Automatic Coarse Space Generation for Robust Domain Decomposition Methods

in honor of Patrick Le Tallec’s 60th birthday

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Outline

- Tire simulation and Domain Decomposition Methods
- Abstract Domain Decomposition Framework (covering most existing methods)
- How to build (in an automatic fashion) a Coarse Space to increase robustness / convergence rate of existing Domain Decomposition Methods
- Application to Schwarz and BDD Decomposition Method
- Conclusion and Perspectives
Tires are made of highly heterogeneous incompressible nonlinear materials in thin layers and are subjected to large deformations, (self) contact, thermo-mechanical coupling, fluid-structure interactions...

Targeted performances imply various scales in space and time (wear, endurance, longitudinal and transversal traction, rolling resistance, noise, handling) ...
Tire design involves an increasing number of 3D simulations to assess and optimize performances.

- A 3D time dependent computation: < 1 million nodes over 500 time steps.
- In a word, quite a reasonable case but scaling matters.

Unfortunately, huge heterogeneities, incompressibility, bad aspect ratios and contact remain quite a challenge for Domain Decomposition Methods.

Handling multi-scale coupling requires good understanding of the solution decomposition. An even better reason to invest on the field.
Abstract Domain Decomposition Framework

Let us find \( u \in V \) such that \( Au = b \), in \( V^* \); \( A \) is symmetric and coercive.

Let \( (V_i) \) be local spaces and \( R_i^\top : V_i \to V \) extension operators such that

\[
\sum_{i=0}^{I} R_i^\top V_i = V.
\]

Let \( \tilde{A}_i : V_i \to V_i^* \) be local invertible operators.

Let \( \tilde{P}_i : V \to V_i \) be the local operators such that, for all \( u \in V \),

\[
\left\langle \tilde{A}_i \tilde{P}_i u, v_i \right\rangle_{V_i^*, V_i} = \left\langle Au, R_i^\top v_i \right\rangle_{V^*, V}, \quad \forall v_i \in V_i.
\]

It defines the best approximation of \( u \) in \( V_i \). Denote \( P_i = R_i^\top \tilde{P}_i : V \to V \).

**Remark 1** \( \tilde{P}_i R_i^\top : V_i \to V_i \) is the identity operator provided \( \tilde{A}_i = R_i A R_i^\top = A_i \).
Abstract Domain Decomposition Framework

Additive Preconditioned Operator:

\[ P_{add} = \sum_{i=0}^{I} R_i^\top \tilde{P}_i = \left( \sum_{i=0}^{I} R_i^\top \tilde{A}_i^{-1} R_i \right) A. \]

Hybrid Preconditioned Operator:

\[ P_{hyb} = P_0 + (I - P_0) \sum_{i=1}^{I} R_i^\top \tilde{P}_i (I - P_0). \]
Abstract Domain Decomposition Framework

Assumption 1 (Strong Cauchy-Schwartz inequality) For all $u_m \in V_m, v_n \in V_n$

$$\left| \left\langle A R_m^T u_m, R_n^T v_n \right\rangle_{V^*,V} \right| \leq \epsilon_{ij} \left\langle A R_m^T u_m, R_m^T u_m \right\rangle_{V^*,V}^{1/2} \left\langle A R_n^T v_n, R_n^T v_n \right\rangle_{V^*,V}^{1/2},$$

for all $1 \leq m, n \leq I$.

Assumption 2 (Stability of local operators) For all $u_m \in \text{Range}(\tilde{P}_m) \subset V_m$,

$$\left\langle A R_m^T u_m, R_m^T u_m \right\rangle_{V^*,V} \leq \omega \left\langle \tilde{A}_m u_m, u_m \right\rangle_{V_m^*,V_m},$$

for all $1 \leq m, n \leq I$.

