

Two-scale Dirichlet-Neumann preconditioners for elastic problems with boundary refinements

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Abstract

The present work deals with the efficient resolution of elastostatics problems on domains with boundary refinements. The proposed approach separates the boundary refinements from the interior of the domain by the mortar method, and uses Dirichlet-Neumann preconditioners to solve the corresponding algebraic system. We prove that the simplest Dirichlet-Neumann algorithm achieves independence of the condition number of the preconditioned system with respect to the number and the size of the small details. Nevertheless, the situation no longer prevails when the refined boundary is clamped. An enhanced preconditioner is then designed by the introduction of a coarse space to mitigate the aforementioned sensitivity. Some numerical tests are performed to confirm the analysis, and the tools are extended by the proposition of a quasi-Newton method in the case of nonlinear elasticity. This paper is an extended version of a work presented at the DD16 conference with proofs and complete numerical results.

Key words: domain decomposition, Dirichlet-Neumann preconditioner, coarse grid, mortar methods

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1 Introduction

The present paper is devoted to the construction of efficient numerical procedures to solve vector elliptic problems with small geometric details on the boundary of the domain, where a *localized fine-scale* behavior of the solution is expected. In particular, the solution for displacements $u \in \mathbb{R}^d$ of the linearized elastostatics problem will be considered, that is for $d = 2, 3$

$$\begin{cases} -\operatorname{div}(\mathbf{E} : \varepsilon(u)) = f & \text{on } \Omega \subset \mathbb{R}^d, \\ u = 0 & \text{on } \Gamma_D, \\ (\mathbf{E} : \varepsilon(u)) \cdot n = g & \text{on } \Gamma_N. \end{cases} \quad (1)$$

Above, the linearized strain tensor is denoted by

$$\varepsilon(u) = \frac{1}{2} (\nabla u + \nabla^\top u),$$

and the fourth order tensor \mathbf{E} is assumed to be elliptic over the set of symmetric matrices

$$\exists \alpha > 0, \forall \xi \in \mathbb{R}^{d \times d}, \xi^\top = \xi, \quad (\mathbf{E} : \xi) : \xi \geq \alpha \xi : \xi.$$

In our framework, we consider solutions that vary rapidly inside the disjoint subsets $(\Omega_k)_{1 \leq k \leq K}$ of Ω . In applications like tire design, one could think of geometric refinements or sculptures along the boundary $\partial\Omega$. Conversely, the solution u has slow variations in the interior subdomain $\Omega_0 = \Omega \setminus \overline{(\cup_{1 \leq k \leq K} \Omega_k)}$.

The strategy proposed in this paper uses a non-conforming mortar formulation for (1) in order to decompose the physical domain into coarse and fine zones. Dirichlet-Neumann preconditioners are proposed, analyzed and tested to solve the corresponding linear system for the approximate cost of inversion of the coarse system, that is the problem set over Ω_0 . To achieve such a purpose, it is crucial that the computational cost of the local solution over each $(\Omega_k)_{1 \leq k \leq K}$ remains of the same order (or less expensive) than the computational cost of the coarse solution over Ω_0 . The present paper is essentially concerned with a two-scale property for such preconditioners, in the sense that the condition number of the preconditioned system should remain independent of the number and the size of the small subdomains.

Mortar methods were introduced in [1,2] as a weak coupling between subdomains with non-conforming meshes, or between subproblems solved with different approximation methods. The main purpose was to overcome the very

sub-optimal “ \sqrt{h} ” error estimate obtained with pointwise matching. The analysis of this method as a mixed formulation can be found in [3]. For the present purpose, various Lagrange-multiplier spaces can be indifferently adopted. For example, one can use the original formulation from [1]. It is worth noticing that because of the disjoint character of the small subdomains, no modification of Lagrange multipliers is necessary on interface boundaries. Indeed, interfaces are only shared by two subdomains: the coarse one, and a fine one. Therefore, no cross-points or cross-lines shared by more than two subdomains are encountered. Other possibilities include the dual variant from [4,5] which provides a diagonal constraint, or discontinuous stabilized formulations [6–9] involving block-diagonal constraints. In the case of a second order approximation in displacements, one can also adopt the proposal from [10], opting for affine Lagrange multipliers. The list is of course not exhaustive.

Moreover, the coercivity constant for non-conforming elastostatics under mortar constraint has been proved in [11,8], to be independent of the number and the size of the subdomains. There is consequently no limitation in considering a high number of small subdomains, error estimates remaining optimal. A brief review of the non-conforming formulation adopted herein to discretize (1) is presented in section 2.

The challenge is then to develop a solver which efficiently handles such situations. In the present framework, the non-symmetric roles played by the coarse and fine subdomains favors Dirichlet-Neumann preconditioners (see for instance [12,13]), rather than symmetric strategies such as Neumann-Neumann [14] or FETI [15], already widely studied in the mortar setting [16–20]. In sections 3 and 4, we start by proposing a basic Dirichlet-Neumann preconditioner and prove that its quality is independent of the number and of the size of the boundary refinements, provided no refinement is clamped. In this sense, we can talk of two-scale preconditioning. Nevertheless, this first preconditioner can only be applied when Ω_0 is clamped, and its quality deteriorates when a displacement boundary condition is imposed on a fine subdomain. We mitigate this drawback by the introduction of local spaces representing the interface rigid motions. An enhanced Dirichlet-Neumann preconditioner insensitive to boundary conditions is then obtained and analyzed. The resulting algorithm is summarized in section 5, and tested in section 6 to confirm the theoretical analysis.

Finally, when considering nonlinear problems with non-stiff geometric refinements on the boundary, section 6 shows from the numerical point of view, the efficiency of quasi-Newton methods based on our preconditioners.

2 A mortar formulation

2.1 Continuous problem

Let $\Omega \subset \mathbb{R}^d$ be an open set partitioned into $K+1$ subsets $(\Omega_k)_{0 \leq k \leq K}$ satisfying $\overline{\Omega} = \cup_{i=0}^K \overline{\Omega}_i$ and $\Omega_k \cap \Omega_l = \emptyset$ if $k, l \geq 1$. We denote by $\Gamma_{0k} = \overline{\Omega}_0 \cap \overline{\Omega}_k$ the interface between Ω_0 and Ω_k , and the skeleton of the internal interfaces is denoted by $\mathcal{S} = \cup_{k=1}^K \Gamma_{0k}$. For the understanding of the situation, let us say that Ω_0 has slowly varying physical properties whereas the disjoint subsets $(\Omega_k)_{1 \leq k \leq K}$ have rapidly varying ones or complex geometries. Moreover, the subdomain $\overline{\Omega}_0$ has a non-empty intersection with all the subdomains $(\Omega_k)_{1 \leq k \leq K}$. We will also assume as a simplification that the intersection between two local subdomains Ω_k , $k \geq 1$ is empty. In other words, for the time being, the inclusions are disconnected. On the part Γ_D of the boundary $\partial\Omega$, an homogeneous Dirichlet boundary condition is imposed. Concerning the coefficients of the fourth order elasticity tensor \mathbf{E} , we assume that the stress tensor is symmetric whatever the deformation in the material is, namely for almost all $x \in \Omega$,

$$\forall \xi \in \mathbb{R}^{d \times d}, \xi^\top = \xi, \quad \mathbf{E}(x) : \xi \text{ is a symmetric matrix.}$$

Moreover, the different materials are spectrally isotropic, namely for all $k \geq 1$, there exists two constants c_k and C_k , such that for almost all $x \in \Omega_k$,

$$\forall \xi \in \mathbb{R}^{d \times d}, \xi^\top = \xi, \quad c_k \xi : \xi \leq (\mathbf{E}(x) : \xi) : \xi \leq C_k \xi : \xi. \quad (2)$$

For homogeneous isotropic materials, if E_k stands for the Young modulus of the material filling Ω_k , both c_k and C_k are proportional to E_k .

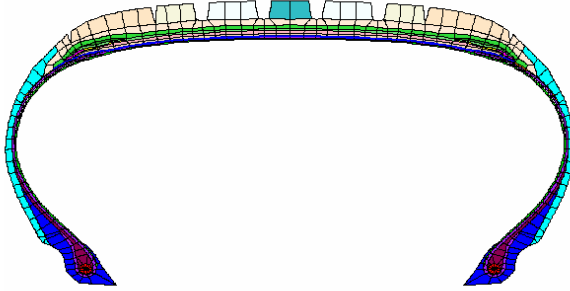


Fig. 1. Example of a structure presenting small geometric refinements on its boundary.

We introduce the following spaces for the displacement fields

$$H_*^1(\Omega) = \{v \in H^1(\Omega)^d, v|_{\Gamma_D} = 0\},$$

$$H_*^1(\Omega_k) = \{v \in H^1(\Omega_k)^d, v|_{\Gamma_D \cap \partial\Omega_k} = 0\},$$

$$X = \left\{ v \in L^2(\Omega)^d, v_k = v|_{\Omega_k} \in H_*^1(\Omega_k), \forall k \right\} = \prod_{k=0}^K H_*^1(\Omega_k),$$

X being endowed with the H^1 broken norm

$$\|v\|_X = \left(\sum_{k=0}^K \|v\|_{H^1(\Omega_k)^d}^2 \right)^{\frac{1}{2}}.$$

We also introduce as interface space

$$M = \prod_{k=1}^K L^2(\Gamma_{0k})^d.$$

In the whole paper, for homogeneity reasons, the H^1 norm is rescaled and defined by

$$\|v\|_{H^1(\Omega_k)^d}^2 = \frac{1}{(L_k)^2} \|v\|_{L^2(\Omega_k)^d}^2 + \|\nabla v\|_{L^2(\Omega_k)^d}^2,$$

where L_k denotes the diameter of Ω_k .

Our elastostatic problem (1) amounts to finding $u \in H_*^1(\Omega)$ such that

$$a(u, v) = l(v), \quad \forall v \in H_*^1(\Omega), \quad (3)$$

where the continuous coercive bilinear form a is defined as

$$a(u, v) = \int_{\Omega} (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in H_*^1(\Omega),$$

and the continuous linear form l as

$$l(v) = \int_{\Omega} f \cdot v + \int_{\Gamma_N} g \cdot v, \quad \forall v \in H_*^1(\Omega),$$

with $f \in L^2(\Omega)^d$ and $g \in L^2(\Gamma_N)^d$. This problem is well-posed from the Lax-Milgram lemma, by using Korn's inequality (see [21]) to prove the coercivity of the bilinear form a on $H_*^1(\Omega)$.

2.2 Discretization

We introduce here a domain-based non-conforming discretization of the problem using mortar elements. Well-posedness results and error estimates are reviewed.

2.2.1 The mesh

For each $0 \leq k \leq K$, let us consider a family of shape regular meshes $(\mathcal{T}_{k;h_k})_{h_k>0}$ defined over each domain Ω_k , and denote $h_k = \sup_{T \in \mathcal{T}_{k;h_k}} \text{diam}(T)$ the maximal local diameter of the triangulation. The mesh $\mathcal{T}_{0;h_0}$ defined on Ω_0 is the coarsest, i.e $h_0 > h_k$ for all $1 \leq k \leq K$, and a non-conforming family of meshes $(\mathcal{T}_h)_{h>0}$ over Ω is obtained as $\mathcal{T}_h = \cup_{k=0}^K \mathcal{T}_{k,h_k}$, denoting $h = \max_{0 \leq k \leq K} h_k$.

For each $1 \leq k \leq K$, the interface Γ_{0k} inherits from the family of meshes $(\mathcal{F}_{k;\delta_k})_{\delta_k>0}$ defined as the trace of the fine meshes $(\mathcal{T}_{k;h_k})_{h_k>0}$ over Γ_{0k} , with $\delta_k = \sup_{F \in \mathcal{F}_{k;\delta_k}} h(F)$. Moreover, each Γ_{0k} is assumed to be the union of faces of elements in $\mathcal{T}_{k;h_k}$ for any discretization. Such an assumption is in general referred to as *geometrical conformity*.

