BV estimates for mortar methods in linear elasticity

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Abstract

This paper is concerned with the convergence of mortar methods applied to linear elasticity. We prove that the conventional mesh-dependent norms used in the analysis of mortar methods are bounded below by the BV norm. When combined with standard results, this bound establishes a decomposition-independent and mesh-independent proof of the convergence of mortar methods in linear elasticity.

Key words: Mortar methods, error estimates, BV space, linear elasticity
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1 Introduction

Mortar methods were introduced by C. Bernardi, Y. Maday and A.T. Patera for the Poisson equation in [1,2] in order to formulate a weak continuity condition at the interface of subdomains in which different variational approximations are used. Relaxing the constraint on the boundaries of the interfaces, the formulation of F. Ben Belgacem [3] with Lagrange multipliers is the standard framework in which the method is understood at present time. One of the key aspects of the method consists of defining appropriate spaces of Lagrange multipliers for enforcing the gluing constraint. Indeed, the original proposal of a modified trace space [1,2] for Lagrange multipliers suffers from a number of shortcomings, such as the non-locality of the constraint over the interfaces, and a necessary special treatment of the boundary of the interfaces. Using the

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concept of bi-orthogonal bases for low order elements, both Wohlmuth [4] and Kim-Lazarov-Pasciak-Vassilevski [5] proposed Lagrange multipliers rendering the constraint diagonal. Earlier approaches used discontinuous Lagrange multipliers to localize constraint [6,7]. The design of bi-orthogonal bases for higher-order approximations is a subject of considerable interest at present (e. g., [8,9] and the references therein). In addition, for second-order approximations Seshaiyer [10] showed that the use of continuous first-order Lagrange multipliers over the interfaces are optimal, with the noticeable advantage of requiring no modification on interface boundaries. An interesting strategy that partakes from the aforementioned advantages is the use of discontinuous Lagrange multipliers of low order and interface stabilization, as proposed by Ben Belgacem [11] for the Stokes problem; and by Hauret and Le Tallec [12,13] for elastostatics and higher order approximations.

Mortar methods have proved a useful tool for domain decomposition and many domain decomposition algorithms have been adapted to this framework, including the Neumann-Neumann approach [14], substructuring approaches to solve the saddle-point problem [15,16]; a variant of the Bramble-Pasciak-Shatz preconditioner in two-dimensional [17]; and also FETI and FETI-DP approaches [18–20]. Multigrid methods [21,22] have also been extensively studied within the mortar framework. Other applications include: the Stokes equation [23–25]; Navier-Stokes equations [26,27]; linearized elasticity [28]; plate and shell problems [29,30]; and electromagnetism [31–35], among others. More recently, the mortar approach has additionally been extended or adapted to contact mechanics [36–39].

An outstanding question in the area of mortar formulations for elliptic problems concerns the difficulty in assembling the constraint. Indeed, the assembly by quadrature has been shown to result in sub-optimal properties [40]. Additional modifications of the method have been developed to enable a pointwise evaluation of the constraint [40–42]. However, the formulation then becomes non-symmetric and still lacks for a proof of coercivity. Another crucial aspect concerns the fact that, in practice, the meshes of the subdomains are generated independently and the underlying interfaces are not known in general. The formulation of the constraint must then be adjusted to enable the gluing of displacements along implicit interfaces. Some promising strategies have been proposed and tested in that regard [43,44], and an analysis of those strategies has been carried out in two dimensions [45,46].

Another outstanding question concerns the choice of the right topology to establish the convergence of the method. For Poisson's equation, the first convergence analysis of the mortar method made use of the broken H^1 norm for the displacements, and of the $H^{-1/2}$ norm–dual of $H^{1/2}_{00}$ norm–for Lagrange multipliers over the interfaces [1,3,22]. However, this norm depends on the particular subdomain decomposition and, therefore, does not allow for the comparison of approximations obtained from different decompositions. In addition, the associated norm for the jump of displacements across interfaces is the broken $H_{00}^{1/2}$ norm, which precludes the discrete displacements from jumping at interface junctions. In particular, the meshes must be conforming on interface boundaries. While this restriction is onerous enough in two dimensions, it becomes unwieldy or unfeasible in three dimensions. To sidestep this problem, Wohlmuth [47] has proposed and used a mesh-dependent norm introduced by Agouzal and Thomas [48] and recently revised by Dahmen *et al.* [49] (cf, also [50]). This mesh-dependent norm is closely related to the norms used in the analysis of Discontinuous Galerkin (DG) methods (cf, e. g., [51] for an overview). The mesh-dependent norm is particularly convenient because no additional constraints are necessary at interface junctions to obtain an estimate, is readily computable and has the same scaling properties as the the broken H^1 norm with respect to the mesh size. Building on work by Gopalakrishnan [52] and Brenner [53] for the scalar case, recent work on the linearized elasticity problem has established the subdomain-decomposition independence of the constants arising in the error estimates [54,12,55], especially the coercivity constant.

These advances notwithstanding, error estimates based on mesh-dependent norms are necessarily formal and do not supply a rigorous proof of convergence. Lew *et al.* [56] have recently obtained mesh-independent error bounds for DG approximations in linear elasticity in terms of bounded variation (BV) norms [56]. These estimates unequivocally proof the convergence in the BV topology of DG methods applied to linear elasticity. The choice of BV topology may at first seem odd in the context of linear elasticity, where existence is expected in H^1 under the usual technical assumptions. However, the BV framework is indeed natural for DG and mortar methods where the approximating functions themselves are discontinuous and do not belong to H^1. In addition, BV topologies are natural in the general context of free-discontinuity problems, such as resulting from fracture, shear banding, and other localization processes [57]. The BV framework may thus be useful in guiding extensions of DG and mortar methods to such problems.

In this paper we adapt the BV estimates of Lew *et al.* [56] to mortar methods. The main result of the paper is a proof that the conventional mesh-dependent norms are bounded below by the BV norm. When combined with standard results [54,12], this bound establishes a decomposition-independent and mesh-independent proof of the convergence of mortar methods in linear elasticity.

2 Formulation of the problem

Whereas mortar methods are by now standard, they may stand a brief review, especially as regards aspects relevant to subsequent developments.

