. \]  

(35)

Multiplying the identity,
\[- \Delta M = \text{curl}^2 M - \nabla (\text{div} M),\]
by $H$, integrating by parts, using the boundary condition $\text{curl} M \times n = 0$ on $\Gamma_T$ and the magnetostatic equations (22) and (23), we get:
\[- \int_D \Delta M \cdot H\, dx = \int_D (\text{div} M)(\text{div} H)\, dx = - \int_D |\text{div} M|^2\, dx + \int F \text{div} M\, dx. \]

Reporting this in (35) and using (29) and (30) we get:
\[- \int_D \text{div}(U \otimes M) \cdot H\, dx = - \frac{1}{d} \frac{d}{dt} \int_D |H|^2\, dx - \int_D \partial_t F \varphi\, dx - \int_D (\Omega \times M) \cdot H\, dx - \frac{1}{\tau} \int_{\Omega} (\chi_0 + 1)|H|^2\, dx - \sigma \int_D |\text{div} M|^2\, dx + \sigma \int D F \text{div} M\, dx. \]  

(36)

Finally, combining (31)–(36) we obtain:
\[
\frac{d}{dt} \int_D \left( \rho \left( \frac{1}{2} |U|^2 + P_c(\rho) \right) + \frac{\mu_0^2}{2} |H|^2 \right)\, dx + \frac{\mu_0 (\chi_0 + 1)}{\tau} \int_D |H|^2\, dx + \mu \int_D |\nabla U|^2\, dx \\
- \int_D p_m(M) \text{div} U\, dx + (\lambda + \mu) \int_D |\text{div} U|^2\, dx + \mu_0 \sigma \int_D |\text{div} M|^2\, dx \\
= - \mu_0 \int_D (\Omega \times M) \cdot H\, dx - \frac{\mu_0}{\tau} \int_D F \varphi\, dx - \mu_0 \int_D \partial_t F \varphi\, dx \\
+ \mu_0 \sigma \int_D F \text{div} M\, dx + 2\zeta \int_D (\text{curl} \Omega) \cdot U\, dx - \zeta \int_D |\text{curl} U|^2\, dx. \]  

(37)

Consider now the angular momentum equation (10). Multiplying (10) by $\Omega$, integrating by parts and using the identities,
\[
\int_D \partial_t (\rho \Omega) \cdot \Omega\, dx = \int_D (\rho U \otimes \Omega) \cdot \nabla \Omega\, dx = \frac{d}{dt} \int_D \frac{1}{2} \rho |\Omega|^2\, dx, \]

and
\[
(\Omega \times M) \cdot H = \Omega \cdot (M \times H), \quad \int_D (\text{curl} \Omega) \cdot U\, dx = \int_D (\text{curl} U) \cdot \Omega\, dx, \]
we find:
\[
\frac{d}{dt} \int_D \frac{1}{2} \rho |\Omega|^2\, dx + \mu' \int_D |\nabla \Omega|^2\, dx + (\lambda' + \mu') \int_D |\text{div} \Omega|^2\, dx \\
= \mu_0 \int_D (\Omega \times M) \cdot H\, dx + 2\zeta \int_D ((\text{curl} \Omega) \cdot U - 2|\Omega|^2)\, dx. \]  

(38)

Then, multiplying the magnetization equation (11) by $\mu_0 M$, integrating by parts and using (29) we find:
\[
\frac{d}{dt} \left( \frac{\mu_0}{2} \|M\|^2 \right) + \frac{\mu_0}{\tau} (\|M\|^2 + \chi_0 \|H\|^2) + \frac{\mu_0}{2} \int_D |M|^2 \text{div} \, U \, dx + \mu_0 \sigma (\|\text{curl} \, M\|^2 + \|\text{div} \, M\|^2) \\
= -\frac{\chi_0}{\tau} \int_D F \varphi \, dx.
\]

Adding (37)–(39) and using the relation \( p_m(M) = \frac{\mu_0}{2} |M|^2 \), we obtain the equality:
\[
\frac{d}{dt} \mathcal{E}(t) + \mathcal{E}^d(t) = S(t), \quad t \in (0, T),
\]
where \( \mathcal{E}(t) \) and \( \mathcal{E}^d(t) \) are defined by (25) and (27), respectively, and
\[
S(t) = -\frac{\mu_0}{\tau} (1 + \chi_0) \int_D F \varphi \, dx - \mu_0 \int_D \partial_t F \varphi \, dx + \mu_0 \sigma \int_D F \text{div} \, M \, dx.
\]
Integration of (40) from 0 to \( t \) gives the energy equality:
\[
\mathcal{E}(t) + \int_0^t \mathcal{E}^d(s) \, ds = \mathcal{E}_0 + \int_0^t S(s) \, ds,
\]
where \( \mathcal{E}_0 \) is defined by (26). Estimating \( S(t) \) with use of the Young inequality and the Poincaré inequality \( \|\varphi\|_{L^2(D)} \leq C \|\nabla \varphi\| = C \|H\| \), we deduce from (41) the energy inequality (24).

**Remark 1.** The attempt to extend the previous calculations to a case of more general magnetic pressure encounters a technical complication. Assume that the magnetic pressure \( p_m(M) \) is in the form \( p_m(M) = p_{0m}(|M|^2) \) where \( p_{0m} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a smooth function. Consider \( \phi_m(M) = \phi_{0m}(|M|^2) \) where \( \phi_{0m} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is also a smooth function. Multiplying the magnetization equation (11) by \( \nabla M \phi(M) \) we get, after elementary calculations,
\[
\partial_t \phi_m(M) + \text{div} \left( \phi_m(M) U \right) + \left( \text{div} \left( \phi_m(M) \right) \right) + \left( \nabla M \phi_m(M) \right) - \sigma H_{\text{ess}}(\phi_m) \nabla M \cdot \nabla M, \quad (42)
\]
where \( H_{\text{ess}}(\phi_m) \) denotes the Hessian matrix of \( \phi_m \). We choose \( \phi_{0m} \) to satisfy the ordinary differential equation:
\[
2 \tau \phi_{0m}'(r) - \phi_{0m}(r) = p_{0m}(r),
\]
that is
\[
\left( \frac{\phi_{0m}(r)}{r^{1/2}} \right)' = \frac{p_{0m}(r)}{2r^{3/2}},
\]
then
\[
\phi_{0m}(r) = \frac{r^{1/2}}{2} \int_r^0 \frac{p_{0m}(s)}{2s^{3/2}} \, ds.
\]
Note that if \( p_{0m}(r) = r \) we recover \( \phi_{0m}(r) = r \). When \( p_{0m}(r) \neq r \) there are technical difficulties to estimate the terms in the right hand-side of (42). We cannot use (as above) the magnetostatic equations to estimate the term \(-\frac{1}{\tau} (M - \chi_0 H) \cdot \nabla M \phi_m(M) \). The boundary conditions on \( M \) cause also complications.

3. A regularized problem

Let \( \varepsilon > 0 \), \( \delta > 0 \) and \( \beta > 0 \) be fixed. We consider the system formed by the coupled equations:

- Continuity equation with vanishing artificial viscosity,
\[ \partial_t \rho + \text{div}(\rho U) = \varepsilon \Delta \rho \quad \text{in } D_T, \]  
\[ \frac{\partial \rho}{\partial n} = 0 \quad \text{on } \partial D, \]  
\[ \rho(0) = \rho_{0, \delta} \quad \text{in } D. \] 

- Linear momentum equation with artificial pressure,
\[ \partial_t (\rho U) + \text{div}(\rho U \otimes U) - \mu \Delta U - (1 + \mu) \nabla (\text{div} U) + \nabla \left( p(\rho, M) + \delta \rho \beta \right) + \varepsilon \nabla U \cdot \nabla \rho = R \quad \text{in } D_T, \]  
\[ U = 0 \quad \text{on } \partial D, \]  
\[ (\rho U)(0) = V_{0, \delta} \quad \text{in } D, \] 

with \( R \) defined by (12).

- Regularized angular momentum equation,
\[ \partial_t (\rho \Omega) + \text{div}(\rho U \otimes \Omega) - \mu' \Delta \Omega - (1 + \mu') \nabla (\text{div} \Omega) + \varepsilon \nabla \Omega \cdot \nabla \rho = S \quad \text{in } D_T, \]  
\[ \Omega = 0 \quad \text{on } \partial D, \]  
\[ (\rho \Omega)(0) = Q_{0, \delta} \quad \text{in } D, \] 

with \( S \) defined by (13).

- Magnetization equation,
\[ \partial_t M + \text{div}(U \otimes M) - \alpha M + \frac{1}{\tau} (M - \chi_0 H) = \Omega \times H \quad \text{in } D_T, \]  
\[ M \cdot n = 0 \quad \text{on } \partial D, \]  
\[ M(0) = M_0 \quad \text{in } D. \] 

- Magnetostatic equations,
\[ H = \nabla \varphi, \quad \text{div}(H + M) = F \quad \text{in } D_T, \]  
\[ H \cdot n = 0 \quad \text{on } \partial D. \] 

The initial data are chosen to satisfy the following properties, similarly as in [15], p. 149. The density \( \rho_{0, \delta} \in C^{2+r}_{0}(\overline{D}), \ r > 0 \), satisfies the homogeneous Neumann boundary condition \( \frac{\partial \rho_{0, \delta}}{\partial n} = 0 \) on \( \Gamma \). Furthermore, we assume:
\[ 0 < \delta \leq \rho_{0, \delta}(x) \leq \delta^{-1/2} \beta \quad \text{for all } x \in D, \]  
\[ \rho_{0, \delta} \to \rho_0 \quad \text{in } L^p(D), \quad \left\| \left\{ x \in D; \rho_{0, \delta}(x) < \rho_0(x) \right\} \right\| \to 0 \quad \text{as } \delta \to 0. \] 

The initial linear momentum \( V_{0, \delta} \) and the initial angular momentum \( Q_{0, \delta} \) are defined as
\[
V_{0, \delta}(x) = \begin{cases} 
V_0 & \text{if } \rho_{0, \delta}(x) \geq \rho_0(x), \\
0 & \text{if } \rho_{0, \delta}(x) < \rho_0(x), 
\end{cases}
\]
and
\[
Q_{0, \delta}(x) = \begin{cases} 
Q_0 & \text{if } \rho_{0, \delta}(x) \geq \rho_0(x), \\
0 & \text{if } \rho_{0, \delta}(x) < \rho_0(x). 
\end{cases}
\]

For later reference, we state the following lemma, see [15], Section 7.3.1 for the proof.

**Lemma 1.** Suppose that the initial condition \( \rho_{0, \delta} \) is positive, belongs to \( C^{2+r}(\overline{D}) \) and satisfies the compatibility condition \( \frac{\partial \rho_{0, \delta}}{\partial n} = 0 \) on \( \Gamma \). Then, for any given \( U \in C([0, T]; C^{2}_{0}(\overline{D})) \), problem (43)–(45) possesses a unique classical solution \( \rho \) satisfying:
\[ \rho \in C([0, T]; C^{2+r}_{0}(\overline{D})), \quad \partial_t \rho \in C([0, T]; C^{r}_{0}(\overline{D})). \]

We also have:
We have the decomposition following result (see [1] for the proof).

\[(\inf_{x \in D} \rho_{0, \delta}(x)) \exp\left(-\int_0^t \| \text{div}\, U(s) \|_{L^\infty(D)} \, ds \right) \leq \rho(t, x) \leq (\sup_{x \in D} \rho_{0, \delta}(x)) \exp\left(\int_0^t \| \text{div}\, U(s) \|_{L^\infty(D)} \, ds \right) \text{ for all } t \in [0, T], \ x \in D. \quad (57)\]

4. The semi-Galerkin approximation

Let \((a_j)_{j \geq 1}\) be a smooth basis of \(H^1_0(D)\). Consider a sequence \((d_j)_{j \geq 1}\) of eigenfunctions of the Laplace operator, associated with the eigenvalues \((\lambda_j)_{j \geq 1}\), satisfying:

\[-\Delta d_j = \lambda_j d_j \text{ in } D, \quad \frac{\partial d_j}{\partial n} = 0 \text{ on } \partial D, \quad \int_D d_j \, dx = 0. \quad (58)\]

The sequence \((d_j)_{j \geq 1}\) is an orthogonal basis of the space \(\{v \in H^1(D): \int_D v \, dx = 0\}\) with respect to the scalar product of \(H^1(D)\). Let \(G\) and \(H\) denote the closed subspaces of the Hilbert space \(M\) defined by:

\[G = \left\{ v \in \left(H^1(D)\right)^3 : \text{div}\, v = 0 \text{ in } D, \ v \cdot n = 0 \text{ on } \partial D \right\},\]

\[H = \left\{ h = \nabla \psi : w \in H^2(D), \ \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial D, \int_D \psi \, dx = 0 \right\}.\]

Let \((c_j)_{j \geq 1}\) denote a smooth orthogonal basis of the space \(G\) with respect to the scalar product \((19)\). We have the following result (see [1] for the proof).

**Lemma 2.** We have the decomposition \(\mathcal{M} = G \oplus H\), with \(G\) and \(H\) orthogonal with respect to the scalar product defined by \((19)\). Moreover, the sequence \((\{c_j\}_{j \geq 1} \cup \{\nabla d_j\}_{j \geq 1}\) is an orthogonal basis of the space \(\mathcal{M}\).

