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Conservation analysis for integration schemes in quasi incompressible nonlinear elastodynamics.

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Abstract

Discrete energy conservation plays a key role for the unconditional stability of time integration schemes in nonlinear elastodynamics. We present a theoretical and numerical study of energy evolution for midpoint, trapezoidal and Hilber-Hughes-Taylor second order schemes, as compared with energy conserving schemes like [Gon00]. We underline that in general, *linearly dissipative* schemes (with spectral radius less than one) do not dissipate mechanical energy, and do not achieve unconditional stability when extended to the nonlinear framework.

1 Introduction

Time integration schemes for elastodynamics have been developed for a long time in a linear framework in which consistency and linear stability insure convergence by time step refinement. Whereas the conditionally stable explicit centered method must be mentioned for its simplicity, the numerical stiffness of such mechanical problems has lead to the development of implicit methods, especially when dealing with incompressible materials: Houbolt, Wilson, Newmark or Hilber-Hughes-Taylor [Bat82, Cri97, GR93, HHT77]. Nevertheless, when considering nonlinear problems, the previous implicit schemes lose their unconditional stability and nonlinear criteria of stability must be found.

In the Hamiltonian framework (i.e. with conservative loadings), a geometrical approach could consist in constructing numerical schemes whose flow is symplectic [HLW02, SSC94], entailing the conservation of the volume in the phase space. Nevertheless, such a condition is not always sufficient to insure the stability of the numerical system for large time steps and for stiff problems. In the compressible case, this statement will be numerically assessed for the symplectic midpoint scheme. A deeper analysis is proposed by J.C.Simo and O.Gonzalez in [SG93]. In particular, the

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authors show that symplecticness is difficult to maintain in the case of kinematically constrained systems.

Another idea to stabilize the discrete solution can consist in imposing energy conservation as a constraint, by projection [NNR77, HLC78] or by Lie group methods [MK99]. In fact, by a mean value argument, Simo and Tarnow have shown in [ST92] that conservation could be achieved by choosing correctly the algorithmic definition of the second Piola-Kirchhoff stress tensor. Such an idea has lead to a very practical conservative scheme proposed by Gonzalez in [Gon00].

Energy dissipation would introduce additional robustness, but two remarks must be done. In the linear framework, the schemes whose spectral radius is strictly less than one, often said to be *linearly dissipative* such as HHT [HHT77, Hug87], do not dissipate the mechanical energy in general. When extended to the nonlinear framework, they do not insure longterm stability as the evolution of energy cannot be controlled. Achieving energy dissipation in this framework is quite difficult: as a heuristic proof, we refer to the work of Armero and Romero in [AR01a, AR01b], where in order to obtain dissipation, the authors introduce a dissipation term in the energy conserving scheme of Gonzalez. Then, we would like to make the reader think about the following statement:

"In the nonlinear framework, conserving energy seems easier than dissipating it."

The purpose of the present paper is to discuss the energy conservation properties of these different schemes in a nonlinear framework. In section 2, we introduce the problem of nonlinear quasi incompressible elastodynamics. Concerning cost efficiency for time integrators, we indicate in section 3 why completely implicit schemes are the only way to explore. In section 4, some implicit time integration schemes are analyzed and discussed. We study the conservation properties of midpoint type schemes (4.2), that generalize Newmark's trapezoidal rule in the nonlinear framework, and of the linearly dissipative HHT scheme (4.3). Numerical tests are presented in section 5. An overview of these results has already been written in [THR02]. The present paper contains complete proofs.

2 Quasi incompressible elastodynamics

2.1 The incompressible model

The open set $\Omega \subset \mathbb{R}^3$ denotes the interior of the reference configuration of a solid body and its time dependent deformation is described by the following mapping:

$$\varphi : [0,T] \times \Omega \to \mathbb{R}^3. \tag{1}$$

The material is assumed to be incompressible in the sense that on $[0,T] \times \Omega$,

$$\det F = \det \nabla\varphi = 1. \tag{2}$$

The density of the material on the reference configuration is denoted by $\rho : \Omega \to \mathbb{R}_+^*$ and the body forces by $f : [0,T] \times \Omega \to \mathbb{R}^3$. On the subsets Γ_0 and Γ_1 of the boundary $\Gamma = \partial\Omega$ of the domain, the displacement $\varphi_d : [0,T] \times \Gamma_0 \to \mathbb{R}^3$ and the traction

$g : [0,T] \times \Gamma_1 \to \mathbb{R}^3$ are prescribed. Moreover $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and n is the outward normal unit vector.

The first Piola-Kirchhoff stress tensor in the material is denoted by Π and is given by the hyperelastic constitutive law:

$$\begin{aligned}
\Pi &= \frac{\partial \hat{\mathcal{W}}}{\partial F} - p \, cof \, F \\
&= F \cdot \left(2 \frac{\partial \mathcal{W}}{\partial C} - \frac{1}{(\det C)^{1/2}} p \, cof \, C \right) \\
&= F \cdot \left(2 \frac{\partial \mathcal{W}}{\partial C} - 2p \frac{\partial \det C^{1/2}}{\partial C} \right) \\
&= F \cdot \Sigma.
\end{aligned}$$

The symmetric tensor Σ is known as the second Piola-Kirchhoff stress tensor, $p : [0,T] \times \Omega \to \mathbb{R}$ denotes the hydrostatic pressure