2.1 Elastostatics

We consider a linear elastic solid occupying a bounded and open domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Let $\Gamma_D \subset \partial\Omega$ be the part of its boundary where zero displacements are prescribed, and $\Gamma_N = \partial\Omega \setminus \Gamma_D$ the complementary part where the tractions $g \in L^2(\Gamma_N; \mathbb{R}^d)$ are applied. In addition, we denote by $f \in L^2(\Omega; \mathbb{R}^d)$ the body forces applied over Ω, and by $\mathbf{E} \in L^\infty(\Omega; \mathbb{R}^{d\times d\times d\times d})$ a distribution of elastic moduli over the material satisfying:

$$\begin{cases} \mathbf{E}(x) : \xi \text{ is a symmetric matrix for any symmetric matrix } \xi \in \mathbb{R}^{d\times d}, \\ \exists c_0 > 0, \quad (\mathbf{E}(x) : \xi) : \xi \geq c_0 \, \xi : \xi, \quad \forall \xi \in \mathbb{R}^{d\times d}, \end{cases}$$

for almost-every $x \in \Omega$. The linearized elastostatics problem then consists of finding $u \in H^1_*(\Omega)$ such that:

$$\int_\Omega (\mathbf{E} : \varepsilon(u)) : \varepsilon(v) = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v, \qquad \forall v \in H^1_*(\Omega). \tag{1}$$

Here and subsequently, $\varepsilon(u)$ denotes the symmetrized part of the gradient of u, i. e., $\varepsilon(u) := \frac{1}{2}(\nabla u + \nabla u^t)$, and:

$$H^1_*(\Omega) = \{v \in H^1(\Omega; \mathbb{R}^d), \quad v|_{\Gamma_D} = 0 \text{ a.e. on } \Gamma_D\}.$$

Alternatively,

$$a(u, v) = \int_\Omega (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in H^1(\Omega; \mathbb{R}^d),$$

$$l(v) = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v, \quad \forall v \in H^1(\Omega; \mathbb{R}^d),$$

and problem (1) can be reformulated as that of finding $u \in H^1_*(\Omega)$ such that:

$$a(u, v) = l(v), \quad \forall v \in H^1_*(\Omega). \tag{2}$$

2.2 Mortar formulation

Let us recall here the finite element non-conforming discretization obtained for problem (2) in the framework of the mortar method, introduced in [1]. For simplicity, let us assume it is based on a *polygonal* non-overlapping partition $(\Omega_k)_{1\leq k\leq K}$ of Ω, i. e.

$$\bigcup_{k=1}^{K} \Omega_k = \Omega, \qquad \Omega_k \cap \Omega_l = \emptyset \quad \text{for all } 1 \leq k < l \leq K.$$

Each Ω_k is endowed with a triangulation $\mathcal{T}_{k;h_k}$ where h_k denotes the maximum diameter of the elements. The associated local discretization space of order q is then defined as

$$X_{k;h_k} = \Big\{ v \in H^1_*(\Omega_k), \quad v|_K \in [\mathbb{P}_q(T)]^d \quad \forall T \in \mathcal{T}_{k;h_k} \Big\} \oplus \mathcal{B}_{k;h_k},$$

where $\mathbb{P}_q(T)$ is the space of polynomials of total order less or equal to q in T, and $\mathcal{B}_{k;h_k}$ is a possible enrichment on the boundary of Ω_k. The early idea of interface bubble stabilization comes from the so-called three-field formulation of Brezzi and Marini [58]. Such an enrichment has been introduced in the mortar setting by Belgacem [11] for the Stokes problem, and by Hauret and Le Tallec [13] for elastostatics and higher order approximations. Then, the product space

$$X_h = \prod_{k=1}^{K} X_{k;h_k},$$

with $h = \max\limits_{1\leq k\leq K} h_k$, defines a non-conforming approximation space. Denoting by X the product space $\prod_{k=1}^{K} H^1_*(\Omega_k)$, we have $X_h \subset X$.

We now proceed to formulate the gluing constraint for the displacements in X_h. Denoting by Γ_{kl} the interface between subdomains Ω_k and Ω_l when it exists, the skeleton-interface between the subdomains is defined as $\mathcal{S} = \cup_{1\leq k<l\leq K}\Gamma_{kl}$. When the direction of the unit normal vector on Γ_{kl} is not smooth, i. e. Γ_{kl} has several faces, it may be important to consider a partition of $\mathcal{S}$ which is finer than the one given by the $(\Gamma_{kl})_{1\leq k<l\leq K}$ (see remark 4). More precisely, we consider the new partition

$$\mathcal{S} = \bigcup_{m=1}^{N} \gamma_m,$$

where

- for any $1 \leq m \leq N$, there exists two subdomains Ω_k and Ω_l such that $\gamma_m \subset \Gamma_{kl}$,
- any Γ_{kl} is the union of an entire number of γ_ms, which are its faces.

For any $1 \leq m \leq N$, γ_m is then included in a pairwise interface Γ_{kl}, and in the framework of mortar methods, one has to choose once for all, a non-mortar or slave side $k(m)$ equal to k or l. This choice made, γ_m inherits a surface mesh $\mathcal{F}_{m;h}$ which is the trace of the mesh $\mathcal{T}_{k(m);h_{k(m)}}$ built on the slave domain $\Omega_{k(m)}$

on γ_m. More precisely

$$\mathcal{F}_{m;h} = \{T \cap \gamma_m, \quad T \in \mathcal{T}_{k(m);h_{k(m)}}\}. \tag{3}$$

In addition, we assume the standard so-called hypothesis of geometrical conformity:

Assumption 1 (Geometrical conformity) *For every* $1 \leq m \leq N$, *and every* $F \in \mathcal{F}_{m;h}$, F *is an entire face of an element* $T \in \mathcal{T}_{k(m);h_{k(m)}}$, *which we denote by* $T(F)$.

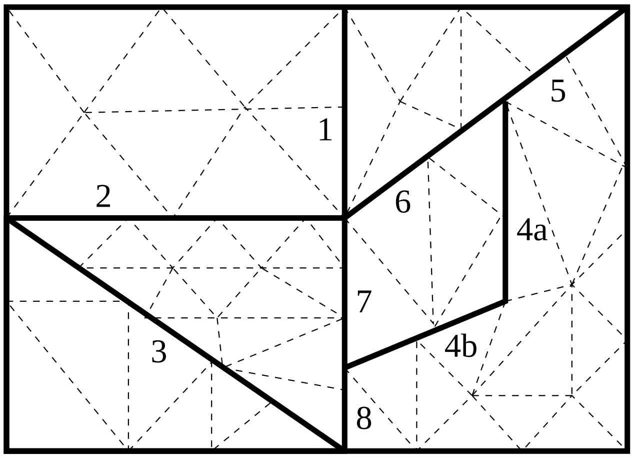

Fig. 1. Decomposition of a two-dimensional domain and meshes of the subdomains. There are 8 interfaces between subdomains, represented by the thick interior lines; for interfaces 1 to 3, the non-mortar side can be chosen arbitrarily. Interface 4 must be split into two interfaces 4a and 4b when using continuous Lagrange multipliers (see remark 4) and the choice of the non-mortar side is arbitrary on 4a and 4b. Concerning interfaces 5 to 8, the non-mortar side must coincide with the side where the numbering of the interface appears in order to satisfy assumption 1.

