

# Energy conservation in fluid structure interactions.

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## Abstract

Two numerical difficulties arise when computing low speed large displacements fluid structure interaction problems. On the one hand, classical time integration schemes which are dissipative, unconditionally stable and accurate for fluids flowing inside fixed domains may lose their long term stability properties when used on moving domains. On the other hand, nonlinear constitutive laws in the structure, and nonlinear constraints such as incompressibility, may lead to unexpected numerical instabilities when using standard midpoint or Newmark schemes on the structure. We prove herein that in contrast, high order schemes using a partially nonconservative formulation associated to the variable  $\sqrt{\rho}JU$  on the fluid, and local nonlinear energy corrections on the structure lead to an overall scheme which properly conserves kinetic and mechanical energy in the system.

## 1 Introduction

We consider herein the numerical solution of low speed large displacement problems where a flexible incompressible elastic structure in large displacements interacts with the flow of an external or internal incompressible fluid, at low viscosity. Hydraulic shock absorbers, biomedical flows in flexible pipes, aeroelastic instabilities of flexible aircrafts or tall bridges, or ocean flows around very long risers are examples of such situations. The numerical challenge is to predict the longterm time evolution and stability of these coupled systems.

The separate analysis of either deformable structures in large displacements for a small time [8], or incompressible viscous fluids flowing inside fixed domains [3] is rather well understood. But as illustrated in this paper, making both models interact for a long time, and introducing nonlinear kinematic constraints is more difficult. This coupling must respect the geometric compatibility of the different domains, the kinematic and

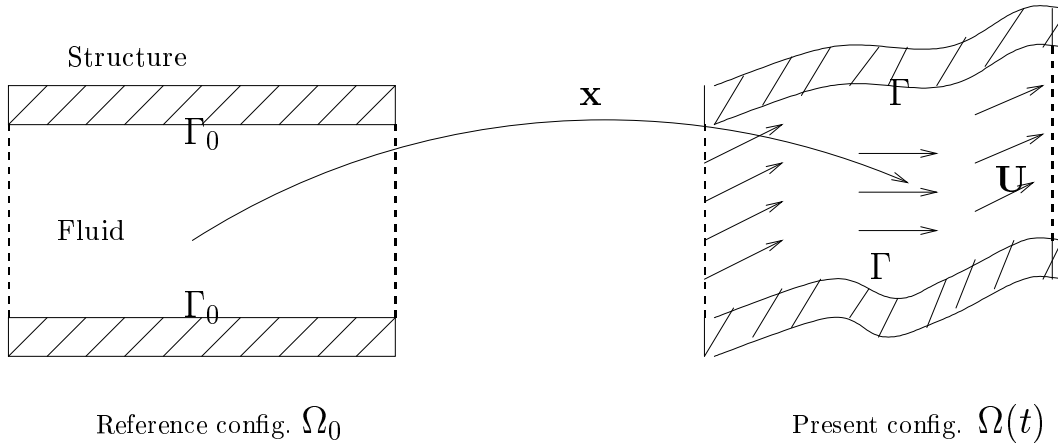


Figure 1: Geometric configurations : the fixed reference configuration  $\Omega_0$  where the fluid domain is delimited by the structure at rest, and the physical configuration  $\Omega(t)$ .

kinetic conditions satisfied by the fluid and the structure inside their own domains, at their common interface, and properly conserve mass AND energy.

The conservation laws written on a global fixed reference configuration define the mechanical problem to be solved (§2). Second order time discretizations are next proposed (§3). We observe there that classical time integration schemes may loose their long term stability properties when used on moving domains. We also produce in §4 new estimates proving that these schemes do properly conserve the energy, if they are used on a partially nonconservative formulation associated to the variable  $\sqrt{\rho J}U$ , and are used with local energy corrections as proposed in [6]. The efficiency of these corrections is illustrated in §5 for a pure structural problem.

## 2 Mechanical problem

The system under study occupies a moving domain  $\Omega(t)$  in its present configuration. It is made of a fluid in motion in a deformable part  $\Omega^f(t)$  of  $\Omega(t)$  and of a deformable structure which lies on the complement  $\Omega^s(t)$  of  $\Omega^f(t)$  in  $\Omega(t)$  (Figure 1). The problem consists in finding both the time evolution of this configuration, and the velocity  $U := \frac{dx}{dt}$  and Cauchy stress tensor  $\sigma$  within the fluid and the structure.

To evaluate the strain field or write the elastic constitutive laws inside the structure, it is very convenient to transport the conservation laws for both the fluid and the structure on a fixed reference configuration  $\Omega_0$ . The choice of the configuration  $\Omega_0$  and of the map  $x : \Omega_0 \rightarrow \Omega(t)$  (and hence of its jacobian  $J = \det \frac{\partial x}{\partial x_0}$  and of the underlying grid velocity

$U_G = \left( \frac{\partial x}{\partial t} \right)_{|x_0}$  may be arbitrary (Arbitrary Lagrangian Eulerian (ALE) formulation), but, on the structure  $\Omega^s$ , the equations are much simpler when the point  $x(x_0, t)$  corresponds to the present position  $x^s(x_0, t)$  of the material point which was located in  $x_0$  at time  $t_0$ . As for the mapping  $x^f$  from  $\Omega_0^f$  onto  $\Omega^f(t)$ , it will be fully characterized by its nodal values on each nodal point of the discretization grid. It can be any reasonable extension  $x^f = \text{Ext}(x_{|\Gamma_0}^s)$  of the material interface deformation, but it must match the structural deformation on this interface.

