Chapitre 4

A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems

Résumé

Dans ce chapitre, nous introduisons et analysons une formulation mortier stabilisée utilisant des multiplicateurs de Lagrange discontinus, pour des approximations aux éléments finis d’ordre 1 et 2 de la solution de problèmes d’élasticité linéarisée. Cette approche s’inscrit dans la continuité de Brezzi et Marini [BM00] qui utilisent de tels multiplicateurs pour une formulation dite à trois champs dans le cadre de problèmes elliptiques scalaires. Dans le cas d’un grand nombre de sous-domaines, nous montrons en outre l’indépendance de la constante de coercivité de la forme bilinéaire associée au problème d’élastostatique non-conforme par rapport au nombre de sous-domaines considérés et à leur taille, par extension au cas d’interfaces courbes des idées de Gopalakrishnan et Brenner [Gop99, Bre03, Bre04], et généralisation de l’opérateur d’interpolation de Scott et Zhang [SZ90]. De plus, nous rappelons la convergence optimale de la méthode en élastostatique linéarisée en utilisant les outils de Wohlmuth [Woh01], et procédons à une extension au cas de l’élastodynamique. Enfin, des choix concrets d’espaces sont proposés, les détails pratiques de mise en œuvre sont indiqués, et des tests numériques viennent illustrer la présente analyse.
Abstract

We introduce and analyze first and second order stabilized discontinuous two-field mortar formulations for linearized elasticity problems, following the stabilization technique of Brezzi and Marini [BM00] introduced in the scalar elliptic case for a three-field formulation. By extension to the curved interfaces case of the ideas from Gopalakrishnan and Brenner [Gop99, Bre03, Bre04], and from the introduction a generalized Scott and Zhang interpolation operator [SZ90], we prove the independence of the coercivity constant of the broken elasticity bilinear form with respect to the number and the size of the subdomains. Moreover, we prove the optimal convergence of the method by mesh refinement by using the tools from Wohlmuth [Woh01] in the elastostatic case, and extend the result to the elastodynamic framework. Finally, we detail practical issues and present numerical tests to illustrate the present analysis.

4.1 Introduction

In this paper, we introduce, analyze and test a non-conforming formulation using stabilized discontinuous mortar elements to find the vector solution $u$ of linearized elasticity problems such as:

$$
\begin{align*}
- \text{div}(E : \varepsilon(u)) &= f, \quad \Omega \subset \mathbb{R}^d, (d = 2, 3) \\
u &= 0, \quad \Gamma_D, \\
(E : \varepsilon(u)) \cdot n &= g, \quad \Gamma_N,
\end{align*}
$$

(4.1)

where the linearized strain tensor is classically given by:

$$
\varepsilon(u) = \frac{1}{2} (\nabla u + \nabla^t u),
$$

and the fourth order elasticity tensor $E$ is assumed to be elliptic over the set of symmetric matrices:

$$
\exists \alpha > 0, \forall \xi \in \mathbb{R}^{d \times d}, \xi^t = \xi, \quad (E : \xi) : \xi \geq \alpha \xi : \xi.
$$

The analysis is also extended to the elastodynamics problem:

$$
\begin{align*}
\rho \frac{\partial^2 u}{\partial t^2} - \text{div}(E : \varepsilon(u)) &= f, \quad [0, T] \times \Omega, \\
u &= 0, \quad [0, T] \times \Gamma_D, \\
(E : \varepsilon(u)) \cdot n &= g, \quad [0, T] \times \Gamma_N, \\
\frac{\partial u}{\partial t} &= u_0, \quad \{0\} \times \Omega, \\
\frac{\partial u}{\partial t} &= \dot{u}_0, \quad \{0\} \times \Omega,
\end{align*}
$$

(4.2)
and we consider this analysis as a theoretical background for using discontinuous mortar elements in nonlinear elastodynamics.

Mortar methods have been introduced for the first time in [BMP93, BMP94] as a weak coupling between subdomains with nonconforming meshes, or between subproblems solved with different approximation methods. The main purpose was to overcome the very sub-optimal \( \sqrt{k} \) error estimate obtained with pointwise matching. The analysis of this method as a mixed formulation was first made in [Bel99].

Nevertheless, in spite of the optimal error convergence obtained with the original mortar elements, some numerical difficulties appear. First, the original space of Lagrange multipliers ensuring the weak coupling is rather difficult to build in 3D on the boundary of the interfaces when more than two subdomains have a common intersection (see [BM97, BD98]). Moreover, the original constrained space has a non-local basis on the non-conforming artificial interfaces, which may lead to small spurious oscillations of the approximate solution.

To overcome the first difficulty, one idea is given in [Ses98] when displacements are at least approximated by second order polynomials. The introduced Lagrange multipliers have a lower order, still enabling optimal error estimates, and no special treatment is needed on the boundary of the interfaces. To overcome the second difficulty, dual mortar spaces are proposed in [Woh00, Woh01], enabling the localization of the mortar kinematical constraint. In order to benefit from the advantages of these two approaches, we propose to introduce stabilized low order discontinuous mortar elements. This idea has already been introduced for a first order three-field mortar formulation in [BM00], and we exploit it herein in the two-field framework for first and second order elements when dealing with elastostatics and elastodynamics problems.

Mortar formulations also provide a natural framework for domain decomposition, as observed by [Tal93, AKP95, AMW99, AAKP99, Ste99] and the references therein. A large number of subdomains and their small size is therefore a basic difficulty to overcome. To get an optimal use of such domain decomposition methods, it is then crucial that the constants arising in the analysis of the mortar formulation remain independent (or at least weakly dependent) on the number and the size of the subdomains. One can readily check that the only potential dependence on such parameters is hidden in the coercivity constant of the broken bilinear form associated to the linearized elastostatics problem. In the framework of elliptic scalar problems, both [Gop99, Bre03] and [BM00] have shown the independence of the coercivity constant with respect to the number and the size of the subdomains, respectively when considering two and three-field mortar formulations with plane interfaces. An extension to the vector elasticity case has been proposed by [Bre04]. By definition of a generalized Scott and Zhang [SZ90] interpolation operator, we simplify and extend herein the result to potentially curved interfaces.

In section 2, the fundamental assumptions and results arising in mortar element methods to approximate the solution of the elastostatics problem (4.1) are recalled. Well-posedness results are recalled in section 3. Moreover, we prove in section 4, the indepen-
dence of the coercivity constant with respect to the number and the size of the subdomains. In section 5, we recall the optimal convergence of the method by mesh refinement, and generalize the analysis to the elastodynamics problem (4.2) in section 6. We propose in section 7 the analysis of stabilized discontinuous mortar elements, proving the satisfaction of the fundamental assumptions. In section 8, some practical issues are pointed out: the choice of an appropriate penalization term, and the exact integration of the constraint. We present numerical tests in section 9 to confirm the previous analysis.

4.2 Nonconforming setting

4.2.1 Position of the problem

Let \( \Omega \subset \mathbb{R}^d \) \( (d = 2, 3) \), be an open set partitioned into \( K \) subsets \( (\Omega_k)_{1 \leq k \leq K} \). We denote by \( \Gamma_{kl} = \overline{\Omega_k} \cap \overline{\Omega_l} \) the interface between \( \Omega_k \) and \( \Omega_l \), and the skeleton of the internal interfaces is denoted by \( \mathcal{S} = \bigcup_{k,l \geq 1} \Gamma_{kl} \). On the part \( \Gamma_D \) of the boundary \( \partial \Omega \), an homogeneous Dirichlet boundary condition is imposed. Concerning the coefficients of the fourth order elasticity tensor \( E \), we assume that the stress tensor is symmetric whatever the deformation is in the material, namely for almost all \( x \in \Omega \):

\[
\forall \xi \in \mathbb{R}^{d \times d}, \xi^t = \xi, \quad E(x) : \xi \text{ is a symmetric matrix.}
\]

Moreover, in the theoretical analysis, we will suppose that for all \( k \geq 1 \), there exists two constants \( c_k \) and \( C_k \), such that for almost all \( x \in \Omega_k \):

\[
\forall \xi \in \mathbb{R}^{d \times d}, \xi^t = \xi, \quad c_k \xi : \xi \leq (E(x) : \xi) : \xi \leq C_k \xi : \xi. \quad (4.3)
\]

If the material of the subdomain \( \Omega_k \) has a Young modulus \( E_k \), both \( c_k \) and \( C_k \) are proportional to \( E_k \).

We introduce the following spaces:

\[
H^1_v(\Omega) = \{ v \in H^1(\Omega)^d, \ v|_{\Gamma_D} = 0 \},
\]

\[
H^1_v(\Omega_k) = \{ v \in H^1(\Omega_k)^d, \ v|_{\Gamma_D \cap \partial \Omega_k} = 0 \},
\]

\[
X = \left\{ v \in L^2(\Omega)^d, \ v_k = v|_{\Omega_k} \in H^1_v(\Omega_k), \forall k \right\} = \prod_{k=0}^{K} H^1_v(\Omega_k),
\]

\( X \) being endowed with the \( H^1 \) broken norm:

\[
\| v \|_X = \left( \sum_{k=0}^{K} \| v \|_{H^1_v(\Omega_k)^d}^2 \right)^{\frac{1}{2}}.
\]
Here, in order to be scale independent when dealing with a large number of subdomains, we use a scale invariant definition of the $H^1$ norm:

$$\|v\|_{H^1(\Omega)}^2 = \frac{1}{(L_k)^2} \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2,$$

$L_k$ being a characteristic length of $\Omega_k$, for instance its diameter.

We are interested in finding $u \in H^1_s(\Omega)$ such that:

$$a(u, v) = l(v), \quad \forall v \in H^1_s(\Omega), \quad (4.4)$$

where the continuous coercive bilinear form $a$ is defined by:

$$a(u, v) = \int_{\Omega} (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in H^1_s(\Omega),$$

and the continuous linear form $l$ by:

$$l(v) = \int_{\Omega} f \cdot v + \int_{\Gamma^N} g \cdot v, \quad \forall v \in H^1_s(\Omega).$$

This problem is classically well-posed by Lax-Milgram lemma, the Korn’s inequality (see [DL72]) ensuring the coercivity of the bilinear form $a$ over $H^1_s(\Omega) \times H^1_s(\Omega)$.

### 4.2.2 Discretization

We introduce here a non-conforming discretization of the problem (4.4) using mortar elements to be further defined later on. The discrete problem is proved to be well-posed and error estimates are derived in the mesh-dependent norms already introduced and used in [AT95, Woh99]. Some useful elementary trace and lifting results for the related mesh-dependent spaces are reviewed and detailed in appendix 4.10.
The mesh

For each $1 \leq k \leq K$, we consider a family of shape regular affine meshes $(T_k; h_k)_{h_k > 0}$ on the subdomain $\Omega_k$. This means that each element $T$ is the image of a reference element $\hat{T}$ by an affine mapping $J_T$. For each $T \in T_k; h_k$, we will denote its diameter:

$$h(T) = \text{diam}(T),$$

and the local mesh size by:

$$h_k = \sup_{T \in T_k; h_k} h(T).$$

Then, a nonconforming family of domain based meshes $(T_h)_{h > 0}$ over $\Omega$ is obtained by:

$$T_h = \bigcup_{k=1}^K T_{k; h_k}, \quad h = \max_{1 \leq k \leq K} h_k.$$

The skeleton $S = \bigcup_{k,l \geq 1} \Gamma_{kl}$ is partitioned into $M$ interfaces $(\Gamma_m)_{1 \leq m \leq M}$, and can then be decomposed as $S = \bigcup_{1 \leq m \leq M} \Gamma_m$. Moreover, we assume that for each $1 \leq m \leq M$, there exists at least one domain $\Omega_k$ with $k \geq 1$ such that $\Gamma_m \subset \partial \Omega_k$, and denote $k(m) := k$ the name of one of these subdomains, taken once for all for each interface. This side will said to be the non-mortar (or slave) side.

For each $1 \leq m \leq M$, $\Gamma_m$ inherits a family of meshes $(F_{m; \delta_m})_{\delta_m > 0}$, obtained as the trace of the volumic mesh $(T_{k(m); h_k(m)})_{h_k(m) > 0}$ of the slave subdomain over $\Gamma_m$. We have denoted by:

$$\delta_m = \sup_{F \in F_{m; \delta_m}} h(F).$$

We also denote by $\bar{\delta}_m$ the size of the mesh on the mortar side:

$$\bar{\delta}_m = \sup_{T \in T_{h_k; l \neq k(m)}} \text{diam}(T \cap \Gamma_m).$$

Then, a family of interface meshes $(F_{\delta})_{\delta > 0}$ can be defined over $S$ by:

$$F_{\delta} = \bigcup_{m=1}^M F_{m; \delta_m}, \quad \delta = \max_{1 \leq m \leq M} \delta_m.$$

For each $F \in F_{m; \delta_m}$, we denote by $T(F) \in T_{k(m); h_k(m)}$ the unique element $T \in T_{h(m); h_k(m)}$ such that $T \cap S = F$.

Moreover, the following assumption is made:

**Assumption 4.1.** $F \in F_{\delta}$ is always an entire face of $T(F) \in T_h$.

In other words, the construction of the interfaces $(\Gamma_m)_{1 \leq m \leq M}$ respects the mesh of the slave sides. An example of situation obeying to assumption 1 is given on figure 4.2.
4.2. Nonconforming setting

Fig. 4.2 – A situation where the mesh $\mathcal{F}_{1,\delta_1}$ of the interface $\gamma_{01}$ is inherited from the mesh $\mathcal{T}_{0,h_0}$ of $\Omega_0$. The assumption 1 would be violated if at the opposite, $\Omega_1$ were the slave side.

Remark 4.1. For simplicity, the mesh is assumed to be affine but the following results are still valid for regular quasi-uniform quadrangular meshes, at least in 2D (see [GR86]). In fact, the only assumptions to satisfy are the following standard inequalities:

\[
\begin{align*}
|\hat{w}|_{H^m(\tilde{K})} & \leq C \text{ diam}(K)^m \text{meas}(K)^{-\frac{1}{2}} |w|_{H^m(K)}, \\
|w|_{H^m(K)} & \leq C \text{ diam}(K)^{-m} \text{meas}(K)^{\frac{1}{2}} |\hat{w}|_{H^m(\tilde{K})},
\end{align*}
\]

between the semi-norms of the function $w$ defined on a mesh-element $K$ and its transformation $\hat{w}$ defined on the corresponding reference element $\tilde{K}$.

Remark 4.2. In the following sections, $C$ will stand for various constants independent of the discretization.

Interface mesh-dependent spaces

We define here some mesh-dependent trace spaces, endowed with useful mesh-dependent norms already introduced and used in [AT95, Woh99]. For each $1 \leq m \leq M$, they are defined by:

\[
\begin{align*}
\mathcal{H}^{1/2}_{\delta}(\Gamma_m) &= \{ \phi \in L^2(\Gamma_m)^d, \| \phi \|_{\delta^{1/2},m}^2 = \sum_{F \in \mathcal{F}_{m,\delta_m}} \frac{1}{h(F)} \| \phi \|_{L^2(F)}^2 < +\infty \}, \\
\mathcal{H}^{-1/2}_{\delta}(\Gamma_m) &= \{ \lambda \in L^2(\Gamma_m)^d, \| \lambda \|_{\delta^{-1/2},m}^2 = \sum_{F \in \mathcal{F}_{m,\delta_m}} h(F) \| \lambda \|_{L^2(F)}^2 < +\infty \},
\end{align*}
\]

endowed respectively with the norms $\| \cdot \|_{\delta^{1/2},m}$ and $\| \cdot \|_{\delta^{-1/2},m}$. The product spaces $\mathcal{W}_\delta = \prod_{k=1}^K \mathcal{H}^{1/2}_{\delta}(\Gamma_m)$ and $\mathcal{M}_\delta = \prod_{k=1}^K \mathcal{H}^{-1/2}_{\delta}(\Gamma_m)$, are then respectively endowed with the
norms :
\[
\|\phi\|_{\delta, \frac{1}{2}} = \left( \sum_{m=1}^{M} \|\phi\|^2_{\delta, \frac{1}{2}, m} \right)^{1/2},
\]
\[
\|\lambda\|_{\delta, -\frac{1}{2}} = \left( \sum_{m=1}^{M} \|\lambda\|^2_{\delta, -\frac{1}{2}, m} \right)^{1/2}.
\]

They can be viewed as dual spaces by means of the the \(L^2\) inner product :
\[
\int_S \phi \cdot \lambda \leq \|\lambda\|_{\delta, -\frac{1}{2}} \|\phi\|_{\delta, \frac{1}{2}}, \quad \forall (\phi, \lambda) \in \mathbb{W}_\delta \times \mathbb{M}_\delta.
\] (4.5)

Some elementary results about these spaces can be found in appendix. Their advantage is that the corresponding norms are easily computable, enabling a posteriori estimates [Woh99] and efficient penalization strategies as shown in section 5.

4.2.3 Approximate problem

Nonconforming formulation

Let us define the discrete subspaces of degree \(q\) inside each subdomain :
\[
X_{k; h_k} = \{p \in H^1_0(\Omega_k) \cap C^0(\Omega_k)^d, \quad p|_T \in \mathcal{P}_q(T), \forall T \in T_{k; h_k} \} \oplus \mathcal{B}_{k; h_k},
\]
with \(\mathcal{P}_q = [\mathbb{P}_q]^d\) or \([\mathbb{Q}_q]^d\). We have denoted by \(\mathcal{P}_q\) (resp. \(\mathcal{Q}_q\)) the space of polynomials of total (resp. partial) degree \(q\), and have introduced the possibility of adding a space \(\mathcal{B}_{k; h_k}\) of interface bubble stabilization to be constructed later on. The corresponding product space is denoted by :
\[
X_h = \prod_{k=0}^{K} X_{k; h_k} \subset X.
\]

We introduce the following trace spaces on the non-mortar side :
\[
W_{m; \delta_m} = \{p|_{\Gamma_m}, p \in X_{k(m); h_k(m)} \}, \quad W^0_{m; \delta_m} = W_{m; \delta_m} \cap H^1_0(\Gamma_m)^d,
\]
and the corresponding product space \(W^0_\delta = \prod_{m=1}^{M} W^0_{m; \delta_m}\) endowed with the mesh-dependent norm \(\| \cdot \|_{\delta, \frac{1}{2}}\).

In order to formulate the weak continuity constraint, the following spaces of discontinuous Lagrange multipliers are defined :
\[
M_{m; \delta_m} = \{p \in L^2(\Gamma_m)^d, \quad p|_F \in \mathcal{P}_{q-1}(F), \forall F \in \mathcal{F}_{m; \delta_m} \},
\] (4.6)
as well as the product space $M_\delta = \prod_{m=1}^{M} M_{m;\delta_m}$, endowed with the mesh-dependent norm $\| . \|_{M;\frac{1}{2}}$ and $M = \prod_{m=1}^{M} L^2(\Gamma_m)^d$. The following continuous bilinear form is then introduced to impose interface weak continuity:

$$b : X \times M \rightarrow \mathbb{R}$$

$$(v, \lambda) \mapsto b(v, \lambda) = \sum_{m=1}^{M} \int_{\Gamma_m} [v]_m \cdot \lambda_m,$$

where $[v]_m = v_{k(m)} - v_l$, on $\gamma_{k(m)} \subset \Gamma_m$. Then, the constrained space of discrete unknowns can be defined as:

$$V_h = \{ u_h \in X_h, b(u_h, \lambda_h) = 0, \forall \lambda_h \in M_\delta \}.$$

In order to formulate the non-conforming approximate problem, it is standard to consider the broken elliptic form:

$$\tilde{a} : X \times X \rightarrow \mathbb{R}$$

$$(u, v) \mapsto \tilde{a}(u, v) = \sum_{k=1}^{K} a_k(u_k, v_k),$$

with:

$$a_k(u_k, v_k) = \int_{\Omega_k} (E : \varepsilon(u_k)) : \varepsilon(v_k).$$

We are then interested in finding $(u_h, \lambda_h) \in X_h \times M_\delta$, such that:

$$\begin{cases} 
\tilde{a}(u_h, v_h) + b(v_h, \lambda_h) = l(v_h), & \forall v_h \in X_h, \\
b(u_h, \mu_h) = 0, & \forall \mu_h \in M_\delta.
\end{cases}$$

In other words, we solve our variational problem on the product space $X_h$ under the kinematic continuity constraint $b(\cdot, \cdot) = 0$.

**Remark 4.3.** The theory proposed herein also applies to situations involving continuous Lagrange multipliers defined on spaces like:

$$M_{m;\delta_m} = \{ p \in C^0(\Gamma_m), \ p|_F \in P_p(F), \forall F \in F_{m;\delta_m} \}.$$
Assumption 4.2. For each interface $1 \leq m \leq M$, there exists an operator:
\[ \pi_m : H^{1/2}_0(\Gamma_m) \rightarrow W^0_{m;\delta_m}, \]
such that for all $v \in H^{1/2}_0(\Gamma_m)$:
\[ \int_{\Gamma_m} (\pi_m v) \cdot \mu = \int_{\Gamma_m} v \cdot \mu, \quad \forall \mu \in M_{m;\delta_m}, \]
with:
\[ \|\pi_m v\|_{\delta_{1/2},m} \leq C_m \|v\|_{\delta_{1/2},m}. \]

This assumption means that the projection perpendicular to the multiplier space onto the trace space $W^0_{m;\delta_m}$ with zero extension is continuous. This assumption will have to be checked for each choice of discretization. Its major consequence lies in the fact that the weak-continuity constraint is onto, as shown in the next section.

Concerning the coercivity of $\tilde{a}$ over $V \times V$, where $V$ is a constrained subspace of $X$ to be defined in that section, we have to consider Lagrange multipliers spaces which are sufficiently rich on the interfaces to kill local rigid motions, which are defined by:

Definition 4.1. A displacement field $r \in H^1(\Omega)^d$ over $\Omega \subset \mathbb{R}^d$ is said to be a rigid motion of $\Omega$ iff:
\[ \int_{\Omega} \varepsilon(r) : \varepsilon(w) = 0, \quad \forall w \in H^1(\Omega)^d, \]
which we denote $r \in \mathcal{R}(\Omega)$.

For that purpose, we introduce the following assumption over the Lagrange multipliers spaces:

Assumption 4.3. For all $1 \leq m \leq M$, we assume that there exists two integers $1 \leq k, l \leq K$ such that $\Gamma_m = \gamma_{kl}$ and a minimal Lagrange multiplier space $M_{kl}$ such that $M_{kl} \subset M_{m;\delta_m}$ independently of the discretization. Moreover, we assume that for all $v \in X$ which is locally a rigid motion both over the subdomains $\Omega_k$ and $\Omega_l$, that is $v|_{\Omega_k} \in \mathcal{R}(\Omega_k)$ and $v|_{\Omega_l} \in \mathcal{R}(\Omega_l)$, we have:
\[ \int_{\gamma_{kl}} [v] \cdot \mu = 0 \quad \forall \mu \in M_{kl} \quad \implies [v]_{kl} = 0, \quad (4.8) \]
where the jump of $v$ over $\gamma_{kl}$ is denoted by $[v]_{kl}$.

Under assumption 4.3, the constrained subspace $V$ of $X$ on which the coercivity of the broken bilinear form $\tilde{a}$ holds, is defined as:
\[ V = \{ v \in X, \int_{\gamma_{kl}} [v] \cdot \mu = 0, \quad \forall \mu \in M_{kl}, \quad 1 \leq k, l \leq K \}. \]
Remark 4.4. In the scalar case, as shown in [BMP94], this assumption would be reduced to impose that constant functions belong to the minimal Lagrange multipliers spaces $\left( M_{kl} \right)_{1 \leq k, l \leq K}$.

4.3 Well-posedness

4.3.1 Inf-sup condition

The main consequence of assumption 4.2 is the inf-sup condition satisfied by the bilinear form $b$ expressing the mortar condition:

**Proposition 4.1.** Under assumption 4.2, there exists a constant $\beta > 0$ such that:

$$\inf_{\lambda_h \in M_k \setminus \{0\}} \sup_{u_h \in X_h \setminus \{0\}} \frac{b(u_h, \lambda_h)}{\|\lambda_h\|_{\delta, -\frac{1}{2}} \|u_h\|_X} \geq \beta, \quad (4.9)$$

where $\beta$ is of the form:

$$\beta = C \min_{1 \leq m \leq M} \frac{1}{(C_m)^2},$$

where the constant $C_m$ is the stability constant of $\pi_m$ defined in assumption 4.2, for all $1 \leq m \leq M$, and $C$ is independent of the discretization and of the number of subdomains.

**Proof:** For completeness, we recall the proof from [Woh01]. Let $\lambda \in M_\delta$. For all $1 \leq m \leq M$, denoting by $\lambda_m := \lambda|_{\Gamma_m}$ we have by construction:

$$\|\lambda_m\|_{\delta, -\frac{1}{2}, m} = \sup_{\phi \in \mathbb{E}_\delta^{1/2}(\Gamma_m)} \frac{\int_{\Gamma_m} \lambda_m \cdot \phi}{\|\phi\|_{\delta, \frac{1}{2}, m}},$$

and by definition of the projection $\pi_m$ and by using assumption 4.2:

$$\|\lambda_m\|_{\delta, -\frac{1}{2}, m} \leq C_m \sup_{\phi \in \mathbb{E}_\delta^{1/2}(\Gamma_m)} \frac{\int_{\Gamma_m} \lambda_m \cdot \pi_m \phi}{\|\pi_m \phi\|_{\delta, \frac{1}{2}, m}} \leq C_m \max_{\phi \in \mathbb{W}_m^0, \delta_m} \frac{\int_{\Gamma_m} \lambda_m \cdot \phi}{\|\phi\|_{\delta, \frac{1}{2}, m}}.$$

Let $\phi_\lambda_m \in \mathbb{W}_m^0, \delta_m$ be the function reaching the maximum with $\|\phi_\lambda_m\|_{\delta, \frac{1}{2}, m} = 1$, hence:

$$\int_{\Gamma_m} \lambda_m \cdot \phi_\lambda_m \geq \frac{1}{C_m} \|\lambda_m\|_{\delta, -\frac{1}{2}, m}. \quad (4.10)$$

We use the discrete extension by zero operator over the Lagrange nodes of the volumic mesh $\mathcal{R}_{m:h_m} : \mathbb{W}_m^0, \delta_m \to X_h$ (introduced in definition 4.2 of the appendix, page 203) to define:

$$u_m = \mathcal{R}_{m:h_m} \phi_\lambda_m.$$
Then, because \( \phi_{\lambda_m} \in W^0_{m;\delta_m} \), its extension \( u_m \) has zero trace on all interfaces except \( \Gamma_m \) and thus it is obtained that:

\[
b(u_m, \lambda_m) = \int_{\Gamma_m} \phi_{\lambda_m} \cdot \lambda_m.
\]

(4.11)

Now, let us define the function \( u_h \in X_h \) by:

\[
u_h = \sum_{m=1}^{M} b(u_m, \lambda_m) u_m.
\]

As a consequence, using (4.11) and (4.10):

\[
b(u_h, \lambda) = \sum_{m=1}^{M} b(u_m, \lambda_m)^2 \geq C \sum_{m=1}^{M} \frac{1}{C_m^2} \| \lambda_m \|_{\delta_m, -\frac{1}{2}, m}^2 \geq \min_{1 \leq m \leq M} \frac{1}{(C_m)^2} \| \lambda \|_{\delta_m, -\frac{1}{2}}^2.
\]

(4.12)

Using the locality of the supports of the \( (u_m)_m \) on disjoint small strips around the \( (\Gamma_m)_m \) and lemma 4.19 of the appendix (page 203):

\[
\| u_h \|_X^2 = \sum_{m=1}^{M} b(u_m, \lambda_m)^2 \| u_m \|_{H^1(\Omega_{\delta_m})}^2
\]

\[
\leq C \sum_{m=1}^{M} b(u_m, \lambda_m)^2 \| \phi_{\lambda_m} \|_{\delta_m, \frac{1}{2}, m}^2 = C \sum_{m=1}^{M} b(u_m, \lambda_m)^2.
\]

Then, by (4.11) and the Cauchy-Schwartz inequality (4.5):

\[
\| u_h \|_X^2 \leq C \sum_{m=1}^{M} \left( \int_{\Gamma_m} \phi_{\lambda_m} \cdot \lambda_m \right)^2 \leq C \sum_{m=1}^{M} \| \lambda_m \|_{\delta_m, -\frac{1}{2}, m}^2 = C \| \lambda \|_{\delta_m, -\frac{1}{2}}^2.
\]

(4.13)

Therefore, by construction, for all \( \lambda \in M_{\delta} \), there exists a \( u_h \in X_h \) such that:

\[
b(u_h, \lambda) \geq C \min_{1 \leq m \leq M} \frac{1}{(C_m)^2} \| \lambda \|_{\delta_m, -\frac{1}{2}},
\]

which ends the proof.

\[\square\]

**Remark 4.5.** In the absence of any triple point on the interface, that is if any function defined on \( \Gamma_m \) has zero trace on all other interfaces \( \Gamma_l \), \( l \neq m \), the previous proposition remains true even if one replaces \( W^0_{m;\delta_m} \) by \( W_{m;\delta_m} \) in assumption 4.2.
4.3.2 Local rigid motions

The set of rigid motions is spanned with translations and elementary rotations, as recalled in the following simple lemma:

**Lemma 4.1.** A tridimensional rigid motion $r \in \mathcal{R}(\Omega)$ with $\Omega \subset \mathbb{R}^3$ is a linear combination of the three translations:

$$t_1(x) = e_1, \quad t_2(x) = e_2, \quad t_3(x) = e_3, \quad x \in \Omega,$$

where $(e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$, and the three elementary rotations:

$$\begin{align*}
    r_1(x) &= e_1 \times x = 
    \begin{pmatrix}
      0 \\
      -x_3 \\
      x_2
    \end{pmatrix}, \\
    r_2(x) &= e_2 \times x = 
    \begin{pmatrix}
      x_3 \\
      0 \\
      -x_1
    \end{pmatrix}, \\
    r_3(x) &= e_3 \times x = 
    \begin{pmatrix}
      -x_2 \\
      x_1 \\
      0
    \end{pmatrix}, \quad x \in \Omega.
\end{align*}$$

A bidimensional rigid motion $r \in \mathcal{R}(\Omega)$ with $\Omega \subset \mathbb{R}^2$ is a linear combination of the translations $t_1, t_2$ and of the elementary rotation:

$$r(x) = \begin{pmatrix}
  -x_2 \\
  x_1
\end{pmatrix}, \quad x \in \Omega.$$

**Proof:** By definition of a rigid motion $v \in \mathcal{R}(\Omega)$, it follows that the symmetrized gradient $\varepsilon(v)$ vanishes almost everywhere in $\Omega$. Let us notice that for all $1 \leq i, j, l \leq d$, the following equality holds in the sense of distributions:

$$\frac{\partial^2 v_i}{\partial x_j \partial x_l} = \frac{\partial}{\partial x_j} \varepsilon_{jl}(v) + \frac{\partial}{\partial x_l} \varepsilon_{ij}(v) - \frac{\partial}{\partial x_i} \varepsilon_{jl}(v), \quad \text{in } \mathcal{D}'(\Omega),$$

where the indices $i, j, l$ denote the components of $v$ and $x$ in $\mathbb{R}^d$. The fact that $\varepsilon(v)$ vanishes almost everywhere in $\Omega$ implies that for all $1 \leq i, j, l \leq d$:

$$\frac{\partial^2 v_i}{\partial x_j \partial x_l} = 0, \quad \text{in } \mathcal{D}'(\Omega).$$

By a classical result from distribution theory ([Sch66], page 60) and provided $\Omega$ is connected, each scalar function $v_i$ for all $1 \leq i \leq d$ is an affine function, namely $v_i(x) =$
$a_i + \sum_{j=1}^{d} b_{ij} x_j$ for almost all $x \in \Omega$. Since $\varepsilon_{ii}(v) = 0$, we get $b_{ii} = 0$ and because $\varepsilon_{ij}(v) = 0$, we get $b_{ij} = -b_{ji}$. As a consequence, we obtain for $d = 3$ :

$$
\begin{cases}
v_1(x) = a_1 - b_{21} x_2 + b_{13} x_3, \\
v_2(x) = a_2 + b_{21} x_1 - b_{32} x_3, \\
v_3(x) = a_3 - b_{13} x_1 + b_{32} x_2, \quad \forall x \in \Omega,
\end{cases}
$$

and for $d = 2$ :

$$
\begin{cases}
v_1(x) = a_1 - b_{21} x_2, \\
v_2(x) = a_2 + b_{21} x_1, \quad \forall x \in \Omega,
\end{cases}
$$

which ends the proof.

\[\square\]

**Remark 4.6.** Another way to understand rigid motions in the linear framework comes from the nonlinear one. Indeed, in nonlinear elasticity, if we denote by $\varphi : \Omega \to \mathbb{R}^3$ the deformation of the reference configuration in the sense that $\varphi(\Omega)$ is the deformed domain, the associated strain tensor is defined by :

$$E(\varphi) = \frac{1}{2} (\nabla^t \varphi \cdot \nabla \varphi - id).$$

As shown in [Cia88], if $\Omega$ is a connected open set in $\mathbb{R}^3$ and $\varphi \in C^1(\Omega; \mathbb{R}^3)$, then $E(\varphi) = 0$ iff $\varphi$ is a rotation or a translation. In particular, let us introduce the rotation of angle $\theta$ with respect to $e_3$ :

$$R_\theta^3(x) = \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \\ x_3 \end{pmatrix}.$$ 

For all $\theta \in \mathbb{R}$, we have $E(R_\theta^3) = 0$ and therefore by differentiation with respect to $\theta$ for $\theta = 0$, it is obtained that :

$$dE_{id} \left. \left( \frac{\partial R_\theta^3}{\partial \theta} \right) \right|_{\theta=0} (x) = 0,$$

which is exactly :

$$\varepsilon(e_3 \times x) = 0.$$ 

We have then justified the fact that elementary rotations are rigid motions in the linearized framework. The case of translations is simpler. Indeed, let $T_k^3$ the translation of vector $ke_3$, that is :

$$T_k^3(x) = x + ke_3.$$ 

For all $k \in \mathbb{R}$, we have $E(T_k^3) = 0$, entailing by differentiation with respect to $k$ for $k = 0$ :

$$dE_{id} \left. \left( \frac{\partial T_k^3}{\partial k} \right) \right|_{k=0} (x) = 0,$$

which is exactly :

$$\varepsilon(e_3) = 0.$$
4.3. Well-posedness

4.3.3 Minimal Lagrange multipliers spaces

For instance, the implication (4.8) of assumption 4.3 is true when the traces of first order polynomial displacements over the interfaces belong to the Lagrange multipliers spaces, as shown in the next lemma.

Lemma 4.2. By choosing $M_{kl}$ as the restriction to $\gamma_{kl}$ of first order polynomial displacements, i.e.:

$$M_{kl} = M_1(\gamma_{kl}) = \mathbb{P}_1(\Omega)^d|_{\gamma_{kl}} := \{v|_{\gamma_{kl}}, \quad v \in \mathbb{P}_1(\Omega)^d\}, \quad 1 \leq k, l \leq K,$$

where $\mathbb{P}_1(\Omega)$ is the space of first order polynomials over $\Omega$, the implication (4.8) of assumption 4.3 holds.

Proof: Let us assume that $v \in X$ is such that its restriction $v_k = v|_{\Omega_k}$ (resp. $v_l = v|_{\Omega_l}$) to $\Omega_k$ (resp. $\Omega_l$) is a local rigid motion. Assuming that:

$$\int_{\gamma_{kl}} [v] \cdot \mu = 0, \quad \forall \mu \in M_1(\gamma_{kl}),$$

and because by construction the jump $[v]_{\gamma_{kl}}$ of $v$ across $\gamma_{kl}$ belongs to $M_1(\gamma_{kl})$, we can choose $\mu = [v]_{\gamma_{kl}}$, so that:

$$\int_{\gamma_{kl}} [v]^2 = 0 \implies [v]_{\gamma_{kl}} = 0 \text{ on } \gamma_{kl}.$$

Hence the proof.

Remark 4.7. When considering second order approximations for the displacements, first order polynomials must belong to the space of Lagrange multipliers in order to achieve an optimal rate of convergence, as shown in the proof of proposition 4.7, page 141. The choice of $M_{kl}$ given by lemma 4.2 is then natural. Nevertheless, when considering first order approximations of the displacements, and when more than two subdomains share a common edge, it is impossible for stability reason to conserve all the affine functions in the spaces of Lagrange multipliers. In particular, the order of Lagrange multipliers should be reduced on the interface elements having a non-empty intersection with the boundary of the interface, as pointed out in [BMP93, BMP94] for the scalar case.

It is possible to weaken the assumption of lemma 4.2, for instance by using piecewise constant Lagrange multipliers, at least over interfaces having a tensor product structure.

Lemma 4.3. We assume that for all $1 \leq k, l \leq K$ such that $\Omega_k$ and $\Omega_l$ have a non-empty intersection, the interface $\gamma_{kl} = \partial \Omega_k \cap \partial \Omega_l$ between the subdomains is planar. Denoting by $G_{kl}$ its center of gravity defined by:

$$G_{kl} = \frac{1}{\text{meas}(\Gamma_{kl})} \int_{\Gamma_{kl}} x \, dx,$$
we can characterize $\gamma_{kl}$ by:

$$\gamma_{kl} = \{ x \in \mathbb{R}^3, \quad x - G_{kl} = \xi_1 f_1 + \xi_2 f_2, \quad (\xi_1, \xi_2) \in [-1, 1]^2 \},$$

where $f_1, f_2 \in \mathbb{R}^3$ are linearly independent. We introduce the following partition over $\gamma_{kl}$:

$$\begin{align*}
\gamma_{kl}^{++} &= \{ \xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [0; 1] \text{ and } \xi_2 \in [0; 1] \}, \\
\gamma_{kl}^{+-} &= \{ \xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [0; 1] \text{ and } \xi_2 \in [-1; 0] \}, \\
\gamma_{kl}^{-+} &= \{ \xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [-1; 0] \text{ and } \xi_2 \in [-1; 0] \}, \\
\gamma_{kl}^{--} &= \{ \xi_1 f_1 + \xi_2 f_2; \quad \xi_1 \in [-1; 0] \text{ and } \xi_2 \in [0; 1] \},
\end{align*}$$

and assume that $M_{kl}$ is made of piecewise constant functions over the sets $\gamma_{kl}^{++}, \gamma_{kl}^{+-}, \gamma_{kl}^{-+}$ and $\gamma_{kl}^{--}$. Then, the assertion (4.8) of assumption 4.3 holds.