For every $1 \leq m \leq N$, the gluing constraint is enforced by means of Lagrange multipliers defined on $\mathcal{F}_{m;h}$ and spanning a space $M_{m;h}$ defined in section 2.3. The displacement solution u of (2) is then approximated on the constrained space

$$V_h = \left\{ u_h \in X_h, \quad \int_{\gamma_m} [u_h] \cdot \mu_h = 0 \quad \forall \mu_h \in M_{m;h}, 1 \leq m \leq N \right\},$$

where $[u_h]$ denotes the jump of u_h across $\mathcal{S}$. Let us introduce the bilinear form $\tilde{a} : X \times X \to \mathbb{R}$ as

$$\tilde{a}(u,v) = \sum_{k=0}^{K} a_k(u_k, v_k),$$

with

$$a_k(u_k, v_k) = \int_{\Omega_k} (\mathbf{E} : \varepsilon(u_k)) : \varepsilon(v_k).$$

In addition, suppose that for any $1 \leq k \leq K$ there exist two positive constants c_k and C_k such that, for almost every $x \in \Omega_k$,

$$c_k\, \xi : \zeta \leq (\mathbf{E}(x) : \xi) : \zeta \leq C_k\, \xi : \zeta, \tag{4}$$

for all symmetric matrices $\xi, \zeta \in \mathbb{R}^{d\times d}$. We then become interested in finding $u_h \in V_h$ such that

$$\tilde{a}(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$

The interpretation of mortar methods in the framework of mixed methods is due to the early work of Belgacem [3]. To provide such an interpretation, introduce bilinear form $b : X \times M \to \mathbb{R}$,

$$b(v, \lambda) = \sum_{m=1}^{N} \int_{\gamma_m} [v] \cdot \lambda,$$

where $M = \prod_{m=1}^{N} L^2(\gamma_m; \mathbb{R}^d)$. With this notation, the approximation problem now concerns the determination of a pair $(u_h, \lambda_h) \in X_h \times M_h$ such that

$$\begin{cases} \tilde{a}(u_h, v_h) + b(v_h, \lambda_h) = l(v_h), \quad \forall v_h \in X_h, \\ b(u_h, \mu_h) = 0, \quad \forall \mu_h \in M_h, \end{cases} \tag{5}$$

where $M_h = \prod_{m=1}^{N} M_{m;h}$.

2.3 Notation and assumptions

In the sequel, the space of displacements X is endowed with the rescaled broken H^1 norm $\|\cdot\|_X$ defined by

$$\|v\|_X^2 = \sum_{k=1}^{K} \frac{1}{diam(\Omega_k)^2} \|v\|^2_{L^2(\Omega_k;\mathbb{R}^d)} + |v|^2_{H^1(\Omega_k;\mathbb{R}^d)} \quad \forall v \in X,$$

whereas M is equipped with the norm $\|\cdot\|_M$ defined by

$$\|\mu\|_M^2 = \sum_{m=1}^{N} \|v\|^2_{L^2(\gamma_m;\mathbb{R}^d)} \quad \forall \mu \in M.$$

In addition, we will use special spaces of interface displacements introduced in [48] and defined by

$$\mathbb{H}_h^{1/2}(\gamma_m) = \left\{ \phi \in L^2(\gamma_m; \mathbb{R}^d), \|\phi\|^2_{h,\frac{1}{2},m} = \sum_{F\in\mathcal{F}_{m;h}} \frac{1}{h(F)} \|\phi\|^2_{L^2(F;\mathbb{R}^d)} < +\infty \right\},$$

$$\mathbb{H}_h^{-1/2}(\gamma_m) = \left\{ \lambda \in L^2(\gamma_m;\mathbb{R}^d),\, \|\lambda\|^2_{h,-\frac{1}{2},m} = \sum_{F\in\mathcal{F}_{m;h}} h(F)\|\lambda\|^2_{L^2(F;\mathbb{R}^d)} < +\infty \right\},$$

endowed with the norms $\|\cdot\|_{h,\frac{1}{2},m}$ and $\|\cdot\|_{h,-\frac{1}{2},m}$, respectively.

As is well-known within the context of mixed formulations, the well-posedness of (5) and the accuracy of the numerical solution requires the satisfaction of the so-called *inf-sup* condition [59,60]. From a heuristic point of view, the *inf-sup* condition amounts to saying that the number of weak-continuity constraints on the interfaces must be less than the number of displacement degrees of freedom on the interfaces. In the mortar framework, to ensure a uniform behavior with respect to the structure of the skeleton $\mathcal{S}$ it is traditionally required that the *inf-sup* condition be satisfied interface by interface. Moreover, independence from the relative position of the meshes on the interfaces requires that the *inf-sup* be checked for conforming meshes, which corresponds to the worse case. In addition, since the nodes on the boundary of the interfaces $(\gamma_m)_{1\le m\le N}$ may be shared by more than two subdomains the *inf-sup* condition must be satisfied for vanishing displacement jumps on interface boundaries. In mathematical terms, the *inf-sup* condition is expressed as follows (see [47,55]):

Assumption 2 *For each interface* $1 \le m \le N$*, denoting*

$$W_{m;h} = \left\{ v|_{\gamma_m}, \quad v \in X_{k(m);h_{k(m)}} \right\} \cap H_0^1(\gamma_m;\mathbb{R}^d),$$

there exists a mapping

$$\pi_m : \mathbb{H}_h^{1/2}(\gamma_m) \to W_{m;h},$$

such that for all $v \in \mathbb{H}_h^{1/2}(\gamma_m)$,

$$\int_{\gamma_m} (\pi_m v)\cdot\mu = \int_{\gamma_m} v\cdot\mu, \quad \forall \mu \in M_{m;h},$$

with

$$\|\pi_m v\|_{h,\frac{1}{2},m} \le C_m \|v\|_{h,\frac{1}{2},m},$$

for a positive constant $C_m > 0$ *independent of the surface mesh* $\mathcal{F}_{m;h}$.