We define an approximate solution \((\rho_n, U_n, \Omega_n, M_n, H_n)\) of problem (43)–(56) by the following scheme. The function \(\rho_n\) will be the solution of

\[\partial_t \rho_n + \text{div}(\rho_n U_n) = \varepsilon \Delta \rho_n \text{ in } D_T, \quad (59)\]

\[\frac{\partial \rho_n}{\partial n} = 0 \text{ on } \partial D, \]

\[\rho_n(0) = \rho_{0, \delta} \text{ in } D, \quad (60)\]

and we look for \(U_n, \Omega_n, M_n, H_n\) in the form:

\[U_n = \sum_{j=1}^n a_j^n(t) a_j, \quad \Omega_n = \sum_{j=1}^n \beta_j^n(t) a_j,\]

\[M_n = \sum_{j=1}^n \gamma_j^n(t) c_j + \sum_{j=1}^n \gamma_{j+n}(t) \nabla d_j,\]

and

\[H_n = \nabla \varphi_n \text{ with } \varphi_n = \sum_{j=1}^n \delta_j^n(t) d_j.\]

Let \(X_n\) (respectively \(Y_n\)) denote the space spanned by \(a_1, \ldots, a_n\) (respectively \(c_1, \ldots, c_n, \nabla d_1, \ldots, \nabla d_n\)). The functions \(a_j^n(t), \beta_j^n(t), \gamma_j^n(t)\) and \(\delta_j^n(t)\) will be found from:

(i) the equation of $U_n$,

\[
\frac{d}{dt} \int_D \rho_n U_n \cdot a_j \, dx - \int_D (\rho_n U_n \otimes U_n) \cdot \nabla a_j \, dx - \int_D (p(\rho_n, M_n) + \delta \rho_n^\beta) \text{div} a_j \, dx \\
+ \mu \int_D \nabla U_n \cdot \nabla a_j \, dx + (\lambda + \mu) \int_D (\text{div} U_n)(\text{div} a_j) \, dx + \varepsilon \int_D \nabla \rho_n \cdot \nabla U_n \cdot a_j \, dx \\
= \mu_0 \int_D M_n \cdot \nabla H_n \cdot a_j \, dx + 2 \zeta \int_D (\text{curl} \, \Omega_n) \cdot a_j \, dx \\
- \zeta \int_D (\text{curl} \, U_n) \cdot \text{curl} a_j \, dx \quad (j = 1, \ldots, n),
\]

with the initial condition

\[U_n |_{t=0} = U_{0,\delta,n};\] (62)

(ii) the equation of $\Omega_n$,

\[
\frac{d}{dt} \int_D \rho_n \Omega_n \cdot a_j \, dx - \int_D (\rho_n U_n \otimes \Omega_j) \cdot \nabla a_j \, dx \\
+ \mu' \int_D \nabla \Omega_n \cdot \nabla a_j \, dx + (\lambda' + \mu') \int_D (\text{div} \, \Omega_n)(\text{div} a_j) \, dx + \varepsilon \int_D \nabla \rho_n \cdot \nabla \Omega_n \cdot a_j \, dx \\
= \mu_0 \int_D (M_n \times H_n) \cdot \Omega_j \, dx + 2 \zeta \int_D (\text{curl} \, U_n - 2 \Omega_n) \cdot \Omega_j \, dx \quad (j = 1, \ldots, n),
\]

with the initial condition

\[\Omega_n |_{t=0} = \Omega_{0,\delta,n};\] (64)

(iii) the equation of $M_n$,

\[
\frac{d}{dt} \int_D M_n \cdot c_j \, dx - \int_D (U_n \otimes M_n) \cdot \nabla c_j \, dx + \sigma \int_D (\text{curl} \, M_n) \cdot (\text{curl} c_j) \, dx \\
= \int_D (\Omega_n \times M_n) \cdot c_j \, dx - \frac{1}{\tau} \int_D (M_n - \chi_0 H_n) \cdot c_j \, dx,
\]

\[\text{with the initial condition}
M_n |_{t=0} = M_{0,n};\] (66)

(iv) the equation of $\varphi_n$,

\[
\int_D \nabla \varphi_n \cdot \nabla d_j \, dx = - \int_D M_n \cdot \nabla d_j \, dx - \int_D F \, d_j \, dx \quad (j = 1, \ldots, n).
\]

(69)
Here $U_{0,\delta,n} \in X_n$ and $\Omega_{0,\delta,n} \in X_n$ are uniquely determined by:

\[
\int_D \rho_{0,\delta} U_{0,\delta,n} \cdot \eta \, dx = \int_D V_{0,\delta} \cdot \eta \, dx \quad \text{for all } \eta \in X_n, \tag{70}
\]

\[
\int_D \rho_{0,\delta} \Omega_{0,\delta,n} \cdot \eta \, dx = \int_D Q_{0,\delta} \cdot \eta \, dx \quad \text{for all } \eta \in X_n, \tag{71}
\]

while $M_{0,n}$ is the orthogonal projection of $M_0$ in the space $L^2(D)$ onto the space $\mathcal{Y}_n$.

4.1. Solvability of problem (59)–(69)

Proposition 1. For any fixed $n$ and $T$, there exist functions:

$$ \rho_n \in C([0, T]; C^{2+r}(\overline{D})) $$

such that $\partial_t \rho_n \in C([0, T]; C^r(\overline{D}))$, $U_n \in C^1([0, T]; X_n)$, $\Omega_n \in C^1([0, T]; X_n)$, $M_n \in C^1([0, T]; \mathcal{Y}_n)$, $H_n \in C^1([0, T]; \mathcal{Y}_n)$, solving problem (59)–(69) on the time interval $[0, T]$.

Proof. It consists in two parts.

(i) Local solvability of problem (59)–(69). Denote $\alpha^n = (\alpha_1^n, \ldots, \alpha_n^n)$, $\beta^n = (\beta_1^n, \ldots, \beta_n^n)$, $\gamma_n = (\gamma_1^n, \ldots, \gamma_2^n)$, $\delta^n = (\delta_1^n, \ldots, \delta_n^n)$ and $\alpha_0 = (\alpha_0^1, \ldots, \alpha_0^n)$, $\beta_0 = (\beta_0^1, \ldots, \beta_0^n)$, $\gamma_0 = (\gamma_0^1, \ldots, \gamma_0^{2n})$ such that

$$ U_{0,\delta,n} = \sum_{j=1}^{n} \alpha_{0j} a_j, \quad \Omega_{0,\delta,n} = \sum_{j=1}^{n} \beta_{0j} a_j, \quad M_{0,n} = \sum_{j=1}^{n} \gamma_{0j} c_j + \sum_{j=n+1}^{2n} \gamma_{0j} \nabla d_{j-n}. $$

Consider the ball of $C([0, T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n})$.

\[
E(T) = \left\{ (\alpha, \beta, \gamma) \in C([0, T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n}) : \max_{0 \leq t \leq T} \left( \| \alpha(t) - \alpha_0 \|_{\mathbb{R}^n} + \| \beta(t) - \beta_0 \|_{\mathbb{R}^n} + \| \gamma(t) - \gamma_0 \|_{\mathbb{R}^{2n}} \right) \leq 1 \right\}
\]

and define the operator:

$$ \mathcal{K}: E(T) \rightarrow C([0, T]; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n}) $$

by the following scheme. Given $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in E(T)$, we first solve problem (59)–(61) for $\rho_n$, with the function $U_n$ substituted by: $\bar{U}_n = \sum_{j=1}^{n} \bar{\alpha}_j(t) a_j$. Then we solve the linear system (69) for $\delta^n$, with the function $M_n$ substituted by:

$$ M_n = \sum_{j=1}^{n} \bar{\gamma}_j(t) c_j + \sum_{j=1}^{n} \bar{\gamma}_{j+n}(t) \nabla d_j. \tag{72} $$

We find $\alpha^n$ from the ordinary differential system (62) and (63), with the functions $M_n$, $\Omega_n$, and $H_n$ substituted respectively by (72), $\bar{\Omega}_n = \sum_{j=1}^{n} \bar{\beta}_j(t) a_j$, and

$$ H_n = \nabla \psi_n = \sum_{j=1}^{n} \delta^n(t) \nabla d_j. \tag{73} $$

We find $\beta^n$ from the ordinary differential system (64) and (65), with $M_n$ and $H_n$ substituted by (72), (73), respectively (and $U_n = \sum_{j=1}^{n} \alpha^n_j(t) a_j$) and we find $\gamma^n$ from the ordinary differential system (66)–(68), with $U_n = \sum_{j=1}^{n} \alpha^n_j(t) a_j$, $\Omega_n = \sum_{j=1}^{n} \beta^n_j(t) a_j$ and $H_n$ given by (73).

By Lemma 1, problem (59)–(61) admits a unique solution $\rho_n$ which is positive. Each of the systems (62), (63), and (64), (65), and (66)–(68) has a maximal solution, then there is $T_n > 0$ such that the operator $\mathcal{K}$ is well defined on $E(T_n)$. Moreover, $\mathcal{K}$ is compact since it maps $E(T_n)$ on a bounded set of $H^1(0, T)$, and $\mathcal{K}$ maps $E(T_n)$ into
itself provided \( T_n \) is small enough. Then one can use the Schauder fixed point theorem to conclude that there exists at least one fixed point \( (\alpha_n, \beta_n, \gamma_n) \). We conclude that problem (59)–(69) has a local solution. Note that \( \rho_n \in C(0, T_n; C^{2+r}(\overline{D})) \), \( \partial_t \rho_n \in C(0, T_n; C^{r}(\overline{D})) \), \( U_n \in C^{1}(0, T_n; X_n) \), \( \Omega_n \in C^1([0, T_n]; X_n) \), \( M_n \in C^1([0, T_n]; Y_n) \) and \( H_n \in C^1(0, T_n; Y_n) \).

(ii) Global solvability of problem (59)–(69). To obtain the global existence we establish a discrete energy estimate from which, together with (57), we derive estimates on \( \rho_n, U_n, \Omega_n, M_n \), and \( H_n \) independent of \( T_n \).

Arguing as usual in the Galerkin procedure in order to obtain a priori estimates and following the calculations in Section 2 we derive the analogue (40):

\[
\frac{d}{dt} \mathcal{E}_n(t) + \mathcal{E}_n^d(t) = \mathcal{S}_n(t), \quad t \in (0, T_n),
\]

where

\[
\mathcal{E}_n(t) = \int_D \rho_n(t) \left( \frac{1}{2}|U_n|^2 + \frac{1}{2} |\Omega_n|^2 + P_e(\rho_n) + \frac{\delta}{\beta - 1} \rho_n^{\beta - 1} \right) \, dx + \frac{\mu_0}{2} \int_D \left( |H_n|^2 + |M_n|^2 \right) \, dx; \tag{75}
\]

\[
\mathcal{E}_n^d(t) = \frac{\mu_0}{\tau} \int_D |M_n|^2(t) \, dx + \frac{\mu_0(2\chi_0 + 1)}{\tau} \int_D |H_n|^2(t) \, dx + \mu \int_D |\nabla U_n|^2(t) \, dx
\]

\[
+ (\lambda + \mu) \int_D |\nabla \rho_n|^2(t) \, dx + \varepsilon \int_D |\nabla \rho_n|^2(t) \left( \frac{P_e(\rho_n)}{\rho_n} + \delta \beta \rho_n^{\beta - 2} \right) \, dx
\]

\[
+ \mu' \int_D |\nabla \Omega_n|^2(t) \, dx + (\lambda' + \mu') \int_D |\nabla \Omega_n|^2(t) \, dx + \zeta \int_D |\nabla U_n - 2\Omega_n|^2 \, dx
\]

\[
+ \mu_0 \sigma \left( \int_D |\nabla M_n|^2(t) \, dx + 2 \int_D |\nabla M_n|^2(t) \, dx \right), \tag{76}
\]

and

\[
\mathcal{S}_n(t) = -\frac{\mu_0}{(1 + \chi_0)} \int_D (F \varphi_n(t) \, dx + \mu_0 \int_D (\partial_t F \varphi_n(t) \, dx + \mu_0 \sigma \int_D (F \nabla M_n)(t) \, dx).
\]

Integration of (74) from 0 to \( t \) gives the approximate energy equality:

\[
\mathcal{E}_n(t) + \int_0^t \mathcal{E}_n^d(s) \, ds = \mathcal{E}_0,5,n + \int_0^t \mathcal{S}_n(s) \, ds, \quad t \in (0, T_n),
\]

where

\[
\mathcal{E}_0,5,n = \int_D \left( \frac{1}{2} \nabla V_{0,5} \cdot U_{0,5,n} + \frac{1}{2} \nabla \Omega_{0,5,n} + \rho_{0,5} P_e(\rho_{0,5}) + \frac{\delta}{\beta - 1} \rho_{0,5}^{\beta - 1} \right) \, dx + \frac{\mu_0}{2} \int_D \left( |H_{0,5}|^2 + |M_{0,5}|^2 \right) \, dx.
\]

Here \( H_{0,5} = \nabla \varphi_{0,5} \) where \( \varphi_{0,5} \) is the unique weak solution in \( H^1(D) \) of

\[-\Delta \varphi_{0,5} = \text{div} \, M_{0,5} - F_0 \quad \text{in } D, \quad \frac{\partial \varphi_{0,5}}{\partial n} = 0 \quad \text{on } \partial D, \quad \int_D \varphi_{0,5} \, dx = 0,
\]

with \( F_0 = F|_{t=0} \). We easily verify that \( H_{0,5} \) is the orthogonal projection of \( H_0 \) in the space \( L^2(D) \) onto the space spanned by \( \nabla d_1, \ldots, \nabla d_n \). Thus \( \|H_{0,5}\| \leq \|H_0\| \).