The structure is supposed to be a nonlinear incompressible elastic structure and to interact with a viscous incompressible fluid of given density  $\rho$  which perfectly sticks to its boundary, meaning that the fluid particles must follow the structure during the motion. In this framework, the mechanical evolution of the global fluid structure system take the final form :

*Find the structural deformation  $x^s \in V^s$ , the fluid density  $\rho^f J$  in initial configuration, the pressure  $p \in Q = L^2(\Omega_0)$ , the fluid velocity  $U^f \in V^f$ , the interface traction  $g_\Gamma \in W_\Gamma = (H^{1/2}(\Omega_0))'$ , and the fluid configuration mapping  $x^f \in V^f$  such that*

$$\int_{\Omega_0^f} \frac{\partial J \rho}{\partial t} \Big|_{x_0} \hat{q} + \int_{x(\Omega_0^f, t)} \text{div}_x(\rho(U - U^G)) \hat{q} + \int_{\Omega_0^s} (\det(C^{1/2}(\nabla x)) - 1) \hat{q} = 0, \\ \forall \hat{q} : \Omega_0 \rightarrow \mathbb{R}, \quad (\text{mass and volume conservation}) \quad (1)$$

$$m^s(\ddot{x}^s, \hat{U}^s) + \int_{\Omega_0^f} \frac{\partial J \rho U^f}{\partial t} \Big|_{x_0} \cdot \hat{U}^f + \int_{x^f(\Omega_0^f, t)} \text{div}_x(\rho U^f \otimes (U^f - U^G)) \cdot \hat{U}^f \\ + a^s(\nabla x^s, \nabla \hat{U}^s) + \int_{x^f(\Omega_0^f, t)} (\mu(\nabla_x U^f + \nabla_x^t U^f) - p Id) : \frac{\partial \hat{U}^f}{\partial x} \\ = \int_{\Omega(t)} f \cdot \hat{U} + \int_{\partial\Omega(t)} g \cdot \hat{U} + \int_{\Gamma_0} g_\Gamma \cdot \left( \text{tr}(\hat{U}^s)_{|\Gamma} - \text{tr}(\hat{U}^f)_{|\Gamma} \right), \\ \forall (\hat{U}^s, \hat{U}^f) \in V^s \times V^f, \quad (\text{momentum conservation}) \quad (2)$$

$$\text{tr}(x^s)_{|\Gamma} = \text{tr}(x_0)_{|\Gamma} + \int_0^t \text{tr}(U^f)_{|\Gamma}(\tau) d\tau, \quad (3)$$

$$x^f = \text{Ext}(x_{|\Gamma_0}^s). \quad (\text{fluid configuration map}) \quad (4)$$

Above, the structural mass operator  $m^s$  has the usual linear expression encountered in lagrangian dynamics. When dealing with incompressible or almost incompressible elastic materials in large deformation, the stiffness term  $a^s(\nabla x, \nabla \hat{U})$  must be defined in mixed form as  $a^s(\nabla x, \nabla \hat{U}) = \int_{\Omega_0^s} F \cdot \Sigma(\nabla x) \cdot \nabla \hat{U}$  with  $\Sigma$  the second Piola-Kirchhoff stress tensor given by :

$$\Sigma = 2 \frac{\partial \mathcal{W}}{\partial C} - 2p \frac{\partial \det C^{1/2}}{\partial C}, \quad (5)$$

where  $p$  denotes the hydrostatic pressure,  $\mathcal{W}$  the stored elastic potential function of the right Cauchy-Green strain tensor  $C = F^t \cdot F$ , and  $F = \nabla x$  the deformation gradient.

The above formulation involve three major nonlinear effects : a transport term  $F = \nabla x = \frac{\partial x}{\partial x_0}$  in factor of the stress tensor  $\Sigma$  in the structure, a nonlinear incompressibility constraint on the structure acting on  $(\det F(\nabla x) - 1) = \left( \det (C^{1/2}(\nabla x)) - 1 \right)$ , and a convection term  $\text{div}_x(\rho U^f \otimes (U^f - U^G)) \cdot \hat{U}^f$  in the fluid. The notation  $\left( \det (C^{1/2}(\nabla x)) - 1 \right)$  instead of  $(\det F(\nabla x) - 1)$  is consistant with the conservative writing (5) of the Kirchhoff stress.

A possible modification consists in using a nonconservative writing of the momentum conservation equation in the fluid. This technique selects an adequate exponent  $0 \leq q \leq 1$  and replaces the initial governing variable  $\rho J U^f$  by  $(\rho J)^q U^f$ , simply by adding  $(q - 1)U^f$  times the mass conservation equation to the momentum conservation equation. The case  $q = 1$  corresponds to the initial conservative formulation, the case  $q = 0$  corresponds to the usual nonconservative formulation used for incompressible viscous fluids. The case  $q = 1/2$  turns out to be the right choice for energy conservation in a fluid structure setting. With this addition, the momentum equation (2) becomes

$$\begin{aligned}
& m^s(\ddot{x}^s, \hat{U}^s) + \int_{\Omega_0^f} (J\rho)^{1-q} \frac{\partial (J\rho)^q U^f}{\partial t} \Big|_{x_0} \cdot \hat{U}^f \\
& + \int_{x^f(\Omega_0^f, t)} \left( \text{div}_x[\rho U^f \otimes (U^f - U^G)] + (q - 1) \text{div}_x[\rho(U^f - U^G)]U^f \right) \cdot \hat{U}^f \\
& + a^s(\nabla x^s, \nabla \hat{U}^s) + \int_{x^f(\Omega_0^f, t)} (\mu(\nabla_x U^f + \nabla_x^t U^f) - pId) : \frac{\partial \hat{U}^f}{\partial x} \\
& = \int_{\Omega(t)} f \cdot \hat{U} + \int_{\partial\Omega(t)} g \cdot \hat{U} + \int_{\Gamma_0} g_\Gamma \cdot \left( \text{tr}(\hat{U}^s)|_\Gamma - \text{tr}(\hat{U}^f)|_\Gamma \right), \\
& \quad \forall (\hat{U}^s, \hat{U}^f) \in V^s \times V^f. \tag{6}
\end{aligned}$$