**Proof:** Let $v \in X$ such that its restriction $v_k = v|\Omega_k$ (resp. $v_l = v|\Omega_l$) to $\Omega_k$ (resp. $\Omega_l$) is a local rigid motion, and:

$$\int_{\gamma_{kl}} (v_k - v_l) \cdot \mu = 0, \quad \forall \mu \in M_{kl}. \quad (4.14)$$

As $[v]_{kl} = (v_k - v_l) \in \mathcal{R}(\gamma_{kl})$ is a rigid motion of the interface $\gamma_{kl}$, there exist constant vectors $t, a \in \mathbb{R}^3$ such that:

$$[v]_{kl}(x) = t + a \times (x - G_{kl}), \quad x \in \gamma_{kl}.$$

If we consider constant vector functions $\mu$ in (4.14), it is obtained that:

$$t \cdot \mu \text{ meas}(\gamma_{kl}) + (\mu \times a) \cdot \int_{\gamma_{kl}} (x - G_{kl}) = t \cdot \mu \text{ meas}(\gamma_{kl}) = 0,$$

for all constant vectors $\mu \in \mathbb{R}^3$, entailing that $t = 0$. 

Now, let us prove that \( a = 0 \). We can decompose it into :

\[
a = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3, \quad \text{with} \quad f_3 = f_1 \times f_2,
\]

and then obtain from (4.14):

\[
\int_{[-1;1]^2} \left( \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \right) \times \left( \xi_1 f_1 + \xi_2 f_2 \right) \cdot \mu \, d\xi_1 \, d\xi_2 = 0,
\]

for all functions \( \mu \) that are constant over the sets \( \gamma_{kl}^{++}, \gamma_{kl}^{+-}, \gamma_{kl}^{-+} \) and \( \gamma_{kl}^{--} \). Then :

\[
\begin{align*}
\alpha_1 \int_{[-1;1]^2} \xi_2 (f_3 \cdot \mu) \, d\xi_1 \, d\xi_2 & - \alpha_2 \int_{[-1;1]^2} \xi_1 (f_3 \cdot \mu) \, d\xi_1 \, d\xi_2 \\
+ \alpha_3 \int_{[-1;1]^2} (\xi_1 ((f_3 \times f_1) \cdot \mu) + \xi_2 ((f_3 \times f_2) \cdot \mu)) \, d\xi_1 \, d\xi_2 & = 0,
\end{align*}
\]

(4.15)

for all functions \( \mu \) that are constant over the sets \( \gamma_{kl}^{++}, \gamma_{kl}^{+-} \), \( \gamma_{kl}^{-+} \) and \( \gamma_{kl}^{--} \). For all \( \xi_1 \in [-1;1] \), let us define :

\[
\mu(x) = \mu(\xi_1 f_1 + \xi_2 f_2) = \begin{cases} 
  f_3, & \xi_2 \in [0;1], \\
  -f_3, & \xi_2 \in [-1;0],
\end{cases}
\]

in (4.15), which leads to :

\[
4\alpha_1 \int_0^1 \xi_2 \, d\xi_2 = 2\alpha_1 = 0 \quad \Rightarrow \quad \alpha_1 = 0.
\]

By choosing for all \( \xi_2 \in [-1;1] \) :

\[
\mu(x) = \mu(\xi_1 f_1 + \xi_2 f_2) = \begin{cases} 
  f_3, & \xi_1 \in [0;1], \\
  -f_3, & \xi_1 \in [-1;0],
\end{cases}
\]

in (4.15), it is obtained that :

\[
-4\alpha_2 \int_0^1 \xi_1 \, d\xi_1 = -2\alpha_2 = 0 \quad \Rightarrow \quad \alpha_2 = 0.
\]

When choosing now for all \( \xi_2 \in [-1;1] \) :

\[
\mu(x) = \mu(\xi_1 f_1 + \xi_2 f_2) = \begin{cases} 
  f_2, & \xi_1 \in [0;1], \\
  -f_2, & \xi_1 \in [-1;0],
\end{cases}
\]

it is obtained that :

\[
4\alpha_3 \int_0^1 \xi_1 \, d\xi_1 = 2\alpha_3 = 0 \quad \Rightarrow \quad \alpha_3 = 0.
\]

We conclude that \([v]_{kl} = 0\), hence the proof.

\[\Box\]

**Remark 4.8.** In the proof of lemma 4.3, the space of Lagrange multipliers we have used to check the implication (4.8) of assumption 4.3, is in fact a subspace of dimension 3 of the proposed space \( M_{kl} \).
4.3.4 Standard result of coercivity

We are now ready to recall the standard coercivity result for the bilinear form:

\[ \tilde{d}(u, v) := \sum_{k=1}^{K} d_k(u, v), \quad \forall u, v \in V \]

with:

\[ d_k(u, v) := \int_{\Omega_k} \varepsilon(u) : \varepsilon(v), \quad \forall u, v \in V. \]

The now standard proof, done by contradiction as in [BMP93] for example in the scalar case, does not guarantee the independence on the number and the size of the subdomains. We recall it nevertheless for completeness, and illustrate the way local rigid motions are controlled.

**Proposition 4.2.** Let \( \Omega \) be a bounded \( C^1 \) connected open set. The assumption \( 4.3 \) is supposed to be satisfied. Then, there exists a constant \( C > 0 \) possibly depending on the number and sizes of subdomains such that for all \( v \in V \), the following inequality holds:

\[ \sum_{k=1}^{K} \int_{\Omega_k} \varepsilon(v) : \varepsilon(v) \geq C \left( \sum_{k=1}^{K} \frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} v^2 + \int_{\Omega_k} \nabla v : \nabla v \right). \]

**Proof:** Let us assume that the inequality is false. Then, there exists a sequence \( (v_n)_{n \geq 1} \) in \( V \) such that:

\[ \sum_{k=1}^{K} \int_{\Omega_k} \varepsilon(v_n) : \varepsilon(v_n) \leq \frac{1}{n}, \quad \sum_{k=1}^{K} \frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} (v_n)^2 + \int_{\Omega_k} \nabla v_n : \nabla v_n = 1. \quad (4.16) \]

From (4.16), \( (v_n)_{n \geq 1} \) is bounded in \( X \), and we can extract a subsequence still denoted by \( (v_n)_{n \geq 1} \) converging to \( v \), weakly in \( X \) and strongly in \( L^2(\Omega)^d \) by the Rellich-Kondrachov theorem ([Bré99], page 169). It comes that \( (v_n)_{n \geq 1} \) is a Cauchy sequence in \( L^2(\Omega)^d \). Moreover, from (4.16), we obtain that for all \( 1 \leq k \leq K \), \( (\varepsilon(v_n))_{n \geq 1} \) is strongly convergent to zero in \( L^2(\Omega_k)^{d \times d} \), and as a consequence, is a Cauchy sequence in \( L^2(\Omega_k)^{d \times d} \). Therefore, for all \( 1 \leq k \leq K \), \( (v_n)_{n \geq 1} \) is a Cauchy sequence for the norm:

\[ \frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} (v_n)^2 + \int_{\Omega_k} \varepsilon(v_n) : \varepsilon(v_n), \]

and then for the norm of \( H^1(\Omega_k)^d \) by the Korn’s inequality (proposition 4.3). We deduce that \( (v_n)_{n \geq 1} \) strongly converges to \( v \) in \( X = \prod_{k=1}^{K} H^1(\Omega_k) \) by completeness of \( X \). From (4.16), it comes that \( \varepsilon(v|_{\Omega_k}) = 0 \), i.e. \( v|_{\Omega_k} \) is a rigid motion for all \( 1 \leq k \leq K \).

Let us prove now that \( v \in V \). We have for all \( 1 \leq m \leq M \), and \( \mu \in M_{m;\delta m} \):

\[ \int_{\Gamma_m} [v] \cdot \mu = \int_{\Gamma_m} [v_n] \cdot \mu + \int_{\Gamma_m} [v - v_n] : \mu, \]
and by using that \( v_n \in V \) and the convergence of \( v_n \) to \( v \) in \( H^1(\Omega_{k(m)})^d \), and therefore the convergence of their traces in \( L^2(\Gamma_m)^d \) by the trace theorem, we get:

\[
\int_{\Gamma_m} [v] \cdot \mu = 0, \quad \forall \mu \in M_{m;\delta_m}.
\]

The assumption 4.3 entails that the jump of \( v \) across all the interfaces \( (\Gamma_m)_{1 \leq m \leq M} \) vanishes, making \( v \) a rigid motion over \( \Omega \). Because \( v = 0 \) on \( \Gamma_D \), we deduce that \( v \) vanishes on the whole domain \( \Omega \), which is in contradiction with:

\[
\sum_{k=1}^{K} \frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} (v_n)^2 + \int_{\Omega_k} \nabla v_n : \nabla v_n = 1,
\]

proving the proposition.

In the previous proof, we have used the well-known:

**Proposition 4.3 (Korn’s inequality).** Let \( \Omega \) be a bounded \( C^1 \) connected open set. There exists a constant \( C_\Omega \) independent of the diameter of \( \Omega \) such that:

\[
\int_\Omega \varepsilon(v) : \varepsilon(v) + \frac{1}{\text{diam}(\Omega)^2} \int_\Omega v^2 \geq C_\Omega \left( \int_\Omega \nabla v : \nabla v + \frac{1}{\text{diam}(\Omega)^2} \int_\Omega v^2 \right),
\]

for all functions \( v \) such that \( \int_\Omega \varepsilon(v) : \varepsilon(v) + \frac{1}{\text{diam}(\Omega)^2} \int_\Omega v^2 \) is bounded.

The Korn’s inequality is a consequence of the following lemma (see [DL72], page 112, for a proof):

**Lemma 4.4.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^1 \) connected open set, and \( f \in \mathcal{D}'(\Omega) \) a real valued distribution such that \( f \in H^{-1}(\Omega) \) and for all \( 1 \leq i \leq d \), the derivative \( \partial f / \partial x_i \) in the sense of distributions belongs to \( H^{-1}(\Omega) \), where \( H^{-1}(\Omega) \) is the dual space of \( H^1_0(\Omega) \). Then the distribution \( f \) can be identified to a function by means of the \( L^2 \) inner product and \( f \in L^2(\Omega) \).

Again following [DL72], we get the proof of the Korn’s inequality.

**Proof:** First, we assume that \( \Omega \) has a unit diameter. The following space

\[
E = \{ v \in L^2(\Omega)^d; \varepsilon(v) \in L^2(\Omega)^{d \times d} \}
\]

is an Hilbert space when endowed with the norm:

\[
\|v\|_E = \left( \int_\Omega \varepsilon(v) : \varepsilon(v) + \int_\Omega v^2 \right)^{1/2}, \quad \forall v \in E.
\]
For all $1 \leq i, j, k \leq d$, the following expression holds in $\mathcal{D}'(\Omega)$:

$$
\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j} \varepsilon_{ik}(v) + \frac{\partial}{\partial x_k} \varepsilon_{ij}(v) - \frac{\partial}{\partial x_i} \varepsilon_{jk}(v).
$$

(4.17)

As $v \in E$, then $\varepsilon_{ij}(v) \in L^2(\Omega)$ and then $\partial \varepsilon_{ij}/\partial x_k \in H^{-1}(\Omega)$. From (4.17) we get that:

$$
\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in H^{-1}(\Omega),
$$

and from lemma 4.4, we conclude that for all $1 \leq i, k \leq d$:

$$
\frac{\partial v_i}{\partial x_k} \in L^2(\Omega),
$$

and then $v \in H^1(\Omega)^d$. We have proved that $E \subset H^1(\Omega)^d$, and then $E = H^1(\Omega)^d$. Let $T : H^1(\Omega)^d \rightarrow E$ the canonical embedding from $H^1(\Omega)^d$ to $E$. The application $T$ is linear, continuous because for all $v \in H^1(\Omega)^d$, $\|v\|_E \leq \|v\|_{H^1(\Omega)^d}$ and onto because we have proved that if $v \in E$, then $v \in H^1(\Omega)^d$. From the theorem of the open application (see [Bre99]), there exists a constant $C$ such that for all $v \in E$:

$$
\|v\|_{H^1(\Omega)^d} \leq C\|v\|_E.
$$

If $\Omega$ has not a unit diameter, there exists an homotethy $\varphi$ and a bounded $C^1$ connected open set $\hat{\Omega} \subset \mathbb{R}^d$ with unit diameter such that $\varphi(\hat{\Omega}) = \Omega$. As a consequence, by the change of variables $\varphi$ and by using the previous result over $\hat{\Omega}$, we get:

$$
\int_{\hat{\Omega}} \varepsilon(\check{v}) : \varepsilon(\check{v}) + \frac{1}{\text{diam}(\Omega)^2} \int_{\hat{\Omega}} \check{v}^2 = \text{diam}(\Omega)^{d-2} \left( \int_{\hat{\Omega}} \hat{\varepsilon}(\hat{v}) : \hat{\varepsilon}(\hat{v}) + \int_{\hat{\Omega}} \hat{v}^2 \right) \\
\geq \text{diam}(\Omega)^{d-2} C_{\hat{\Omega}} \left( \int_{\hat{\Omega}} \hat{\nabla} \check{v} : \hat{\nabla} \check{v} + \int_{\hat{\Omega}} \hat{v}^2 \right) \\
= C_{\hat{\Omega}} \left( \int_{\hat{\Omega}} \nabla v : \nabla v + \frac{1}{\text{diam}(\Omega)^2} \int_{\hat{\Omega}} v^2 \right),
$$

which ends the proof.

\[\square\]

4.4 Uniform coercivity

We improve herein the previous coercivity result by showing the independence of the coercivity constant with respect to the number, the size and the shape of the subdomains. Such a result is known for scalar elliptic problems, when interfaces are plane, as proved in [Gop99, Bre03]. A proof for the vector case is also proposed in a recent publication [Bre04]. The originality of our approach is that it uses a generalization of the Scott and Zhang interpolation [SZ90], and is valid for curved interfaces.
4.4. Uniform coercivity

4.4.1 Fundamental assumptions

Let us introduce the assumptions used in the present section. First, we assume that each subdomain is a “compact deformation” of a reference domain, the reference domains being in finite number. More precisely:

Assumption 4.4. It is assumed that:

1. there exists a finite collection of reference domains \((\hat{\Omega}_j)_{1 \leq j \leq J}\) of unit diameter, of compact sets \((K_j)_{1 \leq j \leq J}\) and maps \(\varphi_j : \hat{\Omega}_j \times K_j \to \mathbb{R}^d\), \(1 \leq j \leq J\) such that for all \(1 \leq j \leq J\):
   \[
   \text{diam} \left( \varphi_j(\hat{\Omega}_j, p) \right) = 1, \quad \forall p \in K_j,
   \]
   and the following application:
   \[
   \begin{array}{c}
   K_j \to W^{1,\infty}(\hat{\Omega}_j)^d, \\
   p \to \varphi_j(\cdot, p),
   \end{array}
   \]
   is continuous;

2. for all \(1 \leq j \leq J\), there exists a constant \(C_j > 0\) such that:
   \[
   \det \frac{\partial \varphi_j}{\partial \hat{x}}(\hat{x}, p) \geq C_j, \quad \forall p \in K_j, \text{ for almost all } \hat{x} \in \hat{\Omega}_j;
   \]
   in other words, for all \(p \in K_j\), \(\varphi_j(\cdot, p)\) is a uniform homeomorphism;

3. for all \((\Omega_k)_{1 \leq k \leq K}\) there exists a \(j\) with \(1 \leq j \leq J\) and an element \(p \in K_j\) such that within a scaling factor:
   \[
   \frac{1}{\text{diam}(\Omega_k)} \Omega_k = \varphi_j(\hat{\Omega}_j, p).
   \]
   Moreover, we consider that:

4. there exists a finite collection of reference interfaces \((\hat{\gamma}_j)_{1 \leq j \leq J}\), with \(\hat{\gamma}_j \subset \partial \hat{\Omega}_j\), \(1 \leq j \leq J\), and that the application:
   \[
   \begin{array}{c}
   K_j \to W^{1,\infty}(\hat{\gamma}_j)^d, \\
   p \to \varphi_j(\cdot, p),
   \end{array}
   \]
   is continuous,

5. for all \(1 \leq j \leq J\), there exists a constant \(C_j > 0\) such that:
   \[
   \det \frac{\partial \varphi_j}{\partial \hat{x}}(\hat{x}, p) \geq C_j, \quad \forall p \in K_j, \text{ for almost all } \hat{x} \in \hat{\gamma}_j,
   \]
   and when \(\gamma\) is a part of the boundary of \(\Omega_k = \varphi_j(\hat{\Omega}_j, p)\), we assume that:

6. \(\frac{1}{\text{diam}(\gamma)} \gamma = \varphi_j(\hat{\gamma}_j, p)\).
7. there exists three constants $\kappa, \kappa', \kappa'' > 0$ such that for all $1 \leq k \leq K$ :

$$\begin{align*}
\rho(\Omega_k) &\geq \kappa \text{ diam}(\Omega_k), \\
\text{diam}(\gamma_{kl}) &\geq \kappa' \text{ diam}(\Omega_k), \quad 1 \leq l \leq K, \\
|\gamma_{kl}| &\geq \kappa'' \text{ diam}(\Omega_k)^{d-1},
\end{align*}$$

(4.18)

where $\rho(\Omega_k)$ denotes the diameter of the largest ball contained in $\Omega_k$. The constants $\kappa, \kappa'$ and $\kappa''$ must remain independent of the number and the size of the subdomains.

As a consequence of (4.18), the number of subdomains sharing a common intersection remains bounded by a fixed integer $P$, independently of the chosen regular decomposition.

The assumptions 1 to 6 are used to show a technical result of shape-independence of the constant in Korn-like inequalities with proper scaling, detailed in appendix B (section 4.11, page 205). Assumption 7 will be used to show our interpolation estimates.

To deal with curved interfaces in the framework of Scott-Zhang like interpolation, we will need the technical assumption 4.5, page 127, precized in the definition of the interpolation operator. The present coercivity result will be shown on the constrained space :

$$V = \{ v \in X, \int_{\gamma_{kl}} [v] \cdot \mu = 0, \quad \forall \mu \in \mathbb{P}_1(\gamma_{kl})^d \}$$

Remark 4.9. In this section, we use the Lagrange multipliers spaces $M_{kl} = \mathbb{P}_1(\gamma_{kl})^d$. Nevertheless, one can adopt any $M_{kl}$ such that for all $v \in L^2(\gamma_{kl})^d$, there exists a solution $\pi_{\gamma_{kl}} v \in \mathbb{P}_1(\gamma_{kl})^d$ of :

$$\int_{\gamma_{kl}} \pi_{\gamma_{kl}} v \cdot \mu = \int_{\gamma_{kl}} v \cdot \mu, \quad \forall \mu \in M_{kl},$$

satisfying :

$$\| \pi_{\gamma_{kl}} v \|_{L^2(\gamma_{kl})^d} \leq C \sup_{\mu \in M_{kl}} \frac{\int_{\gamma_{kl}} v \cdot \mu}{\| \mu \|_{L^2(\gamma_{kl})^d}},$$

(4.19)

with a constant $C$ independent of the interface $\gamma_{kl}$. The statement (4.19) is true when adopting Lagrange multipliers satisfying the assumption 4.3, but the constant a priori depends on the shape of the interface $\gamma_{kl}$.

In this section, we assume that all these assumptions are satisfied.

4.4.2 Generalized Korn’s inequality

We will use hereafter the two following generalized Korn’s inequalities reviewed and detailed in appendix 4.11, page 205, for domains satisfying the assumptions of section 4.4.1.
Lemma 4.5. There exists a constant $C_P$ such that for all $\Omega_k$ and $\gamma_{kl}$ satisfying the conditions defined in section 4.4.1, the following inequality holds for all $v \in H^1(\Omega_k)^d$:

$$\|v\|_{H^1(\Omega_k)^d}^2 \leq C_P \left( \frac{1}{\text{diam}(\Omega_k)} \left( \sup_{\mu \in \mathbb{M}_{kl}} \left\| \frac{\int_{\gamma_{kl}} v \cdot \mu}{\| \mu \|_{L^2(\gamma_{kl})^d}} \right\|^2 + d_k(v,v) \right) \right),$$

where $C_P$ does not depend on $\Omega_k$ and $\gamma_{kl}$.

Lemma 4.6. There exists a constant $C_N$ such that for all $\Omega_k$ and $\gamma_{kl}$ satisfying the conditions defined in section 4.4.1, the following inequality holds for all $v \in H^1(\Omega_k)^d$:

$$\|v\|_{H^1(\Omega_k)^d}^2 \leq C_N \left( \frac{1}{\text{diam}(\Omega_k)^2} \left( \sup_{r \in \mathcal{R}(\Omega_k)} \left\| r \|_{L^2(\Omega_k)^d} \right\| \right)^2 + d_k(v,v) \right),$$

where $C_N$ does not depend on $\Omega_k$ and $\gamma_{kl}$.

Then, we deduce the following trace lemma:

Lemma 4.7. There exists a constant $C_T$ such that for all $\Omega_k$ and $\gamma_{kl}$ satisfying the conditions defined in section 4.4.1, the following inequality holds for all $v \in H^1(\Omega_k)^d$:

$$\frac{1}{\text{diam}(\Omega_k)} \left( \sup_{\mu \in \mathbb{M}_{kl}} \left\| \frac{\int_{\gamma_{kl}} v \cdot \mu}{\| \mu \|_{L^2(\gamma_{kl})^d}} \right\|^2 \right) \leq C_T \left( \frac{1}{\text{diam}(\Omega_k)^2} \left( \sup_{r \in \mathcal{R}(\Omega_k)} \left\| r \|_{L^2(\Omega_k)^d} \right\|^2 + d_k(v,v) \right) \right),$$

where $C_T$ does not depend on $\Omega_k$ and $\gamma_{kl}$.

Proof: By using the Cauchy-Schwarz inequality, the Sobolev trace theorem (with proper scaling) and the lemma 4.6, we get:

$$\frac{1}{\text{diam}(\Omega_k)} \left( \sup_{\mu \in \mathbb{M}_{kl}} \left\| \frac{\int_{\gamma_{kl}} v \cdot \mu}{\| \mu \|_{L^2(\gamma_{kl})^d}} \right\|^2 \right) \leq \frac{1}{\text{diam}(\Omega_k)} \int_{\gamma_{kl}} v^2 \leq C \left( \frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} v^2 + \int_{\Omega_k} |\nabla v|^2 \right) \leq CC_N \left( \frac{1}{\text{diam}(\Omega_k)^2} \left( \sup_{r \in \mathcal{R}(\Omega_k)} \left\| r \|_{L^2(\Omega_k)^d} \right\|^2 \right)^2 + d_k(v,v) \right),$$

hence the proof.

4.4.3 A Scott & Zhang like interpolation operator for mortar methods

The proposed interpolation operator builds a conforming approximation of a non-conforming function defined in the constrained space $V$ of functions whose jump is orthogonal to interface Lagrange multipliers, with the usual stability properties shown in [SZ90], even when considering curved interfaces between the subdomains.
Construction of a coarse conforming basis - Let us introduce a coarse conforming triangulation $\mathcal{T}_H$ of $\Omega$, as shown on figure 4.3, which satisfies the following conditions:

1. Each $T \in \mathcal{T}_H$ is totally included in a subdomain $\Omega_k$.
2. The tetrahedra in $\mathcal{T}_H$ possibly have curved faces along the skeleton interface $S$.
3. The tetrahedra $T \in \mathcal{T}_H$ in $\Omega_k$ are such that $\rho(T) \geq C \text{diam}(\Omega_k)$, with $\rho(T)$ the diameter of the largest ball included in $T$.

![Fig. 4.3 – A coarse conforming triangulation $\mathcal{T}_H$ of $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ satisfying conditions 1 and 2.](image)

We define on $\mathcal{T}_H$ the following conforming approximation space:

$$X_H = \{ v \in H^1(\Omega), \quad v|_T \in \mathbb{P}_1(T), \quad T \in \mathcal{T}_H \},$$

where $\mathbb{P}_1(T)$ denotes the space of affine applications over $T$. The vertices of the coarse conforming triangulation $\mathcal{T}_H$ are denoted by $(M_i)_{1 \leq i \leq I}$, and the associated nodal basis of $X_H$ by $(\phi_i)_{1 \leq i \leq I}$ such that:

$$\phi_i(M_j) = \delta_{ij},$$

using the Kronecker symbol $\delta_{ij} = 1$ for $i = j$ and 0 otherwise.

Set of interfaces - Let us denote by $Z_S$ the set of interfaces $\gamma_{kl}$ between two adjacent subdomains, and by $\tilde{Z}$ the set of internal faces of the triangulation $\mathcal{T}_H$, that is the faces of the triangles $T \in \mathcal{T}_H$, which are not included in the skeleton interface $S$. The total set of interfaces is then defined by:

$$Z = Z_S \cup \tilde{Z}.$$  

To deal with curved interfaces in the framework of Scott-Zhang like interpolation, we need the following assumption:
Assumption 4.5. There exists a constant $C > 0$ such that for each node $M_i$ of the coarse triangulation $T_H$, there exists an interface $\gamma_i \in Z$ with $M_i \in \gamma_i$ such that for all matrix $B \in \mathbb{R}^{d \times d}$, we have:

$$\frac{1}{|\gamma_i|} \int_{\gamma_i} |B \cdot (x - G_{\gamma})|^2 \, dx \geq C \lambda(B, \gamma_i)^2 \text{diam}(\gamma_i)^2,$$  

(4.20)

where $G_{\gamma}$ is the center of gravity of $\gamma$, i.e.:

$$G_{\gamma} = \frac{1}{|\gamma|} \int_{\gamma} x \, dx,$$

and $\lambda(B, \gamma)$ the maximal singular value of $B$ on $\gamma$:

$$\lambda(B, \gamma)^2 = \sup_{x \in \gamma} \frac{|B \cdot (x - G_{\gamma})|^2}{|x - G_{\gamma}|^2}.$$

Remark 4.10. The assumption 4.5 means that for all node $M_i$ of the coarse triangulation $T_H$, there exists an interface sharing $M_i$ and having a finite “length” along the principal direction of displacement for all affine fields of displacements. As a counter-example, let us consider the curved interface depicted in the following picture:

```
\begin{tikzpicture}
    \draw[->] (0,0) -- (5,0) node[below] {x};
    \draw[->] (0,0) -- (0,5) node[left] {y};
    \fill (2,2) circle (2pt);\node at (2,2) {$G$};
    \draw[thick,dotted] (0,0) to [out=90,in=180] (4,5) to [out=0,in=90] (2,2) to [out=-90,in=0] (0,0);
    \draw (2,2) -- (2,3) node[above] {$\epsilon$};
    \draw (2,2) -- (3,2) node[right] {$L$};
    \draw (0,0) -- (1,0) node[below] {$\gamma_i$};
\end{tikzpicture}
```

and the linear function $v(x, y) = \epsilon^{-1}y = B \cdot x$. It follows that $\lambda(B, \gamma)^2 \simeq \frac{1}{L^2}$ and

$$\frac{1}{|\gamma_i|} \int_{\gamma_i} |B \cdot (x - G_{\gamma})|^2 \, dx \simeq \frac{1}{L} \int_0^\epsilon (\epsilon^{-1}y)^2 \, dy \simeq \frac{\epsilon}{L}.$$

As a consequence, the assertion (4.20) is not satisfied on $\gamma$ uniformly in $\epsilon$. The reason is that in this case, $\gamma$ is nearly orthogonal to the principal direction of displacement.

Nevertheless, the assertion (4.20) is satisfied for any plane interface $\gamma$ whatever the matrix $B \in \mathbb{R}^{d \times d}$, as shown in the following lemma:

Lemma 4.8. The assumption 4.5 is satisfied when choosing as $\gamma_i$ any plane interface sharing the node $M_i$, provided $\gamma_i$ is shape regular that is:

$$\rho(\gamma_i) \geq C \text{diam}(\gamma_i).$$

Proof: The present proof is done in three dimensions. Let $\gamma$ be a plane interface, and $Q$ a square of maximal edge length ($= \rho(\gamma)/\sqrt{2}$) included in the largest ball contained in $\gamma$ (as shown in the following picture).
We write \( x - G_\gamma = x_1 e_1 + x_2 e_2 = J \cdot x \), where \( e_1 \) and \( e_2 \) are two orthogonal vectors such that \( G_\gamma + \text{span}\{e_1, e_2\} = \gamma \). As the matrix \( J^t \cdot B^t \cdot B \cdot J \) is symmetric semi-definite positive, it can be diagonalized and we still denote by \( e_1 \) and \( e_2 \) its eigenvectors, associated to the eigenvalues \( \mu_1^2 \) and \( \mu_2^2 \) with \( \mu_1^2 \geq \mu_2^2 \). Finally, we choose among all the possible squares \( Q \), the one whose edges are parallel to the eigenvectors :

\[
Q = \{ x_1 e_1 + x_2 e_2, \quad x_1 \in [X_1 - a, X_1 + a], x_2 \in [X_2 - a, X_2 + a] \},
\]

where the center of the largest ball in \( \gamma \) is \( G_\gamma + X_1 e_1 + X_2 e_2 \), and \( 2a = \rho(\gamma)/\sqrt{2} \). Then, we get :

\[
\frac{1}{|\gamma|} \int_{\gamma} |B \cdot (x - G_\gamma)|^2 \geq \frac{1}{|\gamma|} \int_Q |B \cdot (x - G_\gamma)|^2 \geq \frac{1}{|\gamma|} \int_{X_1 - a}^{X_1 + a} \int_{X_2 - a}^{X_2 + a} \left( \mu_1^2 (x_1)^2 + \mu_2^2 (x_2)^2 \right) dx_1 dx_2 \geq \frac{2a}{3|\gamma|} \left( \mu_1^2 ((X_1 + a)^3 - (X_1 - a)^3) + \mu_2^2 ((X_2 + a)^3 - (X_2 - a)^3) \right).
\]

Moreover, we have :

\[
(X_1 + a)^3 - (X_1 - a)^3 = 2a ((X_1 + a)^2 + (X_1 + a)(X_1 - a) + (X_1 - a)^2) = 2a (3X_1^2 + a^2) \geq 2a^3,
\]

leading to :

\[
\frac{1}{|\gamma|} \int_{\gamma} |B \cdot (x - G_\gamma)|^2 \geq \frac{2a}{3|\gamma|} 2a^3 (\mu_1^2 + \mu_2^2) \geq \frac{2a}{3|\gamma|} 2a^3 \mu_2^2.
\]

From shape regularity, we have \(|\gamma| \leq C \text{diam}(\gamma)^2 \leq C \rho(\gamma)^2 = 2Ca^2 \), and therefore :

\[
\frac{1}{|\gamma|} \int_{\gamma} |B \cdot (x - G_\gamma)|^2 \geq C a^2 \mu_2^2 \geq C \text{diam}(\gamma)^2 \mu_2^2,
\]
4.4. Uniform coercivity

but by definition:

\[ \mu_2^2 = \lambda(B, \gamma)^2, \]

which ends the proof. \qed

The main consequence from assumption 4.5 is the simple:

Lemma 4.9. Under assumption 4.5, there exists a constant \( C > 0 \) such that for all locally affine functions \( v \in \mathbb{P}_1(\Omega)^d \), we can find at each node \( M \) of the coarse mesh \( T_H \), an interface \( \gamma \ni M \) for which:

\[ \|v\|_{L^\infty(\gamma)^d}^2 \leq C \frac{1}{|\gamma|} \|v\|_{L^2(\gamma)^d}^2. \]

Proof: Let \( v \) be locally in \( \mathbb{P}_1(\Omega)^d \). For all \( \gamma \in Z \), there exists a vector \( v(G_\gamma) \in \mathbb{R}^d \) and a matrix \( B \in \mathbb{R}^{d \times d} \) such that:

\[ v(x) = v(G_\gamma) + B \cdot (x - G_\gamma), \quad \forall x \in \gamma, \]

the matrix \( B \) being independent of the choice of \( \gamma \in Z \). From assumption 4.5, we can always find at each node \( M_i \) of the coarse mesh \( T_H \), an interface \( \gamma = \gamma_i \) such that (4.20) is satisfied. Then:

\[ \|v\|_{L^\infty(\gamma)}^2 \leq 2|v(G_\gamma)|^2 + 2\lambda(B, \gamma)^2 \text{diam}(\gamma)^2, \]

and from assumption 4.5, we deduce:

\[
\frac{1}{|\gamma|} \|v\|_{L^2(\gamma)}^2 = v(G_\gamma)^2 + \frac{1}{|\gamma|} \int_{\gamma} |B \cdot (x - G_\gamma)|^2 \\
\geq C (v(G_\gamma)^2 + \lambda(B, \gamma)^2 \text{diam}(\gamma)^2) \\
\geq C \|v\|_{L^\infty(\gamma)}^2.
\]

Conforming approximation - For all functions \( v \in X \), we are now ready to define the conforming approximation \( \mathbf{P}v \in H^1_0(\Omega) \) by:

\[ \mathbf{P}v = \sum_{i \geq 1} p_i v(M_i) \phi_i, \]

where:

\[ p_i v = \pi_{\gamma_i} v, \]

in which \( \pi_\gamma \) is the \( L^2(\gamma)^d \) projection over \( \mathbb{P}_1(\gamma)^d \) (the restrictions to \( \gamma \) of functions in \( \mathbb{P}_1(\Omega)^d \)), and \( \gamma_i \in Z \) is among the interfaces sharing \( M_i \), the one which maximizes:

\[ A(\gamma) = \inf_{B \in \mathbb{R}^{d \times d}} \frac{1}{\lambda(B, \gamma)^2 \text{diam}(\gamma)^2} \frac{1}{|\gamma|} \int_{\gamma} |B \cdot (x - G_\gamma)|^2 dx. \]
Let us notice that in the expression \( \pi_{\gamma_i} v \), we choose arbitrarily the side of \( \gamma_i \) on which the trace of \( v \) is taken. When considering \( v \in V \) with the constrained space :

\[
V = \{ v \in X, \int_{\gamma_{kl}} [v] \cdot \mu = 0, \ \forall \mu \in \mathbb{P}_1(\gamma_{kl})^d, 1 \leq k,l \leq K \},
\]

this choice has no influence because :

\[
\int_{\gamma_i} v^+ \cdot \mu = \int_{\gamma_i} v^- \cdot \mu, \ \forall \mu \in \mathbb{P}_1(\gamma_i)^d,
\]

entailing that \( \pi_{\gamma_i} v^+ = \pi_{\gamma_i} v^- \).

**Remark 4.11.** In this section, we use the Lagrange multipliers spaces \( M_{kl} = \mathbb{P}_1(\gamma_{kl})^d \). Nevertheless, one can adopt any \( M_{kl} \) such that for all \( v \in L^2(\gamma_{kl})^d \), there exists a solution \( \pi_{\gamma_{kl}} v \in \mathbb{P}_1(\gamma_{kl})^d \) of :

\[
\int_{\gamma_{kl}} \pi_{\gamma_{kl}} v \cdot \mu = \int_{\gamma_{kl}} v \cdot \mu, \ \forall \mu \in M_{kl},
\]

satisfying :

\[
\| \pi_{\gamma_{kl}} v \|_{L^2(\gamma_{kl})^d} \leq C \sup_{\mu \in M_{kl}} \int_{\gamma_{kl}} v \cdot \mu.
\]

(4.22)

Such a statement is true when adopting Lagrange multipliers satisfying the assumption 4.3, but the constant a priori depends on the shape of the interface \( \gamma_{kl} \).

**Proposition 4.4.** The interpolation operator \( P : \prod_{k=1}^K H^1(\Omega_k)^d \rightarrow (X_H)^d \) defined by (4.21) satisfies the following local inequality for all \( 1 \leq k \leq K \) :

\[
\| v - Pv \|_{H^1(\Omega_k)^d} \leq C \left( \sum_{l \in \mathcal{N}(\Omega_k)} d_k(v,v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right),
\]

(4.23)

where \( \mathcal{N}(\Omega_k) \) denotes the set of indices of the subdomains sharing a vertex with \( \Omega_k \), and \( d_k \) is the bilinear form over \( H^1(\Omega_k)^d \times H^1(\Omega_k)^d \) defined as :

\[
d_k(u,v) = \int_{\Omega_k} \varepsilon(u) : \varepsilon(v), \ \forall u,v \in H^1(\Omega_k)^d.
\]

Moreover, we have denoted by \( S_k \) the union of the neighboring interfaces of \( \Omega_k \) :

\[
S_k = \bigcup_{l,m \in \mathcal{N}(\Omega_k)} \gamma_{lm}.
\]
Fig. 4.4 – A triangular domain decomposition of $\Omega \subset \mathbb{R}^2$, with illustration of the subdomains $(\Omega_l)_{l \in \mathcal{N}(\Omega_k)}$ sharing a vertex with $\Omega_k$ (inside the dark thick line), and of the reunion $\mathcal{S}_k$ of the neighboring interfaces of $\Omega_k$ (in dotted lines).

and :

$$\pi[v](x) = \pi_\gamma[v](x), \quad \text{for all } x \in \gamma, \text{ with } \gamma \in Z_\mathcal{S}.$$

Moreover, when the decomposition into subdomains satisfies the conditions defined in section 4.4.1, the constant $C$ is independent of the diameter and the shape of the subdomains. The definitions of $\mathcal{N}(\Omega_k)$ and $\mathcal{S}_k$ are illustrated on figure 4.4.

Proof : The proof is decomposed into 4 parts. For convenience, we will denote by $\mathcal{O}_k$ the neighborhood of $\Omega_k$ defined as :

$$\mathcal{O}_k = \bigcup_{l \in \mathcal{N}(\Omega_k)} \Omega_l.$$  

1. Range of $\mathbb{P}_1(\mathcal{O}_k)^d$.

Let us consider the affine displacement $v \in \mathbb{P}_1(\mathcal{O}_k)^d$. For all $\gamma \in Z \cap \mathcal{O}_k$, the trace of $v$ over $\gamma$ belongs to $\mathbb{P}_1(\gamma)^d$ by definition, and therefore :

$$\pi_\gamma v = v, \quad \text{on } \gamma.$$

As a consequence, we obtain for all $i \geq 1$ satisfying $M_i \in \overline{\Omega_k}$, that $p_i v = v$, hence :

$$(\mathbb{P} v)_{|\Omega_k} = \sum_{i \geq 1, M_i \in \overline{\Omega_k}} v(M_i) \phi_i = v|_{\Omega_k},$$

because $v|_{\Omega_k} \in (X_H)^d$. 
2. Stability of $P$ in $L^2(\Omega_k)^d$.