Using (70), (71) and the Hölder inequality we deduce that

\[
\int_D V_{0,5} \cdot U_{0,5,n} \, dx \leq \int_D \frac{|V_{0,5}|^2}{\rho_{0,5}} \, dx, \quad \int_D \rho_{0,5} |U_{0,5,n}|^2 \, dx \leq \int_D \frac{|V_{0,5}|^2}{\rho_{0,5}} \, dx.
\]
and
\[
\int_D Q_{0,\delta} \cdot \Omega_{0,\delta,n} \, dx \leq \int_D \frac{|Q_{0,\delta}|^2}{\rho_{0,\delta}} \, dx, \quad \int_D \rho_{0,\delta} |\Omega_{0,\delta,n}|^2 \, dx \leq \int_D \frac{|Q_{0,\delta}|^2}{\rho_{0,\delta}} \, dx.
\]
We also have:
\[
\int_D \rho_{0,\delta} P_v(\rho_{0,\delta}) \, dx = \frac{a}{\gamma - 1} \int_D (\rho_{0,\delta}^{\gamma} - \rho_{0,\delta}) \, dx.
\]
Therefore,
\[
\tilde{\mathcal{E}}_{0,\delta,n} \leq \mathcal{E}_{0,\delta,n} \leq C, \tag{77}
\]
with
\[
\mathcal{E}_{0,\delta,n} = \int_D \left( \frac{1}{2} \frac{|V_{0,\delta}|^2}{\rho_{0,\delta}} + \frac{1}{2} \frac{|Q_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta} P_v(\rho_{0,\delta}) + \frac{\delta}{\beta - 1} \rho_{0,\delta}^{\beta} \right) \, dx + \frac{\mu_0}{2} \int_D (|H_{0n}|^2 + |M_{0n}|^2) \, dx. \tag{78}
\]
Estimating \( S_n(t) \) with use of the Young inequality and the Poincaré inequality \( \|\varphi_n\|_{L^2(D)} \leq C \|\nabla \varphi_n\| = C \|H_n\| \), we get the discrete energy estimate:
\[
\mathcal{E}_n(t) + C_1 \left( \int_0^t \mathcal{E}_n^d(s) \, ds \right) \leq \mathcal{E}_{0,\delta,n} + C_2 \int_0^t \left( 1 + \|F(s)\|^2 + \|\partial_t F(s)\|^2 \right) \, ds, \quad t \in (0, T_n). \tag{79}
\]
We deduce from (79) that the functions \( U_n \) and \( \Omega_n \) are bounded \( L^2(0, T_n; \|H^1_0(D)\|) \), \( M_n \) is bounded in \( L^\infty(0, T_n; \|H^2_0(D)\|) \) and in \( L^2(0, T_n; \mathcal{M}) \) and \( H_n \) is bounded in \( L^\infty(0, T_n; \|H^2_0(D)\|) \), with bounds independent of \( n \) and \( T_n \leq T \). Using (57) we obtain that \( \rho_n \) is bounded from below and above by a positive constant independent of \( T_n \leq T \) and therefore we deduce from (79) that \( U_n \) and \( \Omega_n \) are bounded in \( L^\infty(0, T_n; \|L^2_0(D)\|) \) by a constant that is independent of \( n \) and \( T_n \leq T \). Thus we are allowed to iterate the previous local existence result to construct a solution \( (\rho_n, U_n, \Omega_n, M_n, H_n) \) on the whole interval \([0, T]\). \( \square \)

4.2. Estimates independent of \( n \)

We have the following result:

**Proposition 2.** Assume \( \beta \geq 4 \). Then the approximate solution \( (\rho_n, U_n, \Omega_n, M_n, H_n) \) constructed above satisfy the following estimates:
\[
\|\rho_n\|_{L^\infty(0, T; L^\gamma_0(D))} \leq C, \quad \|\rho_n\|_{L^\infty(0, T; L^8_0(D))} \leq C(\delta), \quad \sqrt{\epsilon} \|\nabla \rho_n\|_{L^2_0(D_T)} \leq C(\delta), \tag{80}
\]
\[
\|\rho_n\|_{L^{\beta+1}(D_T)} \leq C(\epsilon, \delta), \tag{81}
\]
\[
\|U_n\|_{L^2(0, T; H^1_0(D))} \leq C, \quad \|\Omega_n\|_{L^2(0, T; H^2_0(D))} \leq C, \tag{82}
\]
\[
\|\sqrt{\rho_n} U_n\|_{L^\infty(0, T; L^2_0(D))} \leq C, \quad \|\sqrt{\rho_n} \Omega_n\|_{L^\infty(0, T; L^2_0(D))} \leq C, \tag{83}
\]
\[
\|M_n\|_{L^\infty(0, T; L^2_0(D))} \leq C, \quad \|M_n\|_{L^2(0, T; \mathcal{M})} \leq C, \tag{84}
\]
\[
\|H_n\|_{L^\infty(0, T; L^2_0(D))} \leq C, \quad \|H_n\|_{L^2(0, T; \mathcal{M})} \leq C, \tag{85}
\]
where all constants are independent of \( n \).

**Proof.** Estimates (82)–(84), the first two estimates of (80) and the first estimate of (85) follow directly from the discrete energy inequality (79). The third estimate of (80) and estimate (81) are as in [15], p. 164 and [33], p. 361.

Let us prove the second estimate of (85). Multiplying (69) by \( \lambda_j \) and using integrations by parts and (58) we obtain:
\[
\int_D (\text{div } H_n) \Delta d_j \, dx = - \int_D (\text{div } M_n) \Delta d_j \, dx + \int_D F \Delta d_j \, dx. \tag{86}
\]
Multiplying (86) by $\delta^n(t)$ and adding these equations for $j = 1, \ldots, n$, we obtain, since $H_n = \nabla \varphi_n = \sum_{j=1}^n \delta^n(t) d_j$, 

$$
\int_D |\text{div} \ H_n|^2 \, dx = - \int_D (\text{div} \ M_n) (\text{div} \ H_n) \, dx + \int_D F \cdot H_n \, dx,
$$

from which we deduce, using the Cauchy–Schwarz inequality,

$$
\|\text{div} \ H_n\|_{L^2(0,T;L^2(D))} \leq \|\text{div} \ M_n\|_{L^2(0,T;L^2(D))} + \|F\|_{L^2(0,T;L^2(D))}.
$$

Moreover $\text{curl} \ H_n = 0$ and, due to the boundary condition in (58), $H_n \cdot n = 0$. By a classical result for Maxwell fields (see [7], pp. 244–249) we obtain that $H_n$ belongs to $L^2(0,T;\mathcal{M})$ and satisfies,

$$
\|H_n\|_{L^2(0,T;\mathcal{M})} \leq C\left(\|\text{div} \ M_n\|_{L^2(D_T)} + \|F\|_{L^2(D_T)}\right),
$$

which gives $\|H_n\|_{L^2(0,T;\mathcal{M})} \leq C$, according to (84). □

4.3. Passing to the limit as $n \to \infty$

Our goal now is to establish an existence result for problem (43)–(56). We have the following result:

**Proposition 3.** Assume $\beta \geq \max(\delta, \gamma)$. Then problem (43)–(56) admits at least one weak solution $(\rho, U, \Omega, M, H)$ in the following sense.

(i) The density $\rho$ is such that $\nabla \rho \in L^{p_1}(0,T;L^{p_1}(D)), \rho \in L^{p_2}(0,T;W^{2,p_1}(D))$ and $\partial_t \rho \in L^{p_2}(D_T)$, with $p_1 > 2$ and $p_2 > 1$, $\rho \geq 0$ a.e. in $D_T$, the velocity $U$ belongs to the space $L^2(0,T;\mathbb{H}_0^1(D))$, Eq. (43) holds a.e. in $D_T$, and conditions (44) and (45) are satisfied in the sense of traces. Moreover,

$$
\varepsilon \|\nabla \rho\|^2_{L^2(0,T;L^2(D))} + \|\nabla \rho\|_{L^2(0,T;L^2(D))} \leq C(\delta),
$$

and

$$
\int_D \rho(t) \, dx = \int_D \rho_0 \, dx \quad \text{for any} \ t \geq 0.
$$

(ii) The angular velocity $\Omega$ belongs to $L^2(0,T;\mathbb{H}^1_0(D), Eqs. (46) and (49) hold in $D'(D_T)$, the boundary conditions (47) and (50) are satisfied in the sense of traces, the functions $\rho U$ and $\rho \Omega$ belong to $C([0,T];L^{2\beta/\beta+1}(D))$ and satisfy the initial conditions (48) and (51), respectively.

(iii) The function $M$ belongs to $C([0,T];L^2_{\text{weak}}(D)) \cap L^2(0,T;\mathcal{M})$ and the integral identity,

$$
\frac{d}{dt} \int_D M \cdot q \, dx - \int_D (U \otimes M) \cdot \nabla q \, dx + \sigma \int_D (\text{curl} M) \cdot (\text{curl} q) \, dx + \sigma \int_D (\text{div} M) (\text{div} q) \, dx
$$

holds for every $q \in \mathcal{M}$. Moreover, $M$ satisfies the initial condition (54).

(iv) The function $H$ is such that $H = \nabla \varphi$ where $\varphi \in L^{\infty}(0,T;H^1(D)) \cap L^2(0,T;H^2(D))$ and solves problems (22) and (23).

(v) The energy inequality,

$$
\mathcal{E}(t) + C_1 \int_0^t \mathcal{E}^d(s) \, ds \leq \mathcal{E}_{0,\delta} + C_2 \int_0^t (1 + \|F(s)\|^2 + \|\partial_t F(s)\|^2) \, ds,
$$

holds for a.e. $t \in (0,T)$, where $\mathcal{E}(t), \mathcal{E}^d(t)$ and $\mathcal{E}_{0,\delta}$ are defined by (75), (76) and (78), respectively, in which one drops the index $n$.

We will obtain the existence of a weak solution to problem (43)–(56) by passing to the limit, as \( n \to \infty \), in the sequence of approximate solutions \((\rho_n, U_n, \Omega_n, M_n, H_n)\) constructed above. Due to estimates (80), (82), (84) and (85), there are subsequences (still indexed by \( n \)) and functions \( \rho, U, \Omega, M \) and \( H \) such that

\[
\rho_n \rightharpoonup \rho \quad \text{in} \quad L^\infty(0, T; L^\beta(D_T)),
\]

\[
\rho_n U_n \rightharpoonup \rho U \quad \text{weak-* in} \quad L^\infty(0, T; \mathbb{L}^{2\beta/\beta+1}(D)),
\]

\[
M_n \rightharpoonup M \quad \text{weakly in} \quad L^2(0, T; \mathcal{M}) \quad \text{and in} \quad L^\infty(0, T; \mathbb{L}^2(D)),
\]

\[
H_n \rightharpoonup H \quad \text{weakly in} \quad L^2(0, T; \mathcal{M}) \quad \text{and in} \quad L^\infty(0, T; \mathbb{L}^2(D)).
\]

Note that (94) implies that

\[
\varphi_n \rightharpoonup \varphi \quad \text{in} \quad L^2(0, T; H^2(D)) \quad \text{weak and in} \quad L^\infty(0, T; H^1(D)) \quad \text{weak-*}.
\]

The proof of Proposition 3 will be carried out in the following subsections.

### 4.3.1. Passing to the limit in the continuity equation with vanishing artificial viscosity

We pass to the limit in Eq. (59) exactly as in [15], pp. 166–168 and [33], pp. 362–365. There are subsequences, still indexed by \( n \), such that

\[
\rho_n \rightharpoonup \rho \quad \text{strongly in} \quad L^\beta(D_T),
\]

\[
\rho_n U_n \rightharpoonup \rho U \quad \text{weak-* in} \quad L^\infty(0, T; \mathbb{L}^{2\beta/\beta+1}(D))
\]

and weakly in \( L^2(0, T; \mathbb{L}^{6\beta/\beta+6}(D)) \) we have:

\[
\|\rho_n U_n\|_{L^\infty(0, T; \mathbb{L}^{2\beta/\beta+1}(D))} \leq C(\delta),
\]

with \( p_1 = \frac{10\beta-6}{3(\beta+1)} \) and we have \( p_1 > 2 \) for \( \beta > 3 \). We also have,

\[
\|\nabla \rho_n\|_{L^1(0, T; \mathbb{L}^1(D))} \leq C(\delta),
\]

from which it results that

\[
\|\nabla \rho_n \cdot \nabla U_n\|_{L^p(0, T; \mathbb{L}^p(D))} + \epsilon \|\text{div}(\rho_n U_n)\|_{L^p(D_T)} \leq C(\delta),
\]

with \( p_2 = \frac{2p_1}{p_1+1} \) and we have \( p_2 > 1 \) for \( \beta > 3 \). Moreover,

\[
\|\partial_t \rho_n\|_{L^{p_2}(D_T)} + \|\Delta \rho_n\|_{L^{p_2}(D_T)} \leq C(\epsilon, \delta);
\]

the function \( \rho \) belongs to the same class as \( \rho_n \) and satisfies the following estimates:

\[
\epsilon \|\nabla \rho\|_{L^p(0, T; \mathbb{L}^p(D))} \leq C(\delta), \quad \|\partial_t \rho\|_{L^{p_2}(D_T)} + \|\Delta \rho\|_{L^{p_2}(D_T)} \leq C(\epsilon, \delta).
\]

### 4.3.2. Passing to the limit in the approximate linear momentum equation

We first establish the following compactness properties.

**Lemma 3.** The sequences \((M_n)\) and \((H_n)\) belong to a compact set of \( L^2(0, T; \mathbb{L}^2(D)) \).

**Proof.** To prove the compactness of \((M_n)\) we use the method of J.-L. Lions [27], pp. 64–79 (see also [44], pp. 279–291) for the study of weak solutions to incompressible Navier–Stokes equations.

Let \( \tilde{M}_n : \mathbb{R} \to \mathcal{M} \) be the extension of \( M_n \) by 0 outside \([0, T]\) and let \( \tilde{M}_n \) denote the Fourier transform with respect to the time variable of \( \tilde{M}_n \). Let us show that, for \( 0 < \kappa < 1/4 \),

\[
\int_{-\infty}^{+\infty} |\xi|^{2\kappa} \|\tilde{M}_n(\xi)\|^2 \, d\xi \leq C.
\]
Along with (84), this will imply that \((M_n)\) belongs to a compact set of \(L^2(0, T; L^2(D))\). We first rewrite (66)–(68) in the form,
\[
\frac{d}{dt} \int_D \tilde{M}_n \cdot c_j \, dx = \int_D \tilde{G}_n \cdot c_j \, dx + \left( \int_D M_{0n} \cdot c_j \, dx \right) \delta_0 - \left( \int_D M_n(T) \cdot c_j \, dx \right) \delta_T,
\]
\[
\frac{d}{dt} \int_D \tilde{M}_n \cdot \nabla d_j \, dx = \int_D \tilde{G}_n \cdot \nabla d_j \, dx + \left( \int_D M_{0n} \cdot \nabla d_j \, dx \right) \delta_0 - \left( \int_D M_n(T) \cdot \nabla d_j \, dx \right) \delta_T,
\]
for \(j = 1, \ldots, n\), where \(\delta_0\) and \(\delta_T\) are Dirac distributions at 0 and \(T\), respectively, and \(\tilde{G}_n : \mathbb{R} \to \mathcal{M}'\) denotes the extension of \(G_n\) by 0 outside \([0, T]\) with
\[
G_n = -\text{div}(U_n \otimes M_n) + \sigma \Delta M_n + (M_n \times \Omega_n) - \frac{1}{\tau} (M_n - \chi_0 H_n).
\]