Remark 1 *The terminology "*inf-sup *condition" comes from the following inequalities making use of assumption 2 and the duality of the spaces* $\mathbb{H}_h^{1/2}(\gamma_m)$ *and* $\mathbb{H}_h^{-1/2}(\gamma_m)$ *through the inner product of* $L^2(\gamma_m;\mathbb{R}^d)$*:*

$$
\begin{aligned}
&\inf_{\mu\in M_{m;h}} \sup_{v_h\in W_{m;h}} \frac{\int_{\gamma_m} v_h\cdot\mu}{\|v_h\|_{h,\frac{1}{2},m}\|\mu\|_{h,-\frac{1}{2},m}}\\
&\geq \inf_{\mu\in M_{m;h}} \sup_{v\in\mathbb{H}_h^{1/2}(\gamma_m)} \frac{\int_{\gamma_m} \pi_m v\cdot\mu}{\|\pi_m v\|_{h,\frac{1}{2},m}\|\mu\|_{h,-\frac{1}{2},m}}\\
&\geq \frac{1}{C_m}\inf_{\mu\in M_{m;h}} \sup_{v\in\mathbb{H}_h^{1/2}(\gamma_m)} \frac{\int_{\gamma_m} v\cdot\mu}{\|v\|_{h,\frac{1}{2},m}\|\mu\|_{h,-\frac{1}{2},m}}\\
&\geq \frac{1}{C_m}\inf_{\mu\in M_{m;h}} \frac{\|\mu\|_{h,-\frac{1}{2},m}}{\|\mu\|_{h,-\frac{1}{2},m}} = \frac{1}{C_m}.
\end{aligned}
$$

In addition, in order to insure the coercivity of the bilinear form $\tilde{a}$ over the constrained space V_h, i. e.

$$\tilde{a}(v_h, v_h) \geq \tilde{\alpha}\,\|v_h\|_X^2, \quad \forall v_h \in V_h, \tag{6}$$

for a coercivity constant $\tilde{\alpha} > 0$, it is necessary to consider Lagrange-multiplier spaces that are sufficiently rich as to suppress local rigid motions [47,55]. Indeed, the satisfaction of (6) imposes that $\tilde{a}(v_h, v_h) = 0$ with $v_h \in V_h$ imply $v_h = 0$. More precisely, any field of displacement that is locally a rigid motion in the subdomains and which satisfies the weak-continuity constraint must be globally a rigid motion over Ω, which vanishes from the zero boundary condition on Γ_D. Enforcing such a condition interface by interface, we assume:

Assumption 3 *For all $1 \leq m \leq N$, there exists a minimal Lagrange multiplier space $\mathcal{M}_m$ such that $\mathcal{M}_m \subset M_{m;h}$ independently of the discretization, and such that for every $v \in X$ that is locally a rigid motion over the subdomains Ω_k and Ω_l,*

$$\int_{\gamma_m} [v]\cdot\mu = 0 \quad \forall\mu\in\mathcal{M}_m \qquad \Longrightarrow\; [v] = 0 \quad \textit{on } \gamma_m. \tag{7}$$

Under assumption 3, the bilinear form $\tilde{a}$ is coercive over $V \times V$, where

$$V = \left\{ v\in X, \quad \int_{\gamma_m} [v]\cdot\mu = 0, \quad \forall\mu\in\mathcal{M}_m, \quad 1\leq m\leq N \right\},$$

i. e.,

$$\exists\tilde{\alpha} > 0, \quad \tilde{a}(v, v) \geq \tilde{\alpha}\,\|v\|_X^2 \quad \forall v \in V. \tag{8}$$

A fortiori, $\tilde{a}$ is coercive over $V_h \times V_h$ since $V_h \subset V$. These coercivity properties can be established by means of the contradiction argument of [1], but then the independence of the coercivity constant from the subdomain decomposition is not elucidated by the argument. Proofs of the subdomain-decomposition independence of the coercivity constant may be found in [54] and in [12] for the curved interface case.

Due to the coercivity result (8) guaranteed by assumption 3 and the surjectivity of the constraint coming from assumption 2, the well-posedness of (5) follows from standard theory (cf, e. g., [47,55]). Furthermore, in order to insure optimal convergence rates, each approximation space $M_{m;h}$ of Lagrange multipliers must contain all polynomials of total degree less or equal $q-1$ over each element $T \in \mathcal{F}_{m;h}$. The need to conciliate this requirement with assumptions 2 and 3 has lead to several strategies:

- In the original formulation of the mortar method [1], $M_{m;h}$ is a space of continuous functions over γ_m that are polynomials of degree q on every $T \in \mathcal{T}_{m;h}$ inside γ_m and specially modified into polynomials of order $q-1$ when $T \cap \partial\gamma_m \neq \emptyset$. Such modifications can lead to difficulty in the implementation of the method [61]. Moreover, the constraint is not diagonal and cannot be eliminated. Here, no stabilization of the displacements is needed and $\mathcal{B}_{k;h_k} = \emptyset$ for any $1 \leq k \leq K$. As an illustration, for first order elements ($q = 1$) on a one-dimensional interface, the shape functions of $M_{m;h}$ are represented on figure 2.
- In order to render the mortar constraint diagonal, thus enabling the elimination of the constraint when assembling the matrix of the problem, Wohlmuth has proposed the use of dual Lagrange multipliers [4]. This idea was extended by Kim-Lazarov-Pasciak-Vassilevski [5] to the three-dimensional case, who also propose alternatives based on piecewise constant "finite-volume" multipliers. For one-dimensional interfaces, the concept of duality has recently been adapted and analyzed in the case of higher order approximations [9] or for curved interfaces [46]. Despite its appeal, the dual approach still requires modification of the Lagrange multipliers at interface junctions. As an illustration, for first order elements ($q = 1$) on a one-dimensional interface, the corresponding dual shape functions of $M_{m;h}$ are represented on figure 2.

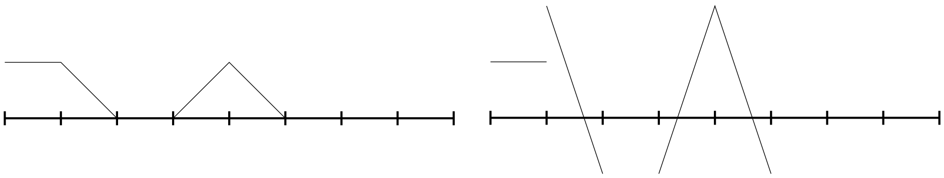

Fig. 2. Primal [1] and dual [4] bases of Lagrange multipliers on the non-mortar side of a straight one-dimensional interface, for a first-order approximation of the displacements ($q = 1$).

- For second-order approximations ($q \geq 2$) Seshaiyer [10] has shown that continuous Lagrange multipliers of degree $q-1$ satisfy all the requisite assumptions. The simple space

$$M_{m;h} = \{\mu \in \mathcal{C}^0(\gamma_m; \mathbb{R}^d), \quad \mu|_F \in [\mathbb{P}_{q-1}(F)]^d \quad \forall F \in \mathcal{F}_{m;h_m}\}$$

  does not require any modification at interface junctions but loses the locality of the constraint ensured by dual approaches [4,5].

- Belgacem [11] and Hauret-Le Tallec [55,12,13] have proposed taking the Lagrange multipliers to be discontinuous of order $q-1$, i. e.,

  $$M_{m;h} = \{\mu \in L^2(\gamma_m;\mathbb{R}^d), \quad \mu|_F \in [\mathbb{P}_{q-1}(F)]^d \quad \forall F \in \mathcal{F}_{m;h_m}\},$$

  This choice results in local gluing constraints without special modifications at interface junctions, albeit at the expense of requiring stabilization on the interface displacements, i. e. $\mathcal{B}_{k;h_k} \neq \emptyset$. For example, when dealing with a one-dimensional interface, all the triangles of the non-mortar mesh must be enriched by an interface bubble, as shown in figure 3.