Let $v \in X$. It is readily obtained from definition (4.21), that :

$$
\|Pv\|^2_{L^2(\Omega_k)^d} \leq \max_{i,M_i \in \Omega_k} \|p_i v(M_i)\|^2 \int_{\Omega_k} \left( \sum_{1 \leq i \leq I} \|\phi_i\|^2 \right).
$$

$$
\leq \max_{i,M_i \in \Omega_k} \|\pi_{\gamma_i} v\|^2_{L^\infty(\gamma_i)^d} \int_{\Omega_k} \left( \sum_{1 \leq i \leq I} \|\phi_i\|^2 \right) .
$$

Under assumption 4.5, we obtain from lemma 4.9 that :

$$
\|\pi_{\gamma_i} v\|^2_{L^\infty(\gamma_i)^d} \leq C \frac{1}{|\gamma_i|} \|\pi_{\gamma_i} v\|^2_{L^2(\gamma_i)^d},
$$

and because $\pi_{\gamma_i}$ is the $L^2(\gamma_i)^d$ projection over $P_1(\gamma_i)^d$, we get :

$$
\|\pi_{\gamma_i} v\|^2_{L^2(\gamma_i)^d} \leq \|v\|^2_{L^2(\gamma_i)^d},
$$

resulting in :

$$
\|Pv\|^2_{L^2(\Omega_k)^d} \leq \max_{i,M_i \in \Omega_k} \frac{1}{|\gamma_i|} \|v\|^2_{L^2(\gamma_i)^d} \int_{\Omega_k} \left( \sum_{1 \leq i \leq I} \|\phi_i\|^2 \right) .
$$

(4.24)

As $\gamma_i$ is a part of the boundary of a domain $\Omega_{l(i)}$ corresponding to the side of $\gamma_i$ on which the trace of $v$ is taken, we get from the Sobolev trace theorem that :

$$
\frac{1}{\text{diam}(\Omega_{l(i)})} \|v\|^2_{L^2(\gamma_i)^d} \leq C \left( \frac{1}{\text{diam}(\Omega_{l(i)})^2} \int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2 \right),
$$

(4.25)

with $C$ uniformly bounded due to the shape regularity of $\Omega_{l(i)}$. Moreover, we have :

$$
\int_{\Omega_k} \left( \sum_{1 \leq i \leq I} \|\phi_i\|^2 \right) = \int_{\Omega_k} dx = |\Omega_k|,
$$

(4.26)

because by construction $\sum_{1 \leq i \leq I} |\phi_i| = 1$. We deduce by exploiting the expressions (4.25) and (4.26) in (4.24) that :

$$
\|Pv\|^2_{L^2(\Omega_k)^d} \leq C \max_{i,M_i \in \Omega_k} \frac{|\Omega_k|}{|\gamma_i|} \text{diam}(\Omega_{l(i)}) \left( \frac{1}{\text{diam}(\Omega_{l(i)})^2} \int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2 \right)
$$

$$
\leq C \max_{i,M_i \in \Omega_k} \frac{|\Omega_k|}{|\Omega_{l(i)}|} \text{diam}(\Omega_{l(i)}) \left( \frac{1}{\text{diam}(\Omega_{l(i)})^2} \int_{\Omega_{l(i)}} v^2 + \int_{\Omega_{l(i)}} |\nabla v|^2 \right)
$$
because from the shape regularity conditions (4.18), we get :

\[ |\gamma_i| \operatorname{diam}(\Omega_{(i)}) \geq \kappa'' \operatorname{diam}(\Omega_{(i)})^{d-1} \operatorname{diam}(\Omega_{(i)}) \]

\[ = \kappa'' \operatorname{diam}(\Omega_{(i)})^d \]

\[ \geq C \kappa'' |\Omega_{(i)}|. \]

Therefore, there exists a subdomain \( \Omega_l \) sharing a node with \( \Omega_k \) such that :

\[ \frac{1}{\operatorname{diam}(\Omega_l)^2} \left\| \mathbf{P} v \right\|_{L^2(\Omega_k)}^2 \leq C \frac{|\Omega_k|}{|\Omega_l|} \left( \frac{1}{\operatorname{diam}(\Omega_l)^2} \int_{\Omega_l} v^2 + \int_{\Omega_l} |\nabla v|^2 \right), \]

after a division of the two sides of the inequality by \( \operatorname{diam}(\Omega_l)^2 \).

Let us show now that \( \operatorname{diam}(\Omega_l) \leq C \operatorname{diam}(\Omega_k) \). From the shape regularity (4.18) of the decomposition, we can build a sequence of (less than) \( P \) adjacent subdomains \((\Omega_{m})_{1 \leq m \leq P}\) such that \( \Omega_m \) and \( \Omega_{m+1} \) share the interface \( \gamma_{m,m+1} \) with \( \Omega_1 = \Omega_k \) and \( \Omega_P = \Omega_l \), as illustrated on the following figure (for triangular subdomains) :

From the shape regularity (4.18) of the decomposition into subdomains, we then have :

\[ \operatorname{diam}(\Omega_{m+1}) \leq \frac{1}{\kappa'} \operatorname{diam}(\gamma_{m,m+1}) \]

\[ \leq \frac{1}{\kappa'} \operatorname{diam}(\Omega_m), \quad (4.27) \]

and by iteration of (4.27), we get :

\[ \operatorname{diam}(\Omega_l) \leq \frac{1}{(\kappa')^P} \operatorname{diam}(\Omega_k). \quad (4.28) \]

Considering that the roles of \( \Omega_k \) and \( \Omega_l \) can be swapped in the previous inequality (4.28), we deduce that \( |\Omega_k| \leq C |\Omega_l| \) from the shape regularity (4.18) of the
decomposition because:

\[|\Omega_k| \leq C \text{diam}(\Omega_k)^d \]
\[\leq C \frac{1}{(k')^d \rho_0} \text{diam}(\Omega_i)^d \]
\[\leq C \frac{1}{(k')^d \rho_0} \frac{1}{Kd} \rho(\Omega_i)^d \]
\[\leq C \frac{1}{(k')^d \rho_0 Kd} |\Omega_i|. \]

As a consequence, we obtain from (4.27) with a still generic use of the constant \(C\), that there exists a subdomain \(\Omega_i\) sharing a node with \(\Omega_k\) such that:

\[\frac{1}{\text{diam}(\Omega_k)^2} \|Pv\|_{L^2(\Omega_k)}^2 \leq C \left( \frac{1}{\text{diam}(\Omega_i)^2} \int_{\Omega_i} v^2 + \int_{\Omega_i} |\nabla v|^2 \right). \quad (4.29)\]

3. Stability of \(P\) in \(H^1(\Omega_k)^d\).

Proceeding as previously, we get for all \(v \in X\) the following bound on the \(H^1(\Omega_k)^d\) semi-norm of the interpolate function \(Pv\):

\[|Pv|_{H^1(\Omega_k)^d}^2 \leq \max_{1 \leq i \leq I} |p_i v(M_i)|^2 \int_{\Omega_k} \left( \sum_{1 \leq i \leq I} |\nabla \phi_i| \right)^2 \]
\[\leq C \max_{i, M_i \in \Omega_k} \frac{\text{diam}(\Omega_{(i)})^2}{|\Omega_{(i)}|} \left( \frac{1}{\text{diam}(\Omega_{(i)})^2} \int_{\Omega_{(i)}} v^2 + \int_{\Omega_{(i)}} |\nabla v|^2 \right) \int_{\Omega_k} \left( \sum_{1 \leq i \leq I} |\nabla \phi_i| \right)^2 . \quad (4.30)\]

Moreover, by decomposing the last integral over \(\Omega_k\) into a sum of integrals over the triangles of the coarse triangulation \(T_H\) belonging to \(\Omega_k\):

\[\int_{\Omega_k} \left( \sum_{1 \leq i \leq I} |\nabla \phi_i| \right)^2 = \sum_{T \in T_H, T \subset \Omega_k} \int_{T} \left( \sum_{1 \leq i \leq I} |\nabla \phi_i| \right)^2 , \]

and using the fact that for all tetrahedra \(T \in T_H\) belonging to \(\Omega_k\), we have the standard result:

\[|\nabla \phi_i| \leq C \frac{1}{\rho(T)} \leq C \frac{1}{\text{diam}(\Omega_k)}, \]

using the assumption 3- made for the coarse triangulation \(T_H\), we conclude that:

\[\int_{\Omega_k} \left( \sum_{1 \leq i \leq I} |\nabla \phi_i| \right)^2 \leq \frac{C}{\text{diam}(\Omega_k)^2} |\Omega_k|. \quad (4.31)\]
4.4. Uniform coercivity

Hence from (4.30) and (4.31), we get by using the same arguments of shape regularity of the decomposition as in the previous part of the proof that there exists a subdomain $\Omega_l$ sharing a node with $\Omega_k$ (the same as in (4.29)) such that:

$$|Pv|_{H^1(\Omega_k)}^2 \leq C \left( \frac{1}{\text{diam}(\Omega_l)^2} \int_{\Omega_l} v^2 + \int_{\Omega_l} |\nabla v|^2 \right).$$

4. Approximation property

For all $v \in X$ the interpolation $Pv \in (X_H)^d$ satisfies from the two previous points of the proof, the following stability property:

$$\|Pv\|_{H^1(\Omega_k)^d}^2 \leq C \|v\|_{H^1(\Omega_l)^d}^2. \quad (4.32)$$

For all rigid motion $p \in R(\mathcal{O}_k)$, which is a fortiori a linear function of $P_1(\mathcal{O}_k)^d$, we have from point 1 that $Pp = p$ on $\Omega_k$, resulting in the following bounds by using the triangular inequality and the stability estimate (4.32):

$$\|v - Pv\|_{H^1(\Omega_k)^d}^2 = \|v - p + P(p - v)\|_{H^1(\Omega_k)^d}^2 \leq 2\|v - p\|_{H^1(\Omega_k)^d}^2 + 2\|P(p - v)\|_{H^1(\Omega_k)^d}^2 \leq C \left( \|v - p\|_{H^1(\Omega_k)^d}^2 + \|v - p\|_{H^1(\Omega_k)^d}^2 \right) \leq C \sum_{l \in \mathcal{N}(\Omega_k)} \|v - p\|_{H^1(\Omega_l)^d}^2.$$

By taking $p$ as the extension over $\Omega$ of the rigid motion projection of $v$ over $\Omega_k$, we get from lemma 4.10, page 135 that:

$$\|v - Pv\|_{H^1(\Omega_k)^d}^2 \leq C \left( \sum_{l \in \mathcal{N}(\Omega_k)} d_l(v, v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right),$$

which is exactly (4.23).

In the previous proof, we have used the following lemma which is a generalization to non-conforming vector functions of the Deny-Lions [DL55] or Bramble-Hilbert [BH70] lemma involving the broken elasticity semi-norm.

**Lemma 4.10.** There exists a constant $C > 0$ such that for all $v \in X$:

$$\sum_{l \in \mathcal{N}(\Omega_k)} \|v - p\|_{H^1(\Omega_l)^d}^2 \leq C \left( \sum_{l \in \mathcal{N}(\Omega_k)} d_l(v, v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right), \quad (4.33)$$
where \( p \in \mathcal{R}(\Omega) \) is the rigid motion satisfying:

\[
\int_{\Omega_k} p \cdot w = \int_{\Omega_k} v \cdot w, \quad \forall w \in \mathcal{R}(\Omega).
\]

Moreover, provided the decomposition into subdomains satisfy the shape regularity condition defined in section 4.4.1, the constant \( C \) is independent of the size and the shape of the neighbor subdomains.

**Proof:** We prove herein the announced upper bound for the quantity

\[
\sum_{l \in \mathcal{N}(\Omega_k)} \| v - p \|^2_{H^1(\Omega_l) \cap \Gamma},
\]

in which the rigid motion \( p \in \mathcal{R}(\Omega) \) is defined by:

\[
\int_{\Omega_k} p \cdot r = \int_{\Omega_k} v \cdot r, \quad \forall r \in \mathcal{R}(\Omega_k).
\]

- First, it follows from lemma 4.6 that:

\[
\| v - p \|^2_{H^1(\Omega_k) \cap \Gamma} \leq C_N \left( \frac{1}{\text{diam}(\Omega_k)^2} \left( \sup_{r \in \mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} (v - p) \cdot r}{\| r \|^2_{L^2(\Omega_k) \cap \Gamma}} \right)^2 + d_k(v - p, v - p) \right) = C_N \, d_k(v, v),
\]

by definition of the local rigid motion projection \( p \).

- If \( \Omega_l \) shares an interface with \( \Omega_k \), we obtain from lemmas 4.5 and 4.7 that:

\[
\begin{align*}
\| v - p \|^2_{H^1(\Omega_l) \cap \Gamma} & \leq 2 \, C_P \left( \frac{1}{\text{diam}(\Omega_l) \cap \Gamma} \left( \sup_{\mu \in M_{kl}} \frac{\int_{\gamma_{kl}} (v - p) \cdot \mu}{\| \mu \|^2_{L^2(\gamma_{kl}) \cap \Gamma}} \right)^2 + d_l(v, v) \right) \\
& \leq 2 \, C_P \left( \frac{1}{\text{diam}(\Omega_l) \cap \Gamma} \left( \sup_{\mu \in M_{kl}} \frac{\int_{\gamma_{kl}} (v - p) \cdot \mu}{\| \mu \|^2_{L^2(\gamma_{kl}) \cap \Gamma}} \right)^2 + d_l(v, v) + \frac{1}{\text{diam}(\Omega_l) \cap \Gamma} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right) \\
& \leq 2 \, C_P \left( \frac{\text{diam}(\Omega_k)}{\text{diam}(\Omega_l) \cap \Gamma} \right) C_T \left( \frac{1}{\text{diam}(\Omega_k)^2} \left( \sup_{r \in \mathcal{R}(\Omega_k)} \frac{\int_{\Omega_k} (v - p) \cdot r}{\| r \|^2_{L^2(\Omega_k) \cap \Gamma}} \right)^2 + d_k(v, v) \right) + d_l(v, v) \\
& + 2 \, C_P \frac{1}{\text{diam}(\Omega_l) \cap \Gamma} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \\
& = 2 \, C_P \left( \frac{\text{diam}(\Omega_k)}{\text{diam}(\Omega_l) \cap \Gamma} \right) C_T \, d_k(v, v) + d_l(v, v) + 2 \, C_P \frac{1}{\text{diam}(\Omega_l) \cap \Gamma} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \\
& \leq C \left( d_k(v, v) + d_l(v, v) + \frac{1}{\text{diam}(\Omega_l) \cap \Gamma} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right),
\end{align*}
\]

(4.34)
because we have $\text{diam}(\Omega_k) \leq C \text{diam}(\Omega_l)$ as in the step 2 of the proof of proposition 4.4.

- For other $l \in \mathcal{N}(\Omega_k)$, we proceed by the same technique used in the step 2 of the proof of proposition 4.4, by reasoning on a sequence of adjacent subdomains, and obtain as above:

\[
\|v - p\|^2_{H^1(\Omega_{m+1})^d} \leq C_P \left( \frac{1}{\text{diam}(\Omega_{m+1})} \left( \sup_{\mu \in M_{lm}^{lm+1}} \int_{\gamma_{lm}^{lm+1}} (v - p) |\Omega_{m+1} \cdot \mu|^2 \right)^2 + d_{m+1}(v, v) \right)
\]

\[
\leq 2 C_P \left( \frac{1}{\text{diam}(\Omega_{m+1})} \left( \sup_{\mu \in M_{lm}^{lm+1}} \int_{\gamma_{lm}^{lm+1}} (v - p) |\Omega_{m+1} \cdot \mu|^2 \right)^2 + d_{m+1}(v, v) \right)
\]

\[
+ 2 C_P \left( \frac{1}{\text{diam}(\Omega_{m+1})} \int_{\gamma_{lm}^{lm+1}} \left( \pi \gamma_{lm}^{lm+1} [v] \right)^2 \right)
\]

\[
\leq 2 C_P \left( \frac{\text{diam}(\Omega_{lm})}{\text{diam}(\Omega_{l+1})} C_T \left( \frac{1}{\text{diam}(\Omega_{lm})} \left( \sup_{r \in \mathcal{R}(\Omega_{lm})} \int_{\Omega_{lm}} (v - p) \cdot r \right)^2 + d_{lm}(v - p, v - p) \right) \right)
\]

\[
+ 2 C_P \left( \frac{d_{m+1}(v, v)}{\text{diam}(\Omega_{m+1})} + \frac{1}{\text{diam}(\Omega_{m+1})} \int_{\gamma_{lm}^{lm+1}} \left( \pi \gamma_{lm}^{lm+1} [v] \right)^2 \right)
\]

\[
\leq 2 C_P \left( \frac{\text{diam}(\Omega_{lm})}{\text{diam}(\Omega_{l+1})} C_T \|v - p\|^2_{H^1(\Omega_{lm})^d} + d_{m+1}(v, v) + \frac{1}{\text{diam}(\Omega_{lm})} \int_{\gamma_{lm}^{lm+1}} \left( \pi \gamma_{lm}^{lm+1} [v] \right)^2 \right),
\]

from Cauchy-Schwarz inequality. From the shape regularity (4.18), it follows that $\text{diam}(\Omega_k) \leq C \text{diam}(\Omega_{lm+1})$ and $\text{diam}(\Omega_{lm}) \leq C \text{diam}(\Omega_{l+1})$ as in the step 2 of the proof of proposition 4.4, and we get:

\[
\|v - p\|^2_{H^1(\Omega_{m+1})^d} \leq C C_P \left( C_T \|v - p\|^2_{H^1(\Omega_{lm})^d} + d_{m+1}(v, v) + \frac{1}{\text{diam}(\Omega_{lm})} \int_{\gamma_{lm}^{lm+1}} \left( \pi \gamma_{lm}^{lm+1} [v] \right)^2 \right).
\]

By induction on $m$ and from (4.34), it is then obtained from $\# \mathcal{N}(\Omega_k) \leq P$ that:

\[
\|v - p\|^2_{H^1(\Omega_l)^d} \leq C (C P C_T)^P C_N \left( \sum_{j \in \mathcal{N}(\Omega_k)} d_j(v, v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right),
\]
and therefore:

\[
\sum_{l \in \mathcal{N}(\Omega_k)} \|v - p\|_{H^1(\Omega_k)^d}^2 \leq C(C_P C_T)^P C_N \sum_{l \in \mathcal{N}(\Omega_k)} \left( \sum_{j \in \mathcal{N}(\Omega_k)} d_j(v, v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right) \leq C P(C_P C_T)^P C_N \left( \sum_{j \in \mathcal{N}(\Omega_k)} d_j(v, v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right),
\]

hence the proof.

Remark 4.12 (Satisfaction of a Dirichlet homogeneous boundary condition). If \( v \in X \) satisfies a Dirichlet homogeneous boundary condition on the part \( \Gamma_D \) of the boundary of the domain \( \Omega \), its interpolation \( P v \) has the same boundary value on \( \Gamma_D \) provided:

- \( T_H \cap \Gamma_D \) is a (possibly curved) triangulation of \( \Gamma_D \),
- the nodes \( M_i \in \Gamma_D \) are associated to faces \( \gamma_i \in Z \) contained in \( \Gamma_D \).

4.4.4 Uniform coercivity result

We improve herein the coercivity result from proposition 4.2 by showing that the coercivity constant is independent of the number and the size of the subdomains:

**Proposition 4.5.** There exists a constant \( C > 0 \) independent of any decomposition of \( \Omega \) into subdomains satisfying the assumptions of section 4.4.1, such that for all displacements fields \( v \in X \):

\[
\|v\|_X^2 \leq C \left( \sum_{k=1}^{K} d_k(v, v) + \sum_{1 \leq k < l \leq K} \frac{1}{\text{diam}(\gamma_{kl})} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}}[v])^2 \right). \tag{4.35}
\]

**Proof:** For all \( v \in V \), the conforming interpolate function \( P v \in (X_H)^d \subset H^1(\Omega)^d \) satisfies the same Dirichlet boundary condition as \( v \) (see remark 4.12) resulting in the usual coercivity result, only depending on the shape of \( \Omega \):

\[
d(Pv, Pv) = d(Pv, Pv) \geq C \|Pv\|_{H^1(\Omega)^d}^2 = C \|Pv\|_X^2. \tag{4.36}
\]

Consequently, we get from (4.36) and proposition 4.4 that:

\[
\|v\|_X^2 = \|v - Pv + Pv\|_X^2 \leq 2 \sum_{k=1}^{K} \|v - Pv\|_{H^1(\Omega_k)^d}^2 + 2 \sum_{k=1}^{K} \|Pv\|_{H^1(\Omega_k)^d}^2,
\]

\[
\leq C \sum_{k=1}^{K} \left( \sum_{l \in \mathcal{N}(\Omega_k)} d_l(v, v) + \frac{1}{\text{diam}(\Omega_k)} \int_{S_k} (\pi[v])^2 \right) + C d(Pv, Pv),
\]
Moreover, we obtain by the triangular inequality and the use of proposition 4.4 that:

\[
\begin{align*}
\tilde{d}(Pv, Pv) & = \tilde{d}(Pv - v, Pv - v) \\
& \leq 2\tilde{d}(Pv - v, Pv - v) + 2\tilde{d}(v, v) \\
& \leq 2 \sum_{k=1}^{K} |Pv - v|_{H^1(\Omega_k)}^2 + 2\tilde{d}(v, v) \\
& \leq C \sum_{k=1}^{K} \left( \sum_{l \in \mathcal{N}(\Omega_k)} d_l(v, v) + \frac{1}{diam(\Omega_k)} \int_{\Omega_k} \left( \pi \varepsilon(v) \right)^2 \right) + 2\tilde{d}(v, v),
\end{align*}
\]

which leads to the final estimate:

\[
\|v\|_X^2 \leq C \left( \sum_{k=1}^{K} d_k(v, v) + \sum_{1 \leq k < l \leq K} \frac{1}{diam(\gamma_{kl})} \int_{\gamma_{kl}} \left( \pi \varepsilon(v) \right)^2 \right), \quad \forall v \in X, \quad (4.37)
\]

by exploiting the fact that \#\mathcal{N}(\Omega_k) \leq P, and \(diam(\Omega_k) \geq diam(\gamma_{kl})\).

### 4.4.5 Existence result for problem (4.7)

From assumption 4.3, we have \(V_h \subset V\) independently of the discretization, and get the uniform coercivity of the bilinear form \(\tilde{a}\) over \(V_h \times V_h\). Indeed, for all \(v_h \in V_h\), we get from (4.35) that:

\[
\tilde{a}(v_h, v_h) = \sum_{k=1}^{K} \int_{\Omega_k} (E : \varepsilon(v_h)) : \varepsilon(v_h)
\]

\[
\geq \min_{k \geq 1} (c_k) \sum_{k=1}^{K} \int_{\Omega_k} \varepsilon(v_h) : \varepsilon(v_h)
\]

\[
\geq C \min_{k \geq 1} (c_k) \left( \sum_{k=1}^{K} \frac{1}{diam(\Omega_k)^2} \int_{\Omega_k} (v_h)^2 + \int_{\Omega_k} |\nabla v_h|^2 \right),
\]

because \(\pi[v_h] = 0\) due to the fact that \(v_h \in V\). The coercivity of the bilinear form \(\tilde{a}\) over \(V_h \times V_h\) is then proved, with independence of the coercivity constant \(\tilde{a} = C \min_{k \geq 1} (c_k)\) with respect to the number and the size of the subdomains. Let us remark that when the Young moduli of the subdomains are multiplied by a constant, \(\tilde{a}\) is multiplicated as well. Since \(\tilde{a}\) is uniformly coercive over \(V_h \times V_h\) and since (4.9) ensures that the weak-continuity constraint \(b\) over the interfaces is onto, the discrete problem (4.7) is well posed by using Babuska and Brezzi’s theory of mixed problems [Bre74, Bab73] summarized in the following proposition:
Proposition 4.6. Let $X_h$ and $M_\delta$ be two real reflexive Banach spaces respectively endowed with the norms $\| \cdot \|_{X_h}$ and $\| \cdot \|_{M_\delta}$. Let $\tilde{a} : X_h \times X_h \to \mathbb{R}$ and $b : X_h \times M_\delta \to \mathbb{R}$ two continuous bilinear forms, and $l : X_h \to \mathbb{R}$ a continuous linear form. Denoting by:

$$V_h = \{ v_h \in X_h, \ b(v_h, \mu_h) = 0, \forall \mu_h \in M_\delta \}$$

the kernel space of $b$, we assume that $\tilde{a}$ is coercive over $V_h \times V_h$ in the sense that:

$$\exists \tilde{\alpha} > 0, \quad \tilde{a}(v_h, v_h) \geq \tilde{\alpha} \| v_h \|_{X_h}^2,$$

and that $b$ satisfies the following inf-sup condition:

$$\exists \beta > 0, \quad \inf_{\mu_h \in M_\delta \setminus \{0\}} \sup_{v_h \in X_h \setminus \{0\}} \frac{b(v_h, \mu_h)}{\| v_h \|_{X_h} \| \mu_h \|_{M_\delta}} \geq \beta.$$

Then there exists a unique solution $(u_h, \lambda_h) \in X_h \times M_\delta$ of:

$$\begin{cases}
\tilde{a}(u_h, v_h) + b(v_h, \lambda_h) = l(v_h), & \forall v_h \in X_h, \\
b(u_h, \mu_h) = 0, & \forall \mu_h \in M_\delta.
\end{cases}$$

4.5 Error estimates in elastostatics

4.5.1 Approximation of displacements

We recall now the standard error estimates in elastostatics under the following assumption:

Assumption 4.6. For all $1 \leq m \leq M$, the family of interface meshes $(\mathcal{F}_m, \delta_m)_{\delta_m > 0}$ over the non-mortar side is quasi-uniform, and $\delta_m / \delta_m$ remains bounded independently of the chosen discretization.

First, we need the following lemma:

Lemma 4.11. For all $1 \leq m \leq M$, there exists an operator:

$$P_m : H^{1/2}_\delta(\Gamma_m) \to W_{m; \delta_m},$$

such that for all $v \in H^{1/2}_\delta(\Gamma_m)$:

$$\int_{\Gamma_m} (P_m v) \cdot \mu = \int_{\Gamma_m} v \cdot \mu, \quad \forall \mu \in M_{m; \delta_m},$$

with:

$$\| P_m v \|_{\delta_m^{1/2}, m} \leq C \| v \|_{\delta_m^{1/2}, m}.$$
Proof: For all $1 \leq m \leq M$, we have by using assumption 4.2:

$$
\inf_{\lambda_h \in M_m \delta_m} \sup_{\phi \in W_m \delta_m} \frac{\int_{\Gamma_m} \lambda_h : \phi_h}{\|\lambda_h\|_{\frac{1}{2} m, \delta} \|\phi_h\|_{\frac{1}{2} m, \delta}} \geq \inf_{\lambda_h \in M_m \delta_m} \sup_{\phi \in W_0 \delta_m} \frac{\int_{\Gamma_m} \lambda_h : \phi_h}{\|\lambda_h\|_{\frac{1}{2} m, \delta} \|\phi_h\|_{\frac{1}{2} m, \delta}} \geq \inf_{\lambda_h \in M_m \delta_m} \sup_{\phi \in E^{1/2}_1(\Gamma_m)} \frac{\int_{\Gamma_m} \lambda_h : \phi_h}{\|\phi\|_{\infty, m, \delta}} = \frac{1}{C_m}.
$$

As $W_m \delta_m$ is reflexive, this condition implies that the map $v_h \in W_m \delta_m \rightarrow l \in M'_m \delta_m$ with $l(\mu) = \int_{\Gamma_m} v_h : \mu$ for all $\mu \in M_m \delta_m$ is onto with a continuous inverse. In other words, it implies that for all $l \in M'_m \delta_m$, there exists a $v_h \in W_m \delta_m$ such that:

$$
\int_{\Gamma_m} v_h : \mu = l(\mu), \quad \forall \mu \in M_m \delta_m,
$$

with

$$
\|v_h\|_{\frac{1}{2} m, \delta} \leq \frac{1}{C_m} \|l\|_{M'_m \delta_m}.
$$

In particular, for all $v \in H^{1/2}_0(\Gamma_m)$, we can define $l(\mu) = \int_{\Gamma_m} v : \mu$ for all $\mu \in M_m \delta_m$, and obtain the announced property. \hfill \Box

Error estimates can then be established by the classical result.

**Proposition 4.7.** If $u \in \prod_{k=1}^{K} H^{q+1}(\Omega_k)^d$ is solution of (4.4) with $(E : \varepsilon(u)) \in \prod_{k=1}^{K} H^{q}(\Omega_k)^{d \times d}$ and $q \geq 1$, and $(u_h, \lambda_h) \in X_h \times M_\delta$ is solution of (4.7), the following error estimate holds:

$$
\|u - u_h\|_X \leq C \left(1 + \max_{1 \leq k \leq K} \frac{C_k}{\delta} \right) \left( \sum_{k=1}^{K} h_k^{2q} u_{q+1, E, \Omega_k}^2 \right)^{1/2},
$$

with:

$$
|u|^{2}_{q+1, E, \Omega_k} = |u|^{2}_{H^{q+1}(\Omega_k)^d} + \frac{1}{C_k} \|E : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d \times d}}. \quad (4.38)
$$

The constant $C$ is independent of the number, the diameter, the Young moduli and the discretization of the subdomains. The coercivity constant of $\delta$ over $V_h \times V_h$ is denoted by $\delta$ and the coefficients $(C_k)_{1 \leq k \leq K}$ characterizing the elasticity tensor $E$ are defined by (4.3).
Remark 4.13. The difference with the original result of Wohlmuth lies in the fact that the
constants appearing in the proof do not depend any more on the number of subdomains.
The result remains true if we replace in (4.38) $q$ by any integer $1 \leq r \leq q$ because it re-
lies on interpolation results which hold for any $1 \leq r \leq q$ given our choice of finite element.

Proof: The proof can be found in [Woh01]. We review here the main steps. By a standard
use of the second Strang’s lemma [Str73], we have :
$$
\| u - u_h \|_X \leq \left( 1 + \frac{C_a}{\alpha} \right) \inf_{v_h \in V_h} \| u - v_h \|_X + \frac{1}{\alpha} \sup_{v_h \in V_h \setminus \{0\}} \frac{\tilde{a}(u, v_h) - l(v_h)}{\| v_h \|_X},
$$
where $C_a = C \max_{1 \leq k \leq K} C_k$ is the continuity constant of $\tilde{a}$, and is independent of $h$. We detail
the estimates for the consistency error (the second term in (4.39)) and the approximation
error (the first term in (4.39)) .

Concerning the consistency error, because $(E : \varepsilon(u)) \in \prod_{k=1}^K H^1(\Omega_k)^{d \times d}$, we have for
all $v_h \in V_h$ by using the divergence formula :

$$
\tilde{a}(u, v_h) - l(v_h) = \sum_{k \geq 1} \int_{\Omega_k} (E : \varepsilon(u)) : \nabla v_h - \int_{\Omega_k} f \cdot v_h - \int_{\Gamma_N} g \cdot v_h
$$
$$
= - \sum_{k \geq 1} \int_{\Omega_k} (\text{div} (E : \varepsilon(u)) + f) \cdot v_h + \sum_{k \geq 1} \int_{\partial \Omega_k} ((E : \varepsilon(u)) \cdot n_k) \cdot v_h
$$
$$
- \int_{\Gamma_N} g \cdot v_h,
$$
which entails by density of $C_c^\infty(\Omega_k)^{d \times d}$ in $L^2(\Omega_k)^d$, that :

$$
div (E : \varepsilon(u)) + f = 0, \quad \text{in } L^2(\Omega_k)^d.
$$
A fortiori, the result (4.41) entails that for all $v \in \bigcap_{k=1}^K H^1_s(\Omega_k)$ :

$$
- \sum_{k=1}^K \int_{\Omega_k} \text{div} (E : \varepsilon(u)) \cdot v = \int_{\Omega} f \cdot v,
$$
which gives by substracting the original problem (4.4) that :

$$
\int_{\Gamma_N} g \cdot v = \int_{\Omega} (E : \varepsilon(u)) : \nabla v + \int_{\Omega} \text{div} (E : \varepsilon(u)) \cdot v, \quad \forall v \in H^1_s(\Omega),
$$
$$
= \int_{\Gamma_N} ((E : \varepsilon(u)) \cdot n) \cdot v, \quad \forall v \in H^1_s(\Omega),
$$
where the normal outward unit vector on \( \Gamma_N \) is denoted by \( n \). As this final expression only depends on the restriction \( v|_{\Gamma_N} \in H^{1/2}_{00}(\Gamma_N)^d \), we conclude that:

\[
\int_{\Gamma_N} g \cdot \phi = \int_{\Gamma_N} ((E : \varepsilon(u)) \cdot n) \cdot \phi, \quad \forall \phi \in H^{1/2}_{00}(\Gamma_N)^d.
\]  

(4.42)

As a consequence, by using (4.41) and (4.42) in (4.40), we get:

\[
\tilde{a}(u, v_h) - l(v_h) = \sum_{1 \leq m \leq M} \int_{\Gamma_m} \lambda \cdot [v_h],
\]

with \( \lambda = (E : \varepsilon(u)) \cdot n \), where the normal unit vector \( n \) is chosen to be outward to \( \Omega_{k(m)} \) on \( \Gamma_m \). Moreover, \( [v_h] \) denotes the jump of \( v_h \) over \( S \). Then, we have to find an upper bound for the following quantity:

\[
\sup_{v_h \in V_h} \frac{\int_S \lambda \cdot [v_h]}{\|v_h\|_X}.
\]

By construction of \( V_h \), we have for all \( \mu_h \in M_\delta \):

\[
\int_S \lambda \cdot [v_h] = \int_S (\lambda - \mu_h) \cdot [v_h] \leq \|\lambda - \mu_h\|_{H^{q-1/2}(\Gamma_m)} [v_h]_{H^{q-1} \Gamma_m}.
\]

Moreover, we can prove that:

\[
\inf_{\mu_h \in M_\delta} \|\lambda - \mu_h\|_{H^{q-1/2}(\Gamma_m)} \leq \delta_m^q \|\lambda\|_{H^{q-1/2}(\Gamma_m)}.
\]

Indeed:

\[
\inf_{\mu_h \in M_\delta} \|\lambda - \mu_h\|_{H^{q-1/2}(\Gamma_m)}^2 = \inf_{\mu_h \in M_\delta} \sum_{F \in F_{\delta,m}} h(F) \|\lambda - \mu_h\|_{L^2(F)^d}^2
\]

\[
\leq \delta_m \inf_{\mu_h \in M_\delta} \|\lambda - \mu_h\|_{L^2(\Gamma_m)^d}^2 \leq C_\delta_m^{2q+1} \|\lambda\|^2_{H^q(\Gamma_m)^d},
\]

because the space of polynomials of degree \( q - 1 \) is included in \( M_{m;\delta_m} \). We have also at order \( q - 1 \):

\[
\inf_{\mu_h \in M_\delta} \|\lambda - \mu_h\|_{H^{q-1/2}(\Gamma_m)}^2 \leq C_\delta_m^{2q} \|\lambda\|^2_{H^{q-1}(\Gamma_m)^d}.
\]

By interpolation between the \( H^{q-1} \) and the \( H^q \) norm (see [LM72]), we obtain:

\[
\inf_{\mu_h \in M_\delta} \|\lambda - \mu_h\|_{H^{q-1/2}(\Gamma_m)}^2 \leq C_\delta_m^{2q} \|\lambda\|^2_{H^{q-1/2}(\Gamma_m)^d},
\]  

(4.43)
As a consequence, by summing the previous estimations over $m \geq 1$:

$$\inf_{\mu_h \in M_k} \| \lambda - \mu_h \|_{H^1/2}^2 = \sum_{m=1}^{M} \inf_{\mu_h \in M_k} \| \lambda - \mu_h \|_{H^1/2}^2 \leq C \sum_{m=1}^{M} \delta_{m}^{2q} \| \lambda \|_{H^{q+1/2}(\partial \Omega_k)}^2 \leq C \sum_{k=1}^{K} h_k^{2q} \| \lambda \|_{H^{q+1/2}(\partial \Omega_k)}^2 \leq C \sum_{k=1}^{K} h_k^{2q} \| \mathbf{E} : \varepsilon(u) \|_{H^q(\Omega_k)}^{d \times d},$$

hence the following estimate:

$$\inf_{\mu_h \in M_k} \| \lambda - \mu_h \|_{H^1/2}^2 \leq C \sum_{k=1}^{K} h_k^{2q} \| \mathbf{E} : \varepsilon(u) \|_{H^q(\Omega_k)}^{d \times d}.$$

(4.44)

Now, let us estimate $\| [v_h] \|_{H^1/2}$, by using the operators $P_m$ introduced above. As $v_h \in V_h$, $[v_h]$ vanishes in the dual space $M'_m;\delta_m$ and therefore $P_m[v_h] = 0$. Then, for all $w_h \in W_m;\delta_m$ and using lemma 4.11:

$$\| [v_h] \|_{H^1/2} \leq \| [v_h] - w_h - P_m ([v_h] - w_h) \|_{H^1/2} \leq C \| [v_h] - w_h \|_{H^1/2}.$$

(4.45)

As the family of meshes over the non-mortar side is quasi-uniform by assumption 4, we have by a standard inverse inequality (see [EG02]):

$$\| [v_h] - w_h \|_{H^1/2} \leq C \| [v_h] - w_h \|_{L^2(\Gamma_m)}^2.$$

Let $w_h$ the $L^2$ projection of $[v_h]$ over $W_m;\delta_m$. We then have:

$$\| [v_h] - w_h \|_{L^2(\Gamma_m)}^2 \leq \| [v_h] \|_{L^2(\Gamma_m)}^2,$$

and by a classical interpolation result:

$$\| [v_h] - w_h \|_{L^2(\Gamma_m)}^2 \leq \| [v_h] - I_h [v_h] \|_{L^2(\Gamma_m)}^2 \leq C \delta_m^2 \| [v_h] \|_{H^1(\Gamma_m)}^2,$$

where $I_h$ is the nodal interpolation over $\Gamma_m$. By interpolation between the $L^2$ and $H^1$ norms (see [LM72]), we obtain:

$$\| [v_h] - w_h \|_{L^2(\Gamma_m)}^2 \leq C \delta_m \| [v_h] \|_{H^{1/2}(\Gamma_m)}^2,$$

resulting in:

$$\| [v_h] - w_h \|_{H^{1/2}}^2 \leq C \| [v_h] \|_{H^{1/2}(\Gamma_m)}^2.$$
It is deduced from (4.45) that:
\[
\|[[v_h]]\|_{\delta, \frac{1}{2}, m} \leq C \|v_h\|_{H^{1/2}(\Gamma_m)}^d, \quad \forall v_h \in V_h.
\] (4.46)

The consistency error is therefore of optimal order, and we get more precisely by using the Cauchy-Schwarz inequality and estimates (4.46) and (4.44):
\[
\left( \int_S \lambda \cdot [v_h] \right)^2 \leq \left( \sum_{m=1}^M \|\lambda - \mu_h\|^2_{\delta, \frac{1}{2}, m} \right) \left( \sum_{m=1}^M \|[[v_h]]\|^2_{\delta, \frac{1}{2}, m} \right), \quad \forall \mu_h \in M_\delta,
\]
\[
\leq C \left( \sum_{k=1}^K h_k^{2q} \|E : \varepsilon(u)\|^2_{H^2(\Omega_k)} d \times d \right) \left( \sum_{m=1}^M \|[[v_h]]\|^2_{H^{1/2}(\Gamma_m)} d \right)
\]
by taking the infimum over \( \mu_h \in M_\delta \). We deduce from the trace theorem that:
\[
\sup_{v_h \in V_h} \left( \int_S \lambda \cdot [v_h] \right) / \|v_h\|_X \leq C \left( \sum_{k=1}^K c_k^{2q} M_{q+1,E,\Omega_k} \right)^{1/2}.
\]

Now, let us consider the approximation error. If \( I_h \) is the standard Lagrange interpolation operator over \( X_h \), \( w_h = I_h u \) does not satisfy the weak constraint on the jump \( [w_h] \) over \( \Gamma_m \). Then we define:
\[
v_h = w_h - \sum_{m=1}^M \bar{R}_{m;hm} \pi_m[w_h]m \in V_h,
\]
with the discrete extension by zero operators \( \bar{R}_{m;hm} : W_{m;\delta} \to X_h \) defined in the definition 4.2 of the appendix, page 203. As a consequence, from lemma 4.19:
\[
\|u - v_h\|^2_X \leq C \left( \|u - w_h\|^2_X + \sum_{m=1}^M \|\pi_m[w_h]m\|^2_{\delta, \frac{1}{2}, m} \right).
\]
Moreover, from assumption 4.2:
\[
\sum_{m=1}^M \|\pi_m[w_h]m\|^2_{\delta, \frac{1}{2}, m} \leq C \sum_{m=1}^M \|[w_h]m\|^2_{\delta, \frac{1}{2}, m} = C \sum_{m=1}^M \|[u - w_h]\|^2_{\delta, \frac{1}{2}, m}.
\]
We have also:
\[
\sum_{m=1}^M \|[u - w_h]\|^2_{\delta, \frac{1}{2}, m}
\]
\[
\leq 2 \sum_{m=1}^M \|[u - w_h]|_{\partial \Omega_k(m)}\|^2_{\delta, \frac{1}{2}, m} + 2 \sum_{m=1}^M \|[u - w_h]|_{\partial \Omega_k \neq k(m)}\|^2_{\delta, \frac{1}{2}, m}.
\]
and use the quasi-uniformity of the non-mortar mesh to obtain:

$$\leq 2 \sum_{m=1}^{M} \| u - w_h \|_{\partial \Omega_k(m)}^2 + 2 \sum_{m=1}^{M} \frac{\delta_m}{h(T)} \| u - w_h \|_{\Omega_k(m)}^2 + \| \nabla (u - w_h) \|_{\Omega_k(m)}^2.$$ 

Using now the lemma 4.18 from the appendix, page 202, we get the following upper bound:

$$\leq 2 \sum_{m=1}^{M} \sum_{T \in T_h, T \subset \Omega_k(m)} \frac{1}{h(T)^2} \| u - w_h \|_{\Omega_k(m)}^2 + \| \nabla (u - w_h) \|_{\Omega_k(m)}^2.$$ 

Then, by using a classical interpolation result and the assumption 4, we get:

$$\sum_{m=1}^{M} \left\| \pi_m [w_h]_{m} \right\|_{H^1_\delta, m}^2 \leq C \left( \frac{1 + \max_{1 \leq m \leq M} \frac{\delta_m}{h(T)}}{\delta_m} \right) \sum_{k=1}^{K} h_k^{2q} |u|_{H^{q+1}(\Omega_k)}^2.$$ 

and the proof is complete.