**Remark 1** *The above formulation is in fact equivalent to three coupled subproblems, which are characteristic of fluid structure interaction problems.*

1. *Solving the mass conservation equation, and choosing in (6)  $\hat{U}^s = 0$  and  $\hat{U}^f$  arbitrary in  $V_0^f = \{\hat{U}^f \in V^f, \text{Tr}(U^f)|_{\Gamma_0} = 0\}$ , we first obtain a standard Navier-Stokes equation written in ALE form on the moving domain  $\Omega^f(t)$ , and formally associated to Dirichlet conditions  $\text{Tr}(U^f)|_{\Gamma_0} = \text{Tr}(U^s)|_{\Gamma_0}$  imposed on the interface and specified by the structural problem.*
2. *By choosing in (6)  $\hat{U}^f = 0$  and  $\hat{U}^s$  arbitrary in  $V^s$ , we obtain a standard structural problem with imposed traction forces on the interface (specified by the fluid problem)*

$$m^s(\ddot{x}^s, \hat{U}^s) + a^s(\nabla x^s, \nabla \hat{U}^s) = \int_{\Omega^s(t)} f \cdot \hat{U}^s + \int_{\partial\Omega(t) \cap \partial\Omega^s(t)} g \cdot \hat{U}^s$$

$$+ \int_{\Gamma_0} g_\Gamma \cdot \text{tr}(\hat{U}^s)|_\Gamma, \forall \hat{U}^s \in V^s.$$

3. The last subproblem defines the grid configuration map inside the fluid by an explicit or implicit equation of the type

$$x^f = \text{Ext}(x^s|_{\Gamma_0}).$$

Its relation to the other subproblems reduces to the local condition relating on the interface, the fluid grid velocity to the local value of the fluid (or structural) physical velocity.

### 3 Time discretization

We now need to construct a time discretization scheme respecting the above kinematic compatibility condition at the fluid structure interface, and conserving energy. We will concentrate here on totally coupled one point time integration schemes. Such techniques solve the full system (including the kinematic compatibility condition (3)) at a sequence of discrete times  $t_n, n = 1, \dots$ , using independent finite difference approximations of the various time derivatives.

#### 3.1 Partially explicit schemes

The cost of an implicit time step can become prohibitive when considering realistic multiscale models. An explicit time stepping is simpler, but for stability reasons, time step has to satisfy a CFL condition, particularly restrictive for quasi-incompressible problems. By analogy with semi explicit schemes commonly used in computational fluid dynamics, a good compromise could consist in only impliciting the pressure terms in the scheme, which is equivalent to project at each time step the result predicted by an explicit time iteration of an associated compressible problem onto the subset of incompressible displacements. It is obvious that the CFL condition to be satisfied for the time step is then far less restrictive, since it is then associated to an underlying soft compressible material. For an isolated structure, the resulting scheme is simply

$$\begin{aligned} & \int_{\Omega} \rho \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t_n^2} \cdot \hat{U} + \int_{\Omega} 2F_n \cdot \frac{\partial \mathcal{W}}{\partial C}(\nabla x_n) : \nabla \hat{U} \\ & + \frac{1}{2} \left( \int_{\Omega} p_{n-1} \text{cof } F_{n-1} : \nabla \hat{U} + \int_{\Omega} p_{n+1} \text{cof } F_{n+1} : \nabla \hat{U} \right) \\ & = \int_{\Omega} f \cdot \hat{U} + \int_{\partial \Omega} g \cdot \hat{U}, \quad \forall \hat{U} \in V, \\ & \int_{\Omega} (\det C_{n+1}^{1/2} - 1) \hat{q} = 0, \quad \forall \hat{q} \in Q. \end{aligned}$$

Unfortunately, because of the nonlinearity in the incompressibility constraint, when solving such a nonlinear problem by a Newton's method, the increments  $\delta x_{n+1}^{(k)}$  and  $\delta p_{n+1}^{(k)}$  are solutions of tangent problems which have the same complexity as a full implicit scheme. The only way to gain in complexity would be to invert only a pressure problem, which requires to adopt a lumped diagonal mass and to neglect all nonlinear coupling terms involving displacements in the calculation of the jacobian  $\det C_{n+1}^{1/2}$ . This modified algorithm does not converge in practice because of the highly nonlinear character of the set of incompressible displacements. Thus, a displacement problem must be inverted at each time step and the associated cost is comparable to the cost of an implicit time step. Therefore, in such constrained nonlinear frameworks involving nonlinear incompressible structures, totally implicit schemes seem to be the only way to explore.