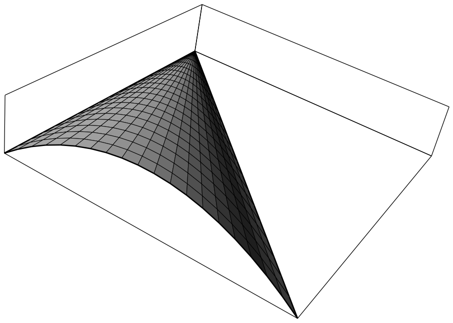

Fig. 3. Interface bubble shape function on a one-dimensional interface. Such a local enrichment is supported by a triangular element on the non-mortar side, as proposed in [11,13].

3 Analysis in BV

We proceed to adapt the BV estimates of Lew *et al.* [56] to mortar methods. The main result of this section is a proof that the conventional mesh-dependent norms are bounded below by the BV norm. This bound establishes a decomposition-independent and mesh-independent proof of the convergence of mortar methods in linear elasticity.

3.1 Mesh-dependent estimate

We begin by reviewing the conventional mesh-dependent error estimates for the mortar method. Denote by $\left(\overline{\mathcal{F}}_{m;h}\right)_{h>0}$ the sequence of meshes defined as in (3) by replacing the non-mortar side by the mortar side, and introduce δ_m (resp. $\overline{\delta}_m$) as the maximal diameter of the elements of $\mathcal{F}_{m;h}$ (resp. $\overline{\mathcal{F}}_{m;h}$). We make the additional assumption:

Assumption 4 *For all* $1 \leq m \leq N$*, the family of interface meshes* $(\mathcal{F}_{m;h})_{h>0}$ *on the non-mortar side is quasi-uniform. In addition, we assume that the ratio* $\overline{\delta}_m/\delta_m$ *is bounded independently of the discretization.*

Remark 2 *These assumptions can be weakened within the framework developed by Dahmen et al. [49].*

In addition, introduce the norm (cf, e. g., [47,55])

$$|||v||| = \|v\|_X + \sum_{m=1}^{N} \|[v]\|_{h,\frac{1}{2},m}, \quad \forall v \in X. \tag{9}$$

the seminorm:

$$|u|^2_{q+1,\mathbf{E},\Omega_k} = |u|^2_{H^{q+1}(\Omega_k;\mathbb{R}^d)} + \frac{1}{C_k^2}\|\mathbf{E} : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d\times d}}, \tag{10}$$

let $\tilde{\alpha}$ be the coercivity constant of $\tilde{a}$ over $V_h \times V_h$ and $(C_k)_{1\leq k\leq K}$ the constants appearing in the inequality (4). Then we have the following error estimate (see [55] for a detailed proof).

Proposition 1 *If* $u \in \prod_{k=1}^{K} H^{q+1}(\Omega_k;\mathbb{R}^d)$ *is the solution of (2) with* $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=1}^{K} H^q(\Omega_k)^{d\times d}$ *and* $q \geq 1$*, and* $(u_h, \lambda_h) \in X_h \times M_h$ *is the solution of (5), the following error estimate holds:*

$$|||u-u_h||| \leq C\left(1 + \max_{1\leq k\leq K}\frac{C_k}{\tilde{\alpha}}\right)\left(\sum_{k=1}^{K} h_k^{2q}\,|u|^2_{q+1,\mathbf{E},\Omega_k} + \sum_{m=1}^{N}\delta_m^{2q}|\lambda|^2_{H^{q-\frac{1}{2}}(\gamma_m;\mathbb{R}^d)}\right)^{1/2}. \tag{11}$$

We have denoted by $\lambda = (\mathbf{E} : \varepsilon(u)) \cdot n$ *the interface flux in which* n *refers to the unit normal vector on the skeleton* $\mathcal{S}$*, outward to the non-mortar side. The constant* C *is independent of the number, the diameter, the Young moduli and the discretization of the subdomains.*

Remark 3 *Following [54,55], it is possible to ensure the independence of the coercivity constant* $\tilde{\alpha}$ *from the discretization, the number, the size and the shape of the subdomains in the decomposition. Consequently, the mesh-dependency of (11) is entirely hidden in the* $|||\cdot|||$ *norm.*

Remark 4 *In estimate (11), it is clear the interfaces* γ_m *have to be regular enough so that we have* $\lambda \in H^{q-\frac{1}{2}}(\gamma_m;\mathbb{R}^d)$*. Indeed, the normal* n *on* γ_m *is part of the definition of* λ*. As a consequence, when using continuous Lagrange multipliers on the* γ_m*s, as in [1,4], it is crucial that the* γ_m*s be regular enough. By way of contrast, for discontinuous Lagrange multipliers as in [5,11,13] the partition of the skeleton* $\mathcal{S}$ *into interfaces is immaterial.*

Remark 5 *The proof of proposition 1 uses Assumption 1. The restriction of geometrical conformity thereby also applies to the result given later in theorem*

1. In the case of fracture along the interfaces, for which the BV framework is natural, the fracture surface may be geometrically nonconforming. However, a careful examination of the proof proposed in [55] reveals that geometrical nonconformity can be violated on the boundary of an interface shared by no other interface. In the definition of the norms $\| \cdot \|_{h,\pm\frac{1}{2},m}$ *(see section 2.3),* $\mathcal{F}_{m;h}$ *then must be replaced by*

$$\mathcal{F}^*_{m;h} = \{F \textit{ face of elements in } \mathcal{T}_{k(m);h_{k(m)}}, \quad \textit{meas}(F \cap \gamma_m) \neq 0\},$$

and the result of proposition 1 still holds, see illustration on figure 4.

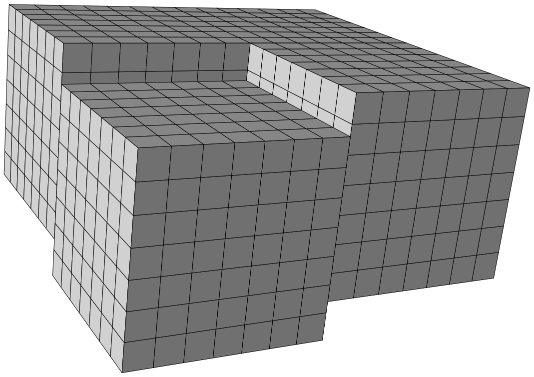

Fig. 4. Example involving fracture where geometrical conformity (Assumption 1) is violated. The proof of Proposition 1 (see [55]) shows that optimality is preserved if assumption 1 is only violated on the boundary of any interface shared by no other interface. Here, the choice of the non-mortar side is arbitrary.