**Remark 4.14.** It can be noticed by reading precisely the previous proof, that a better a priori estimate is obtained when the non-mortar side is taken as the coarsest side (to improve the approximation error) and/or the softer one (to improve the consistency error).

### 4.5.2 Approximation of fluxes

The convergence of Lagrange multipliers uses the inf-sup condition (4.9) and is established by the:

**Proposition 4.8.** If $u \in \prod_{k=1}^{K} H^{q+1}(\Omega_k)^d$ is solution of (4.4) with $(E : \varepsilon(u)) \in \prod_{k=1}^{K} H^{q}(\Omega_k)^{d \times d}$ and $q \geq 1$, and $(w_h, \lambda_h) \in X_h \times M_\delta$ is solution of (4.7), the following error estimate on Lagrange multipliers holds:

$$\| \lambda - \lambda_h \|_{H^{q-1/2}} \leq C \left( \sum_{k=1}^{K} h_k^{2q} |u|_{H^{q+1}(\Omega_k)}^2 \right)^{1/2},$$

with $\lambda = (E : \varepsilon(u)) \cdot n$, where $n$ is the normal unit vector on $S$ which is outward to $\Omega_k(m)$ for all $1 \leq m \leq M$. In more details, the constant $C$ has the following dependence:

$$C = C' \max_{1 \leq k \leq K} C_k \left( 1 + \frac{1}{\beta} \right) + C' \max_{1 \leq k \leq K} C_k \left( 1 + \frac{1}{\beta} \frac{C_k}{\alpha} \right),$$
where the various constants denoted by $C'$ do not depend on the number, the diameter, the Young moduli and the discretization of the subdomains.

**Proof:** As $(E : \varepsilon(u)) \in \prod_{k=1}^{K} H^{1}(\Omega_k)^{d \times d}$, by a simple integration by part over each $\Omega_k$, one can obtain from the continuous problem:

$$\tilde{a}(u, v) + b(v, \lambda) = l(v), \quad \forall v \in X.$$  

Considering this equality with $v = v_h \in X_h \subset X$ and substracting the approximate problem (4.7), we obtain:

$$b(v_h, \mu_h - \lambda_h) = \tilde{a}(u_h - u, v_h) + b(v_h, \mu_h - \lambda), \quad \forall v_h \in X_h, \forall \mu_h \in M_\delta. \quad (4.47)$$

Defining $v_h$ from $\mu_h - \lambda_h$ by the same technique used for constructing $u_h$ in the proof of (4.9), we deduce from (4.12), and using (4.47):

$$\|\mu_h - \lambda_h\|_{\delta, -\frac{1}{2}}^2 \leq C \frac{1}{\beta} b(v_h, \mu_h - \lambda_h)$$

$$= C \frac{1}{\beta} \left( \tilde{a}(u_h - u, v_h) + \sum_{m=1}^{M} \int_{\Gamma_m} [v_h] |_{\Gamma_m} \cdot (\mu_h - \lambda) \right).$$

But by construction:

$$[v_h]_{\Gamma_m} = \phi_{\mu_h - \lambda_h} \int_{\Gamma_m} \phi_{\mu_h - \lambda_h} \cdot (\lambda_h - \mu_h),$$

hence since $\|\phi_{\mu_h - \lambda_h}\|_{\delta, \frac{1}{2}, m} = 1$, we get:

$$\int_{\Gamma_m} [v_h]_{\Gamma_m} \cdot (\mu - \lambda) = \int_{\Gamma_m} \phi_{\mu_h - \lambda_h} \cdot (\lambda_h - \mu_h) \int_{\Gamma_m} \phi_{\mu_h - \lambda_h} \cdot (\mu_h - \lambda)$$

$$\leq \|\lambda_h - \mu_h\|_{\delta, -\frac{1}{2}, m} \|\mu_h - \lambda\|_{\delta, -\frac{1}{2}, m}.$$  

It remains that:

$$\|\mu_h - \lambda_h\|_{\delta, -\frac{1}{2}}^2 \leq \frac{C}{\beta} \max_{1 \leq k \leq K} C_k \|u - u_h\|_X \|v_h\|_X$$

$$+ \frac{C}{\beta} \left( \sum_{m=1}^{M} \|\mu_h - \lambda\|_{\delta, -\frac{1}{2}, m}^2 \right)^{1/2} \left( \sum_{m=1}^{M} \|\mu_h - \lambda_h\|_{\delta, -\frac{1}{2}, m}^2 \right)^{1/2}$$

and recalling that $\|v_h\|_X \leq C \|\mu_h - \lambda_h\|_{\delta, -\frac{1}{2}}$ from (4.13), we obtain after division by $\|\mu_h - \lambda_h\|_{\delta, -\frac{1}{2}}$:

$$\|\mu_h - \lambda_h\|_{\delta, -\frac{1}{2}} \leq C \frac{1}{\beta} \left( \max_{1 \leq k \leq K} C_k \|u - u_h\|_X + \|\mu_h - \lambda\|_{\delta, -\frac{1}{2}} \right).$$
By the triangular inequality, we have:

\[
\|\lambda - \lambda_h\|_{\delta,-\frac{\delta}{2}} \leq \|\lambda - \mu_h\|_{\delta,-\frac{\delta}{2}} + \|\lambda_h - \mu_h\|_{\delta,-\frac{\delta}{2}} \\
\leq C_1 \beta \max_{1 \leq k \leq K} (C_k) \|u - u_h\|_X + \left(1 + C_1 \beta \right) \|\lambda - \mu_h\|_{\delta,-\frac{\delta}{2}}, \quad \forall \mu_h \in M_\delta.
\]

Then, from proposition 4.7 and (4.44), the announced estimate is obtained. \hfill \Box

4.6 Generalization to elastodynamics.

In this section, we analyze the use of mortar elements to solve the linear elastodynamics problem:

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \text{div}(E : \varepsilon(u)) = f, & [0,T] \times \Omega, \\
(E : \varepsilon(u)) \cdot \nu = g, & [0,T] \times \Gamma_N, \\
u = 0, & [0,T] \times \Gamma_D, \\
u = u_0, & \{0\} \times \Omega, \\
\frac{\partial u}{\partial t} = \dot{u}_0, & \{0\} \times \Omega,
\end{cases}
\tag{4.48}
\]

with obvious notation. Let us only notice that the normal outward unit vector over a surface is now denoted by \(\nu\) instead of \(n\) to avoid any possible confusion with the forthcoming numbering of the time steps.

First, the notation of the static case is adapted and a standard result of existence recalled in the elastodynamics framework. In the second subsection, a total approximation in space and time is introduced for the dynamical solution. It uses a mid-point finite difference time integration scheme which is interesting for energy conservation purpose, and a non-conforming finite element space approximation using a mortar weak-continuity constraint over the interfaces. We finally establish the convergence of the approximate solution to the continuous one, which is the main contribution of this section.

Moreover, an important remark has to be done with respect to this analysis. For first order problems in time, Lagrange multipliers are involved in the convergence analysis through the estimation of:

\[
\inf_{\mu_h \in M_\delta} \| (E : \varepsilon(u(t))) \cdot \nu - \mu_h \|_{\delta,-\frac{\delta}{2}},
\]
as shown for example in [BMR01] for an eddy currents model. In the framework of second order problems in time, we underline the idea that the Lagrange multipliers are also involved through the estimation of:

\[
\inf_{\mu_h \in M_\delta} \left\| \left( E : \varepsilon \left( \frac{\partial u}{\partial t} (t) \right) \right) \cdot \nu - \mu_h \right\|_{\delta,-\frac{\delta}{2}},
\]
entailing a higher sensitivity with respect to the choice of Lagrange multipliers. A time discontinuity in interface constraints would lead to a deterioration of convergence.

4.6.1 Position of the problem.

We formulate here the linear elastodynamics problem, using mainly the same notation as in the static case. The body forces are denoted by $f \in L^2(0,T;L^2(\Omega)^d)$, the density of the material by $\rho \in L^\infty(\Omega)$, which is assumed to be greater than a positive constant, and the initial conditions in displacement by $u_0 \in H^1(\Omega)^d$ and in velocity by $\dot{u}_0 \in L^2(\Omega)^d$. A surfacic force $g \in C^1(0,T;L^2(\Gamma_N)^d)$ which is regular in time is applied over the part $\Gamma_N$ of the boundary $\partial \Omega$ and a Dirichlet boundary condition $u = 0$ is imposed on the complementary part $\Gamma_D = \partial \Omega \setminus \Gamma_N$ which can be of zero measure. The elastic properties of the material are the same as in the static case described above.

To give a precise meaning to the system (4.48), we define a solution as a displacement function:

$$ u \in C^0(0,T;H^1_*(\Omega)) \cap C^1(0,T;L^2(\Omega)^d), $$

such that in the sense of distributions on $[0,T]$:

$$ \frac{\partial^2}{\partial t^2} \int_{\Omega} \rho u(t) \cdot v + a(u(t), v) = \int_{\Omega} f(t) \cdot v + \int_{\Gamma_N} g(t) \cdot v, \quad \forall v \in H^1_*(\Omega). \quad (4.49) $$

It is now standard that:

**Proposition 4.9.** Under the previous assumptions, there exists a unique displacement field $u \in C^0(0,T;H^1_*(\Omega)) \cap C^1(0,T;L^2(\Omega)^d)$, such that the equation (4.49) is satisfied in the sense of distributions on $[0,T]$. Moreover, the energy:

$$ E(t) = \frac{1}{2} \int_{\Omega} \rho \left( \frac{\partial u}{\partial t}(t) \right)^2 + \frac{1}{2} a(u(t), u(t)), $$

is conserved, that is for all $t \in [0,T]$:

$$ E(t) = E(0) + \int_{0}^{t} \int_{\Omega} f(s) \cdot \frac{\partial u}{\partial t}(s) ds + \int_{0}^{t} \int_{\Gamma_N} g(s) \cdot \frac{\partial u}{\partial t}(s) ds. $$

We refer to [LM72, RT98] for a proof of the proposition. It is classically done by:

- defining an approximation:

$$ u_m(t) = \sum_{k=1}^{m} c_k(t) \varphi_i, $$

of the solution over the $m$ first eigenmodes $(\varphi_i)_{i \geq 1}$ satisfying:

$$ a(\varphi_i, v) = \omega_i^2 \int_{\Omega} \rho \varphi_i \cdot v, \quad \forall v \in H^1_*(\Omega), $$

with the eigenvalues $(\omega_i^2)_{i \geq 1}$,
proving that the sequence of approximate solutions \( (u_m)_{m \geq 1} \) is a Cauchy sequence in \( C^0(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega)^d) \), resulting in its convergence to a solution \( u \in C^0(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega)^d) \) of the problem (4.49).

The energy estimate for the solution \( u \) comes from the limit of energy estimates for the approximate solutions \( u_m \). The uniqueness of the solution \( u \) is a straightforward consequence of the energy estimate.

### 4.6.2 A midpoint nonconforming fully discrete approximation.

We introduce here a space non-conforming fully discrete approximation of the solution of (4.48). First, at each time \( t \in [0,T] \) the spaces \( H_0^1(\Omega) \) and \( L^2(\Omega)^d \) for the displacements and the velocities are replaced by the non-conforming finite element space \( V_h \) introduced in section 4.2.3, page 110, for the elastostatics problem. We then look for the displacements \( u_h \in C^0(0,T; V_h) \cap C^1(0,T; V_h) \) such that in the sense of distributions on \( [0,T] \):

\[
\frac{\partial^2}{\partial t^2} \int_{\Omega} p u_h(t) \cdot v_h + \tilde{a}(u_h(t), v_h) = \int_{\Omega} f(t) \cdot v_h + \int_{\Gamma_N} g(t) \cdot v_h, \quad \forall v_h \in V_h. \tag{4.50}
\]

The initial conditions in displacement and velocity take the form:

\[
\begin{align*}
\left\{ u_h(0) = \mathcal{P}_h^1 u_0 & \quad \in V_h, \\
\frac{\partial u_h}{\partial t}(0) = \mathcal{P}_h^0 u_0 & \quad \in V_h,
\end{align*}
\tag{4.51}
\]

where \( \mathcal{P}_h^1 \) (resp. \( \mathcal{P}_h^0 \)) denotes a projection from \( H_0^1(\Omega) \) (resp. \( L^2(\Omega)^d \)) to \( V_h \). Now, let \( (t_n)_{n \in \mathbb{N}} \) a sequence of discrete times such that \( t_n = n \Delta t \) for \( n \in \mathbb{N} \). The use of a constant time step \( \Delta t \) enables the optimal time accuracy order established below. The formal integration of (4.50) and of the additional relation:

\[
\frac{\partial}{\partial t} \int_{\Omega} u_h(t) \cdot v_h = \int_{\Omega} \frac{\partial u_h}{\partial t}(t) \cdot v_h, \quad \forall v_h \in V_h,
\]

over \( t \in [t_n, t_{n+1}] \) by the trapezoidal rule gives the following fully discrete system:

\[
\begin{align*}
\int_{\Omega} \frac{\dot{u}_h^{n+1} - \dot{u}_h^n}{\Delta t} \cdot v_h + \tilde{a} \left( \frac{u_h^n + u_h^{n+1}}{2}, v_h \right) & = \frac{L_n(v_h) + L_{n+1}(v_h)}{2}, \quad \forall v_h \in V_h, \\
\frac{u_h^{n+1} - u_h^n}{\Delta t} & = \frac{\dot{u}_h^n + \dot{u}_h^{n+1}}{2}. \tag{4.52}
\end{align*}
\]

We have introduced the virtual work of the applied forces at the discrete time \( t_n \):

\[
L_n(v_h) = \int_{\Omega} f(t_n) \cdot v_h + \int_{\Gamma_N} g(t_n) \cdot v_h, \quad \forall v_h \in V_h,
\]
and have denoted by $u_n^h \in V_h$ (resp. $\dot{u}_n^h \in V_h$) the approximation in time of the space approximation $u(t_n) \in V_h$ of the displacement (resp. $\frac{\partial u}{\partial t}(t_n) \in V_h$ of the velocity), that is the fully discrete approximation of the displacement $u(t_n) \in H^1_*(\Omega)$ (resp. the velocity $\frac{\partial u}{\partial t}(t_n) \in L^2(\Omega)^d$). This trapezoidal finite difference scheme in time has been selected for its exact conservation properties with respect to the energy and to the linear momentum (see [ST92]).

The convergence analysis to come could be extended to other time integrators. The system has to be completed with the initial conditions:

$$
\begin{aligned}
& u_0^h = P_h^1 u_0 \quad \in V_h, \\
& \dot{u}_0^h = P_h^0 \dot{u}_0 \quad \in V_h.
\end{aligned}
$$

Knowing $u_n^h, \dot{u}_n^h \in V_h$ and after elimination of $\dot{u}_{n+1}^h$ by (4.52)-2, we can then determine the fully discrete displacement $u_{n+1}^h \in V_h$ at the discrete time $t_{n+1} \in [0,T]$ by solving:

$$
\int_{\Omega} \frac{2}{\Delta t^2} \rho u_{n+1}^h \cdot v_h + \frac{1}{2} \tilde{a} \left( u_{n+1}^h, v_h \right) = \int_{\Omega} \rho \left( \frac{2}{\Delta t^2} u_n^h + \frac{2}{\Delta t} \dot{u}_n^h \right) \cdot v_h \\
- \frac{1}{2} \tilde{a} \left( u_n^h, v_h \right) + L_n(v_h) + L_{n+1}(v_h),
$$

and the velocity $\dot{u}_{n+1}^h \in V_h$ is obtained by the simple computation:

$$
\dot{u}_{n+1}^h = \frac{2}{\Delta t} (u_{n+1}^h - u_n^h) - \dot{u}_n^h.
$$

The existence of a projection $P_h$ from $H^1_*(\Omega)$ to $V_h$ is detailed in the following lemma:

**Lemma 4.12.** If $\Gamma_D$ has a positive measure, there exists a projection operator:

$$
P_h : H^1_*(\Omega) \rightarrow V_h, \quad u \mapsto P_h u,
$$

such that $P_h u$ is the unique solution $u_h \in V_h$ of:

$$
\tilde{a}(u_h, v_h) = \tilde{a}(u, v_h), \quad \forall v_h \in V_h.
$$

Moreover, for all $u \in H^{r+1}_E(\Omega)$ with $r \geq 1$, we have the following estimates:

$$
\| u - P_h u \|_{X}^2 \leq C \sum_{k=1}^{K} h_k^{2r} |u|_{r+1.\Omega_k}^2.
$$


\[ \|u - \mathbb{P}_h u\|_{L^2(\Omega)}^2 \leq C \left( \sup_{1 \leq k \leq K} h_k^2 \right) \sum_{k=1}^{K} h_k^{2r} |u|_{r+1, \mathbb{E}, \Omega_k}^2. \]

**Observation:** the last inequality holds within a regularity condition, namely that the solution of all elasticity problems over \( \Omega \) be in \( H^2_\mathbb{E}(\Omega) \).

**Remark 4.15.** The constant \( C \) in the estimates of proposition 4.12 is in fact of the form:

\[ C = C' \left( 1 + \max_{k \geq 1} \frac{C_k}{\alpha} \right) \max_{k \geq 1} \frac{C_k}{\alpha}, \]

where \( C' \) is independent of the discretization in space and time, of the number of subdomains, and of the coercivity and continuity constants of the broken bilinear form \( \tilde{a} \). Nevertheless, to simplify the present exposition, we will keep the generic notation \( C \).

**Proof:** The existence of the projection \( \mathbb{P}_h \) is a straightforward consequence of the Lax-Milgram lemma. More precisely, for a given function \( u \in H^1_*(\Omega) \), let us define the continuous linear form \( l \in X' \) by:

\[ l(v) = \tilde{a}(u, v), \quad \forall v \in X. \]

The function \( u \in H^1_*(\Omega) \) is the unique solution of:

\[ a(u, v) = l(v), \quad \forall v \in H^1_*(\Omega), \]

and \( \mathbb{P}_h u \) is the unique solution \( u_h \) of:

\[ \tilde{a}(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h. \]

The error between \( u_h \) and \( u \) in the broken norm \( \| \cdot \|_X \) is given in the proposition 4.7, resulting in the announced estimate. The estimation in the \( L^2(\Omega)^d \) norm can be obtained by a Aubin-Nitsche argument (cf. [Aub87] for example) that we detail here. Let us assume that for all \( \phi \in L^2(\Omega)^d \), there exist a solution \( \zeta_\phi \in H^2_\mathbb{E}(\Omega) \) of:

\[ \tilde{a}(v, \zeta_\phi) = \int_\Omega \phi \cdot v, \quad \forall v \in H^1_*(\Omega). \quad (4.54) \]

Indeed, we have assumed that the solution of all elasticity problems over \( \Omega \) be in \( H^2_\mathbb{E}(\Omega) \). First, because the application:

\[ T : H^2_\mathbb{E}(\Omega) \cap H^1_*(\Omega) \to H^1_*(\Omega)', \quad \zeta \mapsto T\zeta; \quad \langle T\zeta, v \rangle_{H^1_*(\Omega)', H^1_*(\Omega)} = \tilde{a}(v, \zeta), \]

is linear, continuous and bijective, the inverse \( T^{-1} \) is continuous by the open application theorem [Bré99, Yos65]. As a consequence, the solution \( \zeta_\phi \in H^2_\mathbb{E}(\Omega) \) of (4.54) satisfies:

\[ \left( \sum_{k=1}^{K} C_k^2 |\zeta_\phi|_{2, \mathbb{E}, \Omega_k}^2 \right)^{1/2} \leq C \|\phi\|_{H^1_*(\Omega)} \leq C \|\phi\|_{L^2(\Omega)^d}. \quad (4.55) \]
4.6. Generalization to elastodynamics.

Now, let us prove the announced upper bound on \( \|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} \), by the Aubin-Nitsche technique. Namely:

\[
\|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} = \sup_{\phi \in L^2(\Omega)^d \setminus \{0\}} \frac{\int_{\Omega} (u - \mathbb{P}_h u) \cdot \phi}{\|\phi\|_{L^2(\Omega)^d}}
\]

and by definition of \( \mathbb{P}_h \), \( \tilde{a}(u - \mathbb{P}_h u, v_h) = 0 \) for all \( v_h \in V_h \), resulting in the following expression for all \( v_h \in V_h \), and \( \phi \) realizing the supremum in the above inequality:

\[
\|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} \leq \frac{\tilde{a}(u - \mathbb{P}_h u, \zeta_{\phi} - v_h)}{\|\phi\|_{L^2(\Omega)^d}}.
\]

By taking the infimum of the right hand side over \( v_h \in V_h \), and by using the approximation property of \( \mathbb{P}_h \) in \( X \) (proposition 4.7), and the relation (4.55), we get:

\[
\|u - \mathbb{P}_h u\|_{L^2(\Omega)^d} \leq C \left( \sum_{k=1}^{K} h_k^{2r} |u|_{r+1, \mathbf{E}, \Omega_k}^2 \right)^{1/2} \left( \sum_{k=1}^{K} h_k^2 C_k^2 |\zeta_{\phi}|_{2, \mathbf{E}, \Omega_k}^2 \right)^{1/2}
\]

4.6.3 Convergence analysis

Now, we prove the convergence of the fully discrete approximation given by (4.52) to the continuous solution of (4.49). For that purpose, we introduce the following space:

\[
H_{q+1, \mathbf{E}}^1(\Omega) = \{ v \in H^1(\Omega); \|v\|_{q+1, \mathbf{E}, \Omega} < +\infty \},
\]

which is endowed with the following norm:

\[
\|v\|_{q+1, \mathbf{E}, \Omega}^2 = \|v\|_{H^1(\Omega)}^2 + \sum_{k=1}^{K} \left( \|v\|_{H^{q+1}(\Omega_k)^d}^2 + \frac{1}{C_k^2} \|\mathbf{E} : \varepsilon(v)\|_{H^q(\Omega_k)^d \times d}^2 \right).
\]

We also denote as in proposition 4.7:

\[
|v|_{q+1, \mathbf{E}, \Omega_k}^2 = |v|_{H^{q+1}(\Omega_k)^d}^2 + \frac{1}{C_k^2} \|\mathbf{E} : \varepsilon(v)\|_{H^q(\Omega_k)^d \times d}^2,
\]

and state the main result of that section:
Proposition 4.10 (Error estimate). If

\[ u \in C^1(0, T; H^{q+1}_E(\Omega)) \cap C^2(0, T; \prod_{k=1}^{K} H^{r+1}(\Omega_k)^d) \cap C^4(0, T; L^2(\Omega)^d) \]

is solution of (4.49) and \((u_n^h; \dot{u}_n^h)_{n \in \mathbb{N}}\) is the fully discrete solution of (4.52), then the following error estimate holds:

\[ \left\| \sqrt{\rho} \left( \ddot{u}(t_{n+1/2}) - \frac{\dot{u}_n^h + \dot{u}_{n+1}^h}{2} \right) \right\|^2_{L^2(\Omega)^d} + \tilde{\alpha} \left\| u(t_{n+1/2}) - \frac{u_n^h + u_{n+1}^h}{2} \right\|^2_X \leq C \left( \| P_h \dot{u}_0 - \dot{u}_0^h \|^2_{L^2(\Omega)^d} + \| P_h u_0 - u_0^h \|^2_X \right) \]

\[ + C \left( \frac{(\Delta t)}{t_0} \right)^4 \left\{ \tilde{\alpha} \sup_{t \in [0, T]} \| \bar{u}(t) \|^2_{\tilde{X}} + \sup_{t \in [0, T]} \| \sqrt{\rho} t_0 \bar{u}(t) \|^2_{L^2(\Omega)^d} + \frac{T}{t_0} \sup_{t \in [0, T]} \| \sqrt{\rho} t_0 \bar{u}(t) \|^2_{L^2(\Omega)^d} \right\} \]

\[ + h^2 \frac{T}{t_0} \sum_{k=1}^{K} h^2_k \sup_{t \in [0, T]} \left| \sqrt{\rho} t_0 \bar{u}(t) \right|^2_{H^1, \Omega_k} + \sup_{t \in [0, T]} \left| \sqrt{\rho} \ddot{u}(t) \right|^2_{H^{r+1}, \Omega_k} \]

\[ + \tilde{\alpha} \sum_{k=1}^{K} h^2 \sum_{k=1}^{K} \sup_{t \in [0, T]} \left| u(t) \right|^2_{q+1, \Omega_k} + \frac{T}{t_0} \sup_{t \in [0, T]} \left| t_0 \ddot{u}(t) \right|^2_{q+1, \Omega_k} \]

\[ \left( 1 + \frac{\Delta t}{t_0} \right)^n, \]

where \(C\) denotes various constants independent of the discretization in space and time, and \(t_{n+1/2} = \frac{1}{2}(t_n + t_{n+1})\). Moreover, \(P_h\) is the projection \(P_h\) from \(H^1_\ast(\Omega)\) to \(V_h\) given in lemma 4.12 if \(\Gamma_D\) has a positive measure, and is defined by (4.66) if \(\Gamma_D\) has a null measure, and \(r\) is any integer with \(1 \leq r \leq q\). Finally, \(t_0\) is a reference length of time.

In order to simplify the exposition of the proof, we assume that \(\Gamma_D\) has a positive measure so that the bilinear form \(a\) is coercive over \(H^1_\ast(\Omega) \times H^1(\Omega)\). We will enumerate in the remark following the proof the necessary modifications when \(\Gamma_D\) has a null measure. The proof is inspired by the convergence proof introduced in [TM00] for fluid-structure analysis.

**Proof:** For clarity, the proof is decomposed into six parts. The time derivative of \(u\) will be sometimes denoted by \(\dot{u}\) to simplify notation.

1. **The discrete evolution of error.**

Let us define the projection on \(V_h\) of the error in displacements at time \(t_n\) by:

\[ e u_n^h = P_h u(t_n) - u_n^h, \]

and a new approximation \((V_n^h)_{n \geq 0}\) of velocities by:

\[ \frac{1}{2} (V_n^h + V_{n+1}^h) = \frac{1}{\Delta t} (P_h u(t_{n+1}) - P_h u(t_n)), \]
with the initial condition $V_0^h = P_h \tilde{u}_0$. The gap between the fully discrete velocity $\tilde{u}_n^h$ and $V_n^h$ is then defined by:

$$eV_n^h = V_n^h - \tilde{u}_n^h.$$ 

We now establish the equation satisfied by these errors.

To do so, we first show that for all $t \in [0, T]$:

$$
\int_{\Omega} \rho \frac{\partial^2 u}{\partial t^2} (t) \cdot v_h + \tilde{a}(u(t), v_h) = \int_{\Omega} f(t) \cdot v_h + \int_{\Gamma_N} g(t) \cdot v_h + \int_{S} \lambda(t) \cdot [v_h], \quad \forall v_h \in V_h, \quad (4.56)
$$

with $\lambda(t) = (E : \varepsilon(u(t))) \cdot \nu$, where $\nu$ is the normal unit vector on $S$ which is outward to the non-mortar subdomain.

Due to the assumptions that for all $t \in [0, T]$, $(E : \varepsilon(u(t))) \in \prod_{k=1}^K H^1(\Omega_k)^{d \times d}$ and that the time derivatives of $u$ have a classical sense, we obtain from (4.49) that for all $t \in [0, T]$ and all $v \in C_c^\infty(\Omega)^d$:

$$
\int_{\Omega} \left( \rho \frac{\partial^2 u}{\partial t^2} (t) - \text{div} (E : \varepsilon(u(t))) - f(t) \right) \cdot v = 0.
$$

By density of $C_c^\infty(\Omega)^d$ in $L^2(\Omega)^d$ we have then that for all $t \in [0, T]$:

$$
\rho \frac{\partial^2 u}{\partial t^2} (t) - \text{div} (E : \varepsilon(u(t))) - f(t) = 0, \quad \text{in } L^2(\Omega)^d. \quad (4.57)
$$

Then, we can obtain some information about the natural boundary conditions. Indeed, we get a fortiori from (4.57) that:

$$
\int_{\Omega} \left( \rho \frac{\partial^2 u}{\partial t^2} (t) - \text{div} (E : \varepsilon(u(t))) - f(t) \right) \cdot v = 0, \quad \forall v \in H^1_s(\Omega), \quad (4.58)
$$

and by substracting the original problem (4.49) to (4.58), we obtain for all $v \in H^1_s(\Omega)$:

$$
\int_{\Gamma_N} g(t) \cdot v = \int_{\Omega} (E : \varepsilon(u(t))) : \nabla v + \int_{\Omega} \text{div} (E : \varepsilon(u(t))) \cdot v
\quad := \int_{\Gamma_N} ((E : \varepsilon(u(t))) \cdot \nu) \cdot v.
$$

Obviously, this relation does not depend on $v \in H^1_s(\Omega)$ but only on its trace $v|_{\Gamma_N} \in H^{1/2}_{00}(\Gamma_N)^d$, resulting in:

$$
\int_{\Gamma_N} g(t) \cdot \phi = \int_{\Gamma_N} ((E : \varepsilon(u(t))) \cdot \nu) \cdot \phi, \quad \forall \phi \in H^{1/2}_{00}(\Gamma_N)^d. \quad (4.59)
$$
Now, we can show the relation (4.56). By exploiting the divergence formula, and the results (4.57) and (4.59), we get for all \( t \in [0, T] \), and all \( v_h \in V_h \):

\[
\tilde{a}(u, v_h) = \sum_{k=1}^{K} \int_{\Omega_k} (E : \varepsilon(u(t))) : \varepsilon(v_h)
\]

\[
= - \sum_{k=1}^{K} \int_{\Omega_k} \text{div}(E : \varepsilon(u(t))) \cdot v_h + \sum_{k=1}^{K} \int_{\partial \Omega_k} ((E : \varepsilon(u(t))) \cdot \nu) \cdot v_h
\]

\[
= \int_{\Omega} \left( f(t) - \frac{\partial^2 u}{\partial t^2}(t) \right) \cdot v_h + \int_{\Gamma_N} g(t) \cdot v_h + \int_S \lambda(t) \cdot [v_h],
\]

resulting in the announced expression (4.56).