Assumption 3 (Stable decomposition) For all $u \in V$, there exists $u_i \in V_i, 0 \leq i \leq I$ such that

$$\sum_{i=0}^{I} R_i^T u_i = u \quad \text{and} \quad \sum_{i=0}^{I} \left\langle \tilde{A}_i u_i, u_i \right\rangle_{V_i^*,V_i} \leq C_0 \left\langle A u, u \right\rangle_{V^*,V}.$$
Abstract Domain Decomposition Framework

Theorem 1  Under Assumptions 1,2,3, one has for every \( u \in V \),

\[
\frac{1}{C_0} \langle A u, u \rangle_{V^*, V} \leq \langle A P_{add} u, u \rangle_{V^*, V} \leq (1 + \rho(\epsilon) \omega) \langle A u, u \rangle_{V^*, V}.
\]

Under Assumptions 1,2 and Assumption 3 over \( \text{Range}(I - P_0) \), one has for every \( u \in V \),

\[
\frac{1}{\max(1, C_0)} \langle A u, u \rangle_{V^*, V} \leq \langle A P_{hyb} u, u \rangle_{V^*, V} \leq \max(1, \rho(\epsilon) \omega) \langle A u, u \rangle_{V^*, V}.
\]

- For local operators \( A \), the spectral radius \( \rho(\epsilon) \) is only related to the local topology of the decomposition into subdomains.

- To guarantee the robustness of domain decomposition methods, the constants \( C_0 \) and \( \omega \) need to be controlled uniformly.
Coarse Space Generation

Let \( \tilde{A}_n : U_n \rightarrow U_n^* \) and \( R_n^\top : U_n \rightarrow V \) for every \( 1 \leq n \leq I \).

They define a first method to be robustified by the introduction of a coarse-space.

Let us build a new one with \( V_n \subset U_n \), \( V_0 \), \( \tilde{A}_i \), \( R_i^\top \) satisfying improved properties.

**Lemma 1 (Assumption 2 Enhancement)*** Let \( \gamma \) be a positive parameter and \( X_n \subset U_n \) be the subspace of eigenvectors \( u_n \in U_n \) such that

\[
\tilde{A}_n u_n = \lambda_n A_n u_n, \quad \text{with} \quad \lambda_n \leq \gamma,
\]

including \( \ker(\tilde{A}_n) \). The following decomposition holds: \( U_n = X_n \oplus X_n^\perp \), \( 1 \leq n \leq I \).

For all \( u_n \in X_n^\perp \),

\[
\langle A_n u_n, u_n \rangle_{V^*, V} \leq \gamma^{-1} \langle \tilde{A}_n u_n, u_n \rangle_{U_n^*, U_n},
\]

with notation \( A_n = R_n A R_n^\top \).
To proceed with Assumption 3 Enhancement, several ingredients are introduced:

- **Restriction operators** $D_n : V \to \tilde{U}_n$ and $U_n \subset \tilde{U}_n$.
- **Weighting operators** $\Xi_n : \tilde{U}_n \to U_n$ such that $\sum_{n=1}^I \mathbb{R}_n^\top \Xi_n D_n = \text{Id}$.
- **Local additive decomposition**

$$\sum_{n=1}^I \langle A_n D_n u, D_n u \rangle_{U_n^*,U_n} \leq C_1 \langle A u, u \rangle_{V^*,V}, \quad \forall u \in V.$$

**Lemma 2 (Assumption 3 Enhancement)** Let $\theta$ be a positive parameter and $\tilde{Y}_n \subset \tilde{U}_n$ be the subspace of eigenvectors $\tilde{u}_n \in \tilde{U}_n$ such that

$$A_n \tilde{u}_n = \lambda_n \Xi_n^\top \tilde{A}_n \Xi_n \tilde{u}_n \quad \text{in} \ \tilde{U}_n^*, \quad \text{with} \ \lambda_n \leq \theta.$$

The following decomposition holds: $U_n = Y_n \oplus Y_n^\perp$, $1 \leq n \leq I$ with $Y_n = \Xi_n \tilde{Y}_n$. For all $u \in \sum_{n=1}^I \mathbb{R}_n^\top Y_n^\perp$, there exists $u_n = \Xi_n D_n u \in Y_n^\perp$ such that

$$\sum_{n=1}^I \langle \tilde{A}_n u_n, u_n \rangle_{U_n^*,U_n} \leq C_1 \theta^{-1} \langle A u, u \rangle_{V^*,V}.$$
With $Z_n = X_n + Y_n$, let us define the resulting space decompositions

$$U_n = Z_n \oplus V_n, \quad V_0 = \sum_{n=1}^{I} \mathbb{R}_n^T Z_n \subset V.$$  

New extension operators read $R_n^T = \mathbb{R}_n^T$ on $V_n$ and $R_0^T : V_0 \hookrightarrow V$.

New local operators read $\tilde{A}_n = \tilde{\Lambda}_n$ on $V_n$ and $\tilde{A}_0 = R_0 A R_0^T$.