### 3.2 Standard implicit schemes

A standard implicit scheme in elastodynamics uses a trapezoidal rule for time integration with say stress averaging [1]. For nonlinear problems, Simo or Crisfield [12, 2] have proposed to use in addition a transport averaging, which means that each integrand  $(\cdot)_{n+1/2}$  is predicted as follows

$$\begin{aligned} \left( F \cdot \Sigma(\nabla x) \right)_{n+1/2} &= \frac{1}{2}(\nabla x_{n+1} + \nabla x_n) \cdot \Sigma_{n+1/2}, \quad (\text{transport averaging}) \\ \Sigma_{n+1/2} &= \left( \mathcal{W}_{,C}(C(\nabla x_{n+1})) + \mathcal{W}_{,C}(C(\nabla x_n)) \right) \\ &\quad - p_{n+1} \frac{\partial \det(C^{1/2}(\nabla x_{n+1}))}{\partial C} - p_n \frac{\partial \det(C^{1/2}(\nabla x_n))}{\partial C}, \quad (\text{stress averaging}) \\ U_{n+1/2}^s &= \frac{x_{n+1}^s - x_n^s}{\Delta t_n} = \frac{1}{2}(U_{n+1}^s + U_n^s), \\ (\ddot{x}^s)_{n+1/2} &= \frac{U_{n+1}^s - U_n^s}{\Delta t_n}. \end{aligned}$$

For the fluid, the corresponding energy conserving scheme would be Crank Nicholson defined by

$$\left( \frac{\partial(\rho J)^q U^f}{\partial t} \right)_{n+1/2} = \frac{((\rho J)^q U^f)_{n+1} - ((\rho J)^q U^f)_n}{\Delta t_n},$$

and averaging all expressions at time  $t_{n+1/2}$  by

$$(\cdot)_{n+1/2} = \frac{(\cdot)_{n+1} + (\cdot)_n}{2},$$

except for the intermediate velocity and jacobian which, because of conservation arguments to be seen later, must be given by the weighted interpolation rules

$$U_{n+1/2}^f = \frac{((\rho J)^q U^f)_{n+1} + ((\rho J)^q U^f)_n}{(\rho J)_{n+1}^q + (\rho J)_n^q}, \quad (7)$$

$$J_{n+1} = J_n + J_{n+1/2} \operatorname{div}_x (U^G)_{n+1/2}. \quad (8)$$

With this choice, the time discrete problem is :

At each time  $t_{n+1}$ , find the structural deformation  $x_{n+1}^s \in V^s$ , the fluid density  $(\rho^f J)_{n+1}$ , the pressure  $p_{n+1} \in Q$ , the fluid velocity  $((\rho J)^q U^f)_{n+1}$ , the interface traction  $(g_\Gamma)_{n+1} \in W_\Gamma$ , and the fluid configuration mapping  $x_{n+1}^f \in V^f$  such that

$$\begin{aligned} & \int_{\Omega_0^f} \frac{(J\rho)_{n+1} - (J\rho)_n}{\Delta t_n} \hat{q} + \int_{x^f(\Omega_0^f, t_{n+1/2})} \operatorname{div}_x (\rho(U^f - U^G))_{n+1/2} \hat{q} \\ & + \int_{\Omega_0^s} (\det(C^{1/2}(\nabla x)) - 1)_{n+1/2} \hat{q} = 0, \quad \forall \hat{q} : \Omega_0 \rightarrow \mathbb{R}, \end{aligned} \quad (9)$$

for mass and volume conservation,

$$\begin{aligned} & m^s(\ddot{x}_{n+1/2}^s, \hat{U}^s) + \int_{\Omega_0^f} (J\rho)_{n+1/2}^{1-q} \left( \frac{\partial (J\rho)^q U^f}{\partial t} \Big|_{x_0} \right)_{n+1/2} \cdot \hat{U}^f \\ & + \int_{x^f(\Omega_0^f, t_{n+1/2})} \left( \operatorname{div}_x [\rho U_{n+1/2}^f \otimes (U^f - U^G)_{n+1/2}] \right. \\ & \quad \left. + (q-1) \operatorname{div}_x [\rho(U^f - U^G)_{n+1/2}] U_{n+1/2}^f \right) \cdot \hat{U}^f \\ & + \int_{\Omega_0^s} F_{n+1/2} \cdot \Sigma(\nabla x)_{n+1/2} : \nabla \hat{U}^s \\ & + \int_{x^f(\Omega_0^f, t_{n+1/2})} (\mu(\nabla_x U^f + \nabla_x^t U^f) - pId)_{n+1/2} : \frac{\partial \hat{U}^f}{\partial x} \\ & = \int_{\Omega(t_{n+1/2})} f_{n+1/2} \cdot \hat{U} + \int_{\partial\Omega(t_{n+1/2})} g_{n+1/2} \cdot \hat{U} \\ & + \int_{\Gamma_0} (g_\Gamma)_{n+1/2} \cdot \left( \operatorname{tr}(\hat{U}^s)|_\Gamma - \operatorname{tr}(\hat{U}^f)|_\Gamma \right), \quad \forall (\hat{U}^s, \hat{U}^f) \in V^s \times V^f, \end{aligned} \quad (10)$$

for momentum conservation, and for the kinematic interface continuity and the fluid configuration map:

$$\operatorname{tr}(U_{n+1/2}^s)|_{\Gamma_0} = \operatorname{tr}(U_{n+1/2}^f)|_{\Gamma_0}, \quad (11)$$

$$x_{n+1}^f = \operatorname{Ext}((x_{\Gamma_0}^s)_{n+1}). \quad (12)$$

### 3.3 Energy conservation

Energy conservation is a key point in studying fluid structure interactions. A time integration of the principle of momentum conservation taking the real velocity field as test function indicates that the variation of the sum of the kinetic energy of the system and of

the elastic energy of the structure must be equal to the difference between the energy introduced by the external boundary conditions and the energy dissipated by viscous effects inside the fluid. Respecting this energy principle is crucial for preserving stability, and for ensuring the long term accuracy of the numerical predictions. Moreover, this bound on the energy is *the* major tool in the theoretical and numerical analysis of the linearized version of our fluid structure interaction problem following the steps of [3].

In theory, the above time integration schemes have good properties with respect to energy conservation, achieving second order conservation, with an error vanishing at the linear limit. The bad surprise is that these time integration schemes violate this principle of energy conservation when dealing with deformable domains or with nonlinear structures. More precisely, for  $q \neq 1/2$ , and non volume preserving grid configuration maps  $x^f$ , or non quadratic elastic energy, small pollutions terms appear in the energy conservation principle, which may grow exponentially in time.