By computing the half sum of the expressions (4.56) for \( t = t_n \) and \( t = t_{n+1} \) and subtracting the first line of the system (4.52), it comes that for all \( v_h \in V_h \):

\[
\int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h - \int_{\Omega} \rho \frac{\dot{u}_{n+1}^h - \dot{u}_n^h}{\Delta t} \cdot v_h + \tilde{a} \left( \frac{u(t_n) - u_n^h}{2} + \frac{u(t_{n+1}) - u_{n+1}^h}{2}, v_h \right)
\]

\[
= \int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h - \frac{1}{2} \int_{\Omega} \rho \left( \frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1}) \right) \cdot v_h + \int_S \frac{\lambda(t_n) + \lambda(t_{n+1})}{2} \cdot [v_h],
\]

where we have added the term \( \int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h \) on the both sides of the equality. From the lemma 4.12 and the definitions of \( e u_n^h \) and \( e V_n^h \), we deduce that for all \( v_h \in V_h \):

\[
\int_{\Omega} \rho \frac{e V_{n+1}^h - e V_n^h}{\Delta t} \cdot v_h + \tilde{a} \left( \frac{e u_n^h + e u_{n+1}^h}{2}, v_h \right)
\]

\[
= \int_{\Omega} \rho \frac{V_{n+1}^h - V_n^h}{\Delta t} \cdot v_h - \frac{1}{2} \int_{\Omega} \rho \left( \frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1}) \right) \cdot v_h + \int_S \frac{\lambda(t_n) + \lambda(t_{n+1})}{2} \cdot [v_h],
\]

that we sum up in the following expression :

\[
\int_{\Omega} \rho \frac{e V_{n+1}^h - e V_n^h}{\Delta t} \cdot v_h + \tilde{a} \left( \frac{e u_n^h + e u_{n+1}^h}{2}, v_h \right) = E_{n+1/2}^a(v_h) + E_{n+1/2}^c(v_h). \tag{4.60}
\]

We have denoted the approximation error in time and space by :

\[
E_{n+1/2}^a(v_h) = \int_{\Omega} \sqrt{\rho} T_{n+1/2} \cdot v_h, \quad \forall v_h \in V_h,
\]

with :

\[
T_{n+1/2} = \sqrt{\rho} \frac{V_{n+1}^h - V_n^h}{\Delta t} - \frac{1}{2} \sqrt{\rho} \left( \frac{\partial^2 u}{\partial t^2}(t_n) + \frac{\partial^2 u}{\partial t^2}(t_{n+1}) \right),
\]
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and the consistency error by:

\[ E^c_{n+1/2}(v_h) = \frac{1}{2} \int_S (\lambda(t_n) + \lambda(t_{n+1})) \cdot [v_h], \quad \forall v_h \in V_h. \]

It will be convenient to have estimations at midtime steps, and this is why we introduce the midtime quantities:

\[ eV^h_{n+1/2} = \frac{eV^h_n + eV^h_{n+1}}{2}, \quad eu^h_{n+1/2} = \frac{eu^h_n + eu^h_{n+1}}{2}, \]

whose evolution is given by averaging (4.60) between two consecutive time steps. We get for all \( v_h \in V_h \):

\[
\int_\Omega eV^h_{n+1/2} - eV^h_{n-1/2} \cdot v_h + \frac{1}{2} \left( eu^h_{n-1/2} + eu^h_{n+1/2} \right) = E^a_n(v_h) + E^c_n(v_h), \tag{4.61}
\]

where:

\[ E^\square_n(v_h) = \frac{1}{2} \left( E^\square_{n-1/2}(v_h) + E^\square_{n+1/2}(v_h) \right), \quad \forall v_h \in V_h, \]

in which \( \square \) stands for “a” or “c”. In (4.61), we choose:

\[ v_h = \frac{eu^h_{n+1/2} - eu^h_{n-1/2}}{\Delta t} = \frac{eV^h_{n-1/2} + eV^h_{n+1/2}}{2}, \]

by construction of \( (V^h_n)_{n \geq 0} \), which gives by summation on all time steps between 1 and \( n \) the main estimation of this first step of the proof:

\[
\eta^h_{n+1/2} - \eta^h_{1/2} = \Delta t \sum_{i=1}^{n} E^a_i \left( \frac{eV^h_{i-1/2} + eV^h_{i+1/2}}{2} \right) + E^c_i \left( \frac{eu^h_{i+1/2} - eu^h_{i-1/2}}{\Delta t} \right), \tag{4.62}
\]

with:

\[ \eta^h_{n+1/2} = \frac{1}{2} \int_\Omega \rho eV^h_{n+1/2} \cdot eV^h_{n+1/2} + \frac{1}{2} \bar{a}(eu^h_{n+1/2}, eu^h_{n+1/2}). \]

2. An upper bound for \( \eta^h_{1/2} \).

We establish here an upper bound for \( \eta^h_{1/2} \). By definition of \( \eta^h_{1/2} \), we get by using the symmetry of \( \bar{a} \):

\[
\eta^h_{1/2} = \frac{1}{2} \int_\Omega \rho \left( \frac{eV^h_0 + eV^h_1}{2} \right)^2 + \frac{1}{2} \bar{a} \left( \frac{eu^h_0 + eu^h_1}{2}, \frac{eu^h_0 + eu^h_1}{2} \right) \]

\[
\leq \frac{1}{4} \int_\Omega \rho (eV^h_0)^2 + \frac{1}{4} \int_\Omega \rho (eV^h_1)^2 + \frac{1}{4} \bar{a}(eu^h_0, eu^h_0) + \frac{1}{4} \bar{a}(eu^h_1, eu^h_1). \]

Using (4.60) with \( n = 0 \) and:

\[ v_h = \frac{eu^h_1 - eu^h_0}{\Delta t} = \frac{eV^h_0 + eV^h_1}{2} \]
by construction, we obtain:

\[
\frac{1}{2} \int_{\Omega} \rho (e V^h_V)^2 + \frac{1}{2} \bar{a}(e u^h_1, e u^h_1) = \frac{1}{2} \int_{\Omega} \rho (e V^h_V)^2 + \frac{1}{2} \bar{a}(e u^h_1, e u^h_1) + \frac{\Delta t}{2} E_{1/2}^\alpha (e V^h_V) + E_{1/2}^c (e u^h_1 - e u^h_0).
\]

The approximation term in the right hand side can be bounded by using the Cauchy-Schwarz inequality:

\[
\frac{\Delta t}{2} E_{1/2}^\alpha (e V^h_V) \leq \left\| \Delta t T_{1/2} \right\|_{L^2(\Omega)^d} \left\| \frac{1}{2} \sqrt{\rho} e V^h_V \right\|_{L^2(\Omega)^d} \leq \frac{1}{2} \left\| \Delta t T_{1/2} \right\|_{L^2(\Omega)^d}^2 + \frac{1}{8} \int_{\Omega} \rho (e V^h_V)^2.
\]

Moreover:

\[
\Delta t T_{1/2} = \sqrt{\rho} \left( V^h_V - V^h_0 - \frac{\Delta t}{2} (\bar{u}(t_0) + \bar{v}(t_1)) \right) = \sqrt{\rho} \left( \frac{2}{\Delta t} (\mathbb{P}_h u(t_1) - \mathbb{P}_h u(t_0)) - 2 \mathbb{P}_h \bar{u}(t_0) - \frac{\Delta t}{2} (\bar{u}(t_0) + \bar{v}(t_1)) \right) = \sqrt{\rho} \left( \frac{2}{\Delta t} (u(t_1) - u(t_0)) - 2 \bar{u}(t_0) - \frac{\Delta t}{2} (\bar{u}(t_0) + \bar{v}(t_1)) \right) + \sqrt{\rho} (\mathbb{P}_h - id) \left( \frac{2}{\Delta t} (u(t_1) - u(t_0)) - 2 \bar{u}(t_0) \right).
\]

(4.63)

We then use the lemma 4.12 and a Taylor’s expansion with integral remainder to bound the second term in (4.63) as follows:

\[
\left\| \sqrt{\rho} (\mathbb{P}_h - id) \left( \frac{2}{\Delta t} (u(t_1) - u(t_0)) - 2 \bar{u}(t_0) \right) \right\|_{L^2(\Omega)^d}^2 \leq C h^2 \sum_{k=1}^{K} \sum_{r=1}^{h^2} \left| \sup_{t \in [0, \Delta t]} \sqrt{\rho} t_0 \ddot{u}(t) \right|_{r+1, E, \Omega_k}^2.
\]

The first term in (4.63) is also bounded by the use of a Taylor’s expansion with integral remainder, resulting in:

\[
\Delta^2 \left\| T_{1/2} \right\|_{L^2(\Omega)^d}^2 \leq C \left( \frac{\Delta t}{t_0} \right)^4 \sup_{t \in [0, \Delta t]} \left\| \sqrt{\rho} t_0 \ddot{u}(t) \right\|_{L^2(\Omega)^d}^2.
\]
For the consistency term, we use that $e u_1^h - e u_0^h \in V_h$, the Cauchy-Schwarz inequality, the inequality (4.46) page 145, and the error estimate (4.44) page 144:

$$
\| [v_h] \|_{\delta^{-1/2}, m} \leq C \| [v_h] \|_{H^{1/2}((\Gamma_m))}, \quad \forall v_h \in V_h,
$$

(4.64)

$$
\inf_{\mu_h \in M_\delta} \| \lambda - \mu_h \|_{\delta^{-1/2}} \leq C \sum_{k=1}^K \frac{h_1^{2q}}{C_k} \| \mathbf{E} : \varepsilon(u) \|_{H^q(\Omega_\delta)^{d \times d}},
$$

(4.65)

to obtain that:

$$
E_{1/2}^c (e u_1^h - e u_0^h) \leq \max_{i=0,1} \int_S (\lambda(t_i) - [e u_1^h - e u_0^h], \quad \forall \mu_h \in M_\delta
$$

$$
\leq \theta \max_{i=0,1} \inf_{\mu_h \in M_\delta} \| \lambda(t_i) - \mu_h \|_{\delta^{-1/2}} + \frac{1}{\theta^2} \| e u_1^h - e u_0^h \|_X, \quad \forall \theta \in ]0, +\infty[,
$$

$$
\leq C \theta^2 \max_{i=0,1} \inf_{\mu_h \in M_\delta} \| \lambda(t_i) - \mu_h \|_{\delta^{-1/2}}^2 + \frac{1}{\theta^2} \| e u_1^h - e u_0^h \|_X^2
$$

$$
\leq C \theta^2 \sum_{k=1}^K \frac{h_1^{2q} C_k^2}{\alpha} \sup_{t \in [0, \Delta t]} \| u(t) \|_{q+1, E, \Omega_k} + \frac{1}{\theta^2} \| e u_1^h \|_X^2 + \frac{1}{\theta^2} \| e u_0^h \|_X^2.
$$

As $\Gamma_D$ has not a null measure, the bilinear form $\tilde{a}$ is coercive over $V_h \times V_h$. Then, we choose $\theta^2 = 8/\tilde{\alpha}$ where $\tilde{\alpha}$ is the coercivity constant of $\tilde{a}$ over $V_h \times V_h$, and obtain the final estimation:

$$
\frac{3}{8} \int_{\Omega} \rho(e V_1^h)^2 + \frac{3}{8} \tilde{a}(e u_1^h, e u_1^h) \leq \frac{1}{2} \int_{\Omega} \rho(e V_0^h)^2 + \frac{5}{8} \tilde{a}(e u_0^h, e u_0^h)
$$

$$
+ C \left( \frac{\Delta t}{t_0} \right)^2 \sup_{t \in [0, \Delta t]} \| \sqrt{\rho} t_0^2 \ddot{u}(t) \|_{L^2(\Omega)^d}
$$

$$
+ C \left( \frac{\Delta t}{t_0} \right)^2 h_1^2 \sum_{k=1}^K \frac{h_1^{2q}}{C_k} \sup_{t \in [0, \Delta t]} \| \sqrt{\rho} t_0^2 \ddot{u}(t) \|_{r+1, E, \Omega_k}^2
$$

$$
+ C \sum_{k=1}^K \frac{h_1^{2q} C_k^2}{\tilde{\alpha}} \sup_{t \in [0, \Delta t]} \| u(t) \|_{q+1, E, \Omega_k}^2.
$$
We estimate here the space and time approximation error given by:

\[ n_{1/2}^h \leq C \left( \| \sqrt{\rho} e V_0^h \|_{L^2(\Omega)^d}^2 + \tilde{a}(\epsilon u_0^h, \epsilon u_0^h) + \left( \frac{\Delta t}{t_0} \right)^4 \sup_{t \in [0, \Delta t]} \| \sqrt{\rho} t_0^2 \ddot{u}(t) \|_{L^2(\Omega)^d}^2 \right) \]

\[ + C \left( \frac{\Delta t}{t_0} \right)^2 h^2 \sum_{k=1}^K h_k^{2q} \sup_{t \in [0, \Delta t]} \| \sqrt{\rho} t_0 \ddot{u}(t) \|_{r+1, \mathbf{E}, \Omega_k}^2 \]

\[ + C \sum_{k=1}^K h_k^{2q} C_k^2 \sup_{t \in [0, \Delta t]} \| u(t) \|_{q+1, \mathbf{E}, \Omega_k}^2. \]

3. **Time and space approximation error estimate.**

We estimate here the space and time approximation error given by:

\[ A = \Delta t \sum_{i=1}^n E_i^a \left( \frac{e V_{i-1/2}^h + e V_{i+1/2}^h}{2} \right). \]

By applying the Cauchy-Schwarz inequality, we obtain:

\[ A \leq \frac{\Delta t}{t_0} \sum_{i=1}^n \left\{ t_0 \frac{T_{i-1/2} + T_{i+1/2}}{2} \right\} \left\{ \sqrt{\rho} \frac{e V_{i-1/2}^h + e V_{i+1/2}^h}{2} \right\} \]

\[ \leq \frac{\Delta t}{2t_0} \sum_{i=1}^n \left\{ t_0 \frac{T_{i-1/2} + T_{i+1/2}}{2} \right\} \left\{ \sqrt{\rho} \frac{e V_{i-1/2}^h + e V_{i+1/2}^h}{2} \right\}^2 \]

\[ \leq \frac{\Delta t}{2t_0} \sum_{i=1}^n \left\{ t_0 \frac{T_{i-1/2} + T_{i+1/2}}{2} \right\} \left\{ \sqrt{\rho} e V_{i+1/2}^h \right\}^2 \]

Let us remark that:

\[ T_{i+1/2} + T_{i-1/2} = \sqrt{\rho} \frac{V_{i+1}^h - V_{i-1}^h}{\Delta t} - \sqrt{\rho} \frac{\ddot{u}(t_{i-1}) + 2\ddot{u}(t_i) + \ddot{u}(t_{i+1})}{2} \]

\[ = \sqrt{\rho} \frac{V_{i+1}^h + V_{i}^h - V_{i-1}^h}{\Delta t} - \sqrt{\rho} \frac{\ddot{u}(t_{i-1}) + 2\ddot{u}(t_i) + \ddot{u}(t_{i+1})}{2} \]

\[ = 2\sqrt{\rho} \frac{u(t_{i+1}) - 2u(t_i) + u(t_{i-1})}{\Delta t^2} - \sqrt{\rho} \frac{\ddot{u}(t_{i-1}) + 2\ddot{u}(t_i) + \ddot{u}(t_{i+1})}{2} \]

\[ + 2\sqrt{\rho} (\mathcal{P}_h - id) \left( \frac{u(t_{i+1}) - 2u(t_i) + u(t_{i-1})}{\Delta t^2} \right). \]

Proceeding, as in the estimation of the approximation error of the second step of the proof,
we use the lemma 4.12 and Taylor’s expansions with integral remainder to obtain:

\[ \| T_{i+1/2} + T_{i-1/2} \|_{L^2(\Omega)^d} \]

\[ \leq C \left( \Delta t^4 \sup_{t \in [0,T]} \| \sqrt{\rho} \ddot{u}(t) \|_{L^2(\Omega)^d} \right. \]

\[ + h^2 \sum_{k=1}^{K} h_{k}^{2r} \sqrt{\rho} \frac{u(t_{i+1}) - 2u(t_{i}) + u(t_{i-1})}{\Delta t^2} \left|_{r+1,E,\Omega_k} \right. \]

\[ \leq \frac{C}{t_0^2} \left( \left( \frac{\Delta t}{t_0} \right)^4 \sup_{t \in [0,T]} \| \sqrt{\rho} t_0^3 \dddot{u}(t) \|_{L^2(\Omega)^d} + h^2 \sum_{k=1}^{K} h_{k}^{2r} \sup_{t \in [0,T]} \| \sqrt{\rho} t_0 \dddot{u}(t) \|_{r+1,E,\Omega_k} \right). \]

4. Consistency error

We estimate here the consistency error given by:

\[ B = \Delta t \sum_{i=1}^{n} E_i^h \left( \frac{e_{u,h}^{i+1/2} - e_{u,h}^{i-1/2}}{\Delta t} \right). \]

Using a reorganization of the terms (equivalent to a discrete integration by parts in time), we obtain:

\[ B = \Delta t \sum_{i=1}^{n} \int_{S} \left( \frac{\lambda(t_{i-1}) + 2\lambda(t_{i}) + \lambda(t_{i+1})}{4} \right) \cdot \left[ \frac{e_{u,h}^{i+1/2} - e_{u,h}^{i-1/2}}{\Delta t} \right] \]

\[ = \Delta t \sum_{i=1}^{n} \int_{S} \left( \frac{\lambda(t_{i-1}) + \lambda(t_{i}) - \lambda(t_{i+1}) - \lambda(t_{i+2})}{4\Delta t} \right) \cdot \left[ e_{u,h}^{i+1/2} \right] \]

\[ + \int_{S} \left( \frac{\lambda(t_{n-1}) + 2\lambda(t_{n}) + \lambda(t_{n+1})}{4} \right) \cdot \left[ e_{u,h}^{n+1/2} \right] \]

\[ - \int_{S} \left( \frac{\lambda(t_{0}) + 2\lambda(t_{1}) + \lambda(t_{2})}{4} \right) \cdot \left[ e_{u,h}^{1/2} \right] \]

\[ = \Delta t \: D + E - F. \]

Concerning the \( \Delta tD \) term, we proceed exactly as in the estimation of the consistency error of the second step of the proof. More precisely, we use that \( \left[ e_{u,h}^{i+1/2} \right] \in V_h \), the
Cauchy-Schwarz inequality and the inequality (4.46), the estimation (4.43) and a Taylor's expansion to get :

\[
\Delta t \, D = \frac{\Delta t}{t_0} \sum_{i=1}^{n-1} \int_S \left( t_0 \frac{\lambda(t_{i-1}) + \lambda(t_i) - \lambda(t_{i+1}) - \lambda(t_{i+2})}{4\Delta t} - \mu_h \right) \cdot \left[ e u_h^{i+1/2} \right], \quad \forall \mu_h \in M_\delta,
\]

\[
\leq \frac{\Delta t}{2t_0} \theta^2 \sum_{i=1}^{n-1} \left| t_0 \lambda(t_{i-1}) + \lambda(t_i) - \lambda(t_{i+1}) - \lambda(t_{i+2}) \right| \cdot \left[ e u_h^{i+1/2} \right] \quad \forall \theta \in [0, +\infty[ \cap M_\delta,
\]

\[
+ \frac{\Delta t}{2\theta^2 t_0} \sum_{i=1}^{n-1} \| e u_h^{i+1/2} \|_X^2, \quad \forall \theta \in [0, +\infty[.
\]

\[
\lesssim \frac{\Delta t}{t_0} \theta^2 \sum_{i=1}^{n-1} \sum_{k=1}^{K} h_k^{2q} \sup_{t \in [0,T]} \left| t_0 \dot{\lambda}(t) \right| \| \lambda(t) \|_{H^{q+1/2}} + \frac{\Delta t}{2\theta^2 t_0} \sum_{i=1}^{n-1} \| e u_h^{i+1/2} \|_X^2, \quad \forall \theta \in [0, +\infty[.
\]

\[
\lesssim \frac{T}{t_0} \theta^2 \sum_{k=1}^{K} h_k^{2q} C_k^2 \sup_{t \in [0,T]} \left| t_0 \dot{\lambda}(t) \right| \| \lambda(t) \|_{H^{q+1/2}} + \frac{\Delta t}{2\theta^2 t_0} \sum_{i=1}^{n-1} \| e u_h^{i+1/2} \|_X^2, \quad \forall \theta \in [0, +\infty[.
\]

and by choosing \( \theta^2 = 1/\bar{\alpha} \), we obtain :

\[
\Delta t \, D \leq \frac{T}{t_0} \sum_{k=1}^{K} h_k^{2q} C_k^2 \sup_{t \in [0,T]} \left| t_0 \dot{\lambda}(t) \right| \| \lambda(t) \|_{H^{q+1/2}} + \frac{\Delta t}{2t_0} \sum_{i=1}^{n-1} \bar{a}(e u_h^{i+1/2}, e u_h^{i+1/2}).
\]

The terms \( E \) and \( F \) are easily bounded by using the same technique :

\[
E \leq C \sum_{k=1}^{K} h_k^{2q} \alpha \sup_{t \in [0,T]} \left| u(t) \right|_{q+1}^2 E_{\Omega_k} + \frac{1}{4} \bar{a}(e u_h^{n+1/2}, e u_h^{n+1/2}),
\]

\[
F \leq C \left( \sum_{k=1}^{K} h_k^{2q} C_k^2 \sup_{t \in [0,T]} \left| u(t) \right|_{q+1}^2 E_{\Omega_k} + \bar{a}(e u_h^{1/2}, e u_h^{1/2}) \right).
\]

Moreover, the second term \( \bar{a}(e u_h^{1/2}, e u_h^{1/2}) \) in the upper bound of \( F \) can be bounded optimally by the second point of the present proof.

5. Estimate on \( \eta_h^{i+1/2} \).
4.6. Generalization to elastodynamics.

Putting together the estimations from the previous points, we obtain that:

$$\frac{1}{2} \left( 1 - \frac{\Delta t}{t_0} \right) \int_\Omega \rho(eV_n^{h+1/2})^2 + \frac{1}{4} \tilde{a}(eu_n^{h+1/2}, ev_n^{h+1/2}) \leq C \left( \int_\Omega (\rho eV_0^{h+1})^2 + \tilde{a}(eu_0^h, ev_0^h) \right)$$

$$+ C \left( \frac{\Delta t}{t_0} \right) \left\{ \sup_{t \in [0,T]} \| \sqrt{\rho} t_0^2 \cdot \dddot{u}(t) \|^2_{L^2(\Omega)^d} + \frac{T}{t_0} \sup_{t \in [0,T]} \| \sqrt{\rho} t_0^3 \cdot \dddot{u}(t) \|^2_{L^2(\Omega)^d} \right\}$$

$$+ C h^2 \left( \frac{T}{t_0} + \left( \frac{\Delta t}{t_0} \right)^2 \right) \sum_{k=1}^{K} h^{2r}_k \sup_{t \in [0,T]} \| \sqrt{\rho} t_0 \dddot{u}(t) \|^2_{L^2(\Omega)^d}$$

$$+ C \sum_{k=1}^{K} h^{2q}_k \frac{C_k^2}{\alpha} \left\{ \sup_{t \in [0,T]} |u(t)|^2_{q+1, E, \Omega} + \frac{T}{t_0} \sup_{t \in [0,T]} |t_0 \dddot{u}(t)|^2_{q+1, E, \Omega} \right\}$$

$$+ \frac{\Delta t}{2t_0} \sum_{i=0}^{n-1} \| \sqrt{\rho} eV_i^{h+1/2} \|^2_{L^2(\Omega)^d} + \frac{\Delta t}{2t_0} \sum_{i=1}^{n-1} \tilde{a}(eu_{i+1/2}^h, ev_{i+1/2}^h).$$

We deduce by applying the discrete Gronwall’s lemma 4.13, and for sufficiently small time steps ($\Delta t \leq t_0/2$) that:

$$\int_\Omega \rho(eV_n^{h+1})^2 + \tilde{a}(eu_n^{h+1/2}, ev_n^{h+1/2}) \leq C \left( \| \sqrt{\rho} eV_0^h \|^2_{L^2(\Omega)^d} + \tilde{a}(eu_0^h, ev_0^h) \right)$$

$$+ \left[ C \left( \frac{\Delta t}{t_0} \right) \right] \left\{ \sup_{t \in [0,T]} \| \sqrt{\rho} t_0^2 \cdot \dddot{u}(t) \|^2_{L^2(\Omega)^d} + \frac{T}{t_0} \sup_{t \in [0,T]} \| \sqrt{\rho} t_0^3 \cdot \dddot{u}(t) \|^2_{L^2(\Omega)^d} \right\}$$

$$+ C h^2 \frac{T}{t_0} \sum_{k=1}^{K} h^{2r}_k \sup_{t \in [0,T]} \| \sqrt{\rho} t_0 \dddot{u}(t) \|^2_{L^2(\Omega)^d}$$

$$+ C \sum_{k=1}^{K} h^{2q}_k \frac{C_k^2}{\alpha} \left\{ \sup_{t \in [0,T]} |u(t)|^2_{q+1, E, \Omega} + \frac{T}{t_0} \sup_{t \in [0,T]} |t_0 \dddot{u}(t)|^2_{q+1, E, \Omega} \right\} \left( 1 + \frac{\Delta t}{t_0} \right)^n.$$ 

6. Conclusion.

We end this proof by establishing the announced error estimates on velocities and displacements. Concerning the estimate on velocities, let us remark that:

$$\dot{u}(t_n+1/2) - \frac{\dot{u}_n^h + \dot{u}_{n+1}^h}{2} = \dot{u}(t_n+1/2) - \frac{V_n^h + V_{n+1}^h}{2} + eV_n^{h+1/2}.$$ 

We have by definition:

$$\dot{u}(t_n+1/2) - \frac{V_n^h + V_{n+1}^h}{2} = \dot{u}(t_n+1/2) - \frac{\mathcal{P}_h u(t_{n+1}) - \mathcal{P}_h u(t_n)}{\Delta t},$$

$$= \dot{u}(t_n+1/2) - \frac{u(t_{n+1}) - u(t_n)}{\Delta t} + (id - \mathcal{P}_h) \left( \frac{u(t_{n+1}) - u(t_n)}{\Delta t} \right),$$
which entails that:

\[
\left\| \sqrt{\rho} \left( \dot{u}(t_{n+1/2}) - \frac{V_{n}^{h} + V_{n+1}^{h}}{2} \right) \right\|_{L^2(\Omega)^d}^2 \leq C \left( \left( \frac{\Delta t}{t_0} \right)^4 \sup_{t \in [0,T]} \left\| \sqrt{\rho} \dot{u}^2(t) \right\|_{L^2(\Omega)^d}^2 + h^2 \sum_{k=1}^{K} h_k^{2r} \sup_{t \in [0,T]} \left| \sqrt{\rho} \ddot{u}(t) \right|_{\Gamma_{r+1,E,\Omega_k}}^2 \right).
\]

Therefore, we deduce the final estimate on velocities by the triangular inequality:

\[
\left\| \sqrt{\rho} \left( \dot{u}(t_{n+1/2}) - \frac{u_n^{h} + u_{n+1}^{h}}{2} \right) \right\|_{L^2(\Omega)^d}^2 \leq C \left( \left( \frac{\Delta t}{t_0} \right)^4 \sup_{t \in [0,T]} \left\| \sqrt{\rho} \dot{u}^2(t) \right\|_{L^2(\Omega)^d}^2 + T \sup_{t \in [0,T]} \left\| \sqrt{\rho} \ddot{u}(t) \right\|_{L^2(\Omega)^d}^2 \right) + C h^2 T \sum_{k=1}^{K} h_k^{2r} \left( \sup_{t \in [0,T]} \left| \sqrt{\rho} \dot{u}(t) \right|_{\Gamma_{r+1,E,\Omega_k}} + \sup_{t \in [0,T]} \left| \sqrt{\rho} \ddot{u}(t) \right|_{\Gamma_{r+1,E,\Omega_k}} \right)
\]

\[
+ C \sum_{k=1}^{K} h_k^{2r} \left( \sup_{t \in [0,T]} \left| u(t) \right|_{q+1,E,\Omega} + T \sup_{t \in [0,T]} \left| \dot{u}(t) \right|_{q+1,E,\Omega} \right) \left( 1 + \frac{\Delta t}{t_0} \right) ^n.
\]

We end by the estimate on displacements. We remark that:

\[
u(t_{n+1/2}) - \frac{u_n^{h} + u_{n+1}^{h}}{2} = u(t_{n+1/2}) - \frac{\mathbb{P}_h u(t_n) + \mathbb{P}_h u(t_{n+1})}{2} + evh_{n+1/2}.
\]

Moreover, we notice that:

\[
u(t_{n+1/2}) - \frac{\mathbb{P}_h u(t_n) + \mathbb{P}_h u(t_{n+1})}{2} = u(t_{n+1/2}) - \frac{u(t_n) + u(t_{n+1})}{2}
\]

\[+ (id - \mathbb{P}_h) \left( \frac{u(t_n) + u(t_{n+1})}{2} \right),
\]

resulting in:

\[
\left\| u(t_{n+1/2}) - \frac{\mathbb{P}_h u(t_n) + \mathbb{P}_h u(t_{n+1})}{2} \right\|_{X}^2 \leq C \left( \left( \frac{\Delta t}{t_0} \right)^4 \sup_{t \in [0,T]} \left| t_0 \dot{u}(t) \right|_{X}^2 + \sum_{k=1}^{K} h_k^{2q} \sup_{t \in [0,T]} \left| u(t) \right|_{q+1,E,\Omega_k}^2 \right).
\]
and we conclude by the triangular inequality that:

\[
\tilde{\alpha} \left\| u(t_{n+1/2}) - \frac{u^h_n + u^h_{n+1}}{2} \right\|_X^2 
\leq C \left( \| \sqrt{\rho} e V^h_0 \|_{L^2(\Omega)^d}^2 + \tilde{\alpha} (e u^h_0, e u^h_0) \right) 
\]

\[
+ C \left\{ \left( \frac{\Delta t}{t_0} \right)^4 \tilde{\alpha} \sup_{t \in [0, T]} \| t_0^2 \tilde{u}(t) \|_X^2 + \sup_{t \in [0, T]} \| \sqrt{\rho} t_0^2 \ddot{u}(t) \|_{L^2(\Omega)^d}^2 + \frac{T}{t_0} \sup_{t \in [0, T]} \| \sqrt{\rho} t_0^3 \dddot{u}(t) \|_{L^2(\Omega)^d}^2 \right\}
\]

\[
+ h^2 \frac{T}{t_0} \sum_{k=1}^K \frac{h^2}{t_k^2} \sup_{t \in [0, T]} \| \sqrt{\rho} t_0 \ddot{u}(t) \|_{r+1, \Omega_0}^2 + \tilde{\alpha} \sum_{k=1}^K \frac{h^2}{t_k^2} \sup_{t \in [0, T]} |u(t)|_{q+1, \Omega_0}^2 + \frac{T}{t_0} \sup_{t \in [0, T]} |t_0 \ddot{u}(t)|_{q+1, \Omega_0}^2
\]

\[
+ \sum_{k=1}^K \frac{h^2}{t_k^2} C_k^2 \left\{ \sup_{t \in [0, T]} |u(t)|_{q+1, \Omega_0}^2 + \frac{T}{t_0} \sup_{t \in [0, T]} |t_0 \ddot{u}(t)|_{q+1, \Omega_0}^2 \right\} \right) \left( 1 + \frac{\Delta t}{t_0} \right)^n.
\]

The proof is complete. \( \square \)

In the previous proof, we have used the following discrete Gronwall’s lemma:

**Lemma 4.13 (Gronwall).** Let \((w_n)_{n \in \mathbb{N}}, a real valued sequence such that:

\[
w_n \leq a + k \sum_{i=0}^{n-1} w_i, \quad \forall n \geq 0,
\]

with \( a > 0 \), and \( k > 0 \). Then, for all \( n \in \mathbb{N} \):

\[
w_n \leq a(1 + k)^n.
\]

**Proof:** Denoting by \( y_n = \sum_{i=0}^{n-1} w_i \), we obtain for all \( n \geq 1 \):

\[
y_n \leq (1 + k) y_{n-1} + a
\]

and by induction:

\[
y_n \leq (1 + k)^n w_0 + a \sum_{i=0}^{n-1} (1 + k)^i \leq a \sum_{i=0}^{n} (1 + k)^i.
\]

As a consequence, for all \( n \geq 1 \):

\[
w_n \leq a + ky_{n-1} \leq a(1 + k)^n,
\]

which ends the proof. \( \square \)
Remark 4.16. The proof of the convergence has been done in the case where the measure of $\Gamma_D$ was positive. Let us mention the necessary modifications of the proof when it is not the case. The displacements have to be decomposed in the space of rigid motions:

$$
\mathcal{R} = \{ v \in H^1(\Omega)^d, \quad a(v, w) = 0, \forall w \in H^1(\Omega) \},
$$

and in the complementary:

$$
\mathcal{V} = \{ v \in H^1(\Omega)^d, \quad \int_\Omega v \cdot r = 0, \forall r \in \mathcal{R} \},
$$

such that $H^1(\Omega)^d = \mathcal{R} \oplus \mathcal{V}$. The solution $u$ of (4.49) can then be decomposed into $u = \overline{u} + u'$, with $\overline{u} \in C^0(0,T; \mathcal{R}) \cap C^1(0,T; \mathcal{R})$ such that in the sense of distributions over $]0,T[$:

$$
\frac{\partial^2}{\partial t^2} \int_\Omega \rho \overline{u}(t) \cdot \nabla = \int_\Omega f(t) \cdot \nabla + \int_{\Gamma_N} g(t) \cdot \nabla, \quad \forall \nabla \in \mathcal{R},
$$

and $u' \in C^0(0,T; \mathcal{V}) \cap C^1(0,T; \mathcal{W})$ such that in the sense of distributions over $]0,T[$:

$$
\frac{\partial^2}{\partial t^2} \int_\Omega \rho u'(t) \cdot \nabla + a(u', v') = \int_\Omega f(t) \cdot v' + \int_{\Gamma_N} g(t) \cdot v', \quad \forall v' \in \mathcal{V},
$$

with $\mathcal{W} = \{ v \in L^2(\Omega)^d, \quad \int_\Omega v \cdot r = 0, \forall r \in \mathcal{R} \}$. The fully discrete approximation of $u$ at time $t_n$ is $u_{n} = \pi_{n} + u_{n}^h$ in displacements and $\dot{u}^h = \ddot{u}_{n}^h + \dddot{u}_{n}^h$ in velocities. To find $(u_{n}^h; \dot{u}_{n}^h)_{n \geq 1}$, one has to replace $V_h$ by:

$$
V'_h = \{ v_h \in V_h; \quad \int_\Omega v_h \cdot r = 0, \forall r \in \mathcal{R} \}
$$

in (4.52). The previous proof gives an upper bound for:

$$
\left\| \frac{\partial u'}{\partial t}(t_{n+1/2}) - \frac{\dot{u}_{n+1}^h + \ddot{u}_{n+1}^h}{2} \right\|_{L^2(\Omega)^d}^2 + \left\| u'(t_{n+1/2}) - \frac{u_{n}^h + \dot{u}_{n}^h}{2} \right\|_{H^1(\Omega)^d}^2,
$$

because $\dot{a}$ is coercive over $V'_h \times V'_h$. To find $(\pi_{n}^h; \dot{\pi}_{n}^h)_{n \geq 1}$, one has to replace $V_h$ by $\mathcal{R}$ in (4.52). An upper bound on:

$$
\left\| \frac{\partial \pi}{\partial t}(t_{n+1/2}) - \frac{\dot{\pi}_{n}^h + \ddot{\pi}_{n+1}^h}{2} \right\|_{L^2(\Omega)^d}^2,
$$

is then obtained by the previous proof, which still applies. Indeed, it is noticeable that there is no consistency error because $\mathcal{R} \subset V_h$, and then, no need of coercivity. Putting together the estimates concerning the rigid motion part of the solution and the complementary part,
the announced estimate then remains the same when $\Gamma_D$ has a null measure.

In this case, the projection $P_h$ of the proposition 4.10 can be constructed as follows. For all $u \in H^1(\Omega)^d$, we can build the decomposition $u = \overline{\pi} + u'$, with $\overline{\pi} \in \mathcal{R}$ and $u' \in V$. The projection $P_h u$ of $u \in H^1(\Omega)^d$ is then defined by:

$$P_h u = \overline{\pi} + u'_h,$$

where $u'_h \in V_h$ is such that:

$$\tilde{a}(u'_h, v'_h) = \tilde{a}(u', v'_h), \quad \forall v' \in V'_h. \tag{4.66}$$

### 4.7 Analysis of discontinuous mortar spaces

In this section, the fundamental assumption 4.2 on mortar spaces is checked for particular discrete spaces with discontinuous Lagrange multipliers, when a suitable stabilization on the interface is added. Let us mention that this assumption is equivalent to the following interface inf-sup conditions for $1 \leq m \leq M$:

$$\inf_{\mu_m \in M, \delta m} \sup_{\phi_m \in W_m} \frac{\int_{\Gamma_m} \mu_m \cdot \phi_m}{\| \mu_m \|_{\delta_m, \frac{1}{2} m} \| \phi_m \|_{\delta_m, \frac{1}{2} m}} \geq \beta'_m. \tag{4.67}$$

In subsection 3.1, we show that a $P_1/P_0$ approximation with interface bubble stabilization on $u_h$ is compatible for $u=\lambda$, i.e satisfies assumption 4.2. This idea has been introduced in [BM00] for the so-called three-field formulation. In subsection 3.3, we propose a numerical procedure to check the compatibility condition (4.67). In subsection 3.4, we show a useful lemma enabling to check only a local inf-sup condition on the interface in the way of [BN83, Ste84, Ste90] for divergence free problems. We use it in the subsection 3.5 to prove (4.67) for a stabilized $P_2$ or $Q_2/P_1$ – discontinuous formulation.

### 4.7.1 Stabilized first order elements

Here, we assume that $\lambda$ is approximated by piecewise constants ($q = 1$), and $u$ by continuous piecewise linear functions with bubbles on the interface $S$ (see figure 4.5). For each mesh element $F \in \mathcal{F}_h$ on the interface $S$, an interface bubble can be defined on $T(F)$ in the way followed by [BM00]. If $T(F)$ is a triangle or a tetrahedron whose vertices are denoted by $(a_i)_i$ with the associated barycentric coordinates $(\lambda_i)_i$, an interface bubble $b_F$ can be defined as:

$$b_F = \prod_{a_i \in S} \lambda_i.$$

When considering a square or cubic reference element $\tilde{Q} = [-1, 1]^d$, we can also define the face bubble associated with the face $\tilde{F} = [-1, 1]^{d-1} \times \{-1\}$.
Fig. 4.5 – Bubble function $\lambda_2 \lambda_3$ on the interface $S$, in a triangle $T$. (Bidimensional problems)

- for $d = 3$ by :
  $$b_F = \frac{1}{2}(1 - x_1^2)(1 - x_2^2)(1 - x_3), \quad \forall \hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in [-1,1]^3,$$

- for $d = 2$ by :
  $$b_F = \frac{1}{2}(1 - x_1^2)(1 - x_2), \quad \forall \hat{x} = (\hat{x}_1, \hat{x}_2) \in [-1,1]^2.$$

**Proposition 4.11.** With $q = 1$ and a bubble stabilization on the interface, the assumption 4.2 is always satisfied with a stability constant independent of the discretization, whatever the relative configuration of the meshes on the interface $S$.

**Proof:** Let $I_m$ be an approximation operator from $H^1(\Omega_m)$ to $W^{0}_{m;\delta_m}$, to be detailed later. For all $v \in H^1(\Omega_m)$, we define with constants $\gamma_F$ to be computed later :

$$\pi_m v = I_m v + \sum_{F \in \mathcal{F}_{m;\delta_m}} \gamma_F b_F$$

Because Lagrange multipliers are piecewise constant, we must have for all $F \in \mathcal{F}_{m;\delta_m}$ :

$$\int_F \pi_m v = \int_F v,$$

which imposes :

$$\gamma_F = \frac{\int_F (v - I_m v)}{\int_F b_F}.$$
By a classical change of variable on the reference element $\tilde{F}$:

$$\int_F b_F = \frac{\text{meas}(F)}{\text{meas}(\tilde{F})} \int_{\tilde{F}} \tilde{b} = C \text{meas}(F),$$

and then by Cauchy-Schwartz inequality:

$$|\gamma_F| \leq C \frac{\|v - I_m v\|_{L^2(F)}}{\text{meas}(F)^{1/2}}.$$ 

Thus, we obtain the following estimate:

$$\|\pi_k v\|_{\delta_{\frac{1}{2},m}}^2 = \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|\pi_k v\|_{L^2(F)}^2 \leq C \left( \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|I_m v\|_{L^2(F)}^2 + \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|v - I_m v\|_{L^2(F)}^2 \frac{\|b_F\|_{L^2(F)}}{\text{meas}(F)} \right)$$

$$\leq C \left( \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|I_m v\|_{L^2(F)}^2 + \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|v\|_{L^2(F)}^2 \right).$$

By choosing the approximation operator $I_m$ as the projection from $\mathbb{H}^{1/2}_{\delta}(\Gamma_m)$ to $W^0_{m,\delta_m}$ for the inner product:

$$\langle u, v \rangle_{\delta_{\frac{1}{2},m}} = \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \int_F u \cdot v,$$

which ensures that we have:

$$\sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|I_m v\|_{L^2(F)}^2 \leq \sum_{F \in \mathcal{F}_m} \frac{1}{h(F)} \|v\|_{L^2(F)}^2,$$

we conclude:

$$\|\pi_m v\|_{\delta_{\frac{1}{2},m}} \leq C \|v\|_{\delta_{\frac{1}{2},m}},$$

which ends the proof.

### 4.7.2 A counter example

We show here that the assumption 4.2 can be easily violated when a bubble stabilization is not introduced. For example, let us consider an interface $S$ whose non-mortar side is
represented on figure 4.6, and equipped with a uniform square mesh. The diameter of the squares is denoted by $\delta$.

We adopt the classical $\mathbb{Q}_1 \times \mathbb{P}_0$ discretization:

$$
\begin{align*}
M_\delta &= \{ p \in L^2(S)^d, \quad p|_F \in \mathbb{P}_0(F)^d, \forall F \in \mathcal{F}_\delta \}, \\
W^0_\delta &= \{ p \in H^1_0(S)^d \cap C^0(S)^d, \quad p|_F \in \mathbb{Q}_1(F)^d, \forall F \in \mathcal{F}_\delta \}.
\end{align*}
$$

If $\lambda_h^* \in M_\delta$ is taken as a checkaboard (as shown on figure 4.6), that is:

$$
\lambda_h^*|_F = \pm a, \quad a \in \mathbb{R}^d,
$$

depending of $F \in \mathcal{F}_\delta$ in the way indicated by figure 4.6, then we have by point symmetry of each shape function around each node:

$$
\int_S \phi_h \cdot \lambda_h^* = 0, \quad \forall \phi_h \in W^0_\delta.
$$

As a consequence, the inf-sup condition (4.67) cannot be satisfied.