By the Fourier transform we have
\[
2i\pi \xi \int_D \tilde{M}_n(\xi) \cdot c_j \, dx = \int_D \tilde{G}_n(\xi) \cdot c_j \, dx + \int_D M_{0n} \cdot c_j \, dx - \exp(-2i\pi \xi T) \int_D M_n(T) \cdot c_j \, dx, \tag{99}
\]
\[
2i\pi \xi \int_D \tilde{M}_n(\xi) \cdot \nabla d_j \, dx = \int_D \tilde{G}_n(\xi) \cdot \nabla d_j \, dx + \int_D M_{0n} \cdot \nabla d_j \, dx - \exp(-2i\pi \xi T) \int_D M_n(T) \cdot \nabla d_j \, dx, \tag{100}
\]
for \(j = 1, \ldots, n\), where \(\tilde{G}_n\) denotes the Fourier transform of \(G_n\). Let \(\tilde{\delta}_n^j(\xi)\) denote the Fourier transform of \(\delta_n^j(t)\). We multiply (99) by \(\tilde{\delta}_n^j(\xi)\) and (100) by \(\tilde{\delta}_n^{j+n}(\xi)\), respectively, and add the resulting equations for \(j = 1, \ldots, n\). We get:
\[
2i\pi \xi \| \tilde{M}_n(\xi) \|^2 = \int_D \tilde{G}_n(\xi) \cdot \tilde{M}_n(\xi) \, dx + \int_D M_{0n} \cdot \tilde{M}_n(\xi) \, dx - \exp(-2i\pi \xi T) \int_D M_n(T) \cdot \tilde{M}_n(\xi) \, dx. \tag{101}
\]

Due to estimates (82), (84) and (85),
\[
\int_0^T \| G_n(t) \|_{\mathcal{M}'} \, dt \leq C \int_0^T \left( \| U_n(t) \|_{\mathcal{M}'} \| M_n(t) \|_{\mathcal{M}'} + \| \nabla M_n(t) \| + \| \Omega_n(t) \| + \| H_n(t) \| \right) \leq C.
\]
Then
\[
\| \tilde{G}_n(\xi) \|_{\mathcal{M}'} \leq C, \quad \forall \xi \in \mathbb{R}.
\]
Using the bounds \(\| M_n(T) \| \leq C, \| M_{0n} \| \leq C\), we deduce from (101) that
\[
\| \xi \| \tilde{M}_n(\xi) \|^2 \leq C \left( \| \tilde{M}_n(\xi) \|_{\mathcal{H}^1(D)} + \| \tilde{M}_n(\xi) \| \right),
\]
and then, using the Poincaré inequality, we get:
\[
\| \xi \| \tilde{M}_n(\xi) \|^2 \leq C \| \tilde{M}_n(\xi) \|_{\mathcal{H}^1(D)}, \quad \forall \xi \in \mathbb{R}.
\]
This inequality implies (98), as in [27], pp. 77–79. Then, one can apply a compactness theorem involving fractional derivatives, see [27] pp. 60–62, to conclude that \((M_n)\) belongs to a compact set of \(L^2(0, T; L^2(D))\).

The compactness of \((H_n)\) in \(L^2(0, T; L^2(D))\) results from estimates (85), the equalities,
\[
\int_0^T \| H_n \|^2 \, dt = - \int_{D_T} M_n \cdot H_n \, dx \, dt - \int_{D_T} F \varphi_n \, dx \, dt, \tag{102}
\]
\[
\int_0^T \| H \|^2 \, dt = - \int_{D_T} M \cdot H \, dx \, dt - \int_{D_T} F \varphi \, dx \, dt, \tag{103}
\]
and the compactness of \((M_n)\). \(\Box\)
By (84) and (85), together with the Sobolev imbedding $H^1(D) \hookrightarrow L^5(D)$ and the Hölder inequality, we have:
\[
\|M_n \cdot \nabla H_n\|_{L^1(0,T;L^{5/2}(\Omega))} \leq C\|M_n\|_{L^2(0,T;M)}\|H_n\|_{L^2(0,T;M)} \leq C,
\]
then, according to Lemma 3, we have for a subsequence still indexed by $n$,
\[
M_n \cdot \nabla H_n \to M \cdot \nabla H \quad \text{in } \mathcal{D}'(D_T).
\]
By interpolation of the spaces $L^\infty(0,T;\mathbb{L}^2(\Omega))$ and $L^2(0,T;\mathbb{L}^5(\Omega))$ we get that $(M_n)$ and $(H_n)$ are bounded in $L^{10/3}(0,T;\mathbb{L}^{10/3}(\Omega))$. Using the Hölder inequality we get:
\[
\|M_n \cdot \nabla H_n\|_{L^{5/4}(0,T;L^{5/4}(D))} \leq C\|M_n\|_{L^{10/3}(0,T;L^{10/3}(D))}\|H_n\|_{L^2(0,T;M)} \leq C,
\]
then, for a selected sequence still indexed by $n$,
\[
M_n \cdot \nabla H_n \to M \cdot \nabla H \quad \text{weakly in } L^{5/4}(0,T;\mathbb{L}^{5/4}(D)).
\]

Let for each fixed $j (1 \leq j \leq n)$ $R_{n,j}$ denote the right-hand side of (62). According to (82) and (104) we have:
\[
\|R_{n,j}\|_{L^{5/4}(0,T)} \leq C. \tag{106}
\]
We deduce from the estimates of $\rho_n$ and $U_n$, see for instance [33], p. 363, that
\[
\|\rho_n U_n \otimes U_n\|_{L^2(0,T;\mathbb{L}^{6\beta/(4\beta+3)}(D))} \leq C(\delta). \tag{107}
\]

Using (107), the estimates of Proposition 2 and (106) one can deduce from Eq. (62) that the functions $t \mapsto \int_{D} \rho_n U_n \cdot a_j \, dx$ form a precompact system in $C[0,T]$, for any fixed $j (1 \leq j \leq n)$. This implies, using Corollary 2.1 in [15], that there is a subsequence, still denoted by $n$, such that
\[
\rho_n U_n \rightharpoonup \rho U \quad \text{in } C(0,T;\mathbb{L}^{2\beta/(\beta+1)}(D)).
\]
Since $\frac{2\beta}{\beta+1} > \frac{6}{5}$, the space $L^{2\beta/(\beta+1)}(D)$ is compactly embedded in $H^{-1}(D)$ and then we have:
\[
\rho_n U_n \rightharpoonup \rho U \quad \text{in } C(0,T;H^{-1}(D)),
\]
which implies, together with (92) and (107),
\[
\rho_n U_n \otimes U_n \rightharpoonup \rho U \otimes U \quad \text{weakly in } L^2(0,T;\mathbb{L}^{6\beta/(4\beta+3)}(D)). \tag{108}
\]
As in [15], p. 169 or [33], pp. 364–365 we have, for a subsequence still indexed by $n$,
\[
\nabla \rho_n \rightarrow \nabla \rho \quad \text{strongly in } L^2(0,T;\mathbb{L}^2(D)),
\]
and taking (92) into account,
\[
\nabla \rho_n \cdot U_n \rightarrow \nabla \rho \cdot U \quad \text{in } \mathcal{D}'(D_T). \tag{109}
\]
Now we can pass to the limit in (62), (63), as $n \to \infty$ and we see that (46) holds in $\mathcal{D}'(D_T)$. Clearly, conditions (47) and (48) are satisfied in the sense of traces.

4.3.3. End of the proof of Proposition 3

Eq. (59) integrated with respect to the space variable implies,
\[
\int_{D} \rho_n(t) \, dx = \int_{D} \rho_{0,\delta} \, dx \quad \text{for any } t \geq 0,
\]
from which, passing to the limit as $n \to \infty$, we deduce (88).

By similar arguments to those we used in Sections 4.3.1 and 4.3.2, we pass to the limit, as $n \to \infty$, successively in the equation of $\Omega_n$, $M_n$, and $H_n$ and establish the points (ii)–(iv).

To prove (v), we multiply (79) by $k \in D(0,T)$, $k \geq 0$, and integrate over $(0,T)$. We obtain:
\[
\int_{0}^{T} \left( \mathcal{E}_n(t) + C_1 \int_{0}^{T} \mathcal{E}_n^d(s) \, ds \right) k(t) \, dt \leq \mathcal{E}_{0,\delta,n} \int_{0}^{T} k(t) \, dt + C_2 \int_{0}^{T} \left( \int_{0}^{t} \left( \|F(s)\|^2 + \|\partial_t F(s)\|^2 \right) \, ds \right) \, dt. \tag{110}
\]
As \( n \to \infty \), we have:

\[
U_{0,\delta,n} \to U_{0,\delta}, \quad \Omega_{0,\delta,n} \to \Omega_{0,\delta}, \quad M_{0,n} \to M_0, \quad H_{0,n} \to H_0
\]

strongly in \( L^2(D) \), where \( H_0 = \nabla \varphi_0 \) and \( \varphi_0 \) is the unique weak solution in \( H^1(D) \) of (20). Passing to the lower limit in (110), as \( n \to \infty \), using (92)–(95), (108), (109) and Lemma 3, we get:

\[
\int_0^T \left( \mathcal{E}(t) + C_1 \int_0^t \mathcal{E}^d(s) \, ds \right) k(t) \, dt \leq \mathcal{E}_{0,\delta} + C_2 \int_0^T \left( \| F(s) \|^2 + \| \partial_t F(s) \|^2 \right) k(t) \, dt,
\]

for any \( k \in \mathcal{D}(0, T), k \geq 0 \), where \( \mathcal{E}(t), \mathcal{E}^d(t) \) and \( \mathcal{E}_{0,\delta} \) are defined by (75), (76) and (78), respectively, in which one drops the index \( n \). This yields the energy estimate (90). The proof of Proposition 3 is complete.

5. Passing to the limit as \( \varepsilon \to 0 \)

In this section we denote by \((\rho_\varepsilon, U_\varepsilon, \Omega_\varepsilon, M_\varepsilon, H_\varepsilon)\) the weak solution of (43)–(56), the existence of which was stated in Proposition 3. Our next goal is to let \( \varepsilon \to 0 \).

**Proposition 4.** There exists at least one approximate solution \((\rho, U, \Omega, M, H)\) in the following sense:

(i) The density \( \rho \) is a nonnegative function, \( \rho \in C([0, T]; L^\beta_{weak}(D)) \), \( \rho(0) = \rho_{0,\delta} \) in \( D \), and the velocity \( U \) belongs to \( L^2(0, T; H^1_0(D)) \). The density \( \rho \) is a renormalized solution of the continuity equation (8) in \((0, T) \times \mathbb{R}^3 \)

(ii) The momentum \( \rho U \) belongs to \( C([0, T]; L^\beta_{weak}(D)) \), \( \sqrt{\rho} U \) belongs to \( L^\infty(0, T; L^2(D)) \) and satisfies \( (\rho U)(0) = V_{0,\delta} \) in \( D \), and the linear momentum equation (9) holds in \( \mathcal{D}'(D_T) \) with a pressure \( p = \alpha \rho^\gamma + \frac{\mu_0}{2} |M|^2 + \delta \rho^2 \) and a force \( R \) satisfying (12).

(iii) The angular velocity \( \Omega \) belongs to \( L^2(0, T; H^1_0(D)) \), \( \sqrt{\rho} \Omega \) belongs to \( L^\infty(0, T; L^2(D)) \), the angular momentum \( \rho \Omega \) belongs to \( C([0, T]; L^2_{weak}(D)) \), satisfies \( (\rho \Omega)(0) = Q_{0,\delta} \) in \( D \), and the angular momentum equation (10) holds in \( \mathcal{D}'(D_T) \).

(iv) The magnetization \( M \) belongs to \( C([0, T]; L^2_{weak}(D)) \) \& \( L^2(0, T; M) \), satisfies the integral identity (21) and the initial condition \( M(0) = M_0 \) in \( D \).

(v) The function \( H \) is such that \( H = \nabla \varphi \) where \( \varphi \in L^\infty(0, T; H^1(D)) \) \& \( L^2(0, T; H^2(D)) \) and solves problem (22) and (23).

(vi) The energy inequality,

\[
\mathcal{E}(t) + C_1 \int_0^t \mathcal{E}^d(s) \, ds \leq \mathcal{E}_{0,\delta} + C_2 \int_0^t \left( 1 + \| F(s) \|^2 + \| \partial_t F(s) \|^2 \right) \, ds
\]

holds for a.e. \( t \in (0, T) \) where \( \mathcal{E}(t), \mathcal{E}^d(t) \) and \( \mathcal{E}_{0,\delta} \) are defined by,

\[
\mathcal{E}(t) = \int_D \rho(t) \left( \frac{1}{2} |U|^2 + \frac{1}{2} |\Omega|^2 + P_\rho(\rho) + \frac{\delta}{\beta - 1} \rho^{\beta - 1} \right) \, dx + \frac{\mu_0}{2} \int_D (|H|^2 + |M|^2) \, dx;
\]

\[
\mathcal{E}^d(t) = \frac{\mu_0}{\tau} \int_D |M|^2 \, dx + \frac{\mu_0}{\tau} \int_D |H|^2 \, dx + \mu \int_D |\nabla U|^2 \, dx + \mu' \int_D |\nabla \Omega|^2 \, dx
\]
\[ (\lambda' + \mu') \int |\text{div } \Omega|^2(t) \, dx + \int |\text{curl } U - 2\Omega|^2(t) \, dx \]
\[ + \mu_0 \sigma \left( \int |\text{curl } M|^2(t) \, dx + 2 \int |\text{div } M|^2(t) \, dx \right) \]

and

\[ \mathcal{E}_{0,\delta} = \int D \left( \frac{1}{2} \frac{|V_{0,\delta}|^2}{\rho_{0,\delta}} + \frac{1}{2} \frac{|Q_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta} \rho_{\varepsilon}(\rho_{0,\delta}) + \frac{\delta}{\beta - 1} \rho_{0,\delta}^\beta \right) \, dx + \frac{\mu_0}{2} \int D (|H_0|^2 + |M_0|^2) \, dx. \]

The rest of this section is devoted to the proof of Proposition 4.