3.2 Decomposition/discretization-free estimate in BV

In this section we supply a mesh-independent error estimate in the BV norm. Since mortar approximations involve discontinuous displacements across interfaces, the choice of BV topology naturally suggests itself. BV topologies also naturally arise in the analysis of free-discontinuity problems [57]. We recall [57] that the space $BV(\Omega;\mathbb{R}^d)$ of functions of bounded variation is

$$BV(\Omega;\mathbb{R}^d) := \{v \in L^1(\Omega;\mathbb{R}^d); \quad |\nabla u|_{TV(\Omega;\mathbb{R}^{d\times d})} < +\infty\},$$

where

$$|\nabla u|_{TV(\Omega;\mathbb{R}^{d\times d})} = \sup_{\substack{\varphi \in \mathcal{C}^1_c(\Omega;\mathbb{R}^{d\times d}) \\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \leq 1}} \int_\Omega u \cdot div(\varphi).$$

is the total variation of ∇u. The space $BV(\Omega;\mathbb{R}^d)$ is endowed with the following norm:

$$\|v\|_{BV(\Omega;\mathbb{R}^d)} = \frac{1}{diam(\Omega)}\|v\|_{L^1(\Omega;\mathbb{R}^d)} + |\nabla u|_{TV(\Omega;\mathbb{R}^{d\times d})}, \quad \forall v \in BV(\Omega;\mathbb{R}^d),$$

The following lemma adapts to mortar methods the BV estimates derived by [56].

Lemma 1 *There exists a constant* $C > 0$ *independent of* Ω*, of the discretization and the decomposition into subdomains, such that*

$$\|v\|_{BV(\Omega;\mathbb{R}^d)} \leq C\ |\Omega|\ |||v|||, \quad \forall v \in X.$$

Proof. Denoting by n_k the normal outward unit vector on Ω_k we have

$$\begin{aligned}\int_\Omega u \cdot div(\varphi) &= \sum_{k=1}^{K}\int_{\Omega_k} u \cdot div(\varphi)\\ &= -\sum_{k=1}^{K}\int_{\Omega_k}\nabla u : \varphi + \sum_{1\leq m\leq N}\int_{\gamma_m}(\varphi\cdot[u])\cdot n_{k(m)}.\end{aligned}$$

Moreover, we have

$$\sup_{\substack{\varphi\in\mathcal{C}^1_c(\Omega;\mathbb{R}^{d\times d})\\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\leq 1}} \int_{\gamma_m}(\varphi\cdot[u])\cdot n_{k(m)} \leq \|[u]\|_{L^1(\gamma_m;\mathbb{R}^d)},$$

$$\sup_{\substack{\varphi\in\mathcal{C}^1_c(\Omega;\mathbb{R}^{d\times d})\\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\leq 1}} \int_{\Omega_k}\nabla u : \varphi \leq \|\nabla u\|_{L^1(\Omega_k;\mathbb{R}^d)},$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&|\nabla u|_{TV(\Omega;\mathbb{R}^{d\times d})}\\
&\leq \sum_{k=1}^{K} \|\nabla u\|_{L^1(\Omega_k;\mathbb{R}^{d\times d})} + \sum_{1\leq m\leq N} \|[u]\|_{L^1(\gamma_m;\mathbb{R}^d)}\\
&= \sum_{k=1}^{K} \|\nabla u\|_{L^1(\Omega_k;\mathbb{R}^{d\times d})} + \sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} \|[u]\|_{L^1(F;\mathbb{R}^d)}\\
&\leq \sum_{k=1}^{K} |\Omega_k|^{1/2} \|\nabla u\|_{L^2(\Omega_k;\mathbb{R}^{d\times d})} + \sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} |F|^{1/2} \|[u]\|_{L^2(F;\mathbb{R}^d)}.
\end{aligned}$$

By the shape regularity of the mesh, which implies $diam(F)\,|F| \leq C\,|T(F)|$, we additionally have

$$\begin{aligned}
&|\nabla u|^2_{TV(\Omega;\mathbb{R}^{d\times d})}\\
&\leq 2\left(\sum_{k=1}^{K} |\Omega_k|^{1/2} \|\nabla u\|_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\right)^2 +\\
&\quad +2\left(\sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} |F|^{1/2} \frac{diam(F)^{1/2}}{diam(F)^{1/2}} \|[u]\|_{L^2(F;\mathbb{R}^d)}\right)^2\\
&\leq 2\left(\sum_{k=1}^{K} |\Omega_k|\right)\left(\sum_{k=1}^{K} \|\nabla u\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\right) +\\
&\quad +2\left(\sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} diam(F)\,|F|\right)\left(\sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} \frac{1}{diam(F)} \|[u]\|^2_{L^2(F;\mathbb{R}^d)}\right)\\
&\leq 2|\Omega|\left(\sum_{k=1}^{K} \|\nabla u\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\right) +\\
&\quad +C\left(\sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} |T(F)|\right)\left(\sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} \frac{1}{diam(F)} \|[u]\|^2_{L^2(F;\mathbb{R}^d)}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
|\nabla u|^2_{TV(\Omega;\mathbb{R}^{d\times d})} &\leq C\;|\Omega|\left(\sum_{k=1}^{K} \|\nabla u\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\right)\\
&\qquad +C\;|\Omega|\left(\sum_{1\leq m\leq N} \sum_{F\in\mathcal{F}_{m;h}} \frac{1}{diam(F)} \|[u]\|^2_{L^2(F;\mathbb{R}^d)}\right). \qquad (12)
\end{aligned}$$

Finally, since

$$\begin{aligned}\frac{1}{diam(\Omega)^2}\|u\|^2_{L^1(\Omega;\mathbb{R}^d)} &\leq |\Omega| \ \frac{1}{diam(\Omega)^2}\|u\|^2_{L^2(\Omega;\mathbb{R}^d)} \\ &= |\Omega| \ \sum_{k=1}^{K}\frac{1}{diam(\Omega)^2}\|u\|^2_{L^2(\Omega_k;\mathbb{R}^d)} \\ &\leq |\Omega| \ \sum_{k=1}^{K}\frac{1}{diam(\Omega_k)^2}\|u\|^2_{L^2(\Omega_k;\mathbb{R}^d)}, \qquad (13)\end{aligned}$$

we deduce from (13) and (12) that

$$\begin{aligned}\|u\|^2_{BV(\Omega;\mathbb{R}^d)} &\leq 2\left(\frac{1}{diam(\Omega)^2}\|u\|^2_{L^1(\Omega;\mathbb{R}^d)} + |\nabla u|^2_{TV(\Omega;\mathbb{R}^{d\times d})}\right) \\ &\leq C \ |\Omega| \ |||u|||^2.\end{aligned}$$

□

We obtain as a consequence the central result of this paper.