**Remark 4.17.** The standard assumption 4.2 ensures the well-posedness of the approximate problem (4.7) whatever the relative configuration of the mortar and non-mortar meshes. In particular, it is always strictly stronger than the inf-sup condition (4.9), except in the conforming case, where it is equivalent. The instability shown on figure 4.6 entails that (4.7) is not well-posed for conforming meshes on the interface, but the problem (4.7) could be well-posed for strictly non-conforming interfaces. Indeed, in the inf-sup condition (4.9), the displacement over the interface enters through its jump whereas it only enters in the assumption 4.2 through its value on the non-mortar side. Obviously, the space of jumps over the interface can be considerably richer than the space of the displacements on the non-mortar side if the interface is really non-conforming. This enrichment coming from the non-conformity can make the inf-sup condition (4.9) satisfied, but in such cases, there will be no robustness with respect to the relative position of the interfaces.
4.7.3 Numerical validation

We propose here a numerical test to check if the inf-sup condition (4.67) is satisfied for a given discretization by mesh-refinement. This test is a simple variant of a test introduced by [BCI00]. For $1 \leq m \leq M$, let us denote by :

$$
\frac{1}{\delta_m} \int_{\Gamma_m} \lambda_m \cdot \phi_m
$$

the discrete inf-sup constant. Then, we have the following result :

**Proposition 4.12.** Under the assumption that the family of meshes $(\mathcal{F}_m, \delta_m)_{\delta_m > 0}$ on the interface $\Gamma_m$ is quasi-uniform, we have :

$$
\beta'_m; \delta_m = O \left( \frac{\lambda_{min} (B_m B_m^t)^{1/2}}{\delta_m} \right),
$$

where $B_m$ is the matrix associated to the bilinear form $b$ on $W^0_{m; \delta_m} \times M_{m; \delta_m}$ and $\lambda_{min}(M)$ the smallest eigenvalue of the matrix $M$. We remark that $\lambda_{min} (B_m B_m^t)^{1/2}$ is the smallest positive singular value of $B_m$.

**Proof :** We have $\beta'_m; \delta_m = \min_{\lambda_m \in \mathcal{M}_{m; \delta_m} \setminus \{0\}} A_{\lambda_m}$ with :

$$
A_{\lambda_m} = \max_{\phi_m \in W^0_{m; \delta_m} \setminus \{0\}} \frac{\int_{\Gamma_m} \lambda_m \cdot \phi_m}{\| \phi_m \|_{\delta^{-1}_m} \| \lambda_m \|_{\delta^{-1}_m}}.
$$

Using matrices and vectors representing data in the chosen discrete spaces in a given basis, we have :

$$
A_{\lambda_m} = \max_{\Phi_m} \frac{\langle B \Phi_m, \Lambda_m \rangle}{\langle M_\Phi \Phi_m, \Phi_m \rangle^{1/2} \langle M_\Lambda \Lambda_m, \Lambda_m \rangle^{1/2}}.
$$

In particular, the matrix $M_\Phi$ (resp. $M_\Lambda$) is the definite positive matrix representing $\| \cdot \|_{\delta^{-1}_m}$ (resp. $\| \cdot \|_{\delta^{-1}_m}$) in the discrete spaces. Let us remark that $B$, $M_\Phi$ and $M_\Lambda$ depend on $h$. The vector $\Phi_m$ reaches the maximum if :

$$
\langle B \Psi_m, \Lambda_m \rangle - s \langle M_\Phi \Phi_m, \Psi_m \rangle = 0, \quad \forall \Psi_m,
$$

with $\langle M_\Phi \Phi_m, \Phi_m \rangle = 1$. As a consequence :

$$
\Phi_m = \frac{M_\Phi^{-1} B^t \Lambda_m}{\langle B M_\Phi^{-1} B^t \Lambda_m, \Lambda_m \rangle^{1/2}},
$$

and :

$$
\frac{1}{\delta_m} \int_{\Gamma_m} \lambda_m \cdot \phi_m
$$
The last result is a consequence of the inequality:

$$\lambda_{\min} \langle M, A \rangle \leq \langle M \Lambda, A \rangle \leq \lambda_{\max} \langle M, A \rangle .$$

Hence we get:

$$\beta_{m; \delta_m} = \min_{\lambda_m \in M_{m; \delta_m} \setminus \{0\}} A_{\lambda_m} \leq C \frac{1}{\delta_m} \lambda_{\min} (B_{m} B_{m}^t)^{1/2},$$

using the result from lemma 4.14, because the interface mesh is quasi-uniform.

Conversely, proceeding as previously, we deduce that:

$$A_{\lambda_m} = \frac{\langle B M_{\phi}^{-1} B^t \Lambda_m, \Lambda_m \rangle^{1/2}}{\langle M \Lambda \Lambda_m, \Lambda_m \rangle^{1/2}} \leq \frac{1}{\lambda_{\min} (M_{\phi})^{1/2}} \frac{1}{\lambda_{\min} (M_{\Lambda})^{1/2}} \frac{\langle B B^t \Lambda_m, \Lambda_m \rangle^{1/2}}{\langle \Lambda_m, \Lambda_m \rangle^{1/2}}, \quad \forall \Lambda_m .$$

yielding:

$$\beta_{m; \delta_m} \geq C \frac{1}{\delta_m} \lambda_{\min} (B_{m} B_{m}^t)^{1/2},$$

using lemma 4.14 on a quasi-uniform mesh. Hence the proof.

In the previous proof, we have used the following lemma:

**Lemma 4.14.** We assume that $F_{m; \delta_m}$ is a family of uniform meshes. For all $\phi_m \in W_{m; \delta_m}$, the following inequalities hold:

$$C \delta_m^{d-2} \langle \Phi_m, \Phi_m \rangle \leq \langle M_{\phi} \Phi_m, \Phi_m \rangle \leq C \delta_m^{d-2} \langle \Phi_m, \Phi_m \rangle ,$$

where $\Phi_m$ is the vector of the nodal degrees of freedom of $\phi_m$ in $W_{m; \delta_m}$, and $\langle M_{\phi} \Phi_m, \Phi_m \rangle = \| \phi_m \|^2_{\delta_m^{1/2} \phi_m}$. Moreover, for all $\lambda_m \in M_{m; \delta_m}$, we have also:

$$C \delta_m^d \langle \Lambda_m, \Lambda_m \rangle \leq \langle M \Lambda \Lambda_m, \Lambda_m \rangle \leq C \delta_m^d \langle \Lambda_m, \Lambda_m \rangle ,$$
where $\Lambda_m$ is the vector of the degrees of freedom of $\lambda_m$ in $M_{m;\delta_m}$, and $\langle M_A \Lambda_m, \Lambda_m \rangle = \|\lambda_m\|_{\delta_m}^{-1/2}$.

**Proof:** The proof of this lemma can be found in [EG02] for the $L^2$ norm and the adaptation to the weighted $L^2$ norms is straightforward. For completeness, we recall the proof. It proceeds in three steps.

1. First, let us denote by $(\hat{\theta}_1, \ldots, \hat{\theta}_n)$ a basis of the finite-element scalar functions defined on the reference element $\hat{F}$, and define the following application:

$$
\psi : S^n \to \mathbb{R},
\eta \mapsto \left\| \sum_{i=1}^n \eta_i \hat{\theta}_i \right\|^2_{L^2(\hat{F})},
$$

over the unit sphere $S^n$ of $\mathbb{R}^n$. Because $S^n$ is compact, $\psi$ admits a minimum $\hat{c}$ and a maximum $\hat{C}$, which are non-negative, and in fact positive. Indeed, by contradiction, let us assume that $\hat{c} = 0$. It would imply the existence of a $\eta \in S^n$ such that $\sum_{i=1}^n \eta_i \hat{\theta}_i = 0$, and because the functions $(\hat{\theta}_i)_{1 \leq i \leq n}$ are independent, then $\eta = 0$, which is in contradiction with $\eta \in S^n$. As a consequence:

$$0 < \hat{c} \leq \hat{C}.$$

For each finite element function $\hat{v} = \sum_{i=1}^n V_i \hat{\theta}_i \neq 0$ on the reference element $\hat{F}$, we introduce the quantity $\eta = V/\|V\|_n$ with $\|V\|_n^2 = \sum_{i=1}^n V_i^2$. We then have:

$$\psi(\eta) = \frac{\|\hat{v}\|^2_{L^2(\hat{F})}}{\|V\|_n^2},$$

and we conclude from the bounds of $\psi$ that:

$$\hat{c} \frac{\|V\|_n^2}{\|V\|_n^2} \leq \frac{\|\hat{v}\|^2_{L^2(\hat{F})}}{\|V\|_n^2} \leq \hat{C} \frac{\|V\|_n^2}{\|V\|_n^2}.$$

2. Let $F$ be a deformed element of the surfacic mesh of $\Gamma_m$ and $T_F : \hat{F} \to F$ the affine application transforming the reference element $\hat{F}$ into $F$. For all $1 \leq i \leq n$ we define by $\theta_i = \hat{\theta}_i \circ T_F^{-1}$ the $i$-th basis function of the scalar finite-element functions over $F$. For each function $v = \sum_{i=1}^n V_i \theta_i$ on the element $F$, we define $\hat{v} = v \circ T_F$ and get classically:

$$\|v\|_{L^2(F)}^2 \leq C \frac{\text{meas}(F)}{\text{meas}(\hat{F})} \|\hat{v}\|_{L^2(\hat{F})}^2,$$

and:

$$\|v\|_{L^2(F)}^2 \geq C \frac{\text{meas}(F)}{\text{meas}(\hat{F})} \|\hat{v}\|_{L^2(\hat{F})}^2,$$

where $C$ denotes various constants independent of $F$. Using that $\text{meas}(F) \leq Ch(F)^{d-1} \leq C \delta_m^{d-1}$ and by quasi uniformity of the mesh that $\text{meas}(F) \geq C \delta_m^{d-1}$, we conclude from the point 1. that:

$$c \delta_m^{d-1} \frac{\|V\|_n^2}{\|V\|_n^2} \leq \|v\|_{L^2(F)}^2 \leq C \delta_m^{d-1} \frac{\|V\|_n^2}{\|V\|_n^2},$$
where the constants $c$ and $C$ do not depend on $F$. By quasi-uniformity of the mesh, we also have:
\[
c\delta_{m}^{d-2}\|V\|_{n}^{2} \leq \frac{1}{h(F)}\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d-2}\|V\|_{n}^{2},
\]
and:
\[
c\delta_{m}^{d}\|V\|_{n}^{2} \leq h(F)\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d}\|V\|_{n}^{2}.
\]

3. Let us generalize to the entire mesh. Let $(\varphi_{i})_{1 \leq i \leq N}$ be the basis of the scalar Lagrange finite-element functions over $\Gamma_{m}$. The suffix $i$ makes reference to the node of the mesh, and we introduce the number $\zeta_{i}$ of elements $F \in \mathcal{F}_{m;\delta_{m}}$ sharing the node $i$. Obviously, we have $\min_{1 \leq i \leq N} \zeta_{i} \geq 1$ as each node belongs at least to one element, and $\max_{1 \leq i \leq N}$ is bounded independently of $\delta_{m}$ by regularity of the mesh. Moreover, we denote by $\mathcal{T}_{F}$ the set of nodes of the element $F$. For all the scalar finite-element functions $v = \sum_{i=1}^{N} V_{i}\varphi_{i}$ over $\Gamma_{m}$, we deduce from the point 2. of the present proof that:
\[
c\delta_{m}^{d-2}\sum_{i \in \mathcal{T}_{F}} V_{i}^{2} \leq \frac{1}{h(F)}\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d-2}\sum_{i \in \mathcal{T}_{F}} V_{i}^{2},
\]
and:
\[
c\delta_{m}^{d}\sum_{i \in \mathcal{T}_{F}} V_{i}^{2} \leq h(F)\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d}\sum_{i \in \mathcal{T}_{F}} V_{i}^{2}.
\]

By summing these inequalities over $F \in \mathcal{F}_{m;\delta_{m}}$, we get:
\[
c\delta_{m}^{d-2}\sum_{F \in \mathcal{F}_{m;\delta_{m}}} \sum_{i \in \mathcal{T}_{F}} V_{i}^{2} \leq \sum_{F \in \mathcal{F}_{m;\delta_{m}}} \frac{1}{h(F)}\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d-2}\sum_{F \in \mathcal{F}_{m;\delta_{m}}} \sum_{i \in \mathcal{T}_{F}} V_{i}^{2},
\]
and:
\[
c\delta_{m}^{d}\sum_{F \in \mathcal{F}_{m;\delta_{m}}} \sum_{i \in \mathcal{T}_{F}} V_{i}^{2} \leq \sum_{F \in \mathcal{F}_{m;\delta_{m}}} h(F)\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d}\sum_{F \in \mathcal{F}_{m;\delta_{m}}} \sum_{i \in \mathcal{T}_{F}} V_{i}^{2}.
\]

We remark that:
\[
\sum_{F \in \mathcal{F}_{m;\delta_{m}}} \sum_{i \in \mathcal{T}_{F}} V_{i}^{2} = \sum_{i=1}^{N} \zeta_{i}V_{i}^{2},
\]
and because of the bounds on $\zeta_{i}$, we obtain:
\[
c\delta_{m}^{d-2}\sum_{i=1}^{N} V_{i}^{2} \leq \sum_{F \in \mathcal{F}_{m;\delta_{m}}} \frac{1}{h(F)}\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d-2}\sum_{i=1}^{N} V_{i}^{2},
\]
and:
\[
c\delta_{m}^{d}\sum_{i=1}^{N} V_{i}^{2} \leq \sum_{F \in \mathcal{F}_{m;\delta_{m}}} h(F)\|v\|_{L^2(F)}^{2} \leq C\delta_{m}^{d}\sum_{i=1}^{N} V_{i}^{2}.
\]
The proof has been done for scalar functions $v$, but the same estimates hold for vector functions by summing the previous inequalities over their components 1, ..., $d$.

As an illustration, we check numerically the satisfaction of the inf-sup condition (4.67) with piecewise constant $\lambda_h \in M_\delta$ and piecewise linear $\phi_h \in W_\delta^0$ with bubble stabilization. It is done on the same square interface $S$ used in the previous subsection (counter example). On figure 4.7, we present the quantity $\frac{1}{\delta^2} \lambda_{\min}(BB^t)^{1/2}$ as a function of $\delta$. In particular, it remains greater than a positive constant as $\delta$ goes to 0, proving (4.67).

![Figure 4.7](image)

**Fig. 4.7** – Numerical computation of $\frac{1}{\delta^2} \lambda_{\min}(BB^t)^{1/2}$ as a function of $\delta$ when $\delta \to 0$.

### 4.7.4 A useful lemma

It can be useful to check only a local inf-sup condition on macro-elements, in the way of Boland-Nicolaides [BN83] or Stenberg [Ste84, Ste90] for divergence free problems. We assume that the interface $S$ is equipped with a family of macro-meshes $(N_\delta)_{\delta > 0}$ constituted of macro-elements. Each macro-element $\omega \in N_\delta$ is a subset $\omega \subset F_\delta$ of adjacent elements. We assume that each element $F \in F_\delta$ belong to at least one and less than $L$ macro-elements in $N_\delta$, independently of $\delta$.

Moreover, each $\omega = \cup_i F_i \in N_\delta$ is assumed to be the image of a reference macro-element $\tilde{\omega} = \cup_i \tilde{F}_i$ by a mapping $J$, such that the restrictions $J|_{\tilde{F}_i} : \tilde{F}_i \rightarrow F_i$ are bounded transformations. The set of reference macro-elements is denoted by $\tilde{N}$.

**Lemma 4.15.** Let us assume that for all reference macro-element $\tilde{\omega} \in \tilde{N}$, we have with
obvious notations:

\[
\inf_{\lambda \in M_{\delta}(\hat{\omega}) \setminus \{0\}} \sup_{\hat{\phi} \in W^0_0(\hat{\omega}) \setminus \{0\}} \int_{\hat{\omega}} \hat{\phi} \cdot \hat{\lambda} \geq \beta_{\hat{\omega}}. 
\]  \hspace{1cm} (4.68)

Then (4.67) is satisfied for all \( k \geq 1 \), with a stability constant \( \beta_k \geq C \inf_{\hat{\omega} \in \mathcal{N}} \beta_{\hat{\omega}}. \)

**Proof:** Thanks to a change of variable, the local assumptions (4.68) on reference macro-elements can be extended on any macro-element \( \omega = J\hat{\omega} \). Indeed, for \( \lambda \in M_{\delta}(\omega) \) and \( \phi \in W^0_0(\omega) \):

\[
\int_{\omega} \lambda \cdot \phi = \sum_i \int_{F_i} \lambda \cdot \phi = \sum_i \frac{\text{meas}(F_i)}{\text{meas}(\hat{F}_i)} \int_{\hat{F}_i} \hat{\lambda} \cdot \hat{\phi} \geq C \sum_i h(F_i)^{d-1} \int_{\hat{F}_i} \hat{\lambda} \cdot \hat{\phi},
\]

by regularity of the mesh. Using its quasi-uniformity, we obtain:

\[
\int_{\omega} \lambda \cdot \phi \geq C\delta^{d-1} \int_{\hat{\omega}} \hat{\lambda} \cdot \hat{\phi}.
\]

We have also:

\[
\|\phi\|_{\delta,\frac{1}{2},\omega}^2 \leq \sum_i \frac{1}{h(F_i)} \|\phi\|_{L^2(F_i)}^2 = \sum_i \frac{1}{h(F_i)} \text{meas}(F_i) \|\phi\|_{L^2(\hat{F}_i)}^2 \leq C\delta^{d-2} \|\hat{\phi}\|_{L^2(\hat{\omega})}^2,
\]

and similarly:

\[
\|\lambda\|_{\delta,\frac{1}{2},\omega}^2 \leq C\delta^d \|\hat{\lambda}\|_{L^2(\hat{\omega})}^2.
\]

Then, from (4.68), we get for all \( \omega \in \mathcal{N}_{\delta} \):

\[
\inf_{\lambda \in M_{\delta}(\omega) \setminus \{0\}} \sup_{\phi \in W^0_0(\omega) \cap H^1_0(\omega) \setminus \{0\}} \frac{\int_{\omega} \phi \cdot \lambda}{\|\phi\|_{\delta,\frac{1}{2},\omega} \|\lambda\|_{\delta,\frac{1}{2},\omega}} \geq C\beta_{\omega}. 
\]  \hspace{1cm} (4.69)
Now, we will prove the global inf-sup condition (4.67). Let $\lambda \in M_\delta$. For all $\omega \in \mathcal{N}_\delta$, the condition (4.69) proves that there exists a function $\phi_\omega \in W^0_\delta(\omega)$ vanishing outside $\omega$ such that:

$$\int_\omega \lambda \cdot \phi_\omega \geq C \|\lambda\|_{\delta, -\frac{1}{2} \omega}^2,$$

with:

$$\|\phi_\omega\|_{\delta, \frac{1}{2} \omega} \leq \|\lambda\|_{\delta, -\frac{1}{2} \omega}.$$

Let us define:

$$\phi = \sum_{\omega \in \mathcal{N}_\delta} \phi_\omega.$$

Then, because each element is in a macro-element at least and in less than $L$:

$$\int_S \lambda \cdot \phi = \sum_{\omega \in \mathcal{N}_\delta} \int_\omega \lambda \cdot \phi_\omega \geq C \sum_{\omega \in \mathcal{N}_\delta} \|\lambda\|_{\delta, -\frac{1}{2} \omega}^2$$

$$= C \sum_{\omega \in \mathcal{N}_\delta} \sum_{F \in \omega} h(F) \|\lambda\|_{L^2(F)}^2 = C \sum_{F \in \mathcal{F}_\delta} \sum_{\omega \ni F} h(F) \|\lambda\|_{L^2(F)}^2$$

$$\geq C \sum_{F \in \mathcal{F}_\delta} h(F) \|\lambda\|_{L^2(F)}^2 = C \|\lambda\|_{\delta, -\frac{1}{2}}^2,$$

and:

$$\|\phi\|_{\delta, \frac{1}{2}}^2 \leq \sum_{\omega \in \mathcal{N}_\delta} \|\phi_\omega\|_{\delta, \frac{1}{2} \omega}^2 \leq \sum_{\omega \in \mathcal{N}_\delta} \|\lambda\|_{\delta, -\frac{1}{2} \omega}^2$$

$$\leq \sum_{F \in \mathcal{F}_\delta} \sum_{\omega \ni F} h(F) \|\lambda\|_{L^2(F)}^2 \leq L \sum_{F \in \mathcal{F}_\delta} h(F) \|\lambda\|_{L^2(F)}^2 = L \|\lambda\|_{\delta, -\frac{1}{2}}^2,$$

which proves (4.67).

As a consequence, local inf-sup conditions has only to be checked on reference macro-elements to ensure a global inf-sup compatibility.

### 4.7.5 Second order stabilized interface elements

We now introduce some stabilized elements achieving second order approximation in displacements and satisfying the local inf-sup condition (4.68).

#### 1D macroelements

For bidimensional problems, we build on the reference interface element $\hat{\omega} = [-1; 1]$, the following spaces:

$$\begin{aligned}
M_\delta(\hat{\omega}) = \mathbb{P}_1(\hat{\omega})^2, \\
W^0_\delta(\hat{\omega}) = \left(\mathbb{P}_2(\hat{\omega})^2 \oplus \text{span}\{\hat{b}\}^2\right) \cap H^1_0(\hat{\omega})^2,
\end{aligned}$$
where the interface bubble function \( \hat{b} \) is an odd function over \([-1,1]\), which satisfies:

\[
\int_{-1}^{1} x \hat{b}(x) \, dx \neq 0.
\]

Then, the local inf-sup condition (4.68) is satisfied on a macro-element made of the single element \( \hat{\omega} \). Indeed, let \( \lambda \in M_{\delta}(\hat{\omega}) \) be such that:

\[
\int_{\hat{\omega}} \phi \cdot \lambda = 0, \quad \forall \phi \in W^0_\delta(\hat{\omega}).
\]

For all \( 1 \leq i \leq 2 \), denoting by \( \lambda_i \) the \( i \)th component of \( \lambda \), we have \( \lambda_i(x) = \alpha_i x + \beta_i \) for \( x \in \hat{\omega} \), and its integral against any second order polynomial and the bubble \( \hat{b} \) vanishes, which implies:

\[
\left\{
\begin{array}{l}
\int_{-1}^{1} \lambda_i(x)(1 - x^2) \, dx = \frac{4}{3} \beta_i = 0 \quad \Rightarrow \quad \beta_i = 0,
\int_{-1}^{1} \lambda_i(x) \hat{b}(x) \, dx = 2\alpha_i \int_{0}^{1} x \hat{b}(x) \, dx = 0 \quad \Rightarrow \quad \alpha_i = 0.
\end{array}
\right.
\]

Therefore, \( \lambda = 0 \), which proves that the local inf-sup condition (4.68) is satisfied.

As a bubble \( \hat{b} \), one can take:

\[
\hat{b}(x) = x(1 - x^2), \quad x \in \hat{\omega}.
\]

Obviously, \( \hat{b} \) is the trace over \( \hat{\omega} \times \{0\} \) of a bubble function \( \hat{h} \) defined in a reference element \( \hat{K} \subset \mathbb{R}^2 \), whose \( \hat{\omega} \times \{0\} \) is an edge.

In the case where \( \hat{K} = \hat{T} \) is a reference triangle, if \( A = (-1,0) \), \( B = (1,0) \) and \( C = (-1,2) \) are its vertices, the interface bubble function \( h \) can be defined as:

\[
\hat{h}(x,y) = \begin{cases} 
(1 - \frac{y}{2}) \hat{b} \left( \frac{2x + y}{2} - y \right), & \forall (x,y) \in \hat{T} \setminus (-1,2), \\
0, & (x,y) = (-1,2).
\end{cases}
\]

Such a function \( \hat{h} \) is represented on figure 4.9.

In the case where \( \hat{K} = \hat{Q} \) is a reference square, if \( A = (-1,0) \), \( B = (1,0) \), \( C = (1,2) \) and \( D = (-1,2) \) are its corners, the interface bubble function \( h \) can be defined as:

\[
\hat{h}(x,y) = \left(1 - \frac{y}{2}\right) \hat{b}(x), \quad \forall (x,y) \in \hat{Q}.
\]

Such a function \( \hat{h} \) is represented on figure 4.10.
4.7. Analysis of discontinuous mortar spaces

\[ T^B = (1,0) \]
\[ C = (-1,0) \]
\[ A = (-1,0) \]

Fig. 4.9 – A reference triangle \( \hat{T} \) and a corresponding interface bubble function \( \hat{h} \) on the edge \([AB] = \hat{\omega} \times \{0\}\).

\[ D = (-1,2) \]
\[ C = (1,2) \]
\[ A = (-1,0) \]
\[ B = (1,0) \]

Fig. 4.10 – A reference square \( \hat{Q} \) and a corresponding interface bubble function \( \hat{h} \) on the edge \([AB] = \hat{\omega} \times \{0\}\).

2D quadrangular interface macroelement

For tridimensional problems, we introduce the following second order 2D quadrilateral interface element. Let \( \hat{\omega} = \hat{Q} = [-1,1]^2 \) be a reference quadrilateral, on which we build the following spaces:

\[
\begin{align*}
M_\delta(\hat{\omega}) &= \mathbb{P}_1(\hat{Q})^3, \\
W_\delta^0(\hat{\omega}) &= \left( \mathbb{Q}_2(\hat{Q})^3 \oplus \text{span} \{ \hat{b}_1, \hat{b}_2 \}^3 \right) \cap H_0^1(\hat{\omega})^3,
\end{align*}
\]

where the bubble functions are defined as follows:

\[
\hat{b}_k(x_1, x_2) = x_k(1 - x_k^2)(1 - x_l^2), \quad l \neq k, \quad (4.71)
\]

the \((x_k)_{k=1,2}\) being the euclidian coordinates in \(\mathbb{R}^2\). An illustration of such bubble functions defined on the reference square is shown on figure 4.11.
Fig. 4.11 – A bubble function defined by (4.71) on the reference square \( \hat{Q} \).

The corresponding element satisfies the local inf-sup condition (4.68) on a macroelement made of the single element \( Q \). Indeed, let \( \lambda \in M_{\delta}(\hat{\omega}) \) be such that :

\[
\int_{\hat{\omega}} \phi \cdot \lambda = 0, \quad \forall \phi \in W^0_{\delta}(\hat{\omega}).
\]

For all \( 1 \leq i \leq 3 \), denoting by \( \lambda_i \) the \( i \)th component of \( \lambda \), we have \( \lambda_i(x_1, x_2) = \alpha_i x_1 + \beta_i x_2 + \gamma_i \) for \( (x_1, x_2) \in \hat{\omega} \), and its integral against any second partial order polynomial and bubble vanishes, which implies :

\[
\int_{\hat{Q}} \lambda(x_1, x_2)(1 - x_1^2)(1 - x_2^2) \, dx_1 \, dx_2 = \frac{16}{9} \gamma_i = 0 \quad \implies \quad \gamma_i = 0,
\]

\[
\int_{\hat{Q}} \lambda(x_1, x_2)x_1(1 - x_1^2)(1 - x_2^2) \, dx_1 \, dx_2 = \frac{16}{45} \alpha_i = 0 \quad \implies \quad \alpha_i = 0,
\]

\[
\int_{\hat{Q}} \lambda(x_1, x_2)x_2(1 - x_1^2)(1 - x_2^2) \, dx_1 \, dx_2 = \frac{16}{45} \beta_i = 0 \quad \implies \quad \beta_i = 0,
\]

that is \( \lambda = 0 \), which proves that the local inf-sup condition (4.68) is satisfied.

As previously, the interface bubble functions \( (\hat{b}_k)_{k=1,2} \) are the restrictions to \( \hat{Q} \times \{0\} \) of functions \( (\hat{h}_k)_{k=1,2} \) defined on a reference cube \( \hat{Q} = \hat{Q} \times [0;2] \) whose \( \hat{Q} \times \{0\} \) is a face. More precisely, we can define for \( k = 1, 2 \) :

\[
\hat{h}_k(x_1, x_2, x_3) = \left( 1 - \frac{x_3^2}{2} \right) \hat{b}_k(x_1, x_2), \quad \forall (x_1, x_2) \in \hat{Q}, \forall x_3 \in [0, 2].
\]

2D triangular interface macroelement

For tridimensional problems, we introduce the following second order 2D triangular interface element. Let \( \hat{\omega} = \hat{T} \) be a triangular element whose vertices are \( A = (1,0), \)
4.7. Analysis of discontinuous mortar spaces

$B = (0, 1)$ and $C = (0, 0)$. We introduce the following spaces :

\[
\begin{align*}
    &M_\delta(\hat{\omega}) = \mathbb{P}_1(\hat{T})^3, \\
    &W_\delta^0(\hat{\omega}) = (\mathbb{P}_2(\hat{T})^3 \oplus \text{span}\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}) \cap H^1_0(\hat{\omega})^3,
\end{align*}
\]

where the bubble functions are defined by :

\[
\begin{align*}
    &\hat{b}_1 = \left(\hat{\lambda}_1 - \frac{1}{2}\right) \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3, \\
    &\hat{b}_2 = \left(\hat{\lambda}_2 - \frac{1}{2}\right) \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3, \\
    &\hat{b}_3 = \left(\hat{\lambda}_3 - \frac{1}{2}\right) \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3,
\end{align*}
\]

in which $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ are the barycentric coordinates on $\hat{T}$, respectively associated to the vertices $A$, $B$ and $C$. A typical example of such bubbles is given on figure 4.12.

\[
\begin{align*}
    &\text{FIG. 4.12 – A bubble function on the reference interface triangle } \hat{T}.
\end{align*}
\]

The corresponding element satisfies the local inf-sup condition (4.68) on a macro-element made of the single element $\hat{T}$. Indeed, let $\lambda \in M_\delta(\hat{\omega})$ be such that :

\[
\int_{\hat{\omega}} \phi \cdot \lambda = 0, \quad \forall \phi \in W_\delta^0(\hat{\omega}).
\]

For all $1 \leq i \leq 3$, denoting by $\lambda_i$ the $i$th component of $\lambda$, we have $\lambda_i = \alpha_i \hat{\lambda}_1 + \beta_i \hat{\lambda}_2 + \gamma_i \hat{\lambda}_3$, and its integral against any second partial order polynomial and bubble vanishes, which implies :

\[
\begin{align*}
    &M \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4.72)
\end{align*}
\]
with:

\[ M_{kl} = \int_T \hat{b}_k \hat{\lambda}_l, \quad 1 \leq k, l \leq 3. \]

To compute these coefficients, we use the following lemma (see for example [Lar95], page 57):

**Lemma 4.16.** Let \( T \) a non-degenerated triangle in \( \mathbb{R}^2 \) and \( \lambda_1(x), \lambda_2(x), \lambda_3(x) \) the barycentric coordinates of \( x \in \mathbb{R}^2 \) with respect to the vertices of \( T \). Then:

\[
\int_T \lambda_1(x)^k \lambda_2(x)^l \lambda_3(x)^m \, dx = 2 \text{meas}(T) \frac{k! \, l! \, m!}{(k+l+m+2)!}.
\]

Now, let us calculate the coefficients of the matrix \( M \) by using the previous lemma. It is obtained that for \( k = 1, 2, 3 \):

\[
M_{kk} = \int_T \hat{\lambda}_1 - \frac{1}{2} \, \hat{\lambda}_1^2 \hat{\lambda}_2 \hat{\lambda}_3 = \int_T \hat{\lambda}_1^2 \hat{\lambda}_2 \hat{\lambda}_3 - \frac{1}{2} \int_T \hat{\lambda}_1^2 \hat{\lambda}_2 \hat{\lambda}_3, \quad (4.73)
\]

and that for all \( k, l \in \{1, 2, 3\} \) such that \( i \neq j \):

\[
M_{kl} = \int_T \hat{\lambda}_1 - \frac{1}{2} \, \hat{\lambda}_1 \hat{\lambda}_2^2 \hat{\lambda}_3 = \int_T \hat{\lambda}_1^2 \hat{\lambda}_2^2 \hat{\lambda}_3 - \frac{1}{2} \int_T \hat{\lambda}_1 \hat{\lambda}_2^2 \hat{\lambda}_3
\]

Then:

\[
- \frac{7!}{2 \text{meas}(\hat{T})} M = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix},
\]

and the original linear system (4.72) is equivalent to:

\[
\begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

The right hand side matrix is invertible, and the only solution is then \( \alpha_i = \beta_i = \gamma_i = 0 \) for all \( 1 \leq i \leq 3 \), that is \( \lambda = 0 \), which proves that the local inf-sup condition (4.68) is satisfied.

As previously, the interface bubble functions \( (\hat{b}_k)_{k=1,2,3} \) are the restrictions to \( \hat{T} \times \{0\} \) of functions \( (\hat{h}_k)_{k=1,2,3} \) defined on a reference tetrahedron \( \hat{T} \) whose \( \hat{T} \times \{0\} \) is a face. More precisely, if \( \lambda_1, \ldots, \lambda_4 \) are the barycentric coordinates associated to the vertices of \( \hat{T} \), and
assuming that $\lambda_4$ is the barycentric coordinate associated to the node not belonging to $\hat{T} \times \{0\}$, we have for $k = 1, 2, 3$:

$$\hat{h}_k = \lambda_k (\lambda_k - \frac{1}{2}) \lambda_l \lambda_m (1 - \lambda_4), \quad \{l, m\} = \{1, 2, 3\} \setminus \{k\}.$$ 

### 4.8 Some numerical issues

The practical implementation of mortar elements such as those introduced in the above sections, faces a few technical problems outlined in this section.

#### 4.8.1 Penalized formulation.

One can replace the solution of a saddle-point problem by the solution of a positive definite one, by introducing a penalized formulation for (4.7). It is a very standard solution in many academic and industrial implementations for treating kinematic constraints and non-homogenous essential boundary conditions. Herein, we propose a mesh-dependent penalization term. Introducing the following $L^2$ inner product:

$$c(\lambda, \mu) = \int_S \lambda \cdot \mu, \quad \forall \lambda, \mu \in M_\delta,$$

and denoting by $\eta > 0$ a small penetration parameter, we propose to replace the problem (4.7) by the symmetric positive definite one:

$$
\begin{cases}
\bar{a}(u_h^\eta, v_h) + b(v_h, \lambda_h^\eta) = l(v_h), & \forall v_h \in X_h, \\
b(u_h^\eta, \mu_h) = \eta \delta_{\min} c(\lambda_h^\eta, \mu_h), & \forall \mu_h \in M_\delta,
\end{cases}
$$

(4.74)

where the minimum diameter of interface surfacic elements has been denoted by:

$$\delta_{\min} = \min_{F \in F_\delta} h(F).$$

Then, we prove the convergence of the penalized solution of the system (4.74) to the exact constrained solution of (4.7) as $\eta$ goes to zero:

**Proposition 4.13.** We assume that the original mortar formulation (4.7) is well-posed, and denote by $(u_h, \lambda_h) \in X_h \times M_\delta$ its unique solution. Then, for all $\eta > 0$, there exists a unique solution $(u_h^\eta, \lambda_h^\eta) \in X_h \times M_\delta$ of (4.74), and the convergence of the penalized solution to $(u_h, \lambda_h)$ as $\eta \to 0$ holds in the sense that:

$$
\|u_h - u_h^\eta\|_X \leq C \eta, \\
\|\lambda_h - \lambda_h^\eta\|_{\delta_{\min}^{-\frac{1}{2}}} \leq C \eta,
$$

where $C$ denotes various constants independent of the penalization coefficient $\eta$, of the decomposition into subdomains, and the discretization.
Remark 4.18. The main difference with the usual penalization strategy used for incompressibility is that $\tilde{a}$ is not coercive on $X_h \times X_h$.

Proof: The proof is inspired from [EG02], with adequate modifications in order to take the remark 4.18 into account. For convenience, we rewrite (4.74) under the usual operator form with obvious notation:

\[
\begin{cases}
\tilde{A}u_h^\eta + B^t \lambda_h^\eta = \mathcal{L}, & \text{on } X_h', \\
Bu_h^{\eta} = \eta \delta_{\min} \mathcal{C} \lambda_h^{\eta}, & \text{on } M'_\delta.
\end{cases}
\]

(4.75)

and the original problem (4.7) as:

\[
\begin{cases}
\tilde{A}u_h + B^t \lambda_h = \mathcal{L}, & \text{on } X'_h, \\
Bu_h = 0, & \text{on } M'_\delta.
\end{cases}
\]

(4.76)

The present proof is decomposed into 4 parts.

1. Well-posedness of the penalized problem - As the bilinear form $c(\cdot, \cdot)$ is coercive and continuous on $M \times M$, with $M = \prod_{m=1}^M L^2(\Gamma_m)^d$, the Lax-Milgram lemma shows the invertibility of $\mathcal{C}$ on $M_\delta \subset M$, and it is obtained from (4.75) that:

\[
\langle \mathcal{K}_{\eta, \delta_{\min}} u_h, u_h \rangle_{X', X} = \langle \tilde{A}u_h, u_h \rangle_{X', X} + \frac{1}{\eta \delta_{\min}} \langle Bu_h, \mathcal{C}^{-1} Bu_h \rangle_{M', M'}.
\]

Moreover, we prove that $\mathcal{K}_{\eta, \delta_{\min}}$ is uniformly coercive with respect to $\eta$, to the decomposition into subdomains, and to the discretization. Indeed, for all $u_h \in X_h$:

\[
\langle \mathcal{K}_{\eta, \delta_{\min}} u_h, u_h \rangle_{X', X} = \langle \tilde{A}u_h, u_h \rangle_{X', X} + \frac{1}{\eta \delta_{\min}} \langle Bu_h, \mathcal{C}^{-1} Bu_h \rangle_{M', M'}.
\]

and if $\lambda_h^1 \in M_\delta$ is the unique solution of $\mathcal{C} \lambda_h^1 = Bu_h$ in $M'_\delta$, it follows that:

\[
\langle \mathcal{K}_{\eta, \delta_{\min}} u_h, u_h \rangle_{X', X} = \langle \tilde{A}u_h, u_h \rangle_{X', X} + \frac{1}{\eta \delta_{\min}} \langle \mathcal{C} \lambda_h^1, \lambda_h^1 \rangle_{M', M'}
\]

\[
= \langle \tilde{A}u_h, u_h \rangle_{X', X} + \frac{1}{\eta \delta_{\min}} \| \lambda_h^1 \|_M^2
\]

\[
\geq \langle \tilde{A}u_h, u_h \rangle_{X', X} + \frac{1}{L} \| \lambda_h^1 \|_M^2,
\]

when $\eta \leq 1$ and $\delta_{\min} \leq L$ (which is not a restriction because both $\eta$ and $\delta_{\min}$ are expected to tend to zero), in which $L$ denotes the diameter of the smallest interface.