5.1. Estimates independent of \( \varepsilon \)

From the energy inequality (90) we deduce that

\[ \| \rho_\varepsilon \|_{L^\infty(0,T;L^1(D))} \leq C, \quad \| \rho_\varepsilon \|_{L^\infty(0,T;L^p(D))} \leq C(\delta), \] (112)
\[ \| U_\varepsilon \|_{L^1(0,T;\mathbb{H}_0^1(D))} + \| \Omega_\varepsilon \|_{L^2(0,T;\mathbb{H}_0^3(D))} \leq C, \] (113)
\[ \| M_\varepsilon \|_{L^\infty(0,T;L^2(D))} + \| M_\varepsilon \|_{L^2(0,T;\mathcal{M})} \leq C, \] (114)
\[ \| H_\varepsilon \|_{L^\infty(0,T;L^2(D))} \leq C, \] (115)

and

\[ \| \sqrt{\nu} U_\varepsilon \|_{L^\infty(0,T;L^3(D))} + \| \sqrt{\nu} \Omega_\varepsilon \|_{L^\infty(0,T;L^2(D))} \leq C. \] (116)

Arguing as for the approximate solution \((\rho_\alpha, U_\alpha, \Omega_\alpha, M_\alpha, H_\alpha)\) we deduce that

\[ \| \rho_\varepsilon U_\varepsilon \|_{L^\infty(0,T;L^{2\delta/(\delta + 1)}(D))} + \| \rho_\varepsilon U_\varepsilon \|_{L^2(0,T;L^{6\delta/(\delta + 6)}(D))} + \| \rho_\varepsilon U_\varepsilon \|_{L^p(0,T;L^{p_1}(D))} \leq C(\delta), \] (117)

and

\[ \varepsilon \| \nabla \rho_\varepsilon \|_{L^p(0,T;L^{p_1}(D))} + \varepsilon \| \nabla \rho_\varepsilon \cdot \nabla U_\varepsilon \|_{L^p(0,T;L^{p_2}(D))} \leq C(\delta). \] (118)

The function \( H_\varepsilon \) belongs to \( L^2(0, T; \mathcal{M}) \) and satisfies,

\[ \| H_\varepsilon \|_{L^2(0, T; \mathcal{M})} \leq C, \] (119)

the function \( M_\varepsilon \cdot \nabla H_\varepsilon \) belongs to \( L^1(0, T; L^{3/2}(D)) \cap L^{5/4}(0, T; L^{5/4}(D)) \) and satisfies,

\[ \| M_\varepsilon \cdot \nabla H_\varepsilon \|_{L^1(0, T; L^{3/2}(D))} + \| M_\varepsilon \cdot \nabla H_\varepsilon \|_{L^{5/4}(0, T; L^{5/4}(D))} \leq C. \] (120)

With similar arguments, one shows that the function \( \Omega_\varepsilon \times M_\varepsilon \) belongs to \( L^1(0, T; L^3(D)) \cap L^2(0, T; L^{3/2}(D)) \), and

\[ \| \Omega_\varepsilon \times M_\varepsilon \|_{L^1(0, T; L^3(D))} + \| \Omega_\varepsilon \times M_\varepsilon \|_{L^2(0, T; L^{3/2}(D))} \leq C, \] (121)

the function \( M_\varepsilon \times H_\varepsilon \) belongs to \( L^1(0, T; L^3(D)) \) and

\[ \| M_\varepsilon \times H_\varepsilon \|_{L^1(0, T; L^3(D))} + \| M_\varepsilon \times H_\varepsilon \|_{L^2(0, T; L^{3/2}(D))} \leq C. \] (122)

Then we deduce from (89) that \( \| \partial_t M_\varepsilon \|_{L^{5/4}(0, T; \mathbb{H}_0^{3/2}(D))} \leq C \) which together with (114), using a compactness result of Aubin–Lions, yields:

\[ (M_\varepsilon) \text{ belongs to a compact set of } L^2(0, T; L^2(D)). \] (123)

As for the compressible barotropic flows, there is an improved estimate for density which is an essential argument in the theory of P.-L. Lions [29] to prove strong convergence of densities, as \( \varepsilon \rightarrow 0 \).
Lemma 4. There is a constant $C(\delta)$, independent of $\varepsilon$, such that

$$\int_{D_\varepsilon} \rho_\varepsilon^{\beta+1} dx dt \leq C(\delta). \quad (124)$$

Proof. We follow [33] (pp. 366–368). Since the right-hand side $R$ of the linear momentum equation (46) may be singular, the estimate of density in [33] is not directly applicable. Let us denote by $B$ the Bokovskii operator which associates to a function $v \in L^p_0(D) = \{v \in L^p(D); \int_D v dx = 0\}$ $(1 < p < \infty)$ the function $\phi$ satisfying:

$$\phi \in \mathcal{W}^{1,p}_0(D), \quad \text{div} \phi = v \quad \text{in} \, D,$$

$$\|
abla \phi\|_{L^p(D)} \leq C \|v\|_{L^p(D)}. \quad (125)$$

Let $\psi \in D([0, T], \mathcal{M} = \frac{1}{|D|} \int_D \rho_\varepsilon dx$ and $\varphi(t, x) = \psi(t)\phi(t, x)$ with $\phi(t, x) = B(\rho_\varepsilon - \overline{\mu})$. We have

$$\partial_t \varphi = \psi \partial_t \overline{\mu} + \psi \partial_t \rho_\varepsilon = \psi \partial_t \rho_\varepsilon - \psi \partial_t \overline{\mu} = \psi \partial_t \rho_\varepsilon - \psi \partial_t \rho_\varepsilon = \psi \partial_t \rho_\varepsilon - \psi \partial_t \overline{\mu},$$

and

$$\|\partial_\phi\|_{L^p(D)} \leq C \|\partial_\phi\|_{L^p(D)} \leq C \|ho_\varepsilon\|_{L^p(D)} + \|ho_\varepsilon\|_{L^p(D)} + \|ho_\varepsilon\|_{L^p(D)}, \quad 1 < p \leq p_1, \quad (126)$$

with $p_1 = \frac{10\beta - 6}{3\beta + 1} > 2, \, \overline{p} = \frac{3p}{3-p}$ if $p < 3, \, \overline{p}$ arbitrary $\geq 1$ if $p = 3$ and $\overline{p} = \infty$ if $p > 3$. One can verify that $\varphi$ is an admissible test function for the linear momentum equation (46), then taking it as a test function for (46) we obtain after some elementary transformations:

$$\int_0^T \int_D \psi(t) \rho_\varepsilon \rho_\varepsilon dx dt = \overline{m} \int_0^T \psi(t) \int_D \rho_\varepsilon dx dt + (\lambda + \mu) \int_0^T \psi(t) \int_D \rho_\varepsilon \nabla U_\varepsilon dx dt$$

$$- (\lambda + \mu) \overline{m} \int_0^T \psi(t) \int_D \nabla U_\varepsilon dx dt + \mu \int_0^T \psi(t) \int_D \nabla U_\varepsilon \cdot \nabla \varphi dx dt$$

$$- \int_0^T \psi(t) \int_D \rho_\varepsilon U_\varepsilon \cdot \varphi dx dt - \int_0^T \psi(t) \int_D \rho_\varepsilon U_\varepsilon \cdot \nabla \varphi dx dt$$

$$+ \varepsilon \int_0^T \psi(t) \int_D \nabla \rho_\varepsilon \cdot \nabla U_\varepsilon \cdot \varphi dx dt$$

$$- \int_0^T \psi(t) \int_D \rho_\varepsilon U_\varepsilon \cdot \nabla \varphi dx dt - \int_0^T \psi(t) \int_D R \cdot \varphi dx dt$$

$$= \sum_{i=1}^9 I_i.$$  

Here $p_\varepsilon = p(\rho_\varepsilon, M_\varepsilon) + \delta_\rho^\beta = \alpha \rho_\varepsilon^\gamma + \frac{\mu_0}{2} |\nabla \varphi|^2 + \delta_\rho^\beta$. We estimate each of the last terms in the previous sum by $C(\delta)\|\psi\|_{W^{1,1}(0, T)}$. For $I_9$ we have:

$$I_9 = - \int_0^T \psi(t) \int_D \mu_0 M_\varepsilon \cdot \nabla H_\varepsilon \cdot \varphi dx dt + \zeta \int_0^T \psi(t) \int_D \nabla U_\varepsilon \cdot \nabla \varphi dx dt$$

$$- 2\zeta \int_0^T \psi(t) \int_D \nabla \Omega_\varepsilon \cdot \varphi dx dt = \sum_{i=1}^3 I_i.$$
Using the Hölder inequality, (126) and estimates (112), (113) and (120) on $\rho_\varepsilon$, $U_\varepsilon$, $\Omega_\varepsilon$ and $M_\varepsilon \cdot \nabla H_\varepsilon$, we have:

$$|J_1^1| \leq \mu_0 \|\psi\|_{C(0,T)} \|M_\varepsilon \cdot \nabla H_\varepsilon\|_{L^1(0,T;L^{3/2}(D))} \|\varepsilon\|_{L^\infty(0,T;L^\infty(D))}$$

$$\leq C \|\psi\|_{C(0,T)} \|\rho_\varepsilon\|_{L^\infty(0,T;L^3(D))}$$

$$\leq C \|\psi\|_{C(0,T)};$$

$$|J_2^1| \leq \xi \|\psi\|_{C(0,T)} \|\nabla U_\varepsilon\|_{L^1(0,T;L^2(D))} \|\nabla \varphi\|_{L^2(0,T;L^2(D))}$$

$$\leq C \|\psi\|_{C(0,T)} \|U_\varepsilon\|_{L^2(0,T;H^1_0(D))} \|\rho_\varepsilon\|_{L^2(D)}$$

$$\leq C(\delta) \|\psi\|_{C(0,T)};$$

and

$$|J_3^3| \leq 2\xi \|\psi\|_{C(0,T)} \|\varphi\|_{L^1(0,T;L^2(D))} \|\varphi\|_{L^2(0,T;L^2(D))}$$

$$\leq C \|\psi\|_{C(0,T)} \|\varphi\|_{L^2(0,T;L^2(D))}$$

$$\leq C \|\psi\|_{C(0,T)}.$$

Thus $|J_3| \leq C(\delta) \|\psi\|_{C(0,T)}$ and then $\int_0^T \psi(t) \int_D \rho_\varepsilon \rho_\varepsilon \, dx \, dt \leq C(\delta) \|\psi\|_{W^{1,1}(0,T)}$. Using (112), (114) and the inequalities $\int_D \rho_\varepsilon |M_\varepsilon|^2 \, dx \, dt \leq \|\rho_\varepsilon\|_{L^\infty(0,T;L^{3/2}(D))} \int_0^T \|M_\varepsilon\|^2_{L^6(D)} \, dt \leq C$, we deduce that $\int_0^T \psi(t) \int_D \rho_\varepsilon^{\beta+1} \, dx \, dt \leq C(\delta) \|\psi\|_{W^{1,1}(0,T)}$ for any $\psi \in D(0, T)$. Then, taking a sequence of functions $\psi = \psi_m \in W^{1,1}(0, T)$, $0 \leq \psi_m \leq 1$, converging pointwise to 1, we obtain (124).

5.2. Passing to the limit as $\varepsilon \to 0$

In accordance with estimates (112)–(115) and (119), there are functions $\rho$, $U$, $\Omega$, $M$ and $H$ such that, for subsequences still indexed by $\varepsilon$, as $\varepsilon \to 0$,

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{in } L^\infty(0, T; L^\beta(D)) \text{ weak-*},$$

$$U_\varepsilon \rightharpoonup U \quad \text{and } \Omega_\varepsilon \rightharpoonup \Omega \quad \text{weakly in } L^2(0, T; H^1_0(D)),$$

$$M_\varepsilon \rightharpoonup M \quad \text{and } H_\varepsilon \rightharpoonup H \quad \text{weak and in } L^\infty(0, T; \mathbb{L}^2(D)) \text{ weak-*}.$$  

According to (123) we can assume that

$$M_\varepsilon \rightharpoonup M \quad \text{in } L^2(0, T; \mathbb{L}^2(D)).$$  

This, together with the relations satisfied by $\int_0^T \|H_\varepsilon\|^2 \, dt$ and $\int_0^T \|H\|^2 \, dt$, implies that

$$H_\varepsilon \rightharpoonup H \quad \text{in } L^2(0, T; \mathbb{L}^2(D)).$$  

From (115), (119) and (131) we deduce that there is a function $\varphi$ such that $\varphi_\varepsilon \rightharpoonup \varphi$ in $L^2(0, T; H^2(D))$ weak, in $L^\infty(0, T; H^1(D))$ weak-* and in $L^2(0, T; H^1(D))$ strong. Then, according to (120)–(122), for subsequences still indexed by $\varepsilon$, we have:

$$M_\varepsilon \times H_\varepsilon \rightharpoonup M \times H \quad \text{weakly in } L^2(0, T; \mathbb{L}^{3/2}(D)),$$

$$\Omega_\varepsilon \times M_\varepsilon \rightharpoonup \Omega \times M \quad \text{weakly in } L^2(0, T; \mathbb{L}^2(D)),$$

$$M_\varepsilon \cdot \nabla H_\varepsilon \rightharpoonup M \cdot \nabla H \quad \text{weakly in } L^{5/4}(0, T; \mathbb{L}^{5/4}(D)),$$

and $H$ satisfies

$$H = \nabla \varphi, \quad \text{div } H = - \text{div } M + F, \quad \text{curl } H = 0 \quad \text{in } D_T, \quad \text{H} \cdot n = 0 \quad \text{on } \Gamma_T.$$  

Let $\eta \in \mathcal{D}(D)$. We deduce from (43) that

$$\frac{d}{dt} \int_D \rho_\varepsilon \eta \, dx = \int_D \rho_\varepsilon U_\varepsilon \cdot \nabla \eta \, dx - \varepsilon \int_D \nabla \rho_\varepsilon \cdot \nabla \eta \, dx.$$  

According to (87), (117) and the Cauchy–Schwarz inequality, the right-hand side of (133) is bounded in $L^2(0, T)$. Consequently, one can use Corollary 2.1 in [15] to conclude that, for a subsequence still indexed by $\varepsilon$,

$$\rho_\varepsilon \to \rho \quad \text{in } C([0, T]; L^\beta_{\text{weak}}(D))$$

(134)

and, since $L^\beta(D)$ is compactly imbedded in $H^{-1}(D)$,

$$\rho_\varepsilon \to \rho \quad \text{in } L^p(0, T; H^{-1}(D)), \quad 1 \leq p < \infty.$$ 

This, together with (117) and (128), implies that $\rho_\varepsilon U_\varepsilon$ converges towards $\rho U$, weakly-$*$ in $L^\infty(0, T; L^2(D))$ and weakly in $L^2(0, T; [L^6(D)]^3)$ and $L^p(0, T; L^{3p/2}(\Omega))$. Similarly, using the Sobolev imbedding $H^{1}(\Omega) \hookrightarrow L^6(\Omega)$ and the Hölder inequality we show that the sequence $(\rho_\varepsilon U_\varepsilon \otimes U_\varepsilon)$ is bounded in $L^2(0, T; [L^6(D)]^3)$ and from the equation,

$$\frac{d}{dt} \int_D (\rho_\varepsilon U_\varepsilon) \cdot \eta \, dx = \int_D (\rho_\varepsilon U_\varepsilon \otimes U_\varepsilon) \cdot \nabla \eta \, dx + \int_D (p_\varepsilon (\rho_\varepsilon, M_\varepsilon) + \delta \rho_\varepsilon^\beta) \cdot \nabla \eta \, dx$$

$$- \mu \int_D \nabla U_\varepsilon \cdot \nabla \eta \, dx - \mu_0 \int_D (M_\varepsilon \cdot \nabla) H_\varepsilon \cdot \eta \, dx + \gamma \int_D \nabla \eta \cdot \nabla (\text{curl } U_\varepsilon) \cdot \eta \, dx,$n

satisfied for any $\eta \in (D(D))^3$, we deduce that

$$\rho_\varepsilon U_\varepsilon \to \rho U \quad \text{in } C([0, T]; [L^{2/(\beta+1)}(D)]^3),$$

(135)

and we conclude (noting that $\frac{2\beta}{\beta+1} > \frac{3}{2}$) that

$$\rho_\varepsilon U_\varepsilon \to \rho U \quad \text{in } L^p(0, T; [H^{-1}(D)]^3), \quad 1 \leq p < \infty,$n

and then

$$\rho_\varepsilon U_\varepsilon \otimes U_\varepsilon \to \rho U \otimes U \quad \text{weakly in } L^2(0, T; [L^6(D)]^3).$$

(136)

By virtue of (87), (113) and (118) we have:

$$\varepsilon \nabla \rho_\varepsilon \cdot \nabla U_\varepsilon \to 0 \quad \text{weakly in } L^p(0, T; L^2(D)).$$

Denoting $\overline{p}$ a weak limit of $p(\rho_\varepsilon, M_\varepsilon) + \delta \rho_\varepsilon^\beta$ in $L^{(\beta+1)/\beta}(D_T)$ we then obtain:

$$\partial_t (\rho U) + \text{div}(\rho U \otimes U) - \mu \Delta U - (\lambda + \mu) \nabla \text{div} U + \nabla \overline{p} = R \quad \text{in } D'(D_T),$$

with $R$ satisfying (12).