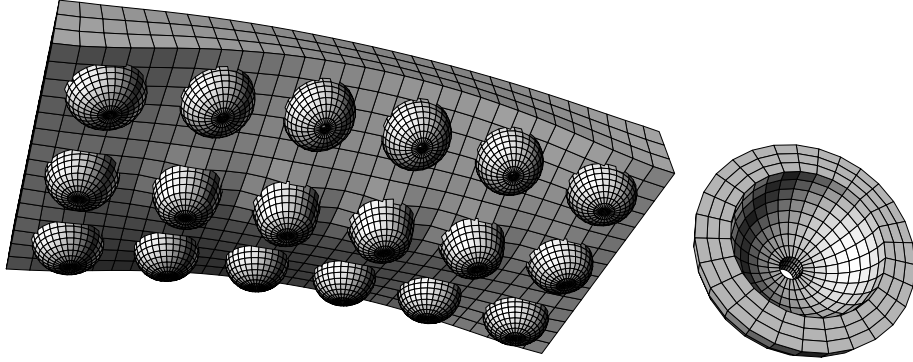


Fig. 2. Mesh of a 3D structure with spherical boundary refinement (left; $\simeq 20,000$ nodes) and zoom on the interior of a boundary detail (right; $\simeq 2,000$ nodes).

2.2.2 Mesh-dependent spaces

We define here some mesh-dependent spaces, endowed with useful mesh-dependent norms already proposed and used in [22,23]. For each $1 \leq k \leq K$, they are defined as

$$\mathbb{H}_\delta^{1/2}(\Gamma_{0k}) = \left\{ \phi \in L^2(\Gamma_{0k})^d; \|\phi\|_{\delta, \frac{1}{2}, k}^2 = \sum_{F \in \mathcal{F}_{k;\delta_k}} \frac{1}{h(F)} \|\phi\|_{L^2(F)^d}^2 < +\infty \right\},$$

$$\mathbb{H}_\delta^{-1/2}(\Gamma_{0k}) = \left\{ \lambda \in L^2(\Gamma_{0k})^d; \|\lambda\|_{\delta, -\frac{1}{2}, k}^2 = \sum_{F \in \mathcal{F}_{k;\delta_k}} h(F) \|\lambda\|_{L^2(F)^d}^2 < +\infty \right\},$$

endowed respectively with the norms $\|\cdot\|_{\delta, \frac{1}{2}, k}$ and $\|\cdot\|_{\delta, -\frac{1}{2}, k}$. The product spaces $\mathbb{W}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{1/2}(\Gamma_{0k})$ and $\mathbb{M}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{-1/2}(\Gamma_{0k})$, are consequently provided

with the product norms

$$\|\phi\|_{\delta, \frac{1}{2}} = \left(\sum_{k=1}^K \|\phi\|_{\delta, \frac{1}{2}, k}^2 \right)^{1/2}, \quad \|\lambda\|_{\delta, -\frac{1}{2}} = \left(\sum_{k=1}^K \|\lambda\|_{\delta, -\frac{1}{2}, k}^2 \right)^{1/2},$$

and can be seen as dual spaces by means of the L^2 inner product.

Remark 1 *The use of such mesh-dependent spaces instead of $H_{00}^{1/2}(\Gamma_{0k})^d$ and its dual $H^{-1/2}(\Gamma_{0k})^d = (H_{00}^{1/2}(\Gamma_{0k})^d)'$ has several advantages. First, these mesh-dependent norms are easily computable, which is convenient as a posteriori estimates (cf. [23]) and penalized formulations (cf. [24]) are concerned. Moreover, such mesh dependent norms are at present time privileged tools for error estimates in the 3D setting.*

2.2.3 Non-conforming approximation

Let us introduce the discrete subspaces of degree q inside each subdomain

$$X_{k;h_k} = \{p \in H_*^1(\Omega_k) \cap C^0(\Omega_k)^d, \quad p|_T \in \mathcal{P}_q(T), \forall T \in \mathcal{T}_{k;h_k}\} \oplus \mathcal{B}_{k;h_k},$$

with $\mathcal{P}_q = [\mathbb{P}_q]^d$ or $[\mathbb{Q}_q]^d$, where \mathbb{P}_q (resp. \mathbb{Q}_q) is the space of polynomials of partial (resp. total) degree q . We have introduced a possible stabilization space $\mathcal{B}_{k;h_k}$ of interface bubbles as in [25,7,24]. The corresponding product space is denoted by $X_h = \prod_{k=0}^K X_{k;h_k} \subset X$. Moreover, let us define the following trace spaces on the non-mortar side (small subdomain side herein):

$$W_{k;\delta_k} = \{p|_{\Gamma_{0k}}, p \in X_{k;h_k}\}, \quad W_{k;\delta_k}^0 = W_{k;\delta_k} \cap H_0^1(\Gamma_{0k})^d,$$

endowed with the mesh-dependent norm $\|\cdot\|_{\delta, \frac{1}{2}, k}$.

In order to formulate the weak continuity constraint, we introduce the spaces $M_{k;\delta_k}$ of (possibly discontinuous) Lagrange multipliers defined on the meshes $\mathcal{F}_{k;\delta_k}$. Achieving optimal approximation requires that such spaces contain all polynomials $[\mathbb{P}_{q-1}]^d$ of degree $q-1$. The product space $M_\delta = \prod_{k=1}^K M_{k;\delta_k}$ is endowed with the mesh-dependent norm $\|\cdot\|_{\delta, -\frac{1}{2}}$. Making use of the bilinear form

$$b(v, \lambda) = \sum_{k=1}^K \int_{\Gamma_{0k}} [v] \cdot \lambda, \quad \forall (v, \lambda) \in X \times M$$

with the jump function $[v] = v_0 - v_k$ on the interfaces Γ_{0k} , we introduce the decomposition

$$\begin{aligned} b(v, \lambda) &= \sum_{k=1}^K \int_{\Gamma_{0k}} v_0 \cdot \lambda_k - \sum_{k=1}^K \int_{\Gamma_{0k}} v_k \cdot \lambda_k \\ &:= \sum_{k=1}^K b_{0k}(v_0, \lambda_k) - \sum_{k=1}^K b_k(v_k, \lambda_k), \end{aligned}$$

and the constrained space of admissible displacements

$$V_h = \{u_h \in X_h, \quad b(u_h, \lambda_h) = 0, \quad \forall \lambda_h \in M_\delta\}.$$

In other words, such admissible displacements are continuous “on average” across the interfaces $(\Gamma_{0k})_{1 \leq k \leq K}$. Then, the problem of interest uses the broken elliptic form

$$\tilde{a}(u, v) := \sum_{k=0}^K a_k(u_k, v_k) = \sum_{k=0}^K \int_{\Omega_k} (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in X,$$

and amounts to finding $(u_h, \lambda_h) \in X_h \times M_\delta$, such that

$$\begin{cases} \tilde{a}(u_h, v_h) + b(v_h, \lambda_h) = l(v_h), & \forall v_h \in X_h, \\ b(u_h, \mu_h) = 0, & \forall \mu_h \in M_\delta. \end{cases} \quad (4)$$

In other words, we solve our variational problem on the product space X_h under the kinematic weak-continuity constraint $b(u_h, \cdot) = 0$.

2.2.4 Fundamental assumptions and error estimates

In order to ensure the well-posedness of the problem (4), some fundamental assumptions have to be made. Concerning the compatibility of X_h and M_δ , we assume (cf. [13,24]):

Assumption 1 *For each $1 \leq k \leq K$, there exists an operator*

$$\pi_k : \mathbb{H}_\delta^{1/2}(\Gamma_{0k}) \rightarrow W_{k;\delta_k},$$

such that for all $v \in \mathbb{H}_\delta^{1/2}(\Gamma_{0k})$,

$$\int_{\Gamma_{0k}} (\pi_k v) \cdot \mu = \int_{\Gamma_{0k}} v \cdot \mu, \quad \forall \mu \in M_{k;\delta_k},$$

with

$$\|\pi_k v\|_{\delta, \frac{1}{2}, k} \leq C \|v\|_{\delta, \frac{1}{2}, k}.$$

This assumption means that the projection perpendicular to the multiplier space onto the trace space $W_{k;\delta_k}$ is continuous. This implies a limitation on the size of M_δ with respect to X_h . If more than two subdomains had a common intersection, the range $W_{k;\delta_k}$ of π_k in Assumption 1 would be replaced by $W_{k;\delta_k}^0$, in order to enable independent projections on each interface.

The coercivity of \tilde{a} over $V_h \times V_h$ is obtained under the following assumption (cf. [13,24]):

Assumption 2 For all $1 \leq k \leq K$, we assume that there exists a subspace \tilde{M}_k of the Lagrange-multiplier space $M_{k;\delta_k}$ such that $\tilde{M}_k \subset M_{k;\delta_k}$ independently of the mesh size δ_k . Moreover, we assume that if $v \in X$ is locally a rigid motion over all the $(\Omega_k)_{k \geq 0}$ in the sense that

$$\tilde{a}(v, w) = 0, \quad \forall w \in X,$$

and satisfies

$$\int_{\Gamma_{0k}} [v] \cdot \mu = 0, \quad \forall \mu \in \tilde{M}_k, \quad k = 1, \dots, K,$$

then $v = 0$.

Various pairs of spaces $X_h \times M_\delta$ can be chosen to satisfy Assumptions 1 and 2.

- The initial formulation from [1,2] proposes discrete displacements of degree q without stabilization, i.e. $\mathcal{B}_{k;h_k} = \emptyset$, and continuous Lagrange multipliers of degree q . In our framework, no modification of the Lagrange multipliers is necessary on the boundaries of the interfaces $(\partial\Gamma_{0k})_{1 \leq k \leq K}$, because they are disjoint. Therefore, with this choice, the trace of fine displacements coincide with Lagrange multipliers, that is $M_{k;\delta_k} = W_{k;\delta_k}$ for all $1 \leq k \leq K$.
- In order to make the mortar weak continuity constraint diagonal, one can adopt the dual Lagrange multipliers from Wohlmuth [4], again without special treatment on the boundaries of the interfaces.
- As shown in [10] for second order approximations of the displacements ($q \geq 2$), the formulation from [1,2] can be modified by using only continuous Lagrange multipliers of degree $q - 1$.
- Discrete displacements of degree q with a proper stabilization are compatible with discontinuous Lagrange multipliers of degree $q - 1$, as proved in [24,7] or in [25] for three-field formulations.

In this framework, we recall the following optimal approximation result (cf. [13,24]):

Proposition 1 Under Assumptions 1 and 2, the problem (4) is well-posed. Moreover, if $u \in \prod_{k=0}^K H^{q+1}(\Omega_k)^d$ is the solution of (3) with $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=0}^K H^q(\Omega_k)^{d \times d}$ in which $q \geq 1$, and $(u_h, \lambda_h) \in X_h \times M_\delta$ is solution of (4), the following error estimate holds:

$$\|u - u_h\|_X + \|\lambda - \lambda_h\|_{\delta, -\frac{1}{2}} \leq C \left(\sum_{k=1}^K h_k^{2q} |u|_{q+1, \mathbf{E}, \Omega_k}^2 \right)^{1/2},$$

with

$$|u|_{q+1, \mathbf{E}, \Omega_k}^2 = |u|_{H^{q+1}(\Omega_k)^d}^2 + \frac{1}{C_k^2} \|\mathbf{E} : \varepsilon(u)\|_{H^q(\Omega_k)^{d \times d}}^2.$$

We have denoted the flux over the artificial interfaces by $\lambda = (\mathbf{E} : \varepsilon(u)) \cdot n$, where n stands for the unit vector on the interfaces $(\Gamma_{0k})_{1 \leq k \leq K}$, which is outward to Ω_0 . Here and now on, C denotes various positive constants independent of the decomposition into subdomains and of the discretization.