**Theorem 2**  The resulting Domain Decomposition Method exhibit robust convergence properties only depending on the connectivity of the decomposition, $\gamma$ and $\theta$.

**Remark 2**  When $\tilde{A}_n$ is not invertible, only the hybrid version makes sense with pseudo-inverses $\tilde{A}_n^\dagger$.

**Remark 3**  Eigen-problems can be simplified most of the time.
An application to the Schwarz Domain Decomposition

Linear Elasticity: Find \( u \in V = H^1_0(\Omega) \) such that

\[
\langle Au, v \rangle_{V^*, V} = \int_{\Omega} E : \nabla^s u : \nabla^s v = \int_{\Omega} f \cdot v, \quad \forall v \in V.
\]

Consider an overlapping decomposition \( \Omega = \bigcup_{n=1}^{I} \Omega_n \).

Take subspaces \( U_n = H^1_0(\Omega_n) \) and \( \mathbb{R}_n^\top : H^1_0(\Omega_n) \to H^1_0(\Omega), 1 \leq n \leq I \).

Work with local exact solvers \( \tilde{A}_n = A_n = \mathbb{R}_n A \mathbb{R}_n^\top \). Assumption 2 is trivial.

Define \( \tilde{U}_n = H^1(\Omega_n) \) and the partition of unity \( \Xi_n : \tilde{U}_n \to U_n \). The eigen-problem of interest is to find \( u_n^{(k)} \in \tilde{U}_n \) such that

\[
\int_{\Omega_n} E : \nabla^s u_n^{(k)} : \nabla^s v_n = \lambda_n^{(k)} \int_{\Omega_n} E : \nabla^s (\Xi_n u_n^{(k)}) : \nabla^s (\Xi_n v_n), \quad \forall v_n \in \tilde{U}_n.
\]

Assumption 3 is enhanced with resulting coarse space

\[
V_0 = \left\{ \mathbb{R}_n^\top \Xi_n u_n^{(k)}, \ 1 \leq n \leq I, \ \lambda_n^{(k)} \leq \theta \right\}.
\]
An application to Balancing Domain Decomposition

Condensed Linear Elasticity; on a non-overlapping partition of $\Omega = \bigcup_{n=1}^{I} \Omega_n$, with $\Gamma = \bigcup_{n=1}^{I} \partial \Omega_n \setminus \partial \Omega$, find $u \in V = \text{Tr}_\Gamma H^1_{0,h}(\Omega)$ such that

\[
\hat{S}u = \sum_{n=1}^{I} D^\top_n S_n D_n u = g, \quad \text{in } V^*.
\]

- $U_n = \tilde{U}_n = \text{Tr}_{\partial \Omega_n \setminus \partial \Omega} H^1_{0,h}(\Omega_n)$ and $D_n : V \to U_n$.
- Introduce $\mathbb{R}^\top_n : U_n \to V$, $1 \leq n \leq I$ by nodal discrete extension.
- Work with local solvers $\tilde{S}_n = \Xi_n^\top S_n \Xi_n^{-1}$ with weighting functions $\Xi_n : U_n \to U_n$.
- Assumption 3 is trivial.
- To enhance Assumption 2, the eigen-problem of interest is to find $u^{(k)}_n \in U_n$ such that

\[
\tilde{S}_n u^{(k)}_n = \lambda^{(k)}_n (\mathbb{R}_n \hat{S} \mathbb{R}^\top_n) u^{(k)}_n.
\]
- Assumption 2 is enhanced by the resulting coarse space

\[
V_0 = \left\{ \mathbb{R}^\top_n u^{(k)}_n, \quad 1 \leq n \leq I, \quad \lambda^{(k)}_n \leq \gamma \right\}.
\]
Some Illustrations

- Highly improved convergence behavior on a highly heterogeneous layered beam (Schwarz)

- Some highly deformed states whose computation is accelerated (Enhanced FETI)
Conclusion

A route towards guaranteed performance of domain decomposition methods at large (multi-scale analysis, space DDM, space-time).

Off-line eigen-problems can be considered in a reduced basis perspective.

Some open questions:

- Building coarse spaces 'on the fly'.
- Extension to non-symmetric problems.
- If the problem additionally depends upon a parameter, efficiently build a coarse space covering computations on the whole parameter range.

Happy Birthday Patrick!