Arguing as above, we recover the angular momentum equation:

$$\partial_t (\rho \Omega) + \text{div}(\rho U \otimes \Omega) - \mu' \Delta \Omega - (\lambda' + \mu') \nabla \text{div} \Omega = S \quad \text{in } D'(D_T),$$

(137)

with $S$ satisfying (13), and the magnetization equation:

$$\partial_t M + \text{div}(U \otimes M) - \sigma \Delta M + \frac{1}{\tau} (M - \chi_0 H) = \Omega \times M \quad \text{in } D'(D_T).$$

(138)

Since $\rho_\varepsilon$ and $U_\varepsilon$ satisfy (43) a.e. in $D_T$ and the boundary conditions (44), one can extend $\rho_\varepsilon$ and $U_\varepsilon$ by 0 outside $D$ to obtain:

$$\partial_t \rho_\varepsilon + \text{div}(\rho_\varepsilon U_\varepsilon) = \varepsilon \text{div}(\chi(D) \nabla \rho_\varepsilon) \quad \text{in } D'(0, T \times \mathbb{R}^3).$$

Clearly,

$$\varepsilon \text{div}(\chi(D) \nabla \rho_\varepsilon) \to 0 \quad \text{in } L^2(0, T; H^{-1}(\mathbb{R}^3)).$$
 according to (134) and (135),

$$\rho_\varepsilon \to \rho \quad \text{in } C([0, T]; L^\beta_{\text{weak}}(\mathbb{R}^3)), $$

and

$$\rho_\varepsilon U_\varepsilon \to \rho U \quad \text{in } C([0, T]; L^{2(\beta+1)/\beta}_{\text{weak}}(\mathbb{R}^3)), $$

then the limit functions $\rho$ and $U$ satisfy the continuity equation:

$$\partial_t \rho + \text{div}(\rho U) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^3).$$

(139)

Since $\rho \in L^2(0, T; L^2(\mathbb{R}^3))$, as a consequence of the DiPerna–Lions transport theory [8], $\rho$ is a renormalized solution of Eq. (139) in $(0, T) \times \mathbb{R}^3$.

As for the compressible barotropic flows, we have a property of weak continuity of the effective viscous flux.

**Lemma 5.** Assume $\beta > \max(4, \gamma)$. Then, for subsequences still indexed by $\varepsilon$,

$$\lim_{\varepsilon \to 0} \int_{D_T} \psi \eta(p(\rho_\varepsilon, M_\varepsilon) + \delta \rho_\varepsilon^\beta - (\lambda + 2\mu) \text{div} U_\varepsilon) \rho_\varepsilon \, dx \, dt = \int_{D_T} \psi \eta(\bar{p} - (\lambda + 2\mu) \text{div} U) \rho \, dx \, dt,$$

for any $\psi \in D(0, T)$ and $\eta \in \mathcal{D}(D)$, where $\bar{p}$ denotes the weak limit in $L^{(\beta+1)/\beta}(D_T)$ of the sequence $(p(\rho_\varepsilon, M_\varepsilon) + \delta \rho_\varepsilon^\beta)$.

**Proof.** Using the relation $-\Delta U_\varepsilon = \text{curl}^2 U_\varepsilon - \nabla(\text{div} U_\varepsilon)$, we rewrite Eq. (46) in the form,

$$\partial_t (\rho_\varepsilon U_\varepsilon) + \text{div}(\rho_\varepsilon U_\varepsilon \otimes U_\varepsilon) - (\mu + \zeta) \Delta U_\varepsilon - (\lambda + \mu + \zeta) \nabla(\text{div} U_\varepsilon) + \nabla p_\varepsilon = \tilde{R}_\varepsilon,$$

with $\tilde{R}_\varepsilon = \mu_0 M_\varepsilon \cdot \nabla H_\varepsilon + 2\zeta \text{curl} \Omega_\varepsilon - \varepsilon \nabla \rho_\varepsilon \cdot \nabla U_\varepsilon$ and $p_\varepsilon = p(\rho_\varepsilon, M_\varepsilon) + \delta \rho_\varepsilon^\beta$. By (118), (128) and (132) we have, for subsequences still indexed by $\varepsilon$,

$$\tilde{R}_\varepsilon \to \tilde{R} \quad \text{weakly in } L^{5/4}(0, T; L^{5/4}(D)), $$

(141)

with $\tilde{R} = \mu_0 (M \cdot \nabla) H + 2\zeta \text{curl} \Omega$. Note here that $\frac{5}{4} > p_2 = \frac{5\beta-3}{4\beta}$. We also have:

$$p_\varepsilon \to \bar{p} \quad \text{weakly in } L^{(\beta+1)/\beta}(D_T).$$

(142)

Let $z = \frac{2\beta}{\beta+4}, r = \frac{\beta+4}{\beta}, s = \frac{5\beta-3}{4\beta}, q = \beta, w = \beta + 1$. By (135), (128), (134), (141), (142) and (124) we have, for subsequences still indexed by $\varepsilon$,

$$\rho_\varepsilon U_\varepsilon \to \rho U \quad \text{in } C([0, T]; L^z_{\text{weak}}(D)), $$

$$U_\varepsilon \to U \quad \text{weakly in } L^2(0, T; L^2(D)), $$

$$\nabla U_\varepsilon \to \nabla U \quad \text{weakly in } \left(L^2(0, T; L^2(D))\right)^3, $$

$$p_\varepsilon \to \bar{p} \quad \text{weakly in } L^r(D_T), $$

$$\tilde{R}_\varepsilon \to \tilde{R} \quad \text{weakly in } L^s(D_T), $$

$$\rho_\varepsilon \to \rho \quad \text{in } C([0, T]; L^q_{\text{weak}}(D)) \text{ and weakly-* in } L^w(D_T), $$

Moreover,

$$\mathcal{A}(\eta \varepsilon \Delta \rho_\varepsilon) \to 0 \quad \text{strongly in } L^2(0, T; L^z(D)), $$

(143)

where $z'$ is the conjugate exponent of $z$, $\mathcal{A}$ stands for $\mathcal{A}_j$, $j = 1, 2, 3$, and the operator $\mathcal{A}_j : S(\mathbb{R}^3) \to S'(\mathbb{R}^3)$ is defined by $\mathcal{A}_j(v) = -\mathcal{F}^{-1}\left[\frac{i\xi_j}{|\xi|^2} \mathcal{F}(v)\right]$ where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ its inverse. The proof of (143) is the same as for the compressible barotropic flows, see for instance [33], pp. 372 and 373. Thus, all assumptions of Proposition 7.36 in [33] are satisfied and its conclusion yields the result of Lemma 5.
5.3. End of the proof of Proposition 4

(i) Strong convergence of densities. Using the equalities $p(\rho_\varepsilon, M_\varepsilon) = p_\varepsilon(\rho_\varepsilon) + p_\varepsilon(M_\varepsilon) = a\rho_\varepsilon^\gamma + \frac{\mu_\varepsilon}{\varepsilon}|M_\varepsilon|^2$ and the strong convergence in $L^1(D_T)$ of $(p_\varepsilon(M_\varepsilon))$ towards $p_\varepsilon(M)$, as a consequence of (130), relation (140) reduces to,

$$\lim_{\varepsilon \to 0} \int_{D_T} \psi (a\rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta - (\lambda + 2\mu) \text{div} \, U_\varepsilon) \rho_\varepsilon \, dx \, dt = \int_{D_T} \psi (\alpha \overline{\rho}^\gamma + \delta \overline{\rho}^\beta - (\lambda + 2\mu) \text{div} \, U) \rho \, dx \, dt,$$

for any $\psi \in \mathcal{D}(0, T)$ and $\eta \in \mathcal{D}(D)$, where $\overline{\rho}^\gamma, \overline{\rho}^\beta$ denote the limits of $(\rho_\varepsilon^\gamma)$ in $L^{\frac{\gamma+1}{\gamma-1}}(D)$ weak and $(\rho_\varepsilon^\beta)$ in $L^{\frac{\beta+1}{\beta-1}}(D)$ weak, respectively.

Now, using the fact that $\rho_\varepsilon$ is a strong solution of problem (43)–(45) we have, $\rho_\varepsilon \text{ is a renormalized solution of the continuity equation (139)}$ in $(0, T) \times \mathbb{R}^3$, we have in particular:

$$\partial_t (\rho \ln \rho) + (\rho \text{div} \, U) + \rho \text{div} \, U = 0 \quad \text{in } D'(0, T) \times \mathbb{R}^3.$$

Since $\rho_\varepsilon$ is a strong solution of problem (43)–(45) we have,

$$\partial_t b(\rho_\varepsilon) + \text{div} (b(\rho_\varepsilon) U_\varepsilon) + (b'(\rho_\varepsilon) \rho_\varepsilon - b(\rho_\varepsilon)) \text{div} U_\varepsilon = \varepsilon \text{div} (\chi(D) \nabla b(\rho_\varepsilon)) - \varepsilon \chi(D) \frac{b''(\rho_\varepsilon)}{\rho_\varepsilon} \nabla \rho_\varepsilon^2 \quad \text{in } D'(0, T) \times \mathbb{R}^3,$$

for any smooth function $b$ such that $b(0) = 0$. If, in addition, $b$ is convex we deduce that

$$\int_{D_T} \psi \left(b'(\rho_\varepsilon) \rho_\varepsilon - b(\rho_\varepsilon)\right) \text{div} U_\varepsilon \, dx \, dt \leq \int_{D} b(\rho) \, dx + \int_{D_T} \partial_t \psi b(\rho_\varepsilon) \, dx \, dt,$$

for any $\psi \in C^\infty[0, T], \; \psi \geq 0$ and such that $\psi(0) = 1, \; \psi(T) = 0$. Taking a suitable choice of $b$ (smooth convex functions approximating the function $z \mapsto z \ln z$) we obtain:

$$\int_{D_T} \psi \rho_\varepsilon \text{div} U_\varepsilon \, dx \, dt \leq \int_{D} \rho_{0,\delta} \ln (\rho_{0,\delta}) \, dx + \int_{D_T} \partial_t \psi \rho_\varepsilon \ln \rho_\varepsilon \, dx \, dt$$

and passing to the limit as $\varepsilon \to 0$ it holds,

$$\int_{D_T} \overline{\rho} \text{div} \overline{U} \, dx \, dt \leq \int_{D} \rho_{0,\delta} \ln (\rho_{0,\delta}) \, dx + \int_{D_T} \partial_t \overline{\rho} \ln \overline{\rho} \, dx \, dt,$$

from which we infer,

$$\int_{D_T} \overline{\rho} \text{div} \overline{U} \, dx \, dt \leq \int_{D} \rho_{0,\delta} \ln (\rho_{0,\delta}) \, dx - \int_{D} \rho \ln \rho(t) \, dx, \quad \text{for any } t \in [0, T]. \quad (146)$$

Here $\overline{\rho} \text{div} \overline{U}, \; \overline{\rho} \ln \overline{\rho}$ denote the limits of $(\rho_\varepsilon \text{div} U_\varepsilon)$ in $L^2(0, T; L^{2\beta/(2+\beta)}(D))$ weak and $(\rho_\varepsilon \ln \rho_\varepsilon)$ in $L^\infty(0, T; L^p(D))$ weak-* $(1 \leq p < \beta)$, respectively.

On the other hand, we deduce from (145) that

$$\int_{D_T} \rho \text{div} U \, dx \, dt = \int_{D} \rho_{0,\delta} \ln (\rho_{0,\delta}) \, dx - \int_{D} \rho \ln \rho(t) \, dx, \quad (147)$$

then subtracting (147) from (146) we get:

$$\int_{D} (\overline{\rho} \ln \overline{\rho} - \rho \ln \rho)(t) \, dx \leq \int_{0}^{t} \int_{D} (\rho \text{div} U - \overline{\rho} \text{div} \overline{U}) \, dx \, dt. \quad (148)$$
We deduce from (144) that
\[
\int_0^t \int_D (\rho \nabla U - \rho \nabla U) \, dx \
\approx \frac{1}{\lambda + 2\mu} \int_0^t \int_D \left( a(\rho^{1/2} \rho - \rho^{1/2}) + \delta(\rho^{\beta/2} \rho - \rho^{\beta/2}) \right) \, dx \, dt.
\]
Since the right-hand side of this inequality is nonpositive, according to the increasing of the functions \( z \mapsto z^\gamma \) and \( z \mapsto z^\beta \), the right-hand side of (148) is nonpositive, whence \( \rho \ln(\rho) = \rho \ln(\rho) \) in \( D_T \) which implies that \( \rho \rightarrow \rho \) in \( L^p(D_T) \) and, then
\[
\rho \rightarrow \rho \quad \text{in} \quad L^p(D_T) \quad \text{for} \quad 1 \leq p < \beta + 1. \tag{149}
\]
Consequently, the limit functions \( \rho, U, \Omega, M, H \) satisfy the momentum equation,
\[
\partial_t (\rho U) + \div(\rho U \otimes U) - \mu \Delta U - (\lambda + \mu) \nabla (\div U) + \nabla p = R \quad \text{in} \quad D'(D_T),
\tag{150}
\]
with \( p = a \rho^\gamma + \frac{\mu}{2} |M|^2 + \delta \rho^\beta \) and \( R = \mu_0(M \cdot \nabla)H - \zeta \curl^2 U + 2\zeta \curl \Omega \).