From the definition of $\pi$ given in section 4.4.3, we get that for all interfaces $\gamma_{kl}$:

\[
\int_{\gamma_{kl}} \pi[u_h] \cdot \mu = \int_{\gamma_{kl}} [u_h] \cdot \mu = \int_{\gamma_{kl}} \lambda_h^1 \cdot \mu, \quad \forall \mu \in M_{kl},
\]
because we have for $\Gamma_m = \gamma_{kl}$, the inclusion $M_{kl} \subset M_m;\delta_m$ from assumption 4.3. As a consequence, taking $\mu = \pi[u_h]$ and using Cauchy-Schwarz inequality, it follows that:

$$\|\pi[u_h]\|_M \leq \|\lambda_h^1\|_M.$$ 

The inequality (4.35) then provides the existence of a coercivity constant $\kappa > 0$ independent of the number of subdomains, of their sizes, and of the discretization, such that:

$$\langle K_{\eta,\delta_{\min}} u_h, u_h \rangle_{X',X} \geq \left\langle \tilde{A} u_h, u_h \right\rangle_{X',X} + \frac{1}{L} \|\pi[u_h]\|^2_M \geq \kappa \|u_h\|^2_X.$$ 

The continuity of $K_{\eta,\delta_{\min}}$ holds because for all $u_h, v_h \in X_h$, we have:

$$\langle K_{\eta,\delta_{\min}} u_h, v_h \rangle_{X',X} = \left\langle \tilde{A} u_h, v_h \right\rangle_{X',X} + \frac{1}{\eta \delta_{\min}} \langle C^{-1} B u_h, B v_h \rangle_{M',M} \leq \|\tilde{A}\| \|u_h\|_X \|v_h\|_X + \frac{1}{\eta \delta_{\min}} \|B u_h\|_{M'} \|B v_h\|_{M'} \leq \|\tilde{A}\| \|u_h\|_X \|v_h\|_X + \frac{1}{\eta \delta_{\min}} \|[u_h]\|_M \|[v_h]\|_M \leq C \left( 1 + \frac{L_{\max}}{\eta \delta_{\min}} \right) \|u_h\|_X \|v_h\|_X,$$

where $\|\tilde{A}\|$ is the continuity constant of $\tilde{A} : X \rightarrow X'$, and $L_{\max}$ the diameter of the largest interface. Then, for each penalization coefficient $\eta$ and each discretization, there exists a unique solution $u_h^\eta \in X_h$ of (4.77) which satisfies the a priori estimate:

$$\|u_h^\eta\|_X \leq \frac{1}{\kappa} \|\mathcal{C}\|_{X'}.$$ (4.78)

It is crucial noticing that even if the continuity constant of $K_{\eta,\delta_{\min}}$ depends on $\eta$ and on the discretization, the coercivity constant does not.

As a consequence of the inf-sup condition (4.9), an upper bound can be established on $\lambda_h^\eta$ since:

$$\beta \delta_{\min}^{1/2} \|\lambda_h^\eta\|_M \leq \beta \|\lambda_h^\eta\|_{\delta_{\min}^{-1}} \leq \|B^* \lambda_h^\eta\|_{X'} \leq \|\mathcal{C} - \tilde{A} u_h^\eta\|_{X'} \leq \|\mathcal{C}\|_{X'} + ||\tilde{A}|| \|u_h^\eta\|_X \leq \left( 1 + \frac{\|\tilde{A}\|}{\kappa} \right) \|\mathcal{C}\|_{X'},$$

resulting in the following estimate:

$$\|\lambda_h^\eta\|_M \leq \frac{1}{\beta \delta_{\min}^{1/2}} \left( 1 + \frac{\|\tilde{A}\|}{\kappa} \right) \|\mathcal{C}\|_{X'} \leq C \frac{1}{\delta_{\min}^{1/2}}.$$ (4.79)
2. First estimate - By substraction of the penalized system (4.74) to the original one (4.76), we get :

\[
\begin{align*}
\tilde{A}(u_h - u_h^\eta) + B^t(\lambda_h - \lambda_h^\eta) &= 0, \quad \text{on } X'_h, \\
B(u_h - u_h^\eta) &= -\eta \delta_{\min} C \lambda_h^\eta, \quad \text{on } M'_h,
\end{align*}
\]  
(4.80)

and deduce by testing the first equation with \(u_h - u_h^\eta\) that :

\[
\langle \tilde{A}(u_h - u_h^\eta), (u_h - u_h^\eta) \rangle_{X', X} = \eta \delta_{\min} \langle C \lambda_h^\eta, \lambda_h - \lambda_h^\eta \rangle_{M', M} \\
\leq \eta \delta_{\min} \| \lambda_h \|_M \| \lambda_h - \lambda_h^\eta \|_M.
\]  
(4.81)

Moreover, the inf-sup condition (4.9) implies :

\[
\beta \delta_{\min}^{1/2} \| \lambda_h - \lambda_h^\eta \|_M \leq \beta \| \lambda_h - \lambda_h^\eta \|_{\delta, -\frac{1}{2}} \\
\leq \| B^t(\lambda_h - \lambda_h^\eta) \|_{X'} \\
\leq \| \tilde{A}(u_h - u_h^\eta) \|_{X'} \quad \text{from (4.80)}, \\
\leq \| \tilde{A} \| \| u_h - u_h^\eta \|_X, \quad \text{by continuity of } \tilde{A},
\]  
(4.82)

and considering (4.79), it follows from (4.81) that :

\[
\langle \tilde{A}(u_h - u_h^\eta), (u_h - u_h^\eta) \rangle_{X', X} \leq C \eta \| u_h - u_h^\eta \|_X.
\]  
(4.83)

Because \(u_h - u_h^\eta \notin V_h\), we cannot conclude directly about the convergence of the displacements.

3. Convergence of displacements - Let us prove now an upper bound for the quantity :

\[
\sum_{1 \leq k < \ell \leq K} \frac{1}{\text{diam}((\gamma_{kl}))} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}} [u_h - u_h^\eta])^2,
\]

with the notation introduced in section 4.4.3. First, because \(u_h \in V_h\), we have \(\pi[u_h - u_h^\eta] = -\pi[u_h^\eta]\), and :

\[
\int_{\gamma_{kl}} \pi[u_h^\eta] \cdot \mu = \int_{\gamma_{kl}} [u_h^\eta] \cdot \mu = \eta \delta_{\min} \int_{\gamma_{kl}} \lambda_h^\eta \cdot \mu, \quad \forall \mu \in M_{kl} \quad (\subset M_{\delta}).
\]

By taking \(\mu = \pi[u_h^\eta]\), and from Cauchy-Schwarz inequality, we get :

\[
\| \pi[u_h^\eta] \|^2_{L^2(\gamma_{kl})^d} \leq \eta \delta_{\min} \| \lambda_h^\eta \|_{L^2(\gamma_{kl})^d},
\]

and therefore :

\[
\| \pi[u_h - u_h^\eta] \|^2_{L^2(\gamma_{kl})^d} \leq \eta \delta_{\min} \| \lambda_h^\eta \|_{L^2(\gamma_{kl})^d}.
\]  
(4.84)
On the other hand, we get:

\[ \int_{\gamma_{kl}} \pi [u_h - u_h^\eta] \cdot \mu = \int_{\gamma_{kl}} [u_h - u_h^\eta] \cdot \mu, \quad \forall \mu \in M_{kl}, \]

resulting as above in:

\[ \| \pi [u_h - u_h^\eta] \|_{L^2(\gamma_{kl})} \leq \| [u_h - u_h^\eta] \|_{L^2(\gamma_{kl})} \leq C \text{diam}(\gamma_{kl})^{1/2} \left( \| u_h - u_h^\eta \|_{H^1(\Omega_h)^d} + \| u_h^\eta \|_{H^2(\Omega_h)^d} \right), \]

from the (rescaled) Sobolev trace theorem, and deduce with (4.84) that:

\[ \frac{1}{\text{diam}(\gamma_{kl})} \| \pi [u_h - u_h^\eta] \|_{L^2(\gamma_{kl})}^2 \leq C \eta \frac{\delta_{\text{min}}}{\text{diam}(\gamma_{kl})^{1/2}} \| \lambda_h^\eta \|_{L^2(\gamma_{kl})} \left( \| u_h - u_h^\eta \|_{H^1(\Omega_h)^d} + \| u_h^\eta \|_{H^2(\Omega_h)^d} \right). \]

From the uniform boundedness with respect to \( \eta \) of \( \delta_{\text{min}}^{1/2} \| \lambda_h^\eta \|_{L^2(\gamma_{kl})} \) shown in (4.79), we deduce:

\[ \sum_{1 \leq k < l \leq K} \frac{1}{\text{diam}(\gamma_{kl})} \int_{\gamma_{kl}} (\pi_{\gamma_{kl}} [u_h - u_h^\eta])^2 \leq C \eta \frac{\delta_{\text{min}}^{1/2}}{\text{diam}(\gamma_{kl})^{1/2}} \| u_h - u_h^\eta \|_X. \quad (4.85) \]

By summing the inequalities (4.83) and (4.85), and using the coercivity result given by proposition 4.35 (page 138), we deduce for sufficiently small values of \( \delta_{\text{min}} \leq \text{diam}(\gamma_{kl}) \) for all \( 1 \leq k < l \leq K \) that:

\[ \| u_h - u_h^\eta \|_X^2 \leq C \eta \| u_h - u_h^\eta \|_X, \]

leading to the expected convergence result in displacements after division by \( \| u_h - u_h^\eta \|_X \).

4. Convergence of Lagrange multipliers - The convergence of Lagrange multipliers is deduced by using the inf-sup condition (4.9) and the first equation in (4.80):

\[ \beta \| \lambda_h - \lambda_h^\eta \|_{\delta, -\frac{1}{2}} \leq \| B'(\lambda_h - \lambda_h^\eta) \|_X' \leq \| \tilde{A}(u_h - u_h^\eta) \|_X' \leq \| \tilde{A} \| \| u_h - u_h^\eta \|_X \leq C \eta. \]

The penalized formulation reinforces the interest in mesh-dependent formulations. We insist on the presence in the penalty term of the minimum diameter of the surfacic interface elements \( \delta_{\text{min}} \), which is of crucial importance to obtain constants independent of the discretization in convergence estimates with respect to the penalization coefficient \( \eta \). In spite of the practical computational interest of such a penalized formulation, it is recalled that the condition number classically explodes like \( O(1/\eta) \), which suggests that a good compromise should be chosen on the value of the penalization coefficient \( \eta \).
Remark 4.19. From the implementation point of view, the penalization strategy classically enables to solve the symmetric positive definite linear system:

\[
\left( \tilde{A} + \frac{1}{\eta \delta_{\text{min}}} \mathcal{B}' \mathcal{C}^{-1} \mathcal{B} \right) u_h^n = \mathcal{L},
\]

with operator notation from (4.75).

4.8.2 Exact integration of the constraint

From the numerical point of view, especially in 3D, the accurate calculation of the integral \( \int_{\Gamma} \phi_h \cdot \lambda_h \) is difficult when \( \phi_h \) and \( \lambda_h \) do not live on the same side of the interface \( \Gamma \), and are therefore defined on completely independent meshes.

The question of approximating this integral by quadrature has been risen in [CLM97, MRW02]. The authors prove that any approximation of this integral by quadrature either on the mortar or non-mortar side is not optimal, leading to a convergence in \( \sqrt{h} \). This bad behavior will be illustrated in the numerical results to follow. A dissymmetric formulation in which this integral is always approximated by quadrature is proposed.

Herein, we have decided to compute exactly such an integral because the simplest quadrature approach does not lead to accurate simulations as illustrated on figure 4.16 of the next section, and have giving it up using the non-symmetric approach from [CLM97, MRW02].

More precisely, let \( \phi_h \) a finite element displacement living on the mortar side of the interface, and \( P \) an interface element on the same side, where \( \phi_h \) does not vanish. Let \( Q \) an interface element of the non-mortar side having a non-empty intersection with \( P \), and where the finite element Lagrange multiplier \( \lambda_h \) does not vanish. To compute the integral \( \int_{P \cap Q} \phi_h \cdot \lambda_h \), we proceed as follows:

1. We compute the exact intersection of the convex polygons \( P \) and \( Q \) (see figure 4.13), for which we refer to the book of Joseph O’Rourke [O’R82] for example. The code source in C can be downloaded on his website. It is originally written in integer precision, but can be modified to deal with double precision, and also to detect the complete inclusion of a polygonal into another.

2. We introduce the barycentre \( G \) of the \( n \) vertices of the intersection polygon \( P \cap Q \), and decompose it into \( n \) triangles sharing the same vertex \( G \) as illustrated on figure 4.13. We denote \( P \cap Q = \bigcup_{i=1}^n T_i \).

3. For all \( i = 1, \ldots, n \), the integral \( \int_{T_i} \phi_h \cdot \lambda_h \) is computed exactly by quadrature, since \( \phi_h \cdot \lambda_h \) is a polynomial over \( T_i \). The exact integration is then obtained by:

\[
\int_{P \cap Q} \phi_h \cdot \lambda_h = \sum_{i=1}^n \int_{T_i} \phi_h \cdot \lambda_h,
\]

the last term being computed thanks to lemma 4.16.
4.9 Numerical tests for discontinuous mortar-elements

First, we consider an homogeneous beam made with a Hooke’s material, whose a tip is clamped on a wall, and whose the other tip is under traction by a uniform negative pressure. All the characteristics are detailed in the table, figure 4.14. For comparison purpose, both non-conforming and conforming meshes are considered, as shown on figure 4.15. They are respectively made of 2926 nodes with 2240 elements and 4225 nodes with 3456 elements.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young modulus $E$</td>
<td>5000 Pa</td>
</tr>
<tr>
<td>Poisson coefficient $\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>density $\rho$</td>
<td>1 kg/m$^3$</td>
</tr>
<tr>
<td>traction pressure $p$</td>
<td>10000 Pa</td>
</tr>
<tr>
<td>length $L$</td>
<td>2 m</td>
</tr>
<tr>
<td>thickness $l$</td>
<td>1 m</td>
</tr>
<tr>
<td>extension under static loading</td>
<td>3.97 m</td>
</tr>
<tr>
<td>period of the first extensional eigenmode</td>
<td>0.1125 s</td>
</tr>
</tbody>
</table>

Fig. 4.14 – Characteristics of the beam and first numerical estimations.

We test the proposed first order formulation by using a $Q_1$ approximation for the displacements on both conforming and non-conforming models, enriched with an interface bubble stabilization (defined on the finer side of the interface) for the non-conforming model together with $P_0$ Lagrange multipliers on the finer side of the interface (non-mortar side) as described in section 4.7.1 (page 167). We start by illustrating the non-optimal results obtained when computing the mortar constraint by quadrature on the finer side of the
interface. The quadrature is exact for computing $\int_{\Gamma} \mu_h \cdot v_h$ when both $\mu_h$ and $v_h$ live on the finer side of the interface. Such a computation leads to interface oscillations of the displacements, as shown on figure 4.16. This result confirms the work of [CLM97, MRW02], and we will definitively use the exact integration technique described in section 4.8.2.

First, we observe the $L^\infty(\Omega)^d$-norm of the error between the displacements obtained on the conforming model and the non-conforming model on which a penalized formulation of the mortar constraint is adopted, that is $\|u_{h,\text{conforming}} - u_{h,\text{non-conforming}}\|_{L^\infty(\Omega)^d}$, as a function of the penalization coefficient $1/\eta$. The convergence process is illustrated on
4.9. Numerical tests for discontinuous mortar-elements

Figure 4.17. By the triangular inequality, we have:

\[ \| u_{h, \text{conforming}} - u_{h, \text{non-conforming}}^\eta \|_{L^\infty(\Omega)} \leq \]
\[ \leq \| u_{h, \text{conforming}} - u_{h, \text{non-conforming}} \|_{L^\infty(\Omega)} + \| u_{h, \text{non-conforming}} - u_{h, \text{non-conforming}}^\eta \|_{L^\infty(\Omega)}. \]

For \( \eta \leq 10^{10} \), the first term appears to be negligible, and the linear convergence proved in section 4.8.1 is observed. At the penalization limit, the error in displacements between the conforming and non-conforming models is about \( 5 \times 10^{-6} \) m in \( L^\infty \) norm. The corresponding relative error is about \( 10^{-6} \). Concerning Cauchy stresses, a \( 4 \times 10^{-4} \) relative gap between the conforming and non-conforming models is observed. This very good agreement is illustrated on figure 4.18, where the computed distribution of \( \sigma_{11} \) stresses is represented.

![Convergence of the penalized formulation](image.png)

**Fig. 4.17** – Error in displacements \( \| u_{h, \text{conforming}} - u_{h, \text{non-conforming}}^\eta \|_{L^\infty(\Omega)} \) as a function of the penalization coefficient \( 1/\eta \), with \( \| u_{h, \text{conforming}} \|_{L^\infty(\Omega)} = 3.97 \) m.

Finally, let us discuss the influence of the choice of the non-mortar side (defining the multipliers either on the coarse side, or on the fine one) on the solution. The relative gap of the displacements (resp. of the \( \sigma_{11} \) stresses) in \( L^\infty \) norm between the non-conforming solutions computed with these choices is \( 2 \times 10^{-6} \) (resp. \( 8 \times 10^{-4} \)). As illustrated on figure 4.20, the relative gap of stresses remains concentrated on the elements sharing the interface. The relative gaps in displacements and stresses have the same order than the relative gaps between the conforming and non-conforming solutions. Therefore, the static analysis is confirmed (at least in a homogeneous model) indicating that the choice of the non-mortar side can be done on both sides without affecting the convergence.

The same simulations have been computed for a \( Q_2 \) approximation of the displacements both on conforming and non-conforming models, using the interface stabilization presented.
Fig. 4.18 – Distribution of $\sigma_{11}$ stresses on the deformed configuration of the non-conforming (top) and conforming (bottom) models, by using a first order approximation for the displacements.
4.9. Numerical tests for discontinuous mortar-elements

in section 4.7.5, and \( P_1 \) Lagrange multipliers. For this second order approximation, we have kept the same number of nodes than the previous first order approximation. Then, the conforming model is made with 4225 nodes and 432 elements, and the non-conforming one with 2926 nodes and 280 elements. We have adopted the value \( 1/\eta = 10^{11} \) of the penalization coefficient. Then, the relative gap of displacements (resp. maximal stresses) in \( L^\infty \) norm between conforming and non-conforming models is \( 3.10^{-6} \) (resp. \( 1.10^{-3} \)). The distribution of \( \sigma_{11} \) stresses for the conforming and non-conforming models is represented on figure 4.19. Moreover, we show on figure 4.20 that the influence of the choice of the non-mortar side (defining the multipliers either on the coarse side, or on the fine one) is again rather small in this case. Indeed, the relative gap of the \( \sigma_{11} \) stresses between the solutions for the two possible choices of the non-mortar side is always smaller than \( 2.10^{-3} \), keeping the same order than the gap in stresses between the conforming and non-conforming solutions. It is worth noticing that whereas the relative gap of displacements between the first and second order models is \( 2.10^{-4} \) in \( L^\infty \) norm, the maximal stress has been increased by 10% in the second order model, due to the presence of a singularity at the corners of the fixed tip of the beam.

Let us now consider the elastodynamics problem associated with the previous beam model, by using the trapezoidal time discretization given by (4.52). For comparison purpose, the first order conforming and non-conforming space discretizations used above in the static case are tested. Here, the non-mortar side is the finer one. A constant traction (identical to the static case) is applied at the tip of the beam. As this solicitation is derived from a potential, oscillations are expected and observed. Some snapshots of the computed dynamics are given on figure 4.21. In order to compare the space non-conforming solution with the conforming one, the horizontal displacement of the central node of the free tip of the beam is represented on figure 4.22 both for non-conforming and conforming approximations when using 20, 50 and 100 time steps per oscillation period. The proximity of the solutions confirms the theoretical result of optimality of the space non-conforming approximation in linear elastodynamics.

Finally, an homogeneous bidimensional cylinder in plane displacements under pressure is considered. It is made with a Hooke’s material and its characteristics are given on table, figure 4.24. As previously, for comparison purpose, we consider both conforming and non-conforming meshes, respectively constituted of 1456 nodes with 1350 elements and 973 nodes with 810 elements, shown on figure 4.23. The displacements are approximated by \( Q_1 \) polynomials, together with a bubble interface stabilization and \( P_0 \) Lagrange multipliers, as presented in section 4.7.1 (page 167). In that case, the non-mortar and mortar interfaces do not geometrically match. Then, to formulate the weak-continuity constraint, the displacements of the mortar side are projected on the non-mortar side by elementary plane projections on the non-mortar faces. Of course, the previous analysis do not take this approximation into account. A better approach would have been to consider a \( Q_2 \) approximation for the displacements, with an isoparametric description of the interface as recently analyzed in [FMW04]. Nevertheless, the bold approach presented proves to provide good
results in that simple case. The distribution of maximal stresses over the deformed configuration is represented on figure 4.25, both for conforming and non-conforming first order approximations. The quality of the non-conforming approximation shows here the small influence of the geometric non-conformity. The influence of the choice of the non-mortar side is also studied, and the relative gap of maximal stresses between the two possible choices is represented on figure 4.26. Because of the homogeneity of the material and because the non-conforming interface is not in a high stress region, such an influence remains very small.

From a practical point of view in the case of discontinuous mortar elements, let us underline that when dealing with a penalized formulation of the mortar constraint, or the elimination of the constraint as well, the assembling of the stiffness matrix of the problem in displacements can be done in a purely local way due to the discontinuity of the Lagrange multipliers. Indeed, in the corresponding stiffness operator:

$$\tilde{A} + \frac{1}{\eta \delta_{\min}} B^t C^{-1} B = \hat{A} + \frac{1}{\eta \delta_{\min}} \sum_{F \in F_s} B_F^t C_F^{-1} B_F,$$

with the notation used in (4.75), the second term can be computed element by element. Moreover, no special treatment is needed on the boundary of the interfaces, which is a great advantage in terms of implementation. The price to pay for these numerical advantages lies in the implementation of the proposed bubble stabilization.

**Remark 4.20.** A major practical problem concerns the case when the discretization of the mortar and non-mortar interfaces do not geometrically match. Sometimes, in the case of second order approximation for the displacements, an isoparametric discretization of the interface enables perfect geometric matching and the work done by [FMW04] ensures optimal properties. Nevertheless, in real life cases, such a matching often proves to be impossible and the reformulation of the interface weak-continuity constraint on a regularized interface is crucial, especially when dealing with non-linear elasticity. Indeed, stress singularity on the non-mortar interface may occur during large deformations if this interface is not regularized, and Newton’s method convergence is then compromised. Some interesting works regarding these aspects have been published by T. Laursen and M. Puso, and propose Gregory or Hermite patch regularization of the interface (see [PL02, PL03, Pus04]). Such contributions deal also with the treatment of contact surfaces. A comparable regularization approach is proposed and tested in the next chapter of the present work, when considering practical industrial implementation.
Fig. 4.19 – Distribution of $\sigma_{11}$ stresses on the deformed configuration of the non-conforming (top) and conforming (bottom) models, by using a second order approximation for the displacements.
Fig. 4.20 – Relative gap of $\sigma_{11}$ stresses between the solutions computed on the non-conforming model for the two possible choices of the non-mortar side, when using a first order (top) and a second order (bottom) approximation for the displacements. The pictures on the right column are zooms on the finer side of the interface.
Fig. 4.21 – Snapshots of the computed dynamics of the beam by using a non-conforming first order approximation of the displacements.
Fig. 4.22 – Horizontal displacement of the central node of the tip of the beam as a function of time, both for the non-conforming and conforming first order space approximation of the beam, together with a trapezoidal approximation in time. Simulations done with 20, 50 and 100 time steps per period. The good agreement confirms the optimality of the non-conforming space approximation.
4.9. Numerical tests for discontinuous mortar-elements

Fig. 4.23 – Conforming (1456 nodes, 1350 elements) and non-conforming (973 nodes, 810 elements) meshes of a cylinder in plane displacements.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young modulus $E$</td>
<td>5000 Pa</td>
</tr>
<tr>
<td>Poisson coefficient $\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>Internal pressure $p$</td>
<td>100 Pa</td>
</tr>
<tr>
<td>Internal radius</td>
<td>1.0 m</td>
</tr>
<tr>
<td>Interface radius</td>
<td>1.33 m</td>
</tr>
<tr>
<td>External radius</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Maximal displacement under loading</td>
<td>0.058 m</td>
</tr>
</tbody>
</table>

Fig. 4.24 – Characteristics of the cylinder.
Fig. 4.25 – Distribution of maximal stresses in a cylinder under pressure both for conforming and non-conforming space approximation.
Fig. 4.26 – Relative gap of the $\sigma_{11}$ stresses between the solutions computed on the non-conforming model for the two possible choices of the non-mortar side, when using a stabilized first order approximation for the displacements and piecewise constant Lagrange multipliers.
4.10 Appendix A: Mesh-dependent norms.

We present here some useful elementary results for the mesh-dependent norms introduced in the text.

First, the duality between the mesh-dependent norms expresses as follows:

**Lemma 4.17.** For all \( \lambda \in \mathbb{H}^{-1/2}(\Gamma_m) \), we have:

\[
\|\lambda\|_{\delta,-\frac{1}{2},m} = \sup_{\phi \in \mathbb{E}^{1/2}(\Gamma_m)} \frac{\int_{\Gamma_m} \lambda \cdot \phi}{\|\phi\|_{\delta,-\frac{1}{2},m}}.
\]

**Proof:** Let \( \lambda \in \mathbb{H}^{-1/2}(\Gamma_m) \). It is straightforward by a standard Cauchy-Schwarz inequality applied on each face of \( \Gamma_m \) that:

\[
\int_{\Gamma_m} \lambda \cdot \phi \leq \|\lambda\|_{\delta,-\frac{1}{2},m} \|\phi\|_{\delta,-\frac{1}{2},m}, \quad \forall \phi \in \mathbb{H}^{1/2}(\Gamma_m),
\]

hence:

\[
\|\lambda\|_{\delta,-\frac{1}{2}} \geq \sup_{\phi \in \mathbb{E}^{1/2}(\Gamma_m)} \frac{\int_{\Gamma_m} \lambda \cdot \phi}{\|\phi\|_{\delta,-\frac{1}{2},m}}.
\]

Conversely, by introducing \( \phi = \sum_{F \in \mathcal{F}_{m;\delta_m}} h(F) \lambda|_F \):

\[
\sup_{\psi \in \mathbb{E}^{1/2}(\Gamma_m)} \frac{\int_{\Gamma_m} \lambda \psi}{\|\psi\|_{\delta,-\frac{1}{2},m}} \geq \frac{\int_{\Gamma_m} \lambda \phi}{\|\phi\|_{\delta,-\frac{1}{2},m}} = \|\lambda\|_{\delta,-\frac{1}{2},m}.
\]

As the mesh of \( \Gamma_m \) is inherited from the non-mortar side mesh \( \mathcal{T}_{k(m);h_{k(m)}} \), we have the following trace result:

**Lemma 4.18.** There exist a constant \( C > 0 \) independent of the discretization such that for all \( u \in H^1(\Omega_{k(m)}) \):

\[
\|u|_{\Gamma_m}\|_{\delta,-\frac{1}{2},m} \leq C \sum_{T \in \mathcal{T}_{k(m);h_{k(m)}}} \frac{1}{h(T)^2} \|u\|^2_{L^2(T)} + \|\nabla u\|^2_{L^2(T \times d)},
\]

**Proof:** Let us denote by \( \phi_m = u|_{\Gamma_m} \). For all \( F \in \mathcal{F}_{m;\delta_m} \), by a standard change of variable onto the reference element \( \tilde{F} \):

\[
\|\phi_m\|^2_{L^2(\tilde{F})} \leq C \, \text{meas}(\tilde{F}) \|\hat{\phi}\|^2_{L^2(\tilde{F})},
\]
and by the standard trace theorem in Sobolev spaces, we have with $T = T(F)$:

$$
\|\phi_m\|_{L^2(F)^d}^2 \leq C \text{meas}(F) \left( \|\nabla \hat{u}\|_{L^2(T)^{d \times d}}^2 + \|\hat{u}\|_{L^2(T)^d}^2\right)
$$

$$
\leq C \frac{\text{meas}(F)}{h(T)^2} \left( \frac{h(T)^2}{\text{meas}(T)} \|\nabla u\|_{L^2(T)^{d \times d}}^2 + \frac{1}{h(T)^2} \|u\|_{L^2(T)^d}^2\right).
$$

By regularity of the mesh, we have $\text{meas}(T) \geq Ch(T)^d$ and also $\text{meas}(F) \leq Ch(F)^{d-1} \leq Ch(T)^{d-1}$ so that:

$$
\|\phi_m\|_{L^2(F)^d}^2 \leq C h(T) \left( \|\nabla u\|_{L^2(T)^{d \times d}}^2 + \frac{1}{h(T)} \|u\|_{L^2(T)^d}^2\right).
$$

Then, by summing over the $F \in \mathcal{F}_{m;\delta_m}$:

$$
\sum_{F \in \mathcal{F}_{m;\delta_m}} \frac{1}{h(T)} \|\phi_m\|_{L^2(F)^d}^2 \leq C \sum_{T \in \mathcal{T}_{k(m)};h_{k(m)}} \frac{1}{h(T)^2} \|u\|_{L^2(T)^{d \times d}}^2 + \|\nabla u\|_{L^2(T)^d}^2.
$$

Conversely, a lifting result can be established on $W_{m;\delta_m} \cap \mathbb{H}^{1/2}(\Gamma_m)$. For that purpose, we introduce the definition of discrete extension by zero operators:

**Definition 4.2.** Let $\phi_m \in W_{m;\delta_m}$, and $(a_i)_i$; the nodes associated to the Lagrange degrees of freedom of the functions in $X_{k(m);h_{k(m)}}$. The discrete extension by zero operator $\mathcal{R}_{m;h_m}$ over $X_{k(m);h_{k(m)}}$ is defined on the non-mortar side by $\mathcal{R}_{m;\delta_m} \phi_k \in X_{k(m);h_{k(m)}}$ such that:

$$
\mathcal{R}_{m;\delta_m} \phi_k(a_i) = \begin{cases} 
\phi_k(a_i), & a_i \in \Gamma_m, \\
0, & a_i \notin \Gamma_m.
\end{cases}
$$

and the discrete extension by zero operator $\mathcal{\tilde{R}}_{m;\delta_m}$ over $X_h$ by:

$$
\mathcal{\tilde{R}}_{m;\delta_m} \phi_m = \begin{cases} 
\mathcal{R}_{m;\delta_m} \phi_m, & \text{on } \Omega_{k(m)}, \\
0, & \text{elsewhere}.
\end{cases}
$$

**Lemma 4.19.** There exist a constant $C > 0$ independent of the discretization such that for all $\phi_m \in W_{m;\delta_m} \cap \mathbb{H}^{1/2}(\Gamma_m)$,

$$
\|\mathcal{R}_{m;\delta_m} \phi_m\|_{H^1(\Omega_{k(m)})} \leq C \|\phi_m\|_{\mathbb{H}^{1/2}(\Gamma_m)}.
$$
Proof: We have:
\[
\|\mathcal{R}_{m;\delta_m} \hat{\phi}_m\|_{H^1(\Omega_{k(m)})}^2 \leq \sum_{F \in \mathcal{F}_{m;\delta_m}} \sum_{T \in \mathcal{T}_{k(m); h_{k(m)}}: T \cap F \neq \emptyset} \|\mathcal{R}_{m;\delta_m} \phi_m\|_{H^1(T)}^2.
\]

For all \( F \in \mathcal{F}_{m;\delta_m} \) the number of elements \( T \in \mathcal{T}_{k(m); h_{k(m)}} \) such that \( T \cap F \neq \emptyset \) is bounded independently of \( h \) by the shape regularity of the mesh. Concerning the tetrahedron \( T = T(F) \) whose face is \( F \), we have:

\[
\|\mathcal{R}_{m;\delta_m} \phi_m\|_{H^1(T)}^2 = \frac{1}{L^2_{k(m)}} \|\mathcal{R}_{m;\delta_m} \phi_m\|_{L^2(T)}^2 + \|\nabla \mathcal{R}_{m;\delta_m} \phi_m\|_{L^2(T)}^2 \leq C \left( \frac{\mathrm{meas}(T)}{L^2_{k(m)}} \|\hat{\mathcal{R}}_{m;\delta_m} \hat{\phi}_m\|_{L^2(T)}^2 + \frac{\mathrm{meas}(T)}{h(T)^2} \|\nabla \hat{\mathcal{R}}_{m;\delta_m} \hat{\phi}_m\|_{L^2(T)}^2 \right),
\]

and since \( h(T) < L_{k(m)} \):

\[
\leq C \frac{\mathrm{meas}(T)}{h(T)^2} \|\hat{\mathcal{R}}_{m;\delta_m} \hat{\phi}_m\|_{H^1(T)}^2.
\]

By equivalence of the norms for discrete functions on \( \hat{T} \), we get:

\[
\|\mathcal{R}_{m;\delta_m} \phi_m\|_{H^1(T)}^2 \leq C \frac{\mathrm{meas}(T)}{h(T)^2} \|\hat{\phi}_m\|_{L^2(\hat{T})}^2 \leq C \frac{h(T)^{d-2}}{h(F)^{d-1}} \|\phi_m\|_{L^2(F)}^2.
\]

Let us consider now a tetrahedron \( T \in \mathcal{T}_{k(m); h_{k(m)}} \) sharing only an edge or a vertex with \( F \). The number of these tetrahedra is bounded by regularity of the mesh. The Lagrange finite element nodes on the reference face \( \hat{F} \) are denoted by \((\hat{a}_i)_i\). We obtain:

\[
\|\mathcal{R}_{m;\delta_m} \phi_m\|_{H^1(T)}^2 = \frac{1}{L^2_{k(m)}} \|\mathcal{R}_{m;\delta_m} \phi_m\|_{L^2(T)}^2 + \|\nabla \mathcal{R}_{m;\delta_m} \phi_m\|_{L^2(T)}^2 \leq C \left( \frac{\mathrm{meas}(T)}{L^2_{k(m)}} \|\hat{\mathcal{R}}_{m;\delta_m} \hat{\phi}_m\|_{L^2(T)}^2 + \frac{\mathrm{meas}(T)}{h(T)^2} \|\nabla \hat{\mathcal{R}}_{m;\delta_m} \hat{\phi}_m\|_{L^2(T)}^2 \right),
\]

and using that \( h(T) < L_{k(m)} \) and the equivalence of the norms for discrete functional spaces:

\[
\leq C \frac{\mathrm{meas}(T)}{h(T)^2} \|\hat{\mathcal{R}}_{m;\delta_m} \hat{\phi}_m\|_{H^1(T)}^2 \leq C \frac{\mathrm{meas}(T)}{h(T)^2} \max_i |\phi_m(\hat{a}_i)|^2
\]

\[
\leq C \frac{\mathrm{meas}(T)}{h(T)^2} \|\hat{\phi}_m\|_{L^2(\hat{T})}^2 \leq C \frac{\mathrm{meas}(T)}{h(T)^2} \frac{1}{\mathrm{meas}(\hat{T})} \|\phi_m\|_{L^2(\hat{T})}^2 \leq C \frac{h(T)^{d-2}}{h(F)^{d-1}} \|\phi_m\|_{L^2(F)}^2.
\]

Then, the announced result is obtained by using the shape regularity of the mesh and summing the previous inequalities. \( \square \)
4.11 Appendix B : Independence of the Korn’s constant with respect to the shape of a domain

In this section, we detail a proof for lemmas 4.5 and 4.6, page 125, for regular domains $\Omega_k \subset \mathbb{R}^d$ satisfying items 1 to 6 in Assumption 4.4, page 123. For clarity, we denote here $\Omega$ instead of $\Omega_k$, and recall these assumptions:

1. there exists a finite collection of reference domains $(\hat{\Omega}_j)_{1 \leq j \leq J}$ of unit diameter, of compact sets $(K_j)_{1 \leq j \leq J}$ and of maps $\varphi_j : \hat{\Omega}_j \times K_j \to \mathbb{R}^d$, $1 \leq j \leq J$ such that for all $1 \leq j \leq J$:
   $$\text{diam} \left( \varphi_j(\hat{\Omega}_j, p) \right) = 1, \quad \forall p \in K_j,$$
   and the following application:
   $$
   \begin{align*}
   K_j & \to W^{1,\infty}(\hat{\Omega}_j)^d, \\
   p & \mapsto \varphi_j(\cdot, p),
   \end{align*}
   $$
   is continuous,

2. for all $1 \leq j \leq J$, there exists a constant $C_j > 0$ such that:
   $$\det \frac{\partial \varphi_j}{\partial \hat{x}}(\hat{x}, p) \geq C_j, \quad \forall p \in K_j, \text{ for almost all } \hat{x} \in \hat{\Omega}_j,$$

3. there exists a $j$ with $1 \leq j \leq J$ and an element $p \in K_j$ such that within a scaling factor:
   $$\frac{1}{\text{diam}(\Omega)} \Omega = \varphi_j(\hat{\Omega}_j, p).$$

Moreover, we consider that:

4. there exists a finite collection of reference interfaces $(\hat{\gamma}_j)_{1 \leq j \leq J}$, with $\hat{\gamma}_j \subset \partial \hat{\Omega}_j$, $1 \leq j \leq J$, and that the application:
   $$
   \begin{align*}
   K_j & \to W^{1,\infty}(\hat{\gamma}_j)^d, \\
   p & \mapsto \varphi_j(\cdot, p),
   \end{align*}
   $$
   is continuous,

5. for all $1 \leq j \leq J$, there exists a constant $C_j > 0$ such that:
   $$\det \frac{\partial \varphi_j}{\partial \hat{x}}(\hat{x}, p) \geq C_j, \quad \forall p \in K_j, \text{ for almost all } \hat{x} \in \hat{\gamma}_j,$$
   and when $\gamma$ is a part of the boundary of $\Omega = \varphi_j(\hat{\Omega}_j, p)$, we assume that:
   $$
   \frac{1}{\text{diam}(\gamma)} \gamma = \varphi_j(\hat{\gamma}_j, p).
   $$

Remark 4.21. Let us notice that the application $\varphi_j$ and the compact set $K_j$ can in fact be different when considering the reference domain $\hat{\Omega}_j$ or the part $\hat{\gamma}_j$ of its boundary.

In this section, $C$ will denote various positive constants independent of the domain $\Omega$. 
4.11.1 Poincaré-Friedrichs inequalities

As a preliminary, let us prove the following:

**Lemma 4.20.** There exists a constant $C$ independent of any domain $\Omega$ satisfying assumptions 1, 2, 3, and of any $\gamma \subset \partial \Omega$ satisfying assumptions 4, 5, 6 such that:

$$
\frac{1}{\text{diam}(\Omega)^2} \|v\|_{L^2(\Omega)^d}^2 \leq C \left( \|v\|_{H^1(\Omega)^d}^2 + \frac{1}{\text{diam}(\Omega)^{2+d}} \left| \int_\Omega v \, dx \right|^2 \right),
$$

(4.86)

$$
\frac{1}{\text{diam}(\Omega)^2} \|v\|_{L^2(\Omega)^d}^2 \leq C \left( \|v\|_{H^1(\Omega)^d}^2 + \frac{1}{\text{diam}(\Omega)^{2+d}} \left| \int_\gamma v \, d\sigma \right|^2 \right),
$$

(4.87)

for all $v \in H^1(\Omega)^d$.