Theorem 1 *If* $u \in \prod_{k=1}^{K} H^{q+1}(\Omega_k;\mathbb{R}^d)$ *is solution of (2) with* $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=1}^{K} H^q(\Omega_k)^{d\times d}$ *and* $q \geq 1$, *and* $(u_h, \lambda_h) \in X_h \times M_h$ *is solution of (5), the following error estimate holds:*

$$\|u-u_h\|_{BV(\Omega;\mathbb{R}^d)} \leq C\left(1 + \max_{1\leq k\leq K}\frac{C_k}{\tilde{\alpha}}\right)\left(\sum_{k=1}^{K} h_k^{2q}\,|u|^2_{q+1,\mathbf{E},\Omega_k} + \sum_{m=1}^{N}\delta_m^{2q}|\lambda|^2_{H^{q-\frac{1}{2}}(\gamma_m;\mathbb{R}^d)}\right)^{1/2}, \qquad (14)$$

with:

$$|u|^2_{q+1,\mathbf{E},\Omega_k} = |u|^2_{H^{q+1}(\Omega_k;\mathbb{R}^d)} + \frac{1}{C_k^2}\|\mathbf{E} : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d\times d}}. \qquad (15)$$

The constant C *is independent of the number, the diameter, the Young moduli and the discretization of the subdomains.*

4 Summary and concluding remarks

Theorem 1 supplies a rigorous and unambiguous proof of convergence of mortar methods applied to linear elasticity in terms of a fixed BV norm independent of the discretization. It would be natural to pursue extensions of the analysis to free-discontinuity problems such as arise in the variational treatment of linear-elastic fracture mechanics. Evidently, mortar and DG methods provide a convenient framework for the approximation of those problems, with interfaces representing cracks and displacement jumps across the interfaces representing the attendant crack opening displacements. In this case, the energy of the solid includes a cohesive term which is supported on the fracture surface. Interestingly, certain numerical stabilization terms proposed in the context of DG methods have the form of a cohesive energy [56]. Because the geometry and topology of the fracture surface can become exceedingly complex, e. g., as a result of branching and fragmentation, weak notions of convergence such as provided by the topology of *strict convergence* in $BV(\Omega)$ naturally suggest themselves [62]. Finally, because of the strongly nonlinear character of fracture and localization problems, the traditional linear-analysis techniques cease to apply and Γ-convergence arises as the tool of choice for establishing the convergence of numerical approximations.

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References

[1] C. Bernardi, Y. Maday, A. Patera, Domain decomposition by the mortar element method, in: H. K. ans M. Garbey (Ed.), Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters, N.A.T.O. ASI, Kluwer Academic Publishers, 1993, pp. 269–286.

[2] C. Bernardi, Y. Maday, A. Patera, Nonlinear partial differential equations and their applications., Pitman, Paris, 1994, Ch. A new nonconforming approach to domain decomposition: the mortar element method., pp. 13–51.

[3] F. B. Belgacem, The mortar finite element method with Lagrange multipliers, Numer. Math. 84 (1999) 173–197.

[4] B. Wohlmuth, A mortar finite element method using dual spaces for the Lagrange multiplier, SIAM J. Numer. Anal. 38 (2000) 989–1012.

[5] C. Kim, R. Lazarov, J. Pasciak, P. Vassilevski, Multiplier spaces for the mortar finite element method in three dimensions, SIAM J. Numer. Anal. 39 (2001) 519–538.

[6] Y. Kuznetsov, M.F.Wheeler, Optimal order substructuring preconditioners for mixed finite element methods on non-matching grids, East-West J. Numer. Math. 3 (1995) 127–143.

[7] T. Arbogast, I. Yotov, A non-mortar mixed finite element method for elliptic problems on non-matching mutli-block grids, Computer Methods in Applied Mechanics and Engineering 149 (1997) 255–265.

[8] P. Oswald, B. Wohlmuth, On polynomial reproduction of dual FE bases, in: Domain Decomposition Methods 13, Lyon, 2000, pp. 85–96.

[9] B. Lamichhane, B. Wohlmuth, Biorthogonal bases with local support and approximation properties, Tech. Rep. 2005-02, IANS, Stuttgart (2005).

[10] P. Seshaiyer, Non-conforming hp finite element methods, Ph.D. thesis, University of Maryland (1998).

[11] F. B. Belgacem, A stabilized domain decomposition method with non-matching grids for the Stokes problem in three dimensions, SIAM Journal of Numerical Analysis 42 (2) (2004) 667–685.

[12] P. Hauret, P. L. Tallec, A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems, part i: abstract framework, Tech. Rep. 553, CMAP (september 2004).

[13] P. Hauret, P. L. Tallec, A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems, part ii: discontinuous Lagrange multipliers, Tech. Rep. 554, CMAP (september 2004).

[14] P. L. Tallec, Neumann-Neumann domain decomposition algorithm for solving 2d elliptic problems with nonmatching grids, East-West J. Numer. Math. 1 (2) (1993) 129–146.

[15] Y. Achdou, Y. Kuznetsov, O. Pironneau, Substructuring preconditioners for the Q_1 mortar element method, Numerische Mathematik 71 (1995) 419–449.

[16] G. Abdulaev, Y. Achdou, Y. Kuznetsov, C. Prud'homme, On a parallel implementation of the mortar element method, M2AN 33 (2).

[17] Y. Achdou, Y. Maday, O. Widlund, Substructuring preconditioners for the mortar method in dimension two, SIAM Journal of Numerical Analysis 36 (2) (1998) 551–580.

[18] D. Stefanica, Domain decomposition methods for mortar finite elements, Ph.D. thesis, Courant Institute of Mathematical Sciences, New York University (1999).

[19] D. Stefanica, A numerical study of FETI algorithms for mortar finite element methods, SIAM Journal of Scientific Computing 23 (4) (2001) 1135–1160.

[20] M. Dryja, K. Proskurowski, On preconditioners for mortar discretization of elliptic problems, Numerical Linear Algebra with Applications 10 (2003) 65–82.

[21] D. Braess, W. Dahmen, Stability estimates of the mortar finite element method for 3-dimensional problems, East-West J. Numer. Math. 6 (1998) 249–263.

[22] D. Braess, W. Dahmen, C. Wieners, A multigrid algorithm for the mortar finite element method, SIAM J. Numer. Anal. 37 (1999) 48–69.

[23] N. Débit, Y. Maday, The coupling of spectral and finite element methods for the approximation of the Stokes problem, in: Proceedings of the 8th Joint France-Italy-U.S.S.R. Symposium of Computational Mathematics and Applications, Pavia, 1987.

[24] N. Débit, La méthode des éléments avec joints dans le cas du couplage des méthodes spectrales et méthodes des éléments finis: Résolution des équation de Navier-Stokes, Ph.D. thesis, Université Pierre et Marie Curie, Paris VI (1992).