To study this energy conservation for the time discrete case, we multiply at each time  $t^{n+1/2}$  the variational equation (6) by  $U_{n+1/2}^f$  on the fluid, and by  $U_{n+1/2}^s$  on the structure. *This choice cancels the action of the interface traction forces  $g_\Gamma$  because of the imposed kinematic compatibility condition (3).*

On the structure, the integration of the inertia terms directly yields the variation of the structural kinetic energy

$$\begin{aligned} \left(\frac{\partial U}{\partial t}\right)_{n+1/2} \cdot U_{n+1/2}^s &= \frac{U_{n+1}^s - U_n^s}{\Delta t_n} \cdot \frac{U_{n+1}^s + U_n^s}{2} \\ &= \frac{|U_{n+1}^s|^2 - |U_n^s|^2}{2\Delta t_n}. \end{aligned}$$

The integration of the stiffness terms (excluding the pressure terms) yields

$$\begin{aligned} &\frac{F_n + F_{n+1}}{2} \cdot \left( \mathcal{W}_{,C}(C_{n+1}) + \mathcal{W}_{,C}(C_n) \right) \cdot \nabla U_{n+1/2}^s \\ &= \left( \mathcal{W}_{,C}(C_{n+1}) + \mathcal{W}_{,C}(C_n) \right) \cdot \frac{C_{n+1} - C_n}{2\Delta t_n} \\ &= \frac{1}{\Delta t_n} [\mathcal{W}(C_{n+1}) - \mathcal{W}(C_n) + C \frac{\partial^3 \mathcal{W}}{\partial C^3}(C_*) \cdot (C_{n+1} - C_n)^3] \\ &= \frac{1}{\Delta t_n} [\mathcal{W}(C_{n+1}) - \mathcal{W}(C_n)] + c\Delta t_n^2, \end{aligned}$$

with a similar expression for the pressure terms. We therefore observe there a first source of non conservativity at order  $\Delta t_n^2$ .

On the fluid, a direct integration of the inertia terms yields

$$I_{n+1/2}^f := \int_{\Omega_0^f} (J\rho)_{n+1/2}^{1-q} \left( \frac{\partial (J\rho)^q U^f}{\partial t} \right)_{n+1/2} \cdot U_{n+1/2}^f$$

$$\begin{aligned}
& + \int_{x^f(\Omega_0, t_{n+1/2})} \left( \operatorname{div}_x [\rho U_{n+1/2}^f \otimes (U^f - U^G)_{n+1/2}] \right. \\
& \quad \left. + (q-1) \operatorname{div}_x [\rho (U^f - U^G)_{n+1/2}] U_{n+1/2}^f \right) \cdot U_{n+1/2}^f \\
& = \int_{\Omega_0^f} \left( \frac{\partial(\rho J)^q U^f}{\partial t} \right)_{n+1/2} \cdot (\rho J)_{n+1/2}^{1-q} U_{n+1/2}^f \\
& + \int_{x^f(\Omega_0, t_{n+1/2})} \frac{1}{2} |U_{n+1/2}^f|^2 \operatorname{div}_x [\rho (U^f - U^G)]_{n+1/2} \\
& + \int_{x^f(\Omega_0, t_{n+1/2})} \operatorname{div}_x \left[ \frac{1}{2} \rho |U_{n+1/2}^f|^2 (U^f - U^G)_{n+1/2} \right] \\
& + (q-1) \int_{x^f(\Omega_0, t_{n+1/2})} |U_{n+1/2}^f|^2 \operatorname{div}_x [\rho (U^f - U^G)]_{n+1/2}. \tag{13}
\end{aligned}$$

For a conservative formulation ( $q=1$ ), and taking into account the boundary conditions imposed to the velocity  $(U^f - U^G)_{n+1/2}$  on the interface, we have for example

$$I_{n+1/2}^f = \int_{\Omega_0^f} \left( \frac{\partial J \rho U^f}{\partial t} \right)_{n+1/2} \cdot U_{n+1/2}^f + \int_{x^f(\Omega_0, t_{n+1/2})} \frac{1}{2} |U_{n+1/2}^f|^2 \operatorname{div}_x [\rho (U^f - U^G)]_{n+1/2}.$$

In this case, using direct algebraic manipulations and subtracting the weak equation of mass, we can reduce the inertia terms integral to

$$\begin{aligned}
I_{n+1/2}^f & = \int_{\Omega_0^f} \frac{1}{2} \left( \frac{\partial J \rho |U^f|^2}{\partial t} \right)_{n+1/2} \\
& + \int_{\Omega_0^f} \left( \frac{\partial J \rho U^f}{\partial t} \right)_{n+1/2} \cdot U_{n+1/2}^f - \frac{1}{2} |U_{n+1/2}^f|^2 \left( \frac{\partial J \rho}{\partial t} \right)_{n+1/2} - \frac{1}{2} \left( \frac{\partial J \rho |U^f|^2}{\partial t} \right)_{n+1/2} \\
& + \int_{x^f(\Omega_0, t_{n+1/2})} \left( \frac{1}{2} |U_{n+1/2}^f|^2 - \hat{q} \right) \left( \frac{1}{J_{n+1/2}} \left( \frac{\partial J \rho}{\partial t} \right)_{n+1/2} + \operatorname{div}_x [\rho (U^f - U^G)]_{n+1/2} \right).
\end{aligned}$$

Two error terms therefore appear at this level. The last line corresponds to a truncation error

$$e_h = \int_{\Omega^f(t_{n+1/2})} \frac{\Delta t}{2} \inf_{q_h \in Q_h} \left( \frac{1}{2} |U_{n+1/2}^f|^2 - q_h \right) \left( \frac{1}{J_{n+1/2}} \left( \frac{\partial J \rho}{\partial t} \right)_{n+1/2} + \operatorname{div}_x [\rho (U^f - U^G)]_{n+1/2} \right)$$

which can be made very small by a careful choice of the space of pressure test functions  $Q_h$ . This error disappears for the space continuous problem, and for spatially uniform flows approximated by schemes satisfying the Discrete Geometric Conservation Law.