**Proof:** The two inequalities can be easily proved on $\hat{\Omega}_j$ by a contradiction argument for any function $\hat{v} \in H^1(\hat{\Omega}_j)^d$ and any $1 \leq j \leq J$. Proofs can be found in [Nec67, Wlo87].

For any $v \in H^1(\Omega)^d$, there exists an integer $j$ and a function $\hat{v} \in H^1(\hat{\Omega}_j)^d$ such that $v \circ \varphi_j = \hat{v}$, and by classical changes of variable, it follows from assumptions 2 and 5 that:

$$
\begin{align*}
&\left\{ \begin{array}{l}
\|\hat{v}\|_{L^2(\hat{\Omega}_j)^d}^2 \leq \|\nabla \varphi_j\|_{L^\infty(\Omega)} \|\hat{v}\|_{L^2(\hat{\Omega}_j)^d}^2,

\|\hat{v}\|_{H^1(\hat{\Omega}_j)^d}^2 = \int_{\hat{\Omega}_j} \left| \frac{\partial v}{\partial x} \right|^2 \left| \frac{\partial \varphi_j}{\partial x} \right|^2 \det(\nabla \varphi_j)^{-1} \\
&\quad \leq \|(\det \nabla \varphi_j)^{-1}\|_{L^\infty(\Omega)} \|\nabla \varphi_j\|_{L^\infty(\Omega)} \|\nabla \varphi_j\|_{L^\infty(\Omega)} \left| \int \nabla v \, dx \right| \|v\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \},

\|v\|_{L^2(\Omega)^d} \leq \|\nabla \varphi_j\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)^d} \|v\|_{L^2(\Omega)^d} \|v\|_{L^2(\Omega)^d} \|v\|_{L^2(\Omega)^d} \|v\|_{L^2(\Omega)^d} \},

\int_{\hat{\Omega}_j} \hat{v} \, d\bar{x} \leq \left( C_j \right)^{-1} \|\nabla \varphi_j\|_{L^\infty(\hat{\Omega}_j)} \left| \int \nabla v \, dx \right| \leq \left( C_j \right)^{-1} \|v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \},

\int_{\tilde{\gamma}_j} \hat{v} \, d\bar{\sigma} \leq \left( C_j \right)^{-1} \|\nabla \varphi_j\|_{L^\infty(\hat{\Omega}_j)} \left| \int_{\gamma} v \, d\sigma \right| \leq \left( C_j \right)^{-1} \|v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \},
\end{array} \right.
\end{align*}
$$

this latest statement being justified in remark 4.22. Moreover, from the continuity assumption 1 (resp. 4) and the fact that $(K_j)_{1 \leq j \leq J}$ are compact sets, we obtain the uniform boundedness of $\|\nabla \varphi_j(\cdot, p)\|_{L^\infty(\hat{\Omega}_j)}$ (resp. $\|\nabla \varphi_j(\cdot, p)\|_{L^\infty(\tilde{\gamma}_j)}$) with respect to $p \in K_j$. We obtain as a consequence that:

$$
\begin{align*}
&\left\{ \begin{array}{l}
\|v\|_{L^2(\Omega)^d} \leq C \text{diam}(\Omega)^d \|\hat{v}\|_{L^2(\hat{\Omega}_j)^d}^d,

\|v\|_{H^1(\Omega)^d} \leq C \text{diam}(\Omega)^{2-d} \|v\|_{H^1(\Omega)^d}^2,

\left| \int_{\hat{\Omega}_j} \hat{v} \, d\bar{x} \right| \leq C \text{diam}(\Omega)^{-d} \left| \int_{\Omega} v \, dx \right|,

\left| \int_{\tilde{\gamma}_j} \hat{v} \, d\bar{\sigma} \right| \leq C \text{diam}(\Omega)^{-1-d} \left| \int_{\gamma} v \, d\sigma \right|.
\end{array} \right.
\end{align*}
$$

(4.88)
This yields, using the inequalities (4.86) and (4.87) written on $\hat{\Omega}_j$:

$$\frac{1}{\text{diam}(\Omega)^2} \|v\|_{L^2(\Omega)^d}^2 \leq C \text{diam}(\Omega)^{d-2} \|\hat{v}\|_{L^2(\hat{\Omega}_j)^d}^2$$

$$\leq C \hat{C}_j \left( \|\hat{v}\|_{H^1(\hat{\Omega}_j)^d}^2 + \left[ \int_{\hat{\Omega}_j} \hat{v} \, d\hat{x} \right]^2 \right) \text{diam}(\Omega)^{d-2}$$

$$\leq C \hat{C}_j \left( \|v\|_{H^1(\Omega)^d}^2 + \frac{1}{\text{diam}(\Omega)^{2+d}} \left[ \int_{\Omega} v \, dx \right]^2 \right),$$

and also that:

$$\frac{1}{\text{diam}(\Omega)^2} \|v\|_{L^2(\Omega)^d}^2 \leq C \text{diam}(\Omega)^{d-2} \|\hat{v}\|_{L^2(\hat{\Omega}_j)^d}^2$$

$$\leq C \hat{C}_j \left( \|\hat{v}\|_{H^1(\hat{\Omega}_j)^d}^2 + \left[ \int_{\hat{\Omega}_j} \hat{v} \, d\hat{x} \right]^2 \right) \text{diam}(\Omega)^{d-2}$$

$$\leq C \hat{C}_j \left( \|v\|_{H^1(\Omega)^d}^2 + \frac{1}{\text{diam}(\Omega)^{d}} \left[ \int_{\Omega} v \, dx \right]^2 \right),$$

hence the proof. \qed

**Remark 4.22.** By construction of the jacobian $J = \det \nabla \varphi_j$, we have for all $dM \in \mathbb{R}^d$:

$$J \, d\hat{M} \cdot \hat{n} \, d\hat{\sigma} = dM \cdot n \, d\sigma,$$

where $n$ (resp. $\hat{n}$) denotes the outward normal unit vector on $\gamma$ (resp. $\hat{\gamma}$), and $d\sigma$ (resp. $d\hat{\sigma}$) is the surfacic measure over $\gamma$ (resp. $\hat{\gamma}$). Moreover:

$$J \, d\hat{M} \cdot \hat{n} \, d\hat{\sigma} = \hat{dM} \cdot n \, d\sigma = (\nabla \varphi_j \cdot dM) \cdot n \, d\sigma,$$

yielding by identification:

$$J \, \hat{n} \, d\hat{\sigma} = (\nabla \varphi_j)^t \cdot n \, d\sigma,$$

yielding:

$$d\hat{\sigma} = J^{-1} |(\nabla \varphi_j)^t \cdot n| \, d\sigma \leq C_j^{-1} \|\nabla \varphi_j\|_{L^\infty(\hat{\Omega}_j)^{d \times d}} \, d\sigma.$$
Lemma 4.21. There exists a constant $C$ such that for any domain $\Omega$ satisfying the above assumptions 1, 2, 3, the following inequality holds:

$$|v|_{H^1(\Omega)^d} \leq C \left( \|\varepsilon(v)\|_{L^2(\Omega)^{d \times d}} + \frac{1}{diam(\Omega)^{d/2}} \left| \int_{\Omega} \nabla \times v \right| \right),$$

for all $v \in H^1(\Omega)^d$.

Remark 4.23. The scaling $diam(\Omega)^{1-d}$ instead of $diam(\Omega)^{-d/2}$ which appears in [Bre04] seems to be a mistake. Indeed, it is then straightforward by a rescaling argument that doing so, the best constant $C$ is not independent of $diam(\Omega)$.

Proof : Because of scale invariance, we can suppose that $diam(\Omega) = 1$. Let us first observe that the inequality is true, and can be proved by a contradiction argument and Korn’s first inequality (cf. [LM72], page 110). Moreover, the resulting constant is independent of $diam(\Omega)$ from the adopted scaling of the two sides of the inequality, but a priori depends on the shape of $\Omega$, and we denote it by $C(\Omega)$.

Now, let us fix $1 \leq j \leq J$, and consider the closed subset:

$$\tilde{W}^{1,\infty}(\Omega_j)^d = \{ \phi \in W^{1,\infty}(\Omega_j)^d, \ diam(\phi(\Omega_j)) = 1, \ det \tilde{\nabla} \phi \geq C_j \text{ almost everywhere on } \Omega_j \},$$

endowed with the usual norm of $W^{1,\infty}(\Omega_j)^d$. Let us first show that $C(\phi(\Omega_j))$ is continuous with respect to $\phi \in \tilde{W}^{1,\infty}(\Omega_j)^d$, where $C(\phi(\Omega_j))$ is given by:

$$C(\phi(\Omega_j)) = \sup \frac{|\nabla \phi^{-1}|_{H^1(\Omega)^d}}{|\nabla \phi^{-1}|_{H^1(\Omega_j)^d} = 1} \left( \varepsilon(\phi^{-1})\|_{L^2(\Omega)^{d \times d}} + |\int_{\Omega_j} \nabla \times (\phi^{-1})| \right).$$

$$= \sup \frac{R_j(\tilde{\nabla} \phi)}{\sup |\nabla \phi^{-1}|_{H^1(\Omega_j)^d} = 1} \frac{N_j(\tilde{\nabla} \phi)}{D_j(\tilde{\nabla} \phi)}.$$

For this purpose, let us detail that both $(N_j(\tilde{\nabla} \phi))_\phi$ and $(D_j(\tilde{\nabla} \phi))_\phi$ are equicontinuous sets of functions with respect to $\phi \in \tilde{W}^{1,\infty}(\Omega_j)^d$. Let $\hat{F}(\phi)$ be given by:

$$\hat{F}(\phi) = (\nabla \phi)^{-1} \left( \det \nabla \phi \right)^{1/2} = \left( \frac{\text{cof } \nabla \phi}{\det \nabla \phi} \right)^{1/2}.$$

By construction, the map $\hat{F}$ is continuous with respect to $\phi$ on $W^{1,\infty}(\Omega_j)^d$. From the changes of variable $\phi^{-1}$ and $\psi^{-1}$, the triangular inequality, the equivalence of the norms $|A|_2 = \sup_{x \in \mathbb{R}^d, |x| = 1} |Ax|_2$ and $|A| = (A : A)^{1/2}$ for any matrix $A \in \mathbb{R}^{d \times d}$, and the fact that
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\[ |A \cdot B|_2 \leq |A|_2 |B|_2, \] we obtain that:

\[
\left| \hat{v} \circ \phi^{-1} \right|_{H^1(\phi(\tilde{\Omega}_j))^d}^2 - \left| \hat{v} \circ \psi^{-1} \right|_{H^1(\psi(\tilde{\Omega}_j))^d}^2 = \left\| \nabla \hat{v} \cdot F(\phi) \right\|_{L^2(\tilde{\Omega}_j)^{d \times d}} - \left\| \nabla \hat{v} \cdot F(\psi) \right\|_{L^2(\tilde{\Omega}_j)^{d \times d}}
\]

\[ \leq \left\| \nabla \hat{v} \cdot (F(\phi) - F(\psi)) \right\|_{L^2(\tilde{\Omega}_j)^{d \times d}}
\]

\[ = \left( \int_{\tilde{\Omega}_j} \left| \nabla \hat{v} \cdot (F(\phi) - F(\psi)) \right|^2 \, d\tilde{x} \right)^{1/2}
\]

\[ \leq C \left( \int_{\tilde{\Omega}_j} \left| \nabla \hat{v} \right|^2 \left| F(\phi) - F(\psi) \right|^2 \, d\tilde{x} \right)^{1/2}
\]

\[ \leq C \left\| F(\phi) - F(\psi) \right\|_{L^\infty(\tilde{\Omega}_j)^{d \times d}},
\]

because \( |\hat{v}|_{H^1(\tilde{\Omega}_j)^{d \times d}} = 1. \) By the same estimation and the fact that \( |A| = |A^t| \) for any matrix \( A \in \mathbb{R}^{d \times d}, \) we also get:

\[
\left\| \varepsilon(\hat{v} \circ \phi^{-1}) \right\|_{L^2(\phi(\tilde{\Omega}_j))^{d \times d}} - \left\| \varepsilon(\hat{v} \circ \psi^{-1}) \right\|_{L^2(\psi(\tilde{\Omega}_j))^{d \times d}}
\]

\[ = \left\| \frac{1}{2} \left( \nabla \hat{v} \cdot F(\phi) + F(\phi)^t \cdot (\nabla \hat{v})^t \right) \right\|_{L^2(\tilde{\Omega}_j)^{d \times d}} - \left\| \frac{1}{2} \left( \nabla \hat{v} \cdot F(\psi) + F(\psi)^t \cdot (\nabla \hat{v})^t \right) \right\|_{L^2(\tilde{\Omega}_j)^{d \times d}}
\]

\[ \leq \left\| \frac{1}{2} \left( \nabla \hat{v} \cdot (F(\phi) - F(\psi)) + (F(\phi) - F(\psi))^t \cdot (\nabla \hat{v})^t \right) \right\|_{L^2(\tilde{\Omega}_j)^{d \times d}}
\]

\[ \leq C \left\| F(\phi) - F(\psi) \right\|_{L^\infty(\tilde{\Omega}_j)^{d \times d}}.
\]
Finally, denoting by $\tilde{F}(\phi) = F(\phi) (\det \nabla \phi)^{1/2}$, we get:

$$\left\| \int_{\phi(\hat{\Omega}_j)} \nabla \times (\hat{\nu} \circ \phi^{-1}) - \int_{\psi(\hat{\Omega}_j)} \nabla \times (\hat{\nu} \circ \psi^{-1}) \right\| \leq \left\| \int_{\phi(\hat{\Omega}_j)} \nabla \times (\hat{\nu} \circ \phi^{-1}) - \int_{\psi(\hat{\Omega}_j)} \nabla \times (\hat{\nu} \circ \psi^{-1}) \right\| = \frac{1}{2} \left\| \int_{\phi(\hat{\Omega}_j)} \nabla (\hat{\nu} \circ \phi^{-1}) - (\nabla (\hat{\nu} \circ \phi^{-1}))^* \right\| + \frac{1}{2} \left\| \int_{\psi(\hat{\Omega}_j)} \nabla (\hat{\nu} \circ \psi^{-1}) - (\nabla (\hat{\nu} \circ \psi^{-1}))^* \right\| \leq \frac{1}{2} \left\| \int_{\phi(\hat{\Omega}_j)} \nabla (\hat{\nu} \circ \phi^{-1}) \right\| + \frac{1}{2} \left\| \int_{\psi(\hat{\Omega}_j)} \nabla (\hat{\nu} \circ \psi^{-1}) \right\| = \left\| \int_{\hat{\Omega}_j} \nabla \hat{\nu} \cdot (\tilde{F}(\phi) - \tilde{F}(\psi)) \right\| \leq C \int_{\hat{\Omega}_j} \left\| \nabla \hat{\nu} \right\| \left\| \tilde{F}(\phi) - \tilde{F}(\psi) \right\| \leq C \left\| \tilde{F}(\phi) - \tilde{F}(\psi) \right\|_{L^\infty(\hat{\Omega}_j)^{d \times d}} \leq C \left\| \tilde{F}(\phi) - \tilde{F}(\psi) \right\|_{L^\infty(\hat{\Omega}_j)^{d \times d}}.$$

As a consequence, from the continuity of $F$ and $\tilde{F}$ on $W^{1,\infty}(\hat{\Omega}_j)$, $(N_j(\hat{\nu}, \phi))_{\hat{\nu}}$ and $(D_j(\hat{\nu}, \phi))_{\hat{\nu}}$ are equicontinuous sets of functions in $\hat{\nu}$ with respect to $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$. We now prove that the ratios $(R_j(\hat{\nu}, \phi))_{\hat{\nu}}$ are an equicontinuous set of functions with respect to $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$:

- By equicontinuity of $(D(\hat{\nu}, \phi))_{\hat{\nu}}$ with respect to $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$, inf$_{\hat{\nu}} D(\hat{\nu}, \phi)$ is continuous with respect to $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$. Now, let us fix $\phi \in W^{1,\infty}(\hat{\Omega}_j)^d$. The standard Korn’s second inequality shows that with a shape dependent constant $C_\phi > 0$:

$$D(\hat{\nu}, \phi) \geq C_\phi \left\| \hat{\nu} \circ \phi^{-1} \right\|_{H^1(\phi(\hat{\Omega}_j))^d} \geq C_\phi \left\| \nabla \phi (\det \nabla \phi)^{-1/2} \right\|_{L^\infty(\hat{\Omega}_j)^{d \times d}},$$

and entails that inf$_{\hat{\nu}} D(\hat{\nu}, \phi) > 0$. By continuity of inf$_{\hat{\nu}} D(\hat{\nu}, \psi)$ with respect to $\psi \in W^{1,\infty}(\hat{\Omega}_j)^d$, there exists a neighborhood $\mathcal{V}_\phi$ of $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$ in $W^{1,\infty}(\hat{\Omega}_j)^d$, such that there exists a constant $C^D_\phi > 0$ such that:

$$\forall \psi \in \mathcal{V}_\phi, \quad \text{inf}_{\hat{\nu}} D(\hat{\nu}, \phi) > C^D_\phi.$$

- By equicontinuity of $(N(\hat{\nu}, \phi))_{\hat{\nu}}$ with respect to $\phi \in W^{1,\infty}(\hat{\Omega}_j)^d$, sup$_{\hat{\nu}} N(\hat{\nu}, \psi)$ is continuous with respect to $\psi \in W^{1,\infty}(\hat{\Omega}_j)^d$, and is therefore bounded by $C^N_\phi$ on a neighborhood $\mathcal{V}_\phi$ of $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$ in $W^{1,\infty}(\hat{\Omega}_j)^d$, for all $\phi \in \tilde{W}^{1,\infty}(\hat{\Omega}_j)^d$. 


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We then get:

\[ |R_j(\hat{v}, \phi) - R_j(\hat{v}, \psi)| = \left| \frac{N_j(\hat{v}, \phi)}{D_j(\hat{v}, \phi)} - \frac{N_j(\hat{v}, \psi)}{D_j(\hat{v}, \psi)} \right| \]

\[ = \left| \frac{N_j(\hat{v}, \phi)(D_j(\hat{v}, \phi) - D_j(\hat{v}, \psi))}{D_j(\hat{v}, \phi)D_j(\hat{v}, \psi)} - \frac{N_j(\hat{v}, \phi) - N_j(\hat{v}, \psi)}{D_j(\hat{v}, \psi)} \right| \]

\[ \leq \frac{C_N}{C_D} |D_j(\hat{v}, \phi) - D_j(\hat{v}, \psi)| + \frac{1}{C_D} |N_j(\hat{v}, \phi) - N_j(\hat{v}, \psi)|. \]

Hence, the ratios \( (R_j(\hat{v}, \phi))_\hat{v} \) are an equicontinuous set of functions with respect to \( \phi \in \dot{W}^{1,\infty}(\Omega_j)^d \). The supremum \( C(\phi(\Omega_j)) \) of \( R_j(\hat{v}, \phi) \) over \( \hat{v} \) is therefore a continuous function of \( \phi \in \dot{W}^{1,\infty}(\Omega_j)^d \).

Because the application:

\[ \mathcal{K}_j \rightarrow \dot{W}^{1,\infty}(\Omega_j)^d, \]

\[ p \mapsto \varphi_j(\cdot, p), \]

is continuous from assumption 1, the function \( C(\varphi_j(\Omega_j, p)) \) is a continuous function of \( p \in \mathcal{K}_j \) and since \( \mathcal{K}_j \) is a compact, \( C(\varphi_j(\Omega_j, p)) \) reaches its maximum value for a \( p \in \mathcal{K}_j \), entailing the existence of a constant \( C_j' \) such that:

\[ C(\varphi_j(\Omega_j, p)) \leq C_j', \quad \forall p \in \mathcal{K}_j. \]

Hence the proof, and the constant \( C = \max_{1 \leq j \leq J} C_j' \) in lemma 4.21.

4.11.3 Semi-norm estimates

As introduced in [Bre04], let \( \mathfrak{P} : H^1(\Omega)^d \rightarrow \mathcal{R}(\Omega) \) be the rigid motion projection such that for all \( \mathfrak{v} \in H^1(\Omega)^d \), \( \mathfrak{P} \mathfrak{v} \in \mathcal{R}(\Omega) \) is the unique rigid motion such that:

\[ \int_{\Omega} (\mathfrak{P} \mathfrak{v} - \mathfrak{v}) = 0, \quad \int_{\Omega} \nabla \times (\mathfrak{P} \mathfrak{v} - \mathfrak{v}) = 0. \]

The existence and uniqueness comes from the straightforward implication:

\[ \mathfrak{v} \in \mathcal{R}(\Omega), \quad \mathfrak{P} \mathfrak{v} = 0 \Rightarrow \mathfrak{v} = 0. \]

Lemma 4.22. Let \( \Phi \) be a semi-norm satisfying:

\[ |\mathfrak{v}|_{H^1(\Omega)^d} \leq C \Phi(\mathfrak{v}), \quad \forall \mathfrak{v} \in \mathcal{R}(\Omega), \quad (4.89) \]

\[ \Phi(\mathfrak{v} - \mathfrak{P} \mathfrak{v}) \leq C \|\varepsilon(\mathfrak{v})\|_{L^2(\Omega)^{d\times d}}, \quad \forall \mathfrak{v} \in H^1(\Omega)^d, \quad (4.90) \]
with a constant $C$ independent of any $\Omega$ satisfying assumptions 1, 2, 3. Then there exists a constant $C$ independent of $\Omega$ such that:

$$|v|_{H^1(\Omega)^d} \leq C \left( \|\varepsilon(v)\|_{L^2(\Omega)^{d\times d}}^2 + \Phi(v)^2 \right)^{1/2}, \quad \forall v \in H^1(\Omega)^d.$$ 

**Proof:** By using the triangular inequality, the property (4.89) of the semi-norm $\Phi$, and lemma 4.21, we get:

$$|v|_{H^1(\Omega)^d}^2 \leq |\mathcal{P}v + v - \mathcal{P}v|_{H^1(\Omega)^d}^2$$

$$\leq 2|\mathcal{P}v|_{H^1(\Omega)^d}^2 + 2|v - \mathcal{P}v|_{H^1(\Omega)^d}^2$$

$$\leq C\Phi(\mathcal{P}v)^2 + C\|\varepsilon(v - \mathcal{P}v)\|_{L^2(\Omega)^{d\times d}}^2$$

$$\leq C\Phi(\mathcal{P}v)^2 + C\|\varepsilon(v)\|_{L^2(\Omega)^{d\times d}}^2, \quad (4.91)$$

because $\varepsilon(\mathcal{P}v) = 0$, $\mathcal{P}v$ being a rigid body motion. Observing that the triangular inequality and assumption (4.90) entail:

$$\Phi(\mathcal{P}v) \leq \Phi(v) + \Phi(v - \mathcal{P}v)$$

$$\leq \Phi(v) + C\|\varepsilon(v - \mathcal{P}v)\|_{L^2(\Omega)^{d\times d}},$$

it is obtained from (4.91) that:

$$|v|_{H^1(\Omega)^d}^2 \leq C \left( \|\varepsilon(v)\|_{L^2(\Omega)^{d\times d}}^2 + \Phi(v)^2 \right),$$

hence the proof. \qed

We then have to check the assumptions (4.89) and (4.90) for the particular semi-norms $\Phi_1$ and $\Phi_2$ defined by:

$$\Phi_1(v) = \frac{1}{\text{diam}(\Omega)} \sup_{r \in \mathcal{R}(\Omega) \setminus \{0\}, \int_\Omega r = 0} \frac{\int_\Omega v \cdot r}{\|r\|_{L^2(\Omega)^d}}, \quad \forall v \in L^2(\Omega)^d,$$

and:

$$\Phi_2(v) = \frac{1}{\text{diam}(\Omega)^{1/2}} \sup_{r \in \mathcal{R}(\Omega) \setminus \{0\}, \int_\Omega r = 0} \frac{\int_{\Gamma} v \cdot r}{\|r\|_{L^2(\gamma)^d}}, \quad \forall v \in H^1(\Omega)^d.$$

We obtain indeed the:

**Lemma 4.23.** Under assumptions 1, 3 (resp. assumption 4, 6), the semi-norms $\Phi_1$ (resp. $\Phi_2$) satisfy the criterion (4.89).
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Proof : Because of scaling, we can restrict ourselves to the case where $\Omega$ has a unit diameter, since $\|\varepsilon(v)\|_{L^2(\Omega)^{d\times d}}^2$, $|v|^2_{H^1(\Omega)^d}$ and $\Phi_1^2(v)$ all scale like $diam(\Omega)^{d-2}$. The proof is done in three dimensions ($d = 3$), and can be adapted very simply to the bidimensional case ($d = 2$). Let $t, a \in \mathbb{R}^3$. Thus, $v(x) = t + a \times x$, for all $x \in \Omega$ is a rigid motion. It is a simple exercise to check that $|v|^2_{H^1(\Omega)^3} = 2|a|^2|\Omega| \leq C|a|^2$, because $\Omega$ has a unit diameter.

For all $1 \leq j \leq J$, the following application:

$$\mathcal{M}_1^j : \mathbb{R}^3 \times W^{1,\infty}(\hat{\Omega}_j)^d \to \mathbb{R}$$

$$(a, \varphi) \mapsto \mathcal{M}_1^j(a, \varphi) := \int_{\hat{\Omega}_j} |a \times (\varphi(\hat{x}) - G_{\varphi(\hat{\Omega}_j)}|^2 \det \nabla \varphi(\hat{x}) d\hat{x},$$

where $G_{\varphi(\hat{\Omega}_j)}$ is the center of gravity of $\varphi(\hat{\Omega}_j)$ defined as:

$$G_{\varphi(\hat{\Omega}_j)} = \frac{1}{|\varphi(\hat{\Omega}_j)|} \int_{\hat{\Omega}_j} \varphi(\hat{x}) \det \nabla \varphi(\hat{x}) d\hat{x},$$

is continuous. By introducing the compact set $\mathbb{S}^3 := \{a \in \mathbb{R}^3, |a| = 1\} \subset \mathbb{R}^3$, and from assumption 1, it follows by composition that the positive application:

$$\mathbb{S}^3 \times \mathcal{K}_j \to \mathbb{R}$$

$$(e, p) \mapsto \mathcal{M}_1^j(e, \varphi_j(\cdot, p)),$$

is also continuous, and because both $\mathbb{S}^3$ and $\mathcal{K}_j$ are compact sets, we deduce the existence of a lower bound. Therefore, there exists a constant $C_j > 0$ such that for all $e \in \mathbb{S}^3$ and all $p \in \mathcal{K}_j$:

$$\mathcal{M}_1^j(e, \varphi_j(\cdot, p)) \geq C_j.$$

By an homogeneity argument, we deduce for all $a \in \mathbb{R}^3$, and all $p \in \mathcal{K}_j$, that:

$$\mathcal{M}_1^j(a, \varphi_j(\cdot, p)) \geq C_j |a|^2 \geq C C_j |v|^2_{H^1(\Omega)^3}.$$

Hence the satisfaction of the criterion (4.89) for $\Phi_1$ with a constant $C' = C \max_{1 \leq j \leq J} C_j$.

We proceed the same concerning $\Phi_2$, by defining the following application for all $1 \leq j \leq J$:

$$\mathcal{M}_2^j : \mathbb{R}^3 \times W^{1,\infty}(\hat{\gamma}_j)^d \to \mathbb{R}$$

$$(a, \varphi) \mapsto \mathcal{M}_2^j(a, \varphi) := \int_{\hat{\gamma}_j} |a \times (\varphi(\hat{x}) - G_{\varphi(\hat{\gamma}_j)}|^2 m_{\varphi}(\hat{x}) d\hat{\sigma}(\hat{x}),$$

where $G_{\varphi(\hat{\gamma}_j)}$ is the center of gravity of $\varphi(\hat{\gamma}_j)$ defined as:

$$G_{\varphi(\hat{\gamma}_j)} = \frac{1}{|\varphi(\hat{\gamma}_j)|} \int_{\hat{\gamma}_j} \varphi(\hat{x}) m_{\varphi}(\hat{x}) d\hat{\sigma}(\hat{x}),$$
\( d\hat{\sigma} \) is the surfacic measure over \( \hat{\gamma}_j \), and \( m_\varphi(\hat{x}) \) is the metric defined as:

\[
m_\varphi(\hat{x}) = \det \nabla \varphi \left| (\nabla \varphi)^{-t} \cdot \hat{n} \right|,
\]

in which \( \hat{n} \) is the outward normal unit vector on \( \hat{\gamma}_j \). The application \( M^j_2 \) is continuous.

By introducing the compact set \( \mathbb{S}^3 := \{ a \in \mathbb{R}^3, \ |a| = 1 \} \subset \mathbb{R}^3 \), and from assumption 6, it follows by composition that the positive application:

\[
\mathbb{S}^3 \times \mathcal{K}_j \rightarrow \mathbb{R} \quad (e, p) \mapsto M^j_2(e, \varphi_j(\cdot, p)),
\]

is also continuous, and because both \( \mathbb{S}^3 \) and \( \mathcal{K}_j \) are compact sets, we deduce the existence of a lower bound. Therefore, there exists a constant \( C_j > 0 \) such that for all \( e \in \mathbb{S}^3 \) and all \( p \in \mathcal{K}_j \):

\[
M^j_2(e, \varphi_j(\cdot, p)) \geq C_j.
\]

By an homogeneity argument, we deduce for all \( a \in \mathbb{R}^3 \), and all \( p \in \mathcal{K}_j \), that:

\[
M^j_2(a, \varphi_j(\cdot, p)) \geq C_j |a|^2 \geq C_j |v|^2_{H^1(\Omega)^3}.
\]

Hence the proof.

The satisfaction of assumption (4.90) for the semi-norms \( \Phi_1 \) and \( \Phi_2 \) is obtained in [Bre04]:

**Lemma 4.24.** The semi-norms \( \Phi_1 \) and \( \Phi_2 \) satisfy the assumption (4.90).

**Proof:** Using the Cauchy-Schwarz inequality, Friedrichs inequality (lemma 4.20) and lemma 4.21, we get:

\[
\Phi_1(v - \mathcal{P}v) \leq \frac{1}{diam(\Omega)} \| v - \mathcal{P}v \|_{L^2(\Omega)^d} \\
\leq C \| v - \mathcal{P}v \|_{H^1(\Omega)^d} \\
\leq C \| \varepsilon(v - \mathcal{P}v) \|_{L^2(\Omega)^{d\times d}} = C \| \varepsilon(v) \|_{L^2(\Omega)^{d\times d}}.
\]

Concerning \( \Phi_2 \), the same result follows by using the Cauchy-Schwarz inequality, the Sobolev trace theorem (lemma 4.25), Friedrichs inequality (lemma 4.20) and lemma 4.21:

\[
\Phi_2(v - \mathcal{P}v) \leq \frac{1}{diam(\Omega)^{1/2}} \| v - \mathcal{P}v \|_{L^2(\gamma)^d} \\
\leq C \| v - \mathcal{P}v \|_{H^1(\Omega)^d} \\
\leq C \| v - \mathcal{P}v \|_{H^1(\Omega)^d} \\
\leq C \| \varepsilon(v - \mathcal{P}v) \|_{L^2(\Omega)^{d\times d}} = C \| \varepsilon(v) \|_{L^2(\Omega)^{d\times d}}.
\]
Hence the proof. □

The Sobolev trace theorem with shape independence of the constant is given by the:

**Lemma 4.25.** There exists a constant $C$ independent of any domain $\Omega$ satisfying assumptions 1, 2, 3, and of the part $\gamma$ of its boundary satisfying assumptions 4, 5, 6, such that:

$$
\frac{1}{\text{diam}(\Omega)} \int_{\gamma} v^2 \leq C \left( |v|_{H^1(\Omega)^d}^2 + \frac{1}{\text{diam}(\Omega)^2} \|v\|_{L^2(\Omega)^d}^2 \right),
$$

(4.92)

for all $v \in H^1(\Omega)^d$.

**Proof:** The inequality (4.92) is true for $\Omega = \hat{\Omega}_j$ and $\gamma = \hat{\gamma}_j$ for any $1 \leq j \leq J$, as the standard Sobolev trace inequality. For any $v \in H^1(\Omega)^d$, there exists an integer $j$ and a function $\hat{v} \in H^1(\hat{\Omega}_j)^d$ such that $v \circ \varphi_j \equiv \hat{v}$, and by classical changes of variable, we get as in the proof of lemma 4.20 that:

$$
\begin{align*}
\int_{\gamma} v^2 &\leq \|\text{cof} \nabla \varphi_j\|_{L^\infty(\hat{\Omega}_j)^{d \times d}} \int_{\hat{\gamma}_j} \hat{v}^2, \\
|\hat{v}|_{H^1(\hat{\Omega}_j)^d}^2 &\leq \|(\det \nabla \varphi_j)^{-1}\|_{L^\infty(\hat{\Omega}_j)^{d \times d}} \|\nabla \varphi_j\|_{L^\infty(\hat{\Omega}_j)^{d \times d}}^2 |v|_{H^1(\Omega)^d}^2 \leq C^{-1}_j \|\nabla \varphi_j\|_{L^\infty(\hat{\Omega}_j)^{d \times d}}^2 |v|_{H^1(\Omega)^d}^2, \\
|\hat{v}|_{L^2(\hat{\Omega}_j)^d}^2 &\leq \|(\det \nabla \varphi_j)^{-1}\|_{L^\infty(\hat{\Omega}_j)^{d \times d}} |v|_{L^2(\Omega)^d}^2 \leq C^{-1}_j |v|_{L^2(\Omega)^d}^2.
\end{align*}
$$

Moreover, from the continuity assumption 1 (resp. 4) and the fact that $(K_j)_{1 \leq j \leq J}$ are compact sets, we obtain the uniform boundedness of $\|\nabla \varphi_j(\cdot, p)\|_{L^\infty(\hat{\Omega}_j)}$ (resp. $\|\nabla \varphi_j(\cdot, p)\|_{L^\infty(\hat{\gamma}_j)}$) with respect to $p \in K_j$. We obtain as a consequence that:

$$
\begin{align*}
\int_{\gamma} v^2 &\leq C \text{diam}(\Omega)^{d-1} \int_{\hat{\gamma}_j} \hat{v}^2, \\
|v|_{L^2(\Omega)^d}^2 &\leq C \text{diam}(\Omega)^{-d} \|\hat{v}\|_{L^2(\hat{\Omega}_j)^d}^2, \\
|v|_{H^1(\Omega)^d}^2 &\leq C \text{diam}(\Omega)^{2-d} |\hat{v}|_{H^1(\hat{\Omega}_j)^d}^2.
\end{align*}
$$

This yields, using the inequality (4.92) written on $\hat{\Omega}_j$:

$$
\begin{align*}
\frac{1}{\text{diam}(\Omega)} \int_{\gamma} v^2 &\leq C \text{diam}(\Omega)^{d-2} \int_{\hat{\gamma}_j} \hat{v}^2 \\
&\leq C \hat{C}_j \text{diam}(\Omega)^{d-2} \left( |\hat{v}|_{L^2(\hat{\Omega}_j)^d}^2 + |\hat{v}|_{H^1(\hat{\Omega}_j)^{d \times d}}^2 \right) \\
&\leq C \hat{C}_j \left( \frac{1}{\text{diam}(\Omega)^2} \|v\|_{L^2(\Omega)^d}^2 + |v|_{H^1(\Omega)^d}^2 \right).
\end{align*}
$$

Hence the proof. □

**4.11.4 Conclusion**

Lemmas 4.5 and 4.6, page 125, are now proved as a consequence of lemmas 4.22 and 4.20. We recall them in the :
Lemma 4.26. There exists two constants $C_P$ and $C_N$, such that for all $\Omega_k$ and $\gamma_{kl}$ satisfying the assumptions 1, 2, 3, 4, 5, 6, the following inequality holds for all $v \in H^1(\Omega_k)^d$:

$$\|v\|_{H^1(\Omega_k)^d}^2 \leq C_P \left( \|\varepsilon(v)\|_{L^2(\Omega_k)^d}^2 + \frac{1}{\text{diam}(\Omega_k)} \left( \sup_{\mu \in M_k} \|\mu\|_{L^2(\gamma_{kl})^d} \right)^2 \right),$$  (4.93)

$$\|v\|_{H^1(\Omega_k)^d}^2 \leq C_N \left( \|\varepsilon(v)\|_{L^2(\Omega_k)^d}^2 + \frac{1}{\text{diam}(\Omega_k)^2} \left( \sup_{r \in \partial(\Omega_k)} \left( \int_{\Omega_k} v \cdot r \right)^2 \right) \right).$$  (4.94)

Proof: From lemma 4.22, which holds due to lemmas 4.23 and 4.24, we have:

$$|v|_{H^1(\Omega_k)^d}^2 \leq C \left( \|\varepsilon(v)\|_{L^2(\Omega_k)^d}^2 + \Phi_{1,\Omega_k} (v)^2 \right),$$  (4.95)

and using here a unit translation $r$, we deduce:

$$\frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} v dx \leq C \left( |v|_{H^1(\Omega_k)^d}^2 + \frac{1}{\text{diam}(\Omega_k)^{2+d}} \right).$$  (4.96)

On the other hand, using lemma 4.20, we have:

$$\frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} v dx \leq C \left( |v|_{H^1(\Omega_k)^d}^2 + \frac{1}{\text{diam}(\Omega_k)^{2+d}} \right).$$  (4.97)

Hence the proof of (4.93) by substituting (4.95) and (4.97) in (4.96) and adding the result to (4.95).

From lemma 4.22, which holds due to lemmas 4.23 and 4.24, we also have:

$$|v|_{H^1(\Omega_k)^d}^2 \leq C \left( \|\varepsilon(v)\|_{L^2(\Omega_k)^d}^2 + \Phi_{2,\Omega_k} (v)^2 \right),$$  (4.98)

and using here a unit translation $r$, we deduce:

$$\frac{1}{\text{diam}(\Omega_k)^2} \int_{\Omega_k} v dx \leq C \left( \sup_{r \in \partial(\Omega_k)} \left( \int_{\gamma_{kl}} v \cdot r \right)^2 \right).$$  (4.99)

Hence the proof of (4.94) by substituting (4.95) and (4.97) in (4.98) and adding the result to (4.95).
and using here a unit translation \( r \), we deduce:

\[
\frac{1}{\text{diam}(\Omega_k)^d} \left| \int_{\gamma_{kl}} v \, dx \right|^2 \leq \frac{1}{\text{diam}(\Omega_k)^2} \sup_{r \in \mathcal{R}(\Omega_k)} \frac{\left( \int_{\gamma_{kl}} v \cdot r \right)^2}{\| r \|_L^2(\gamma_{kl})^d}.
\]  

(4.101)

Hence the proof of (4.94) by substituting (4.98) and (4.101) in (4.100) and adding the result to (4.98). \( \square \)