(ii) Energy inequality. The energy inequality (111) follows from (90) by arguing as in Section 4.3.3, using (136), (149) and similar convergence results for \( (\rho, \Omega \otimes \Omega) \) and \( (\rho, U \otimes U) \), (130), (131) and lower semi-continuity of norms. The proof of Proposition 4 is finished.

6. Proof of the main theorem

In this section we denote by \( (\rho_\delta, U_\delta, \Omega_\delta, M_\delta, H_\delta) \) the weak solution constructed in the previous section. The functions \( \rho_\delta, U_\delta, \Omega_\delta, M_\delta \) and \( H_\delta \) satisfy the equation of continuity (139), the momentum equation (150) with \( p_\delta = a \rho_\delta^\gamma + \frac{\mu}{2} |M_\delta|^2 + \delta \rho_\delta^\beta \) and \( R_\delta = \mu_0(M_\delta \cdot \nabla)H_\delta - \zeta \curl^2 U_\delta + 2\zeta \curl \Omega_\delta \), the angular momentum equation (137), the magnetization equation (138) and the magnetostatic equations (132). Our goal now is to let \( \delta \rightarrow 0 \).

6.1. Estimates independent of \( \delta \)

From the energy inequality (111) we deduce that
\[
\|\rho_\delta\|_{L^\infty(0,T;L^\gamma(D))} \leq C, \tag{151}
\]
\[
\|U_\delta\|_{L^2(0,T;\Xi_2(D))} + \|\Omega_\delta\|_{L^2(0,T;\Xi_3(D))} \leq C, \tag{152}
\]
\[
\|M_\delta\|_{L^\infty(0,T;L^2(D))} + \|M_\delta\|_{L^2(0,T;M)} \leq C, \tag{153}
\]
\[
\|H_\delta\|_{L^\infty(0,T;L^2(D))} \leq C, \tag{154}
\]
and
\[
\|\sqrt{\rho_\delta} U_\delta\|_{L^\infty(0,T;L^2(D))} + \|\sqrt{\rho_\delta} \Omega_\delta\|_{L^\infty(0,T;L^2(D))} \leq C.
\]

Then, arguing as in the previous section we derive the estimates:
\[
\|\rho_\delta U_\delta\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(D))} + \|\rho_\delta U_\delta\|_{L^2(0,T;L^{2\gamma/(\gamma+1)}(D))} \leq C, \tag{155}
\]
\[
\|H_\delta\|_{L^2(0,T;M)} \leq C, \tag{156}
\]
\[
\|M_\delta \cdot \nabla H_\delta\|_{L^1(0,T;\Xi^{3/2}(D))} + \|M_\delta \cdot \nabla H_\delta\|_{L^{5/4}(0,T;\Xi^{5/4}(D))} \leq C, \tag{157}
\]
\[
\|\Omega_\delta \times M_\delta\|_{L^1(0,T;\Xi^1(D))} + \|\Omega_\delta \times M_\delta\|_{L^2(0,T;\Xi^{3/2}(D))} \leq C, \tag{158}
\]
\[
\|M_\delta \times H_\delta\|_{L^1(0,T;\Xi^1(D))} + \|M_\delta \times H_\delta\|_{L^2(0,T;\Xi^{3/2}(D))} \leq C, \tag{159}
\]
and show that \( (M_\delta) \) and \( (H_\delta) \) belong to a compact set of \( L^2(0,T;\Xi^2(D)) \).

As for barotropic flows we have an improved estimate for the density.

**Lemma 6.** We have:
\[
\int_{D_T} \rho_\delta^{\gamma + \theta} \, dx \, dt + \delta \int_{D_T} \rho_\delta^{\beta + \theta} \, dx \, dt \leq C, \quad \theta = \frac{2}{\gamma} - 1. \tag{160}
\]
Proof. We follow the lines of the proof of improved estimate for density in the case of barotropic flows [33], pp. 376–381. Consider the function $b_k, k > 0$ and $\theta > 0$, defined as

$$b_k(s) = \begin{cases} s^\theta & \text{for } 0 \leq s < k, \\ k^\theta & \text{for } s \geq k. \end{cases}$$

We have:

$$\partial_t (b_k(\rho)) + \text{div}(b_k(\rho) U_k) + ((\rho - (b_k)_+''(\rho) - b_k(\rho)) \text{ div } U_k) = 0$$

in $(0, T) \times \mathbb{R}^3$.

Denoting $S_\alpha (\alpha > 0)$ the regularizing operator by convolution with respect to the time variable, we have:

$$\partial_t S_\alpha (b_k(\rho)) + \text{div}(S_\alpha (b_k(\rho) U_k)) + S_\alpha ((\rho - (b_k)_+''(\rho) - b_k(\rho)) \text{ div } U_k) = 0$$

in $J \times \mathbb{R}^3$,

where $J$ is an open interval, $J \subset T \subset (0, T)$. Consider the function $\psi(t, x) = \psi(t)/\phi(t, x)$ with $\psi \in D(T)$ and $\phi(t, x) = B(S_\alpha (b_k(\rho)) - \frac{1}{|D|} \int_D S_\alpha (b_k(\rho)) dx)$ where $B$ is the Bokovskii operator defined by (125). We have:

$$\|\psi(t)\|_{L^p(D)} + \|\nabla \psi(t)\|_{L^p(D)} \leq C \|S_\alpha (b_k(\rho))(t)\|_{L^p(D)}, \quad t \in (0, T), 1 < p < \infty,$$

$$\|\partial_t \psi(t)\|_{L^p(D)} \leq C \|S_\alpha ((\rho - (b_k)_+''(\rho) - b_k(\rho)) \text{ div } U_k)(t)\|_{L^p(D)} + C \|S_\alpha (b_k(\rho) U_k)(t)\|_{L^p(D)}, \quad t \in (0, T), 1 < p \leq 2,$$

(161)

with $\bar{p} = \frac{3p}{3 - p}$ if $p < 3$, $\bar{p}$ arbitrary $\geq 1$ if $p = 3$, and $\bar{p} = \infty$ if $p > 3$, and $p = \frac{3p}{3 + p}$ if $p > \frac{3}{2}$, $\bar{p}$ arbitrary $> 1$ if $p \leq \frac{3}{2}$.

Taking $\psi$ as a test function for the linear momentum equation (150) we obtain:

$$\int_0^T \psi(t) \int_D (p(\rho, M) + \delta p_\delta) S_\alpha (b_k(\rho)) dx dt$$

$$= \frac{1}{|D|} \int_0^T \psi(t) \int_D S_\alpha (b_k(\rho)) \left(\int_D (p(\rho, M) + \delta p_\delta) dx\right) dx dt$$

$$+ (\lambda + \mu) \int_0^T \psi(t) \int_D \text{ div } U_k \text{ div } \varphi dx dt + \mu \int_0^T \psi(t) \int_D \nabla U_k \cdot \nabla \varphi dx dt$$

$$- \int_0^T \psi(t) \int_D \rho_k U_k \cdot \varphi dx dt - \int_0^T \psi(t) \int_D \rho_k U_k \cdot \partial_t \varphi dx dt$$

$$- \int_0^T \psi(t) \int_D \rho_k U_k \cdot U_k \cdot \nabla \varphi dx dt - \int_0^T \psi(t) \int_D \Omega_k \cdot \varphi dx dt$$

$$= \sum_{i=1}^7 I_i.$$

We estimate each term $I_i$ ($1 \leq i \leq 7$). The first sixth terms are handled as in [33], pp. 378–380. Passing to the limit, as $\alpha \to 0$ and $k \to \infty$, we get $|I_i| \leq C\|\psi\|_{W^{1,1}(0, T)}, 1 \leq i \leq 6$, provided $0 < \theta \leq \frac{2}{3}$. As for $I_7$ we have

$$I_7 = - \int_0^T \psi(t) \int_D \mu_0 M \cdot \nabla H_k \cdot \varphi dx dt + \zeta \int_0^T \psi(t) \int_D \text{ curl } U_k \cdot \text{ curl } \varphi dx dt$$

$$- 2\zeta \int_0^T \psi(t) \int_D \text{ curl } \Omega_k \cdot \varphi dx dt = \sum_{i=1}^7 I_i.$$
Using the Hölder inequality, (161), (151), (152 and (157) on $\rho_\delta$, $U_\delta$, $\Omega_\delta$ and $M_\delta \cdot \nabla H_\delta$, we have:

$$|J_1^2| \leq \mu_0 \|\psi\|_{C[0,T]} \|M_\delta \cdot \nabla H_\delta\|_{L^1(0,T;L^{3/2}(D))} \|\psi\|_{L^\infty(0,T;L^3(D))}$$

$$\leq C \|\psi\|_{C[0,T]} \|\rho_\delta^6\|_{L^\infty(0,T;L^{3/2}(D))}$$

$$\leq C \|\psi\|_{C[0,T]}, \quad \text{provided} \ 0 < \theta \leq \frac{2\gamma}{3};$$

$$|J_2^2| \leq \zeta \|\psi\|_{C[0,T]} \|\text{curl} U_\varepsilon\|_{L^2(0,T;L^2(D))} \|\text{curl} \psi\|_{L^2(0,T;L^2(D))}$$

$$\leq C \|\psi\|_{C[0,T]} \|U_\delta\|_{L^2(0,T;H^1_0(D))} \|\rho_\delta^6\|_{L^2(D_T)}$$

$$\leq C \|\psi\|_{C[0,T]}, \quad \text{provided} \ 0 < \theta \leq \frac{\gamma}{2},$$

and

$$|J_3^2| \leq 2\zeta \|\psi\|_{C[0,T]} \|\Omega_\delta\|_{L^2(0,T;L^2(D))} \|\psi\|_{L^2(0,T;L^2(D))}$$

$$\leq C \|\psi\|_{C[0,T]} \|\Omega_\delta\|_{L^2(0,T;H^1_0(D))} \|\rho_\delta^6\|_{L^2(0,T;L^{5/2}(D))}$$

$$\leq C \|\psi\|_{C[0,T]}, \quad \text{provided} \ 0 < \theta \leq \frac{5\gamma}{6}.$$
\[ \rho_\delta U_\delta \otimes U_\delta \to \rho U \otimes U \quad \text{weakly in } L^2(0, T; L^{6/(4\gamma+3)}(D)), \]
\[ \rho_\delta U_\delta \otimes \Omega_\delta \to \rho U \otimes \Omega \quad \text{weakly in } L^2(0, T; L^{6/(4\gamma+3)}(D)), \]
\[ \rho_\delta \Omega_\delta \otimes \Omega_\delta \to \rho \Omega \otimes \Omega \quad \text{weakly in } L^2(0, T; L^{6/(4\gamma+3)}(D)). \]

We also have, for subsequences still indexed by \( \delta \),
\[ M_\delta \times H_\delta \to M \times H \quad \text{weakly in } L^2(0, T; L^2(D)), \]
\[ \Omega_\delta \times M_\delta \to \Omega \times M \quad \text{weakly in } L^2(0, T; L^{3/2}(D)), \]
\[ M_\delta \cdot \nabla H_\delta \to M \cdot \nabla H \quad \text{weakly in } L^{5/4}(0, T; L^{5/4}(D)), \]
and \( H = \nabla \varphi \) and \( \varphi \) satisfies (22), (23). Note also that \( \delta p_\delta^\beta \to 0 \) weakly in \( L^{(\beta+\gamma)/\beta}(DT) \).

As for the compressible barotropic flows, we have a property of weak continuity of the effective viscous flux. Denote by \( T_k, k > 0 \), the cut-off functions given by \( T_k(z) = kT(\frac{z}{k}) \) where \( T \in C^\infty(\mathbb{R}) \) is chosen such that
\[ T(z) \equiv \begin{cases} T(z) = z & \text{for } z \in (0, 1), \\ T(z) = 1 & \text{concave on } (0, \infty), \\ T(z) = 2 \quad & \text{for } z \geq 3, \\ T(z) = -T(-z) & \text{for } z(-\infty, 0). \end{cases} \]

**Lemma 7.** We have, for subsequences still indexed by \( \delta \),
\[ \lim_{\delta \to 0} \int_D \psi \eta(p(\rho_\delta, M_\delta) + \delta p_\delta^\beta - (\lambda + 2\mu) \text{div} U_\delta) T_k(\rho_\delta) \, dx \, dt \]
\[ = \int_D \psi \eta\left( p(\rho, M) - \frac{\lambda + 2\mu}{\mu} \text{div} U \right) \frac{T_k(\rho)}{\rho} \, dx \, dt, \] (168)
for any \( \psi \in \mathcal{D}(0, T) \) and \( \eta \in \mathcal{D}(D) \), where \( p(\rho_\delta, M_\delta) \to p(\rho, M) \) weakly in \( L^1(DT) \) and \( T_k(\rho_\delta) \to \frac{T_k(\rho)}{\rho} \) weakly-* in \( L^\infty(DT) \).