The second line is proportional to the truncation error induced by the time discretization scheme, but the coefficient of proportionality depends on the regularity in time of

the map  $\rho J$ , that is in particular on the time regularity of the grid configuration  $x^f$ . In other words, any abrupt changes of  $J$  can lead to large local errors. For a Crank Nicholson scheme, we have in more details

$$\begin{aligned} & \left( \frac{\partial J \rho U^f}{\partial t} \right)_{n+1/2} \cdot U_{n+1/2}^f - \frac{1}{2} |U_{n+1/2}^f|^2 \left( \frac{\partial J \rho}{\partial t} \right)_{n+1/2} - \frac{1}{2} \left( \frac{\partial J \rho |U^f|^2}{\partial t} \right)_{n+1/2} = \\ & - \frac{1}{8 \Delta t} \left( (\rho J)_{n+1} - (\rho J)_n \right) |U_{n+1}^f - U_n^f|^2 = O(\Delta t_n^2). \end{aligned}$$

In practice, such a second order conservation is not good enough for nonlinear structures, even in the absence of fluid, as observed on figure 2, on a typical nonlinear incompressible cantilever beam, of length  $L = 1m$ , density  $\rho = 1000kg/m^3$  and stiffness constants  $c_1 = 2MPa$ ,  $c_2 = 0.2MPa$ . Moreover, HHT schemes [7] are not very convincing either, being overdissipative for small time steps and slightly unstable for large time steps.

## 4 Energy conserving schemes

The above analysis indicate that energy corrections are needed. Different choices are proposed in the litterature. We have tested and adopted a nonlinear and non symmetric correction term proposed by Gonzalez [6], which we extend to quasi incompressible problems in mixed formulations and to fluid structure interaction problems. The proposed correction consists in replacing in (10) the averaged stress tensor by :

$$\begin{aligned} \Sigma_{n+1/2} &= 2 \frac{\partial \mathcal{W}}{\partial C} (C_{n+1/2}) \\ &+ 2 \left( \mathcal{W}(C_{n+1}) - \mathcal{W}(C_n) - \frac{\partial \mathcal{W}}{\partial C} (C_{n+1/2}) : \delta C_{n+1/2} \right) \frac{\delta C_{n+1/2}}{\delta C_{n+1/2} : \delta C_{n+1/2}} \\ &- (p_{n+1} + p_n) \left[ \frac{\partial \det C^{1/2}}{\partial C} (C_{n+1/2}) + \right. \\ &\left. \left( \det C_{n+1}^{1/2} - \det C_n^{1/2} - \frac{\partial \det C^{1/2}}{\partial C} (C_{n+1/2}) : \delta C_{n+1/2} \right) \frac{\delta C_{n+1/2}}{\delta C_{n+1/2} : \delta C_{n+1/2}} \right], \quad (14) \end{aligned}$$

with  $\delta C_{n+1/2} = C_{n+1} - C_n$ . This correction amounts to replace derivatives by divided differences. By construction, we have

$$\frac{1}{2} \Sigma_{n+1/2} : \delta C_{n+1/2} = W(C_{n+1}) - W(C_n) - (p_{n+1} + p_n) (\det C_{n+1}^{1/2} - \det C_n^{1/2}).$$

But, from the incompressibility condition (9) written with  $\hat{q}_{|\Omega_0^f} = 0$ , and assuming that the initial condition satisfies also the incompressibility condition, we obtain for all  $\hat{q}$  and

all  $n$ , that  $\int_{\Omega_0^s} (\det C_n^{1/2} - 1) \hat{q} = 0$ , implying exact energy conservation :

$$\int_{\Omega_0^s} \frac{1}{2} \Sigma_{n+1/2} : \delta C_{n+1/2} = \int_{\Omega_0^s} (W(C_{n+1}) - W(C_n)). \quad (15)$$

We then deduce the following conservation theorem

**Theorem 1** *Let us consider an isolated system made of a hyperelastic structure coupled to a viscous fluid, governed by the coupled system (1)-(4), using the partially nonconservative formulation corresponding to  $q = 1/2$  and Gonzalez energy correction. For either the space continuous or the space discretized problem, the total energy variation is then equal to the energy dissipated by viscous effects in the fluid before and after time discretization.*

**Proof.** The time continuous result is obtained by letting  $\Delta t$  go to zero in the following proof. To obtain this energy conservation for the time discrete case, we multiply at each time  $t_{n+1/2}$  the variational equation (10) by  $U_{n+1/2}^f$  on the fluid, and by  $U_{n+1/2}^s$  on the structure. *This choice again cancels the action of the interface traction forces  $g_\Gamma$  because of the imposed kinematic compatibility condition (3).*

The integration of the inertia terms on the structure yields as before the time variation of the kinetic energy of the structure. For the stiffness part, we have from (15)

$$\begin{aligned} & \int_{\Omega_0^s} \frac{F_n + F_{n+1}}{2} \cdot \Sigma_{n+1/2} \cdot \nabla U_{n+1/2}^s \\ &= \int_{\Omega_0^s} \Sigma_{n+1/2} : \left( \frac{F_n^t + F_{n+1}^t}{2} \cdot \frac{F_{n+1} - F_n}{\Delta t_n} \right) \\ &= \frac{1}{\Delta t_n} \int_{\Omega_0^s} \frac{1}{2} \Sigma_{n+1/2} : \delta C_{n+1/2} \\ &= \frac{1}{\Delta t_n} \int_{\Omega_0^s} (W(C_{n+1}) - W(C_n)). \end{aligned}$$