Remark 2 (Choice of the non-mortar side) *In the preceding discretization, we have chosen the non-mortar side, which defines the multipliers, as the fine-scale side of the interface \mathcal{S} . In this case, the standard analysis of mortar methods ensures the satisfaction of Assumptions 1 and 2. Nevertheless, this choice is by no means the only possible one, as long as these assumptions hold. In particular, it is possible to define the Lagrange multipliers over the coarse mesh $\mathcal{T}_{0;h_0}$, and restrict them to each interface Γ_{0k} . In this case, Assumption 1 will be satisfied in practice provided the mesh $\mathcal{T}_{k;h_k}$ is fine enough as compared to $\mathcal{T}_{0;h_0}$. Assumption 2 will be always satisfied when using for instance \mathbb{Q}_1 Lagrange multipliers. This alternative choice of the non-mortar side will be tested in section 6.1.*

3 Two-scale preconditioners.

3.1 Notation

The previous discretization leads to a well-posed discrete linear problem with optimal error estimates. In this section, we propose and analyze preconditioners to solve this linear system for the approximate computational cost of the coarse-scale problem on Ω_0 , provided the solution of the local problem over each $(\Omega_k)_{1 \leq k \leq K}$ be at a sufficiently low-cost. That is why we have assumed that the $(\Omega_k)_{1 \leq k \leq K}$ are small and disjoint. Then, the inversions of the fine-scale problems on the boundary can be efficiently computed in parallel.

Some notation and remarks must first be introduced:

- In this section, all quantities live in finite dimensional spaces. If a is a bilinear form, then \mathbf{A} represents the matrix of a in the discrete space. If u is a function, then U is the vector of its degrees of freedom in the chosen discrete space.
- For all $0 \leq k \leq K$, the bilinear form $a_k(\cdot, \cdot)$ is continuous in $H^1(\Omega_k)^d \times H^1(\Omega_k)^d$ and its continuity constant is C_k , already defined in (2).
- When $\Gamma_D \cap \partial\Omega_0$ has a positive measure, $a_0(\cdot, \cdot)$ is coercive in $H_*^1(\Omega_0) \times H_*^1(\Omega_0)$. We denote by α_0 its constant of coercivity, which is proportional to c_0 defined in (2), within a shape-dependent constant.
- For all $1 \leq k \leq K$ such that Ω_k is fixed on a part of its boundary, the bilinear form $a_k(\cdot, \cdot)$ is coercive over $H_*^1(\Omega_k) \times H_*^1(\Omega_k)$. The coercivity constant, α_k ,

is proportional to c_k defined in (2), within a constant which depends continuously on the shape of Ω_k but not on its size. Indeed, a_k and the rescaled $H^1(\Omega_k)$ -norm have the same dependence with respect to the diameter L_k of Ω_k .

3.2 Schur complement system

With obvious notation, the discrete problem (4) leads to the following linear system to solve

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^K \mathbf{B}_{0k}^\top \Lambda_k = F_0, \\ \mathbf{A}_k U_k - \mathbf{B}_k^\top \Lambda_k = F_k, \quad 1 \leq k \leq K, \\ \mathbf{B}_{0k} U_0 - \mathbf{B}_k U_k = 0, \quad 1 \leq k \leq K. \end{cases} \quad (5)$$

Defining the local extended stiffness matrix of the k -th ($k \geq 1$) subproblem by

$$\mathbf{K}_k = \begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^\top \\ -\mathbf{B}_k & 0 \end{pmatrix},$$

the problem (5) can be rewritten as

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^K \mathbf{B}_{0k}^\top \Lambda_k = F_0, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K. \end{cases} \quad (6)$$

The algebraic elimination of Λ_k in (6) leads to

$$\begin{cases} \mathbf{D}_0 U_0 = \overline{F}_0, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K. \end{cases} \quad (7)$$

Denoting by $R_k = (0, I_{M_k; \delta_k})$ the canonical restriction from $X_{k; h_k} \times M_{k; \delta_k}$ to $M_{k; \delta_k}$, we have introduced the Schur complement matrix

$$\mathbf{D}_0 = \mathbf{A}_0 - \sum_{k=1}^K \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k} \quad (8)$$

resulting from the elimination of small scales and the equivalent coarse force

$$\overline{F}_0 := F_0 - \sum_{k=1}^K \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} (F_k, 0)^\top.$$

The problem is now split into a coarse problem defined on Ω_0 , and into fine problems defined on $(\Omega_k)_{1 \leq k \leq K}$ using the coarse solution U_0 . Of course, the elimination of small scales is still hidden in the definition of \mathbf{D}_0 , and making the scale separation effective imposes an iterative resolution of (7) where \mathbf{D}_0^{-1} will be replaced by a coarse-scale approximate inverse $\hat{\mathbf{D}}_0^{-1}$.

3.3 Two possible definitions for $\hat{\mathbf{D}}_0$

3.3.1 A symmetrized Dirichlet-Neumann preconditioner

The simplest idea replaces the Schur complement \mathbf{D}_0 by the stiffness of the coarse problem $\hat{\mathbf{D}}_0 = \mathbf{A}_0$, which reduces the proposed preconditioning to a symmetrized Dirichlet-Neumann iteration. Indeed, determining the approximate solution requires solving

- (1) Dirichlet problems on $(\Omega_k)_{1 \leq k \leq K}$ with zero weak trace on the interfaces $(\Gamma_{0k})_{1 \leq k \leq K}$ to obtain the resultant interface forces and therefore \overline{F}_0 ,
- (2) a Neumann problem on Ω_0 with the force \overline{F}_0 to compute U_0 ,
- (3) Dirichlet problems on $(\Omega_k)_{1 \leq k \leq K}$ to compute the $(U_k)_{1 \leq k \leq K}$ with forces $(F_k)_{1 \leq k \leq K}$ and U_0 as a weak trace over the interfaces $(\Gamma_{0k})_{1 \leq k \leq K}$.

Observe that steps 1 and 3 are compatible with a parallel implementation. In section 4.2, we shall prove that the condition number of the associated preconditioned system is independent of the number and of the size of the fine-scale subdomains, $(\Omega_k)_{k \geq 1}$. We also prove that the method is efficient as long as the ratio of stiffnesses between Ω_0 and $(\Omega_k)_{k \geq 1}$ does not become small, and when the small subdomains are not fixed on a part of their boundary.

3.3.2 An enhanced symmetrized Dirichlet-Neumann preconditioner

The previous simplest choice of preconditioner may lack efficiency when

- the substructure Ω_k is of small size and is fixed on a part of its boundary. In this situation, because of its size, the substructure will have a rather large stiffness to interface rigid body displacements.
- the substructure Ω_k may have other preferred directions of large stiffness to interface motions (rigid links, incompressibility).

Assuming that these directions of localized interface stiffness are few (this is indeed the case for interface rigid body motions), we denote by N_k their number. We then propose a modification of the previous preconditioner to correct such a lack of efficiency.

For all $k \geq 1$ such that Ω_k is fixed on a part of its boundary, we denote by $(e_k^i)_{1 \leq i \leq N_k}$ (with $N_k = 6$ in general) the interface rigid motions of Γ_{0k} or rigid links and introduce $\mathring{W}_k = \text{span}\{e_k^i; i = 1, \dots, N_k\}$. To each interface rigid body motion e_k^i , we associate its local a_k -harmonic extension $(u_k^i, \lambda_k^i) \in X_{k;h_k} \times M_{k;\delta_k}$ as the solution of

$$\begin{cases} a_k(v, u_k^i) - \int_{\Gamma_{0k}} v \cdot \lambda_k^i = 0, & \forall v \in X_{k;h_k}, \\ - \int_{\Gamma_{0k}} u_k^i \cdot \mu = - \int_{\Gamma_{0k}} e_k^i \cdot \mu, & \forall \mu \in M_{k;\delta_k}. \end{cases} \quad (9)$$

These solutions span two small local spaces

$$\mathring{X}_k = \text{span}\{u_k^i; i = 1, \dots, N_k\} \subset X_{k;h_k}, \quad \mathring{M}_k = \text{span}\{\lambda_k^i; i = 1, \dots, N_k\} \subset M_{k;\delta_k}.$$

If $k \geq 1$ is such that Ω_k is not fixed on its boundary, the convention that $\mathring{W}_k = \mathring{M}_k = \{0\}$ and $N_k = 0$ is adopted.

Now, instead of finding U_0 such that $\mathbf{D}_0 U_0 = \overline{F}_0$, we propose to compute $u_0 \in X_{0;h_0}$, $(u_k) \in (\mathring{X}_k)_{1 \leq k \leq K}$ and $(\lambda_k) \in (\mathring{M}_k)_{1 \leq k \leq K}$ solution of the coupled problem

$$\begin{cases} a_0(u_0, v_0) + \sum_{k=1}^K \int_{\Gamma_{0k}} v_0 \cdot \lambda_k = \overline{l}_0(v_0), & \forall v_0 \in X_{0;h_0}, \\ a_k(u_k, v_k) - \int_{\Gamma_{0k}} v_k \cdot \lambda_k = 0, & \forall v_k \in \mathring{X}_k, \quad 1 \leq k \leq K, \\ - \int_{\Gamma_{0k}} u_k \cdot \mu_k = - \int_{\Gamma_{0k}} u_0 \cdot \mu_k, & \forall \mu_k \in \mathring{M}_k, \quad 1 \leq k \leq K, \end{cases} \quad (10)$$

where \overline{l}_0 is the linear form associated with the coarse vector, \overline{F}_0 .

We introduce the matrix $\mathbf{I}_{0k} \in \mathbb{R}^{N_k \times \dim X_{0;h_0}}$ defined for all $v_0 \in X_{0;h_0}$ by

$$(\mathbf{I}_{0k} V_0)_i = \int_{\Gamma_{0k}} v_0 \cdot \lambda_k^i = \langle \mathbf{B}_{0k} V_0, \Lambda_k^i \rangle, \quad \forall i = 1, \dots, N_k,$$

that is $\mathbf{I}_{0k} = [\Lambda_k^1, \dots, \Lambda_k^{N_k}]^\top \mathbf{B}_{0k} = \mathbf{\Lambda}_k^\top \mathbf{B}_{0k}$, and the restriction $\mathring{\mathbf{A}}_k$ of the displacement stiffness matrix \mathbf{A}_k on the local space \mathring{X}_k . Thus

$$(\mathring{\mathbf{A}}_k)_{ij} = (U_k^i)^\top \mathbf{A}_k U_k^j = a_k(u_k^j, u_k^i) = \int_{\Gamma_{0k}} u_k^j \cdot \lambda_k^i,$$

and keeping (9)-1 in mind, the system (10) reads

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^K \mathbf{I}_{0k}^\top \Theta_k = \overline{F}_0, \\ \mathring{\mathbf{A}}_k Z_k - \mathring{\mathbf{A}}_k^\top \Theta_k = 0, \\ -\mathring{\mathbf{A}}_k Z_k = -\mathbf{I}_{0k} U_0, \quad 1 \leq k \leq K. \end{cases} \quad (11)$$

The new vector Θ_k (resp. Z_k) denotes the component of λ_k (resp. u_k) in \mathring{M}_k (resp. \mathring{X}_k) in (10). From the elimination of Θ_k and Z_k in (11), one obtains that $\hat{\mathbf{D}}_0 U_0 = \overline{F}_0$, with the new approximate Schur complement given by

$$\hat{\mathbf{D}}_0 = \mathbf{A}_0 + \sum_{k=1}^K \mathbf{I}_{0k}^\top \mathring{\mathbf{A}}_k^{-\top} \mathbf{I}_{0k} \quad (12)$$

$$= \mathbf{A}_0 + \sum_{k=1}^K \mathbf{B}_{0k}^\top \mathbf{\Lambda}_k \mathring{\mathbf{A}}_k^{-\top} \mathbf{\Lambda}_k^\top \mathbf{B}_{0k}. \quad (13)$$

Of course, factorizing $\hat{\mathbf{D}}_0$ can be achieved at low computational cost as compared to factorizing the exact Schur complement \mathbf{D}_0 . Indeed, the local problem (11)-2,(11)-3 for $k \geq 1$ used in the construction of $\hat{\mathbf{D}}_0$, is only of dimension $N_k = 6$.