[25] F. B. Belgacem, The mixed mortar finite element method for the incompressible Stokes problem: convergence analysis, SIAM J. Numer. Analysis 37 (4) (2000) 1085–1100.

[26] G. Anagnostou, Non conforming sliding spectral elements methods for unsteady incompressible Navier-Stokes equations, Ph.D. thesis, Massachusetts Institute of Technology (1991).

[27] Y. Achdou, O. Pironneau, A fast solver for Navier Stokes equations in the laminar regime using mortar finite element and boundary element methods, SIAM J. Numer. Anal. 32 (1995) 985–1016.

[28] P. L. Tallec, T. Sassi, Domain decomposition with nonmatching grids: augmented Lagrangian approach, Math. of Comp. 64 (212) (1995) 1367–1396.

[29] C. Lacour, Analyse et résolution numérique de méthodes de sous-domaines non conformes pour des problèmes de plaques, Ph.D. thesis, Universit Pierre et Marie Curie, Paris (1997).

[30] C. Lacour, Non-conforming domain decomposition method for plate and shell problems, Contemp. Math. 218 (1998) 304–310.

[31] F. Rapetti, Approximations des quations de la magntodynamique en domaine tournant par la mthode des lments avec joints, Ph.D. thesis, Laboratoire J.-L. Lions, Paris 6 University (2000).

[32] A. Buffa, Some numerical and theoretical problems in computational electromagnetism, Ph.D. thesis, University of Milano (2000).

[33] A. Buffa, F. B. Belgacem, Y. Maday, The mortar element method for 3D Maxwell's equations: first results, SIAM J. Numer. Anal. 39 (3) (2001) 880–881.

[34] A. Buffa, Y. Maday, F. Rapetti, A sliding mesh-mortar method for two dimensional eddy currents model for electric engines, M2AN 35 (2) (2001) 191–228.

[35] A. B. Abdallah, F. B. Belgacem, Y. Maday, F. Rapetti, Mortaring the two-dimensional edge finite elements for the discretization of some electromagnetic models, Mathematical Models and Methods in Applied Sciences 14 (11) (2004) 1635–1656.

[36] F. B. Belgacem, P. Hild, P. Laborde, Extension of the mortar finite element to a variational inequality modeling unilateral contact, Math. Models Appl. Sci 9 (1999) 287–303.

[37] G. Bayada, M. Chambat, K. Lhalouani, T. Sassi, Eléments finis avec joints pour des problèmes de contact avec frottement de Coulomb non local, C.R. Acad. Sci. Paris "Analyse numérique" 325 (1997) 1323–1388.

[38] P. Hild, Numerical implementation of two nonconforming finite element methods for unilateral contact, Computer Methods in Applied Mechanics and Engineering 184 (1) (2000) 99–123.

[39] M. Puso, T. Laursen, A 3d contact smoothing method using gregory patches, International Journal for Numerical Methods in Engineering 54 (2002) 1161–1194.

[40] L. Cazabeau, C. Lacour, Y. Maday, Numerical quadratures and mortar methods, in: J. Wiley, Sons (Eds.), Computational Science for the 21st Century, 1997, pp. 119–128.

[41] Y. Maday, F. Rapetti, B. Wohlmuth, The influence of quadrature formulas in 3d mortar methods, in: Recent Developments in Domain Decomposition Methods, Lecture Notes in Computational Science and Engineering, Vol. 23, Springer, 2002, pp. 203–221.

[42] S. Bertoluzza, S. Falleta, V. Perrier, Wavelet/FEM coupling by the mortar method, in: Lecture Notes in Computational Science and Engineering, Springer, 2002, pp. 119–132.

[43] M. Puso, T. Laursen, Mesh tying on curved surfaces in 3d, Engineering Computations 20 (2003) 305–139.

[44] M. Puso, A 3d mortar method for solid mechanics, Int. J. Num. Meth. Engr. 59 (2004) 315–336.

[45] B. Flemisch, J. Melenk, B. Wohlmuth, Mortar methods with curved interfaces, Tech. rep., Max Planck Institute, preprint. (2004).

[46] B. Flemisch, M. Puso, B. Wohlmuth, A new dual mortar method for curved interfaces: linear elasticity, Tech. rep., Lawrence Livermore National Laboratory, CA, (2004).

[47] B. Wohlmuth, Discretization methods and iterative solvers based on domain decomposition, Springer, 2001.

[48] A. Agouzal, J. Thomas, Une méthode d'éléments finis hybrides en décomposition de domaines, RAIRO M2AN 29 (1995) 749–764.

[49] W. Dahmen, R. Faermann, I. Graham, W. Hackbush, S. Sauter, Inverse inequalities on non-quasi-uniform meshes and application to the mortar element methods, Mathematics of Computation 73 (247) (2003) 1107–1138.

[50] D. Braess, W. Dahmen, The mortar element method revisited -what are the right norms ?, in: R. Hoppe, J. Périaux, Y. Kuznetsov, N. Débit, M. Garbey (Eds.), Thirteen International Conference on Domain Decomposition Methods, 2001.

[51] D. Arnold, F. Brezzi, C. Cockburn, D.Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 39 (5) (2002) 1749–1779.

[52] J. Gopalakrishnan, On the mortar finite element method, Ph.D. thesis, Texas A and M University (August 1999).

[53] S. Brenner, Poincaré-Friedrichs inequalities for piecewise H^1 functions, SIAM J. Numer. Anal. 41 (1) (2003) 306–324.

[54] S. Brenner, Korn's inequalities for piecewise H^1 vector fields, Mathematics of Computation 73 (2004) 1067–1087.

[55] P. Hauret, Méthodes numériques pour la dynamique des structures non-linéaires incompressibles à deux échelles (Numerical methods for the dynamic analysis of two-scale incompressible nonlinear structures), Ph.D. thesis, Ecole Polytechnique (2004).

[56] A. Lew, P. Neff, D. Sulsky, M. Ortiz, Optimal BV estimates for a discontinuous Galerkin method for linear elasticity, Applied Mathematics Research eXpress 3.

[57] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems (Oxford Mathematical Monographs), Clarendon Press, 2000.

[58] F. Brezzi, D. Marini, Error estimates for the three-field formulation with bubble stabilization, Math. Comp. 70 (2000) 911–934.

[59] I. Babuska, The finite-element method with Lagrangian multipliers, Numerische Mathematik 20 (1973) 179–182.

[60] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, RAIRO Analyse Numérique, Série Rouge 8 (1974) 129–151.

[61] F. B. Belgacem, Y. Maday, The mortar element method for three dimensional finite element, M2AN 31 (1997) 289–303.

[62] P. Bělík, M. Luskin, Approximation by piecewise constant functions in a BV metric, Mathematical Models and Methods in Applied Sciences 3 (2003) 373–393.