For the fluid, in the inertia term calculated in (13), two divergence terms cancel when one uses the choice  $q = 1 - q = \frac{1}{2}$ . Due to the boundary conditions imposed on  $(U^f - U^G)$ , the last divergence term also disappears. The fluid inertia terms finally reduce to the simple time integration of the variable  $\sqrt{\rho J} U^f$  multiplied by (the finite difference approximation of) its time derivative

$$I_{n+1/2}^f = \int_{\Omega_0^f} \left( \frac{\partial \sqrt{\rho J} U^f}{\partial t} \right)_{n+1/2} \cdot (\sqrt{\rho J})_{n+1/2} U_{n+1/2}^f.$$

This is exactly the variation of the fluid kinetic energy *if we use a weighted intermediate velocity* as defined in (7). Indeed, we have

$$I_{n+1/2}^f = \int_{\Omega_0^f} \frac{1}{2\Delta t_n} \left( (\sqrt{\rho J} U^f)_{n+1} - (\sqrt{\rho J} U^f)_n \right) \cdot \left( (\sqrt{\rho J} U^f)_{n+1} + (\sqrt{\rho J} U^f)_n \right)$$

which is exactly

$$I_{n+1/2}^f = \int_{\Omega_0^f} \frac{(\rho J |U^f|^2)_{n+1} - (\rho J |U^f|^2)_n}{2\Delta t_n}.$$

Observe that it was very important here to work with  $q = 1/2$ . In fact, any other formulation induces a first order time integration error on the kinetic energy transport equation.

The energy conservation principle follows after multiplication by  $\Delta t$  and summation over all time steps. The pressure term disappears because of the mass conservation equation (9) written with  $\hat{q} = p_{n+1/2}$ , and from the weighted definition of the jacobian in (8). In the absence of external load and of any inflow, we obtain that the variation of the sum of the total kinetic energy and of the structural elastic energy, reduces then as expected to a viscous dissipation term.

**Remark 2** *The stability proved above is stronger than what is usually checked in the numerical analysis for time discretization schemes. It proves that the energy is unconditionally and uniformly bounded in time.*

## 5 Numerical experiment

In this section, we illustrate numerically the previous analysis on the case of a bidimensional quasi incompressible cantilever beam in plane displacements. The elastic potential is given by the Mooney-Rivlin constitutive law:

$$\mathcal{W}(C) = c_1 (\text{tr } C - 3) + c_2 (\text{tr } \text{cof } C - 3).$$

Data is presented on figure 3, and a  $Q_1$  finite-element space approximation is performed on the mesh presented on figure 4.



Figure 2: A cantilever beam with constant force F.

As F is a constant force, we ideally expect to observe the conservation of the following discrete quantity:

$$H_n = \int_{\Omega} f \cdot \varphi_n - \mathcal{E}_n,$$

length	1 m
width	0.1 m
F	1000 N
$\rho$	1000 kg/m <sup>3</sup>
$c_1$	2 MPa
$c_2$	0.2 MPa
$1/\epsilon$	2.E12 Pa
T	10 s
$\Delta t$	0.02 s
Newton's tolerance	1E-7

Figure 3: Physical data and numerical choices.

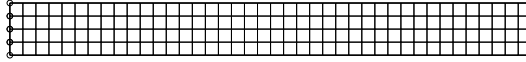


Figure 4: A 250 elements mesh of the beam.

with the energy:

$$\mathcal{E}_n = \frac{1}{2} \int_{\Omega} \rho^s (U^s)^2 + \int_{\Omega} \mathcal{W}(C_n).$$

The constant time step  $\Delta t$  is chosen such that there is approximately 20 time steps per oscillation of the cantilever beam. With this value of  $\Delta t$ , 4 or 5 Newton's iterations per time step are necessary to solve the problem with the required precision ( $10^{-7}$  m). Due to numerical instabilities, the number of Newton's iterations per time step can grow up; the simulation is stopped when exceeding 20.

Our first observation is that when  $H_n$  decreases (global increase of the energy of the system), the number of Newton's iterations per time step grows up until the method does not converge any more. As a consequence, trapezoidal, midpoint and HHT [7] schemes cannot achieve the simulation on the whole time interval  $[0, T]$  for the specified parameters. Only the conservative Gonzalez scheme can achieve long term time integration without such an overcost.

Energy evolution is presented on figure 6. Energy conservation does not hold for usual schemes in the nonlinear framework, as shown in the previous analysis. With the chosen parameters, even HHT which is proved to be linearly dissipative, is globally energy growing. Gonzalez scheme is quasi exactly conservative up to a very small error term depending on Newton's tolerance.

Concerning midpoint and trapezoidal schemes, energy growing goes with numerical instabilities on the speed, as shown on figure 7.