**Proof.** We follow the proof of Proposition 6.1 in [15]. Using the relation \(-\Delta U_\delta = \text{curl}^2 U_\delta - \nabla(\text{div } U_\delta)\), we rewrite Eq. (150) in the form,
\[ \partial_t (\rho_\delta U_\delta) + \text{div}(\rho_\delta U_\delta \otimes U_\delta) - (\mu + \gamma) \Delta U_\delta - (\lambda + \mu - \gamma) \nabla(\text{div } U_\delta) + \nabla p_\delta = \tilde{R}_\delta, \] (169)
where \( p_\delta = a\rho_\delta^\gamma + \frac{\mu_0}{2} |M_\delta|^2 + \delta p_\delta^\beta \) and \( \tilde{R}_\delta = \mu_0(M_\delta \cdot \nabla)H_\delta + 2\zeta \text{curl } \Omega_\delta \). Let \( S_\delta \) denote the viscous stress tensor given by,
\[ S_\delta = (\mu + \gamma)(\nabla U_\delta + \nabla U_\delta^T) + (\lambda - 2\gamma) \text{div } U_\delta, \]
then (169) can be written in the form:
\[ \partial_t (\rho_\delta U_\delta) + \text{div}(\rho_\delta U_\delta \otimes U_\delta) + \nabla p_\delta = \text{div } S_\delta + \tilde{R}_\delta. \] (170)

We take as a test function for Eq. (170) the function \( \phi \) defined by:
\[ \phi(t, x) = \psi(t) \eta(x) \mathcal{A}\left[ \xi(t) T_k(\rho_\delta)(t, \cdot) \right](t, x), \quad \psi \in \mathcal{D}(0, T), \quad \xi \in \mathcal{D}(D), \]
and \( \mathcal{A} \) is the operator introduced in the proof of Lemma 5. Using the equation,
\[ \partial_t T_k(\rho_\delta) + \text{div}(T_k(\rho_\delta)U_\delta) + \left( T_k(\rho_\delta) - T_k(\rho_\delta) \right) \text{div } U_\delta = 0 \quad \text{in } D'(DT), \] (171)
and the equalities,
\[ \text{div } \phi = \psi \nabla \eta \cdot \mathcal{A}\left[ \xi T_k(\rho_\delta) \right] + \psi \eta \xi T_k(\rho_\delta), \]
\[ \partial_t \phi_i = \psi \partial_t \eta_i \mathcal{A}\left[ \xi T_k(\rho_\delta) \right] + \psi \eta_i \partial_t \mathcal{A}\left[ \xi T_k(\rho_\delta) \right] = \psi \partial_t \eta_i \mathcal{A}\left[ \xi T_k(\rho_\delta) \right] - \psi \eta R_j R_i \left[ \xi T_k(\rho_\delta) \right], \]
\[ \partial_t \phi = \partial_t \psi A \left[ \xi T_k(\rho) \right] + \psi \eta A \left[ \xi \partial_t (T_k(\rho)) \right] = \partial_t \psi A \left[ \xi T_k(\rho) \right] + \psi \eta A \left[ T_k(\rho) \nabla \xi \cdot U_3 \right] - \psi \eta A \left[ \text{div} (\xi T_k(\rho) U_3) \right] - \psi \eta A \left[ (T_k' \rho - T_k(\rho)) \text{div} U_3 \right], \]

we can write:

\[ \int \int_{\mathbb{R}^3} \psi (\xi p T_k(\rho) - S_8 : (\nabla \Delta^{-1}) \text{bi} g l[\xi T_k(\rho)]) \, dx \, dt = \sum_{j=1}^{8} I_j^8, \] (172)

with

\[ I_1^8 = \int \int_{\mathbb{R}^3} \psi (S_8 \nabla \eta) \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_2^8 = - \int \int_{\mathbb{R}^3} \psi p \nabla \eta \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_3^8 = - \int \int_{\mathbb{R}^3} \psi \partial_t \psi A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_4^8 = - \int \int_{\mathbb{R}^3} \psi \partial_t \psi A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_5^8 = \int \int_{\mathbb{R}^3} \psi \eta p U_3 \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_6^8 = \int \int_{\mathbb{R}^3} \psi \eta p U_3 \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_7^8 = \int \int_{\mathbb{R}^3} \psi \eta p U_3 \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_8^8 = \int \int_{\mathbb{R}^3} \psi p T_k(\rho) \left( \nabla \Delta^{-1} \nabla \right) \left[ \eta p U_3 \right] - \left( \nabla \Delta^{-1} \nabla \right) \left[ \xi T_k(\rho) \right] \eta p U_3 \, dx \, dt, \]

\[ \mathcal{R}_i \] is the Riesz operator defined via the Fourier transform by \( \mathcal{R}_i (v) = F^{-1} \left[ \frac{\partial}{\partial \xi} F(v) \right] \) and \( (\nabla \Delta^{-1} \nabla)_{i,j} = -\mathcal{R}_i \mathcal{R}_j \).

Following again the proof of Proposition 6.1, the limit functions satisfy the equality:

\[ \int \int_{\mathbb{R}^3} \psi (\xi p T_k(\rho) - S : (\nabla \Delta^{-1}) \text{bi} g l[\xi T_k(\rho)]) \, dx \, dt = \sum_{j=1}^{8} I_j, \] (173)

where

\[ I_1 = \int \int_{\mathbb{R}^3} \psi (S \nabla \eta) \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_2 = - \int \int_{\mathbb{R}^3} \psi p \nabla \eta \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_3 = - \int \int_{\mathbb{R}^3} \psi \partial_t \psi A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_4 = - \int \int_{\mathbb{R}^3} \psi \partial_t \psi A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_5 = \int \int_{\mathbb{R}^3} \psi \eta p U_3 \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_6 = \int \int_{\mathbb{R}^3} \psi \eta p U_3 \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_7 = \int \int_{\mathbb{R}^3} \psi \eta p U_3 \cdot A \left[ \xi T_k(\rho) \right] \, dx \, dt, \]
\[ I_8 = \int \int_{\mathbb{R}^3} \psi p T_k(\rho) \left( \nabla \Delta^{-1} \nabla \right) \left[ \eta p U_3 \right] - \left( \nabla \Delta^{-1} \nabla \right) \left[ \xi T_k(\rho) \right] \eta p U_3 \, dx \, dt, \]
Moreover, as in the proof of Proposition 6.1 in [15] we have:

\[
I_5 = - \int \int \psi \eta p T_k(\rho) \nabla \psi \cdot \eta \, dx \, dt,
\]

\[
I_6 = - \int \int \partial_t \psi \eta U \cdot A[\xi T_k(\rho)] \, dx \, dt,
\]

\[
I_7 = \int \int \psi \eta u U \cdot A[\xi (T_k(\rho) - T_k(\rho))] \nabla U \, dx \, dt,
\]

\[
I_8 = \int \int \psi (\xi T_k(\rho)) (\nabla \Delta^{-1} \nabla) [\eta U] - (\nabla \Delta^{-1} \nabla) [\xi T_k(\rho)] \eta U \, dx \, dt;
\]

and

\[\tilde{R} = \mu_0 (M \cdot \nabla) H + 2 \xi \nabla \Delta \psi \eta \Omega\]

We pass to the limit, as \(\delta \to 0\), in each of the terms \(I_7^j\), \(j \neq 3\), exactly as in the proof of Proposition 6.1 in [15], and obtain \(I_7^\delta \to I_7\).

Consider now \(I_3^\delta\). Since \(T_k(\rho_\delta)\) satisfies the renormalized Eq. (171), we have:

\[
T_k(\rho_\delta) \to T_k(\rho) \quad \text{in} \quad C([0, T]; L_{\text{weak}}(D)), \quad \text{for any} \ p \geq 1,
\]

(174)

and then

\[
A[\xi T_k(\rho_\delta)] \to A[\xi T_k(\rho)] \quad \text{in} \quad C(O), \quad \text{for any compact} \ O \subset D_T.
\]

According to (163) and (167) we have:

\[
\tilde{R}_\delta \to \tilde{R} \quad \text{in} \quad L^{5/4}(0, T; \mathbb{L}^{5/4}(D)),
\]

therefore \(I_3^\delta \to I_3\) as \(\delta \to 0\) and we conclude that

\[
\lim_{\delta \to 0} \int \int \psi \eta (\xi T_k(\rho_\delta) - S_\delta) : (\nabla \Delta^{-1} \nabla) [\xi T_k(\rho_\delta)] \, dx \, dt = \int \int \psi \eta (\xi \partial_t T_k(\rho) - S) : (\nabla \Delta^{-1} \nabla) [\xi T_k(\rho)] \, dx \, dt.
\]

(175)

Moreover, as in the proof of Proposition 6.1 in [15] we have:

\[
\int \int \psi \eta S_\delta : (\nabla \Delta^{-1} \nabla) [\xi T_k(\rho_\delta)] \, dx \, dt = (\lambda + 2 \mu) \int \int \psi \xi (\text{div} U_\delta) T_k(\rho_\delta) \, dx \, dt
\]

\[
+ 2(\mu + \xi) \int \int \psi \eta (U_\delta \otimes \nabla \eta) + U_\delta \cdot \nabla \eta \, dx \, dt,
\]

(176)

and similarly,

\[
\int \int \psi \eta S : (\nabla \Delta^{-1} \nabla) [\xi T_k(\rho)] \, dx \, dt = (\lambda + 2 \mu) \int \int \psi \xi (\text{div} U) T_k(\rho) \, dx \, dt
\]

\[
+ 2(\mu + \xi) \int \int \psi \eta T_k(\rho) : (U \otimes \nabla \eta) + U \cdot \nabla \eta \, dx \, dt.
\]

(177)
Taking (174) into account, the last term in (176) converges, as $\delta \to 0$, towards the last term of (177), and then from (175)–(177) we deduce (168). Lemma 7 is established. 

**Remark 2.** The vortex viscosity coefficient $\zeta$, although it is present in the viscous stress tensor, does not appear in the property of weak continuity of the effective viscous flux (see also Lemma 5).

### 6.3. End of the proof of the main theorem

Recall first the concept of oscillations defect measure introduced by E. Feireisl [15], Section 6.4. The oscillations defect measure associated with a sequence $\rho_m \to \rho$ weakly in $L^1(O)$, $O \subset (0, T) \times \mathbb{R}^3$ and compact, is defined by:

$$ \text{osc}_p[\rho_m \to \rho](O) \equiv \sup_{k \geq 1} \left( \limsup_{m \to \infty} \int_O |T_k(\rho_m) - T_k(\rho)|^p \, dx \, dt \right), $$

where $T_k$ are the cut-off functions introduced in Section 6.2.

According to the strong convergence in $L^2(0, T; L^2(D))$ of $(M_\delta)$ towards $M$, which implies $p_m(M_\delta) \to p_m(M)$ in $L^1(D_T)$ strong, and the equalities $p(\rho_\delta, M_\delta) = p(\rho_\delta) + p_m(M_\delta) = \alpha \rho_\delta + \frac{\mu}{2} |\nabla \rho_\delta|^2$, relation (168) reduces to

$$ \lim_{\delta \to 0} \int_{D_T} \psi \eta (p(\rho_\delta) - (\lambda + 2\mu) \text{div} \, U_\delta) \, T_k(\rho_\delta) \, dx \, dt = \int_{D_T} \psi \eta (p(\rho) - (\lambda + 2\mu) \text{div} \, U) \, T_k(\rho) \, dx \, dt, $$

(178)

where as above the bar denotes a weak limit. Consider now an arbitrary compact $O \subset D_T$. We deduce from (178) that

$$ \lim_{\delta \to 0} \int_O (p(\rho_\delta) T_k(\rho_\delta) - p(\rho) T_k(\rho)) \, dx \, dt = (\lambda + 2\mu) \lim_{\delta \to 0} \int_O ((\text{div} \, U_\delta) T_k(\rho_\delta) - (\text{div} \, U) T_k(\rho)) \, dx \, dt.$$

(179)

By following the proof of Proposition 6.2 in [15] we have:

$$ \lim_{\delta \to 0} \int_O (p(\rho_\delta) T_k(\rho_\delta) - p(\rho) T_k(\rho)) \, dx \, dt \geq a \limsup_{\delta \to 0} \int_O |T_k(\rho_\delta) - T_k(\rho)|^{\gamma + 1} \, dx \, dt $$

(180)

and

$$ \lim_{\delta \to 0} \int_O ((\text{div} \, U_\delta) T_k(\rho_\delta) - (\text{div} \, U) T_k(\rho)) \, dx \, dt \leq C \limsup_{\delta \to 0} \|T_k(\rho_\delta) - T_k(\rho)\|_{L^2(O)}.$$  

(181)

Then we deduce from (179)–(181), together with the Hölder inequality, that

$$ \text{osc}_{\gamma + 1}[\rho_\delta \to \rho](O) \leq C(|O|).$$

(182)

Consequently, in accordance with Proposition 6.3 in [15], the limit function $\rho$ is a renormalized solution of the continuity equation (8); we have in particular,

$$ \partial_t L_k(\rho) + \text{div}(L_k(\rho) U) + T_k(\rho) \, \text{div} \, U = 0 \quad \text{in} \ D'((0, T) \times \mathbb{R}^3),$$

(183)

provided $\rho$ and $U$ were extended by 0 outside $D$, where $L_k(\rho) = \rho \int_1^\rho \frac{T_k(z)}{z^2} \, dz$. Since we also have,

$$ \partial_t L_k(\rho_\delta) + \text{div}(L_k(\rho_\delta) U_\delta) + T_k(\rho_\delta) \, \text{div} \, U_\delta = 0 \quad \text{in} \ D'((0, T) \times \mathbb{R}^3),$$

we deduce that

$$ \partial_t L_k(\rho) + \text{div}(L_k(\rho) U) + T_k(\rho) \, \text{div} \, U = 0 \quad \text{in} \ D'((0, T) \times \mathbb{R}^3).$$

(184)

Now, following Section 6.6 in [15] we take the difference of (183) and (184) and use (178) and the monotonicity of $p_e$ to derive:

$$ \int_D (\overline{L_k(\rho)}(t) - L_k(\rho)(t)) \, dx \, dt \leq \int_0^t \int_D (T_k(\rho) - \overline{T_k(\rho)}) \, dx \, dt \quad \text{for all} \ t \in [0, T].$$
According to (182), the right-hand side tends to 0 as \( k \to \infty \) then it holds \( \frac{\rho \ln(\rho)}{\rho} = \ln(\rho) \) in \( DT \) which implies that \( \rho_0 \to \rho \) in \( L^1(D_T) \) which can be improved to \( \rho_0 \to \rho \) in \( C([0, T]; L^1(D)) \). Moreover, \( \rho \geq 0 \) a.e. in \( DT \), the momentum equation (9), with \( p = a\rho^\gamma + \frac{\mu}{\rho} |\nabla M|^2 \) and \( R = \mu_0 (M \cdot \nabla) H - \zeta \nabla \times \times U + 2\zeta \nabla \times \Omega \), and the angular momentum equation (10) hold in \( D'(DT) \), and the magnetization \( M \) satisfies the integral identity (21).

Arguing similarly as in Section 4.3.3 we prove (24) and we verify all the points of Definition 1. This ends the proof of the main theorem.

Uncited references

[17]

References