For analysis purposes, let us observe that this enhanced Dirichlet-Neumann preconditioner corresponds to a Dirichlet-Neumann decomposition where the Dirichlet substructures are defined by

$$\mathring{X}_{k;h_k}^\perp = \{u_k \in X_{k;h_k}, \quad \int_{\Gamma_{0k}} u_k \cdot \mu = 0, \quad \forall \mu \in \mathring{M}_k\}, \quad 1 \leq k \leq K,$$

and the Neumann substructure by

$$\mathring{X}_h = \{u \in X_h, \quad b(u, \mu) = 0, \quad \forall \mu \in \mathring{M}_k\}.$$

The analysis of this preconditioner is performed in section 4.3, proving now an independence with respect to essential boundary conditions imposed over the small subdomains $(\Omega_k)_{k \geq 1}$.

In order to state a preliminary remark, the following definition is required.

Definition 1 For any $v_0 \in X_{0;h_0}$, its “rigid body projection” over Ω_k denoted by $\mathring{\pi}_k v_0 \in \mathring{X}_k$ is defined as the solution of (11)-2,(11)-3 for the subproblem k . More precisely $(\mathring{\pi}_k v_0, \mathring{\lambda}_k) \in \mathring{X}_k \times \mathring{M}_k$ is such that:

$$\begin{cases} a_k(\mathring{\pi}_k v_0, v_k) - \int_{\Gamma_{0k}} \mathring{\lambda}_k \cdot v_k = 0, & \forall v_k \in \mathring{X}_k, \\ - \int_{\Gamma_{0k}} \mathring{\pi}_k v_0 \cdot \mu_k = - \int_{\Gamma_{0k}} v_0 \cdot \mu_k, & \forall \mu_k \in \mathring{M}_k. \end{cases} \quad (14)$$

In matrix form, we have $\mathring{\pi}_k v_0 = \sum_{j=1}^{N_k} z_j u_k^j$ with $-\mathring{\mathbf{A}}_k Z = -\mathbf{I}_{0k} V_0$, yielding

$$\mathring{\mathbf{\Pi}}_k V_0 = [U_k^1, \dots, U_k^{N_k}] \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} V_0 = \mathbf{U}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} V_0,$$

or equivalently $\mathring{\mathbf{\Pi}}_k = \mathbf{U}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k}$. By construction of $\mathring{\mathbf{A}}_k$, one is now able to deduce that

$$\begin{aligned}
\mathring{\Pi}_k^\top \mathbf{A}_k \mathring{\Pi}_k &= \mathbf{I}_{0k}^\top \mathring{\mathbf{A}}_k^{-\top} \mathbf{U}_k^\top \mathbf{A}_k \mathbf{U}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} \\
&= \mathbf{I}_{0k}^\top \mathring{\mathbf{A}}_k^{-\top} \mathring{\mathbf{A}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} \\
&= \mathbf{I}_{0k}^\top \mathring{\mathbf{A}}_k^{-\top} \mathbf{I}_{0k},
\end{aligned}$$

and therefore the new preconditioner (12) takes the form

$$\hat{\mathbf{D}}_0 = \mathbf{A}_0 + \sum_{k=1}^K \mathring{\Pi}_k^\top \mathbf{A}_k \mathring{\Pi}_k. \quad (15)$$

Also observe from (9) that when a_k is symmetric,

$$\mathring{\pi}_k e_k^i = u_k^i, \quad 1 \leq i \leq N_k. \quad (16)$$

4 Condition number analysis

In this section, we establish upper bounds for the condition number of the preconditioned systems when using the two symmetrized Dirichlet-Neumann preconditioners defined in the subsections 3.3.1 and 3.3.2 respectively. First, the same factorized form for the original linear system and the preconditioner is introduced. Then, we show the spectral equivalence between $\hat{\mathbf{D}}_0$ and \mathbf{D}_0 , detailing the dependence of the constants on the size of the domains, the stiffness of the materials, and on the mesh sizes. We finally deduce estimates on the condition number of the preconditioned system.

4.1 Factorization

The original system to solve is $\mathbf{A}(U_0, U_1, \Lambda_1, \dots, U_K, \Lambda_K)^\top = (F_0, F_1, 0, \dots, F_K, 0)^\top$, under the notation

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_0 & 0 & \mathbf{B}_{01}^\top & \dots & 0 & \mathbf{B}_{0K}^\top \\ 0 & \mathbf{A}_1 & -\mathbf{B}_1^\top & & & \\ \mathbf{B}_{01} & -\mathbf{B}_1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & \mathbf{A}_K & -\mathbf{B}_K^\top \\ \mathbf{B}_{0K} & & & & -\mathbf{B}_K & 0 \end{pmatrix}.$$

Introducing the triangular and block diagonal matrices

$$T = \begin{pmatrix} I & 0 & \dots & 0 \\ \mathbf{K}_1^{-1} R_1^\top \mathbf{B}_{01} & I & & \\ \vdots & & \ddots & \\ \mathbf{K}_K^{-1} R_K^\top \mathbf{B}_{0K} & 0 & \dots & I \end{pmatrix}, \quad H = \begin{pmatrix} \mathbf{D}_0 & 0 & \dots & 0 \\ 0 & \mathbf{K}_1 & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{K}_K \end{pmatrix},$$

a simple calculus shows that the factorization $\mathbf{A} = T^\top H T$ holds.

The matrix of our preconditioners can be written under the similar form $\mathbf{C} = T^\top \hat{H} T$, where \hat{H} is the block diagonal matrix obtained when replacing \mathbf{D}_0 by $\hat{\mathbf{D}}_0$ in the definition of H . The approximate solution is then defined by

$$\mathbf{C}(\tilde{U}_0, \tilde{U}_1, \tilde{\Lambda}_1, \dots, \tilde{U}_K, \tilde{\Lambda}_K)^\top = (F_0, F_1, 0, \dots, F_K, 0)^\top.$$

Observe that both matrices \mathbf{A} and \mathbf{C} remain definite when discrete unknowns belong to the space of weakly continuous displacements

$$\mathcal{E} = \{U = (U_0, U_1, \Lambda_1, \dots, U_K, \Lambda_K)^\top; \mathbf{B}_{0k} U_0 = \mathbf{B}_k U_k, 1 \leq k \leq K\}.$$

The sequel is devoted to the derivation of upper bounds for the condition number $\kappa_{\mathbf{A}, \mathcal{E}}(\mathbf{C}^{-1} \mathbf{A})$ in \mathbf{A} -norm over \mathcal{E} .

4.2 Spectral equivalence for the simple Dirichlet-Neumann

We show herein the spectral equivalence between the Schur complement \mathbf{D}_0 and its approximation $\hat{\mathbf{D}}_0 = \mathbf{A}_0$ for the symmetrized Dirichlet-Neumann preconditioner presented in subsection 3.3.1. We prove:

Proposition 2 *Assuming that \mathbf{A}_0 is invertible, that is $\Gamma_D \cap \partial\Omega_0$ has a positive measure, the following spectral equivalence holds for all U_0 :*

$$W_{1,h} \langle \mathbf{D}_0 U_0, U_0 \rangle \leq \langle \mathbf{A}_0 U_0, U_0 \rangle \leq \langle \mathbf{D}_0 U_0, U_0 \rangle,$$

with

$$\frac{1}{W_{1,h}} = 1 + C \left(\max_{k \in I_1} \frac{C_k}{c_0} + \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right),$$

where I_1 (resp. I_2) is the set of indices $k \geq 1$ such that Ω_k is not fixed on its boundary (resp. is fixed on a part of its boundary). The constant C is independent of the number K and the size of the subdomains.

Observe that the condition number deteriorates for a small fixed subdomain of diameter $L_k \ll L_0$, $k \in I_2$, and for very stiff subdomains, i.e. $C_k \gg c_0$, $k \geq 1$.

Let us recall first the following lemma (see for instance [26], page 158).

Lemma 1 *Let us assume that Γ_{0k} is of class \mathcal{C}^1 . Then, there exists an open set $\Omega'_k \subset \Omega_0$ which is the restriction of a neighborhood of Γ_{0k} to Ω_0 , and a linear extension operator $\mathcal{D}_k : H^1(\Omega'_k)^d \rightarrow H^1(\Omega_k)^d$ such that for all $u \in H^1(\Omega_0)^d$, $\mathcal{D}_k u = u$ on Γ_{0k} and*

$$\int_{\Omega_k} (\mathcal{D}_k u)^2 \leq C \int_{\Omega'_k} u^2, \quad \int_{\Omega_k} (\nabla \mathcal{D}_k u)^2 \leq C \int_{\Omega'_k} (\nabla u)^2,$$

where the constant C does not depend on Ω_k .

Proof : [of the proposition] Let U_0 be given. For all $k \geq 1$, define (U_k, Λ_k) by a local harmonic lifting of U_0 , obeying

$$\begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^\top \\ -\mathbf{B}_k & 0 \end{pmatrix} \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}. \quad (17)$$

In other words, we have $\Lambda_k = -R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k} U_0$ and then by construction of U_k and Λ_k

$$-\langle \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k} U_0, U_0 \rangle = \langle \mathbf{B}_{0k}^\top \Lambda_k, U_0 \rangle = \langle \Lambda_k, \mathbf{B}_k U_k \rangle = \langle \mathbf{A}_k U_k, U_k \rangle \geq 0.$$

By summing over $k \geq 1$, and adding $\langle \mathbf{A}_0 U_0, U_0 \rangle$ on both sides of the inequality, it follows in view of (8) that $\langle \mathbf{D}_0 U_0, U_0 \rangle \geq \langle \mathbf{A}_0 U_0, U_0 \rangle$.

Let us now bound \mathbf{A}_0 from below. Let $u_0 \in H_*^1(\Omega)$ be given. For all $k \geq 1$ such that Ω_k has an empty intersection with Γ_D (we denote $k \in I_1$), we decompose u_0 on Ω'_k (as defined in lemma 1) into $u_0 = r_k + w_k$ in Ω'_k , where r_k belongs to the space $\mathcal{R}(\Omega'_k)$ of rigid motions over Ω'_k and

$$\int_{\Omega'_k} w_k \cdot r = 0, \quad \forall r \in \mathcal{R}(\Omega'_k). \quad (18)$$

We define the displacements $u_k = r_k + u'_k$ in Ω_k where $r_k \in \mathcal{R}(\Omega_k)$ is the natural extension to Ω_k of $r_k \in \mathcal{R}(\Omega'_k)$ (thus $r_k \in \mathcal{R}(\Omega_k \cup \Omega'_k)$), and

$$u'_k = \mathcal{I}_{k;h_k} \mathcal{D}_k w_k + \mathcal{R}_{k;\delta_k} \pi_k (w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k),$$

where $\mathcal{I}_{k;h_k}$ denotes the Scott-Zhang [27] interpolation over $X_{k;h_k}$, and $\mathcal{R}_{k;\delta_k}$ is the extension by zero operator over the grid points of Ω_k . By construction,

the mortar condition holds

$$\int_{\Gamma_{0k}} u_k \cdot \mu = \int_{\Gamma_{0k}} u_0 \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}.$$

Moreover, by using the stability of the extension operator $\mathcal{R}_{k;\delta_k}$ from $W_{k;\delta_k}$ to $H^1(\Omega_k)^d$, Assumption 1, the stability of $\mathcal{I}_{k;h_k}$ from $H^1(\Omega_k)^d$ to $H^1(\Omega_k)^d$, the classical estimate (see [27]) $\|u - \mathcal{I}_{k;h_k} u\|_{\delta, \frac{1}{2}, k} \leq C|u|_{H^1(\Omega_k)^d}$, and the stability property of \mathcal{D}_k from lemma 1, we obtain

$$\begin{aligned} & a_k(u_k, u_k) \\ & \leq C_k \int_{\Omega_k} |\nabla u'_k|^2 = C_k |u'_k|_{H^1(\Omega_k)^d}^2 \\ & \leq 2C_k |\mathcal{I}_{k;h_k} \mathcal{D}_k w_k|_{H^1(\Omega_k)^d}^2 + 2C_k |\mathcal{R}_{k;\delta_k} \pi_k(\mathcal{D}_k w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k)|_{H^1(\Omega_k)^d}^2 \\ & \leq 2C_k |\mathcal{I}_{k;h_k} \mathcal{D}_k w_k|_{H^1(\Omega_k)^d}^2 + 2CC_k \|\pi_k(\mathcal{D}_k w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k)\|_{\delta, \frac{1}{2}, k}^2 \\ & \leq 2C_k |\mathcal{I}_{k;h_k} \mathcal{D}_k w_k|_{H^1(\Omega_k)^d}^2 + 2CC_k \|\mathcal{D}_k w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k\|_{\delta, \frac{1}{2}, k}^2 \\ & \leq CC_k |\mathcal{D}_k w_k|_{H^1(\Omega_k)^d}^2 \leq CC_k |w_k|_{H^1(\Omega'_k)^d}^2. \end{aligned} \quad (19)$$

Moreover, denoting by $\mathfrak{R}_k = \{r \in \mathcal{R}(\Omega'_k), \int_{\Omega'_k} r = 0, \|r\|_{L^2(\Omega'_k)^d} = 1\}$ the space of rigid rotations, the following inequality holds for all $v \in H^1(\Omega'_k)^d$:

$$|v|_{H^1(\Omega'_k)^d}^2 \leq C \left(\int_{\Omega'_k} \varepsilon(v) : \varepsilon(v) + \frac{1}{\text{diam}(\Omega'_k)^2} \left(\sup_{r \in \mathfrak{R}_k} \int_{\Omega'_k} v \cdot r \right)^2 \right). \quad (20)$$

For a proof, the reader is referred to [11] in the case of polyhedral shape regular domains, or to [24] for curved domains. The positive constant C is proved to be independent of the size and the shape of Ω'_k . Therefore, we have from (18) by definition of w_k that

$$|w_k|_{H^1(\Omega'_k)^d}^2 \leq C \int_{\Omega'_k} \varepsilon(w_k) : \varepsilon(w_k).$$

Keeping (19) in mind, a summation over $k \in I_1$ provides

$$\sum_{k \in I_1} a_k(u_k, u_k) \leq C \sum_{k \in I_1} C_k \int_{\Omega'_k} \varepsilon(u_0) : \varepsilon(u_0), \quad (21)$$

with a constant C independent of the size of the subdomains. Since by construction $\cup_{k \in I_1} \Omega'_k \subset \Omega_0$, and since there is a bounded number of domains Ω'_k overlapping at a given point, we deduce

$$\sum_{k \in I_1} a_k(u_k, u_k) \leq C \max_{k \in I_1} (C_k) \int_{\Omega_0} \varepsilon(u_0) : \varepsilon(u_0) \leq \frac{C}{c_0} \max_{k \in I_1} (C_k) a_0(u_0, u_0).$$

For all $k \geq 1$ such that Γ_D is fixed on a part of its boundary (that is $k \in I_2$), we cannot use the extension operator \mathcal{D}_k because it will not satisfy the Dirichlet boundary condition on Γ_D . By a lifting argument, there exists a function \tilde{u}_k whose trace is u_0 on Γ_{0k} and such that

$$\frac{1}{(L_k)^2} \int_{\Omega_k} |\tilde{u}_k|^2 + \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \leq C \left(\frac{1}{L_k} \int_{\Gamma_{0k}} |u_0|^2 + |u_0|_{H^{1/2}(\Gamma_{0k})}^2 \right). \quad (22)$$

The positive constant C is independent of the diameter L_k of the domain Ω_k , but *a priori* depends on the ratio between L_k and the distance from Γ_{0k} to Γ_D .

We modify \tilde{u}_k to obtain a discrete function satisfying the weak-continuity constraint on Γ_{0k} , and define using our previous notation $u_k = \mathcal{I}_{k;h_k} \tilde{u}_k + \mathcal{R}_{k;\delta_k} \pi_k(\tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k)$. By construction, the mortar condition is satisfied

$$\begin{aligned} \int_{\Gamma_{0k}} u_k \cdot \mu &= \int_{\Gamma_{0k}} (\mathcal{I}_{k;h_k} \tilde{u}_k + \tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k) \cdot \mu \\ &= \int_{\Gamma_{0k}} u_0 \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}. \end{aligned}$$

From the same argument (19) as in the case $k \in I_1$ and (22), we get

$$\begin{aligned} a_k(u_k, u_k) &\leq CC_k \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \\ &\leq CC_k \left(\frac{1}{L_k} \int_{\Gamma_{0k}} |u_0|^2 + |u_0|_{H^{1/2}(\Gamma_{0k})}^2 \right) \end{aligned} \quad (23)$$

$$\leq CC_k \frac{L_0}{L_k} \left(\frac{1}{L_0} \int_{\Gamma_{0k}} |u_0|^2 + |u_0|_{H^1(\Omega'_k)}^2 \right). \quad (24)$$

By summation of (24) for $k \in I_2$, one gets

$$\begin{aligned} \sum_{k \in I_2} a_k(u_k, u_k) &\leq C \max_{k \in I_2} \left(C_k \frac{L_0}{L_k} \right) \left(\frac{1}{L_0} \int_{\Gamma_0} u_0^2 + |u_0|_{H^1(\partial\Omega_0)}^2 \right) \\ &\leq C \max_{k \in I_2} \left(C_k \frac{L_0}{L_k} \right) \|u_0\|_{H^1(\Omega_0)}^2 \\ &\leq C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} a_0(u_0, u_0). \end{aligned}$$

As a consequence, with this choice of u_k

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k U_k, U_k \rangle \leq \left(1 + C \max_{k \in I_1} \frac{C_k}{c_0} + C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right) \langle \mathbf{A}_0 U_0, U_0 \rangle.$$

To conclude the proof, let us show that for all $(V_k)_{k \geq 1}$ such that $\mathbf{B}_k V_k = \mathbf{B}_{0k} U_0$, we have:

$$\langle \mathbf{D}_0 U_0, U_0 \rangle \leq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k V_k, V_k \rangle. \quad (25)$$

For all $k \geq 1$, we decompose V_k into $V_k = U_k + \delta U_k$, where (U_k, Λ_k) denotes the solution of (17), and $\mathbf{B}_k \delta U_k = 0$. One has $\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k U_k, U_k \rangle = \langle \mathbf{D}_0 U_0, U_0 \rangle$, and by symmetry of \mathbf{A}_k

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k V_k, V_k \rangle = \langle \mathbf{D}_0 U_0, U_0 \rangle + \sum_{k=1}^K 2 \langle \mathbf{A}_k U_k, \delta U_k \rangle + \langle \mathbf{A}_k \delta U_k, \delta U_k \rangle.$$

Furthermore, $\langle \mathbf{A}_k U_k, \delta U_k \rangle = \langle \mathbf{B}_k^\top \Lambda_k, \delta U_k \rangle = \langle \Lambda_k, \mathbf{B}_k \delta U_k \rangle = 0$ which implies

$$\begin{aligned} \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k V_k, V_k \rangle &= \langle \mathbf{D}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k \delta U_k, \delta U_k \rangle \\ &\geq \langle \mathbf{D}_0 U_0, U_0 \rangle. \end{aligned}$$

Hence (25) which completes the claim. \square

4.3 Spectral equivalence for the enhanced Dirichlet Neumann

For the enhanced Dirichlet-Neumann preconditioner presented in section 3.3.2, we prove that:

Proposition 3 *For all U_0 , the following spectral equivalence holds:*

$$W_{1,h} \langle \mathbf{D}_0 U_0, U_0 \rangle \leq \langle \hat{\mathbf{D}}_0 U_0, U_0 \rangle \leq \langle \mathbf{D}_0 U_0, U_0 \rangle,$$

with

$$\frac{1}{W_{1,h}} = C \left(1 + \max_{k \in I_1 \cup I_2} \frac{C_k}{c_0} \right),$$

where I_1 (resp. I_2) is the set of indices $k \geq 1$ such that Ω_k is not fixed on its boundary (resp. is fixed on a part of its boundary). The constant C is independent of the number K and the size of the subdomains.

Proof : Let U_0 be given. We proceed as in the last part of the previous proof, and introduce (U_k, Λ_k) satisfying (17). We introduce the decomposition $U_k = \mathring{U}_k + W_k$ with $\mathring{U}_k = \Pi_k U_0$, and by construction of U_k , we obtain

$$\begin{aligned}
\langle \mathbf{D}_0 U_0, U_0 \rangle &= \left\langle \left(\mathbf{A}_0 - \sum_{k=1}^K \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k} \right) U_0, U_0 \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k U_k, U_k \rangle \\
&\geq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k \dot{U}_k, \dot{U}_k \rangle + 2 \langle \mathbf{A}_k \dot{U}_k, W_k \rangle. \tag{26}
\end{aligned}$$

Exploiting the decomposition $\dot{U}_k = \dot{\Pi}_k U_0 = \sum_{j=1}^{N_k} z_j U_k^j$ provides

$$\begin{aligned}
\langle \mathbf{A}_k W_k, \dot{U}_k \rangle &= \sum_{j=1}^{N_k} z_j a_k(w_k, u_k^j) = \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} \lambda_k^j \cdot w_k, \quad \text{from (9)-1,} \\
&= \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} (u_k - \dot{u}_k) \cdot \lambda_k^j, \quad \text{by construction of } w_k, \\
&= \sum_{j=1}^{N_k} z_j \left[\int_{\Gamma_{0k}} u_k \cdot \lambda_k^j - \int_{\Gamma_{0k}} u_0 \cdot \lambda_k^j \right], \quad \text{from (14)-2,} \\
&= 0 \quad \text{from (17).}
\end{aligned}$$

Plugging the latter result into (26) gives

$$\begin{aligned}
\langle \mathbf{D}_0 U_0, U_0 \rangle &\geq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k \dot{U}_k, \dot{U}_k \rangle \\
&= \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^K \langle \mathbf{A}_k \dot{\Pi}_k U_0, \dot{\Pi}_k U_0 \rangle \\
&= \left\langle \left(\mathbf{A}_0 + \sum_{k=1}^K \dot{\Pi}_k^\top \mathbf{A}_k \dot{\Pi}_k \right) U_0, U_0 \right\rangle \\
&= \langle \hat{\mathbf{D}}_0 U_0, U_0 \rangle \quad \text{from (15).}
\end{aligned}$$

Let us prove now a lower bound for $\hat{\mathbf{D}}_0$. For U_0 given and for all $1 \leq k \leq K$, as in the proof of the previous proposition, we build a particular function $u_k \in W_{k;\delta_k}$ satisfying the weak continuity constraint on the interface Γ_{0k} . When Ω_k is not fixed on a part of its boundary, which we denote by $k \in I_1$, we take u_k as defined in the previous proof, by ‘‘reflexion’’ with respect to Γ_{0k} . When Ω_k is fixed on a part of its boundary, namely $k \in I_2$, we proceed differently, and define here $\langle u_0 \rangle_k \in \mathcal{R}(\Gamma_{0k})$ (the trace over Γ_{0k} of a rigid motion) such that

$$\int_{\Gamma_{0k}} \langle u_0 \rangle_k \cdot r = \int_{\Gamma_{0k}} u_0 \cdot r, \quad \forall r \in \mathcal{R}(\Gamma_{0k}).$$

Then, we introduce $u_k = \mathcal{I}_{k;h_k} \tilde{u}_k + \mathcal{R}_{k;\delta_k} \pi_k [\tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k] + \hat{\pi}_k \langle u_0 \rangle_k$, where \tilde{u}_k is a function whose trace is zero on Γ_D and is $u_0 - \langle u_0 \rangle_k$ on Γ_{0k} satisfying from a lifting argument and equivalence of norms as in [11,24], the following estimate

$$\int_{\Omega_k} |\nabla \tilde{u}_k|^2 \leq C \left[\frac{1}{L_k} \langle u_0 - \langle u_0 \rangle_k \rangle_k + |u_0 - \langle u_0 \rangle_k|_{H^{1/2}(\Gamma_{0k})^d}^2 \right] \quad (27)$$

$$= C |u_0 - \langle u_0 \rangle_k|_{H^{1/2}(\Gamma_{0k})^d}^2, \quad \text{by construction of } \langle u_0 \rangle_k. \quad (28)$$

The mortar condition is indeed satisfied because

$$\begin{aligned} \int_{\Gamma_{0k}} u_k \cdot \mu &= \int_{\Gamma_{0k}} (\mathcal{I}_{k;h_k} \tilde{u}_k + \tilde{u}_k - \mathcal{I}_{k;h_k} \tilde{u}_k) \cdot \mu + \int_{\Gamma_{0k}} \hat{\pi}_k \langle u_0 \rangle_k \cdot \mu \\ &= \int_{\Gamma_{0k}} \tilde{u}_k \cdot \mu + \int_{\Gamma_{0k}} \hat{\pi}_k \langle u_0 \rangle_k \cdot \mu \\ &= \int_{\Gamma_{0k}} (u_0 - \langle u_0 \rangle_k + \hat{\pi}_k \langle u_0 \rangle_k) \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}, \end{aligned}$$

and because, since $\langle u_0 \rangle_k$ is a linear combination of rigid body motions e_k^i , we have from (16)

$$\int_{\Gamma_{0k}} (\langle u_0 \rangle_k - \hat{\pi}_k \langle u_0 \rangle_k) \cdot \mu = 0, \quad \forall \mu \in M_{k;\delta_k}.$$

On the other hand, we have for $k \in I_2$

$$\begin{aligned} a_k(u_k, u_k) &\leq 2a_k(u_k - \hat{\pi}_k \langle u_0 \rangle_k, u_k - \hat{\pi}_k \langle u_0 \rangle_k) \\ &\quad + 2a_k(\hat{\pi}_k \langle u_0 \rangle_k, \hat{\pi}_k \langle u_0 \rangle_k). \end{aligned} \quad (29)$$

Using the same argument as in (19), we get by construction of u_k

$$\begin{aligned} a_k(u_k - \hat{\pi}_k \langle u_0 \rangle_k, u_k - \hat{\pi}_k \langle u_0 \rangle_k) &\leq CC_k \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \\ &\leq CC_k |u_0 - \langle u_0 \rangle_k|_{H^{1/2}(\Gamma_{0k})^d}^2, \quad \text{from (28),} \\ &\leq CC_k \int_{\Omega'_k} \varepsilon(u_0) : \varepsilon(u_0), \end{aligned} \quad (30)$$

from the Sobolev trace theorem and the inequality (20). On the other hand, we have from lemma 2

$$\begin{aligned}
a_k(\mathring{\pi}_k \langle u_0 \rangle_k, \mathring{\pi}_k \langle u_0 \rangle_k) &\leq 2a_k(\mathring{\pi}_k(u_0 - \langle u_0 \rangle_k), \mathring{\pi}_k(u_0 - \langle u_0 \rangle_k)) \\
&\quad + 2a_k(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0) \\
&\leq CC_k |u_0 - \langle u_0 \rangle_k|_{H^{1/2}(\Gamma_{0k})}^2 + 2a_k(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0) \\
&\leq CC_k \int_{\Omega'_k} \varepsilon(u_0) : \varepsilon(u_0) + 2a_k(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0). \quad (31)
\end{aligned}$$

Equations (21), (29), (30) and (31) enable us to obtain

$$\begin{aligned}
a_0(u_0, u_0) + \sum_{k=1}^K a_k(u_k, u_k) &\leq a_0(u_0, u_0) + C \sum_{k=1}^K C_k \int_{\Omega'_k} \varepsilon(u_0) : \varepsilon(u_0) \\
&\quad + 4a_k(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0) \\
&\leq \left(4 + \frac{C}{c_0} \max_{k \geq 1}(C_k)\right) \left[a_0(u_0, u_0) + \sum_{k \in I_2} a_k(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0) \right] \\
&= \left(4 + \frac{C}{c_0} \max_{k \geq 1}(C_k)\right) \langle \hat{\mathbf{D}}_0 U_0, U_0 \rangle.
\end{aligned}$$

Finally, from (25) and the mortar conditions satisfied by the $(u_k)_{k \geq 1}$, the following upper bound holds

$$\langle \mathbf{D}_0 U_0, U_0 \rangle \leq \left(4 + \frac{C}{c_0} \max_{k \geq 1}(C_k)\right) \langle \hat{\mathbf{D}}_0 U_0, U_0 \rangle.$$

□

In the above proof, we have used the following lemma:

Lemma 2 *If a_k is symmetric, the projection operator $\mathring{\pi}_k$ satisfies:*

$$a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) \leq CC_k \left[\frac{1}{L_k} \int_{\Gamma_{0k}} \langle w \rangle_k^2 + |w|_{H^{1/2}(\Gamma_{0k})}^2 \right]$$

Proof : Let \tilde{w} be a lifting function of w with zero trace on Γ_D , with $\tilde{w} = w$ on Γ_{0k} and satisfying, similarly to (27),

$$\int_{\Omega_k} |\nabla \tilde{w}|^2 \leq C \left[\frac{1}{L_k} \int_{\Gamma_{0k}} \langle w \rangle_k^2 + |w|_{H^{1/2}(\Gamma_{0k})}^2 \right].$$

Let us define as before $\tilde{w}_k = \mathcal{I}_{k;h_k} \tilde{w} + \mathcal{R}_{k;\delta_k} \pi_k(\tilde{w} - \mathcal{I}_{k;h_k} \tilde{w})$ which belongs to $X_{k;h_k}$ and satisfies by construction

$$\int_{\Gamma_{0k}} \tilde{w}_k \cdot \mu = \int_{\Gamma_{0k}} \tilde{w} \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}. \quad (32)$$

We then have on one hand

$$\begin{aligned}
a_k(\tilde{w}_k, \tilde{w}_k) &= a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) + a_k(\mathring{\pi}_k w - \tilde{w}_k, \mathring{\pi}_k w - \tilde{w}_k) \\
&\quad + 2a_k(\mathring{\pi}_k w, \mathring{\pi}_k w - \tilde{w}_k). \quad (33)
\end{aligned}$$

Developing $\mathring{\pi}_k w_k$ into $\mathring{\pi}_k w_k = \sum_{j=1}^{N_k} z_j u_k^j$, we have from (9)-1

$$\begin{aligned}
& a_k(\mathring{\pi}_k w_k - \tilde{w}_k, \mathring{\pi}_k w_k) \\
&= \sum_{j=1}^{N_k} z_j a_k(\mathring{\pi}_k w - \tilde{w}_k, u_k^j) = \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} (\mathring{\pi}_k w - \tilde{w}_k) \cdot \lambda_k^j \\
&= \sum_{j=1}^{N_k} z_j \left[\int_{\Gamma_{0k}} w \cdot \lambda_k^j - \int_{\Gamma_{0k}} w \cdot \lambda_k^j \right], \quad \text{from (14)-2 and (32)} \\
&= 0.
\end{aligned}$$

Plugged back in (33), this implies $a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) \leq a_k(\tilde{w}_k, \tilde{w}_k)$. But on the other hand, proceeding as in (19), we get

$$a_k(\tilde{w}_k, \tilde{w}_k) \leq CC_k \int_{\Omega_k} |\nabla \tilde{w}|^2 \leq CC_k \left[\frac{1}{L_k} \int_{\Gamma_{0k}} \langle w \rangle_k^2 + |w|_{H^{1/2}(\Gamma_{0k})^d}^2 \right]$$

the last inequality coming from the Sobolev lifting theorem. This concludes the proof. \square

4.3.1 Bound on condition number

Referring to the Matsokin-Nepomniaschik framework [28], the spectral equivalences proved for our preconditioners in the previous subsections imply the main result of the section:

Proposition 4 *For the symmetrized Dirichlet-Neumann preconditioner given in section 3.3.1, we have*

$$\kappa_{\mathbf{A}, \mathcal{E}}(\mathbf{C}^{-1} \mathbf{A}) \leq 1 + C \left(\max_{k \in I_1} \frac{C_k}{c_0} + \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right),$$

and for the enhanced Dirichlet-Neumann preconditioner given in section 3.3.2

$$\kappa_{\mathbf{A}, \mathcal{E}}(\mathbf{C}^{-1} \mathbf{A}) \leq C \left(1 + \max_{k \in I_1 \cup I_2} \frac{C_k}{c_0} \right).$$

Both condition numbers are independent of the number K of fine-scale subdomains. Additionally, the performance of the second preconditioner proves to be independent of their sizes. It is also the case for the first preconditioner when no small subdomain is clamped ($I_2 = \emptyset$). In this sense, we can reasonably talk of two-scale preconditioners.

The simplest symmetrized Dirichlet-Neumann preconditioner, which imposes the invertibility of \mathbf{A}_0 (through a Dirichlet boundary condition on Ω_0 for

example), is strongly affected by the presence of small subdomains that are fixed on a part of their boundary, through the ratio L_0/L_k . The enhanced symmetrized Dirichlet-Neumann preconditioner avoids this dependence, and its use is not limited to the case where $\Gamma_D \cap \partial\Omega_0$ has a positive measure. Nevertheless, both condition numbers are affected by the presence of stiff fine subdomains in comparison with the stiffness of the coarse domain, through the ratio E_k/E_0 where E_k (resp. E_0) is the Young modulus of the material in Ω_k (resp. Ω_0). We acknowledge that this limitation is a natural limitation of Dirichlet-Neumann preconditioners. It is nevertheless a significant advantage in the case where the large subdomain Ω_0 is the stiffest.

5 Algorithm

Before testing these two preconditioners, we briefly summarize the way of implementing the associated algorithms. The action of a preconditioner on a right hand side $(F_0, F_1, \dots, F_K)^\top$ in the dual of the constrained space \mathcal{E} leads to the following sequence of operations :

- (1) Compute the equivalent coarse-scale force on Ω_0

$$\overline{F}_0 = F_0 - \sum_{k=1}^K \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ 0 \end{pmatrix},$$

by solving, in parallel, one Dirichlet problem by small subdomain. On each subdomain, the force F_k is imposed and a weak zero trace prescribed on the interface.

- (2) Use the equivalent coarse-scale operator $\hat{\mathbf{D}}_0$ to determine $\tilde{U}_0 = \hat{\mathbf{D}}_0^{-1} \overline{F}_0$. In the case of the enhanced preconditioner, we effectively implement the expression given by (13).
- (3) Solve the local problems for $1 \leq k \leq K$

$$\mathbf{K}_k \begin{pmatrix} \tilde{U}_k \\ \tilde{\Lambda}_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_0 \tilde{U}_0 \end{pmatrix}.$$

It amounts to finding the mechanical equilibrium of the structure Ω_k under loading F_k , the displacement \tilde{U}_0 being imposed in the weak sense on the interface Γ_{0k} .

For the sake of efficiency of the aforementioned algorithm, it is crucial that for every $k \geq 1$, the cost of applying the inverse \mathbf{A}_k^{-1} remains of the same order (or less expensive) than the cost of applying the inverse of the approximate

Schur complement, $\hat{\mathbf{D}}_0^{-1}$. Again, this requirement justifies the assumption that the details should be small and disjoint.

The proposed preconditioners are multiplicative in the sense that the two scales cannot be solved simultaneously. Nevertheless, the solutions over the small details can be performed in parallel.

6 Numerical tests

6.1 A two-scale model

Let us consider the two-scale geometry whose deformed state is represented in Figure 2. The system is clamped at its tips, as well as on the middle row of spherical details, and the materials are supposed to be linearly elastic. The Young modulus and the Poisson coefficient are taken to be constant over the coarse and the fine subdomains, with values (E_0, ν_0) and (E', ν') respectively. In the sequel, we denote by $r = E'/E_0$ the ratio of Young moduli, which is proportional to C_k/c_0 and C_k/α_0 with $k \geq 1$ (appearing in proposition 4) up to a constant which is fixed for the chosen model. In view of proposition 4, this parameter has a crucial influence on the efficiency of the introduced preconditioners.

A uniform negative pressure is applied on the spherical details, and on the face of the plate opposite to these geometrical details. An illustration of the loading is represented in Figure 3, and Figure 2 shows the state of deformation of the structure under these conditions. A \mathbb{Q}_1 approximation is adopted for displacements. We adopt \mathbb{Q}_1 Lagrange multipliers, and note that they can actually be defined either on the mesh of the spherical details, or on the coarse mesh of the plate (cf. Remark 2). Concerning the latter choice, particular attention should be paid to the accuracy of the obtained solution. The definition of the Lagrange multipliers (\mathbb{Q}_1 , or \mathbb{Q}_0 provided a stabilization is added to the displacements as in [24]) is not an issue, as long as the fundamental assumptions of section 2.2.4 are satisfied. The mortar weak-continuity constraint is penalized using a penalization coefficient $\kappa = 10^6 E'$.

We now use our preconditioners in a Conjugate Gradient algorithm to compute the solution of the model problem. Note that if the coarse plate was not fixed on a part of its boundary, the simpler preconditioner couldn't be used, since the invertibility of \mathbf{A}_0 is required. The convergence of the method for the proposed preconditioners is illustrated in Figures 4 and 5, displaying the L^2 norm of the residual along the iterations. The behavior has been computed both when Lagrange multipliers are defined on the fine and coarse sides of the

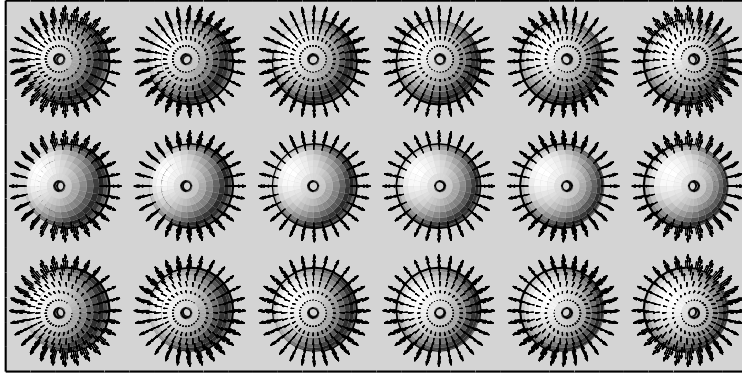


Fig. 3. Loading of the model problem. The spherical details are loaded with constant negative pressure, as well as the hidden face of the plate. Additionally, on the above drawing, the left and right faces are clamped, as well as the center of the spherical details in the middle row (where no arrow appears).

interface. As shown, the influence of this choice on the efficiency of the preconditioners remains small, even though the convergence for coarse Lagrange multipliers is always slightly better. The enhanced Dirichlet-Neumann preconditioner exhibits improved results as compared to the simpler preconditioner, especially as the ratio r becomes large. For $r > 1$, the number of required iterations to reach a given precision is roughly divided by 2. Nevertheless, both preconditioners are affected by the growth of r , which seems unavoidable for non-symmetric strategies.

6.2 Extension to a quasi-Newton method

When dealing with nonlinear problems, the previous methods can be applied to solve each tangent problem in the Newton-Raphson's method. Nevertheless, we show that for non-stiff geometrical details, the previous preconditioners define efficient quasi-Newton methods. More precisely, we propose to replace the Newton's tangent elasticity operator by the approximation provided by a preconditioner. The possible choices are

- the simplest Dirichlet-Neumann preconditioner (simple quasi-Newton),
- the enhanced Dirichlet-Neumann preconditioner pre-computing the lifting of rigid motions and their equivalent stiffness $-\hat{\mathbf{A}}_k$ in (13)– once for all on the undeformed configuration (enhanced quasi-Newton 1),
- the enhanced Dirichlet-Neumann preconditioner fully implemented on each tangent problem (enhanced quasi-Newton 2).

These methods are compared with the full Newton-Raphson's method, where each Newton's tangent problem is fully solved by a Preconditioned Conjugate Gradient algorithm making use of our Enhanced Dirichlet-Neumann precon-

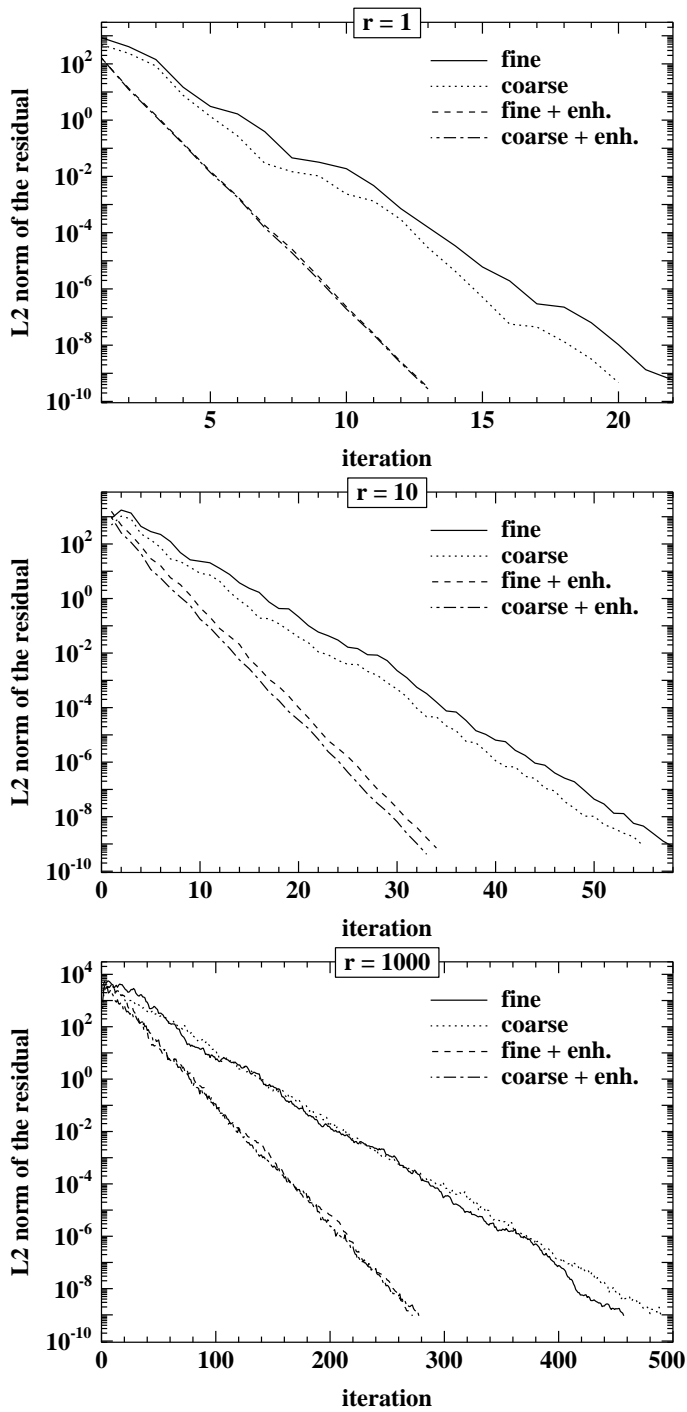


Fig. 4. Convergence of the L^2 norm of the residual along the iterations for $r = 1, 10, 1000$, with or without the enhancement proposed in section 3.3.2. The Lagrange multipliers are defined either on the fine mesh, or on the coarse one.

ditioner. The Lagrange multipliers are defined on the fine mesh.

Let us consider the model problem from Section 6.1 represented in Figure 2, where the linear law is replaced by a Saint-Venant/Kirchhoff material charac-

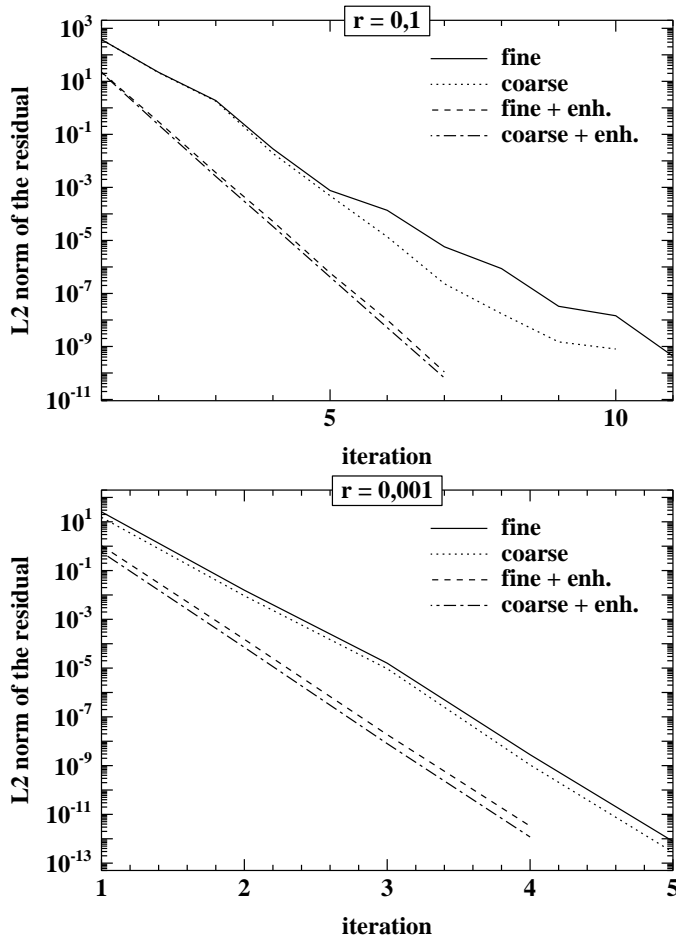


Fig. 5. Convergence of the L^2 norm of the residual along the iterations for $r = 0.1, 0.001$, with or without the enhancement proposed in section 3.3.2. The Lagrange multipliers are defined either on the fine mesh, or on the coarse one.

terized by the hyperelastic energy

$$\hat{\mathcal{W}}(F) = \frac{\lambda}{4} \left[\text{tr}(F^\top \cdot F - id) \right]^2 + \frac{\mu}{8} \text{tr} \left[(F^\top \cdot F - id)^2 \right],$$

in which $F = id + \nabla u$. The L^2 norm of the residual after the consecutive tangent problems is represented in Figures 6 and 7 for various values of the stiffness ratio r . Because some of the sculptures of the problem are clamped, remark that our enhanced preconditioner is required to define an efficient quasi-Newton strategy. Moreover, for large deformation problems, it appears important to resort to the fully implemented variant (enhanced quasi-Newton 2). When relaxing the clamping constraint on the geometrical details (see deformation in Figure 8), the simple quasi-Newton method exhibits good convergence properties for non-stiff details, as illustrated in Figure 9. Finally, let us acknowledge that the efficiency of our quasi-Newton strategies is restricted to the case where boundary details remain non-stiff ($r \lesssim 1$).

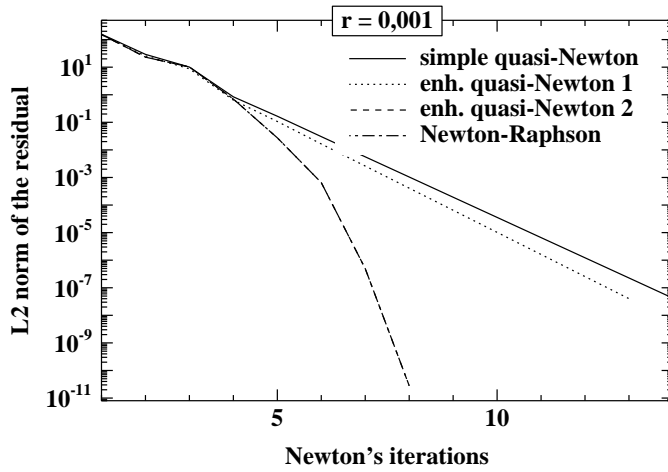


Fig. 6. Convergence of the L^2 norm of the residual along the Newton's iterations for $r = 0.001$.

7 Conclusion

In this paper, we have analyzed and tested two symmetrized Dirichlet-Neumann preconditioners that can be used efficiently in conjunction with a non-conforming mortar formulation to solve elliptic problems with small geometrical details on the boundary. This method is well-adapted to the case where the details are localized enough to make their resolution relatively cheap. In the case where the small structures would not be so localized as to satisfy this assumption, one could imagine another level of hierarchy in the preconditioner through which the Dirichlet problem would be split. Finally, we have derived a quasi-Newton method which is well suited for non-stiff details in the framework of nonlinear problems.

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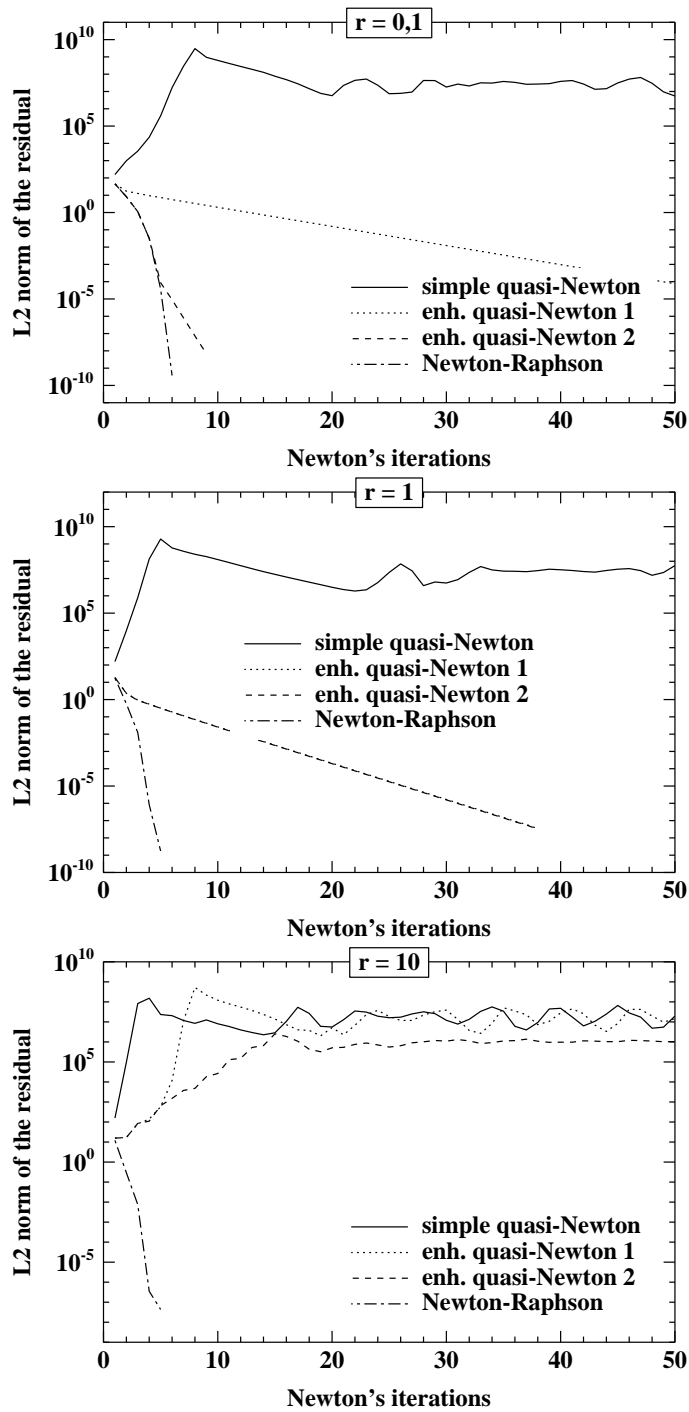


Fig. 7. Evolution of the L^2 norm of the residual along the Newton's iterations for $r = 0.1, 1, 10$.

References

- [1] C. Bernardi, Y. Maday, A. Patera, Domain decomposition by the mortar element method, in: H. Kaper, M. Garbey (Eds.), Asymptotic and Numerical

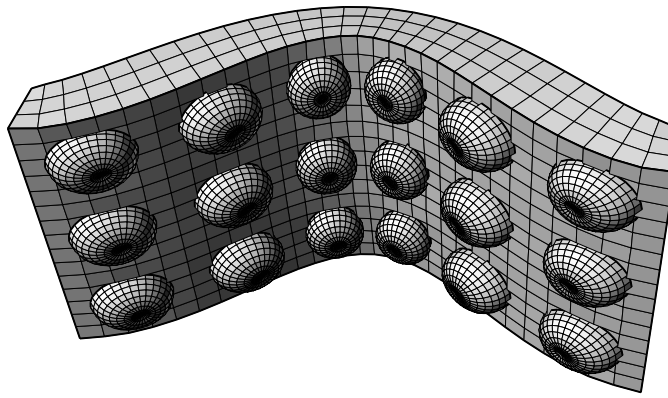


Fig. 8. Deformation of the model problem without clamping on the geometrical details.

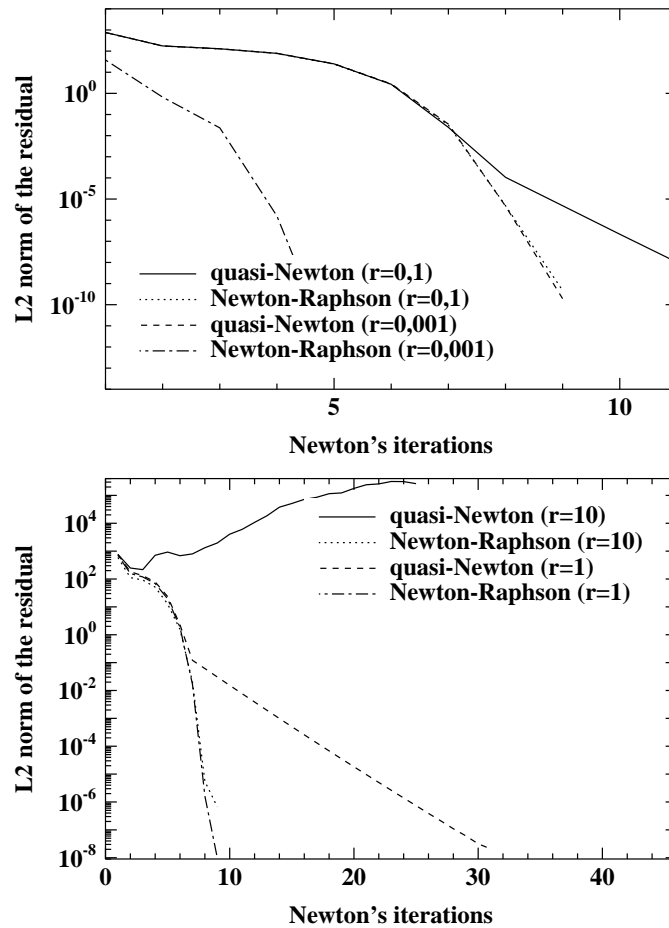


Fig. 9. Evolution of the L^2 norm of the residual along the Newton's iterations for $r = 0.001, 0.1, 1, 10$ in the case of the unclamped model. The Newton-Raphson method and the simple quasi-Newton methods are plotted.

Methods for Partial Differential Equations with Critical Parameters, N.A.T.O. ASI, Kluwer Academic Publishers, 1993, pp. 269–286.

- [2] C. Bernardi, Y. Maday, A. Patera, Nonlinear partial differential equations and their applications. Collège de France Seminar XI, Pitman, Paris, 1994, Ch. A new nonconforming approach to domain decomposition: the mortar element method., pp. 13–51.
- [3] F. B. Belgacem, The mortar finite element method with Lagrange multipliers, *Numer. Math.* 84 (1999) 173–197.
- [4] B. Wohlmuth, A mortar finite element method using dual spaces for the Lagrange multiplier, *SIAM J. Numer. Anal.* 38 (2000) 989–1012.
- [5] C. Kim, R. Lazarov, J. Pasciak, P. Vassilevski, Multiplier spaces for the mortar finite element method in three dimensions, *SIAM J. Numer. Anal.* 39 (2001) 519–538.
- [6] M. Azaiez, F. B. Belgacem, H. E. Fekih, M. Ismael, Numerical simulation of the wave equation with discontinuous coefficients by nonconforming finite elements, *Numer. Methods Partial Differ. Equations* 15 (6) (1999) 637–656.
- [7] F. B. Belgacem, A stabilized domain decomposition method with non-matching grids for the Stokes problem in three dimensions, *SIAM Journal of Numerical Analysis* 42 (2) (2004) 667–685.
- [8] P. Hauret, P. L. Tallec, A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems, part i: abstract framework, Tech. Rep. 553, CMAP (september 2004).
- [9] P. Hauret, P. L. Tallec, A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems, part ii: discontinuous Lagrange multipliers, Tech. Rep. 554, CMAP (september 2004).
- [10] P. Seshaiyer, Non-conforming hp finite element methods, Ph.D. thesis, University of Maryland (1998).
- [11] S. Brenner, Korn’s inequalities for piecewise H^1 vector fields, *Mathematics of Computation* 73 (2004) 1067–1087.
- [12] A. Quarteroni, A. Valli, Domain Decomposition Methods for Partial Differential Equations, Oxford University Press, 1999.
- [13] B. Wohlmuth, Discretization methods and iterative solvers based on domain decomposition, Springer, 2001.
- [14] P. L. Tallec, Y. D. Roeck, M. Vidrascu, Domain decomposition methods for large linearly elliptic three dimensional problems, *J. Comput. Appl. Math.* 34 (1991) 93–117.
- [15] C. Farhat, F.-X. Roux, A method of finite element tearing and interconnecting and its parallel solution algorithm, *Int. J. Num. Meth. Engr.* 32 (1991) 1205–1228.
- [16] P. L. Tallec, Neumann-Neumann domain decomposition algorithm for solving 2d elliptic problems with nonmatching grids, *East-West J. Numer. Math.* 1 (2) (1993) 129–146.

- [17] Y. Achdou, Y. Kuznetsov, O. Pironneau, Substructuring preconditioners for the Q_1 mortar element method, *Numerische Mathematik* 71 (1995) 419–449.
- [18] Y. Achdou, Y. Maday, O. Widlund, Substructuring preconditioners for the mortar method in dimension two, *SIAM Journal of Numerical Analysis* 36 (2) (1998) 551–580.
- [19] G. Abdulaev, Y. Achdou, Y. Kuznetsov, C. Prud'homme, On a parallel implementation of the mortar element method, *M2AN* 33 (2).
- [20] D. Stefanica, Domain decomposition methods for mortar finite elements, Ph.D. thesis, Courant Institute of Mathematical Sciences, New York University (1999).
- [21] G. Duvaut, J.-L. Lions, *Les inéquations en Mécanique et en Physique*, Dunod, 1972.
- [22] A. Agouzal, J. Thomas, Une méthode d'éléments finis hybrides en décomposition de domaines, *RAIRO M2AN* 29 (1995) 749–764.
- [23] B. Wohlmuth, Hierarchical a posteriori error estimators for mortar finite element methods with Lagrange multipliers, *SIAM J. Numer. Anal.* 36 (1999) 1636–1658.
- [24] P. Hauret, *Méthodes numériques pour la dynamique des structures non-linéaires incompressibles à deux échelles* (Numerical methods for the dynamic analysis of two-scale incompressible nonlinear structures), Ph.D. thesis, Ecole Polytechnique (2004).
- [25] F. Brezzi, D. Marini, Error estimates for the three-field formulation with bubble stabilization, *Math. Comp.* 70 (2000) 911–934.
- [26] H. Brézis, *Analyse fonctionnelle*, Dunod, 1999.
- [27] L. Scott, S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, *Math. Comp.* 54 (190) (1990) 483–493.
- [28] A. Matsokin, S. Nepomniaschik, A Schwarz alternating method in a subspace, *Sov. Math.* 29 (10) (1985) 78–84.