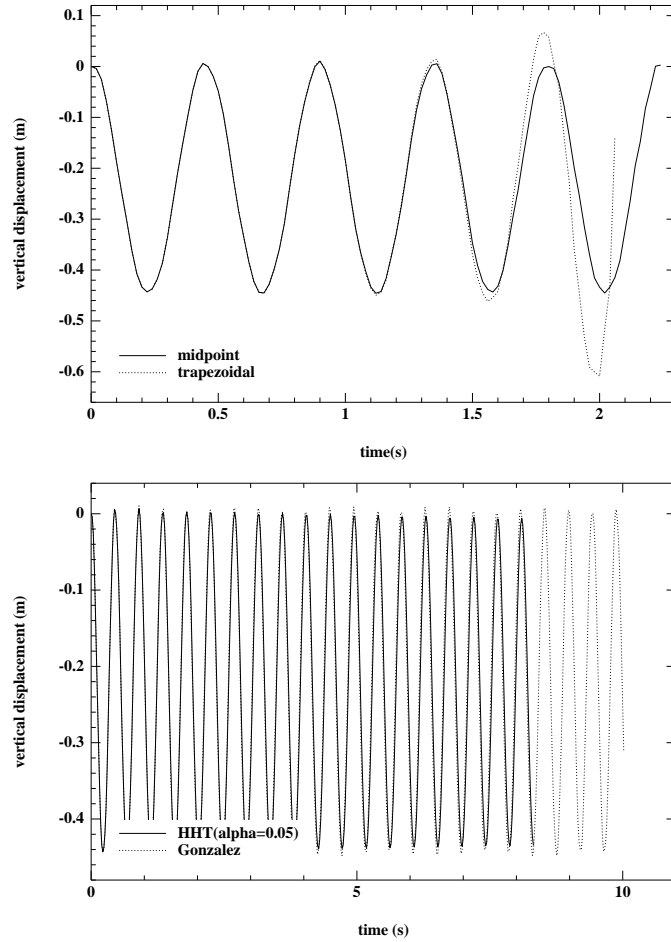


Figure 5: Vertical displacement of the tip of the cantilever beam. Simulation stops when the time step calculation exceeds 20 Newton's iterations.

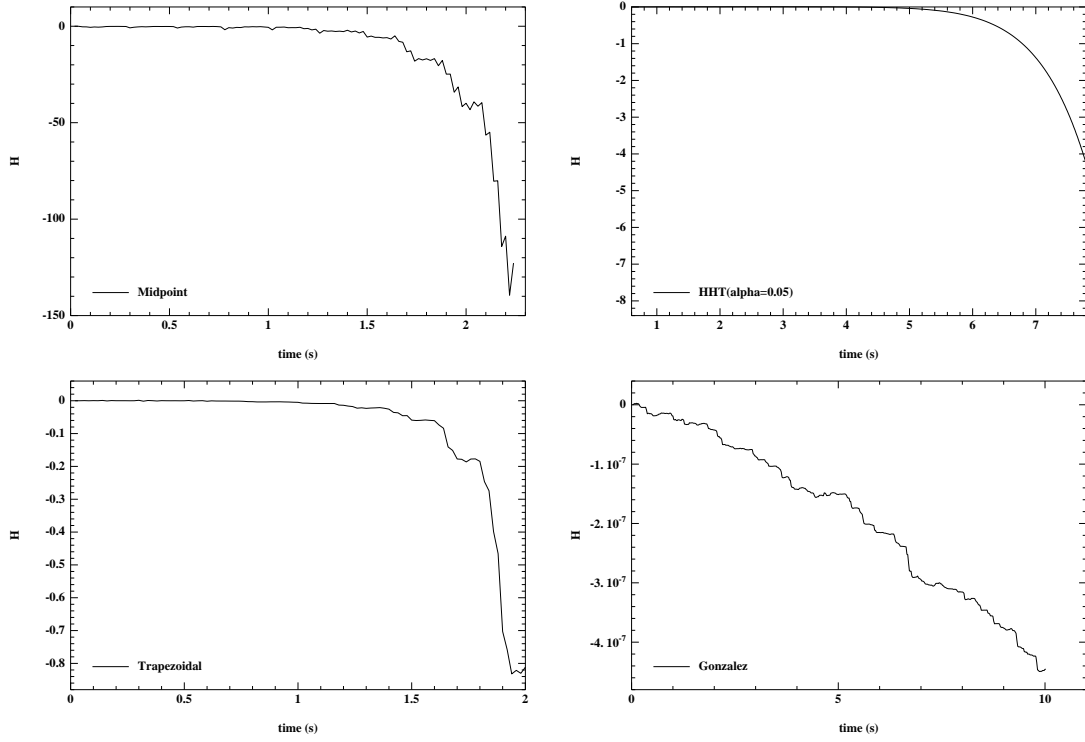


Figure 6: Evolution of the discrete total potential  $H$  (in Joules) as a function of time. As an indication, the maximal value of the deformation potential  $\int_{\Omega} \mathcal{W}(C)$  is about 0.5 Joules.

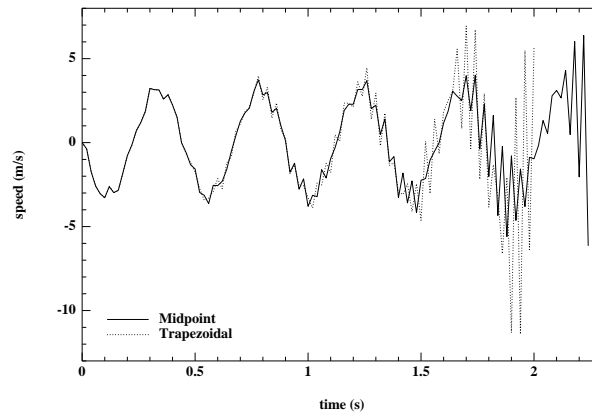


Figure 7: Instability of the vertical speed at the tip of the cantilever beam, for midpoint and trapezoidal schemes.

## 6 Conclusion

The analysis presented in this paper may give a better insight on the stability properties of the different time integration schemes used in fluid structure interaction. We have observed that the classical conservative or nonconservative formulations were not natural candidates for respecting long term energy conservation, and that second order schemes had better stability properties when using a somewhat mixed formulation, and nonlinear energy correction terms on the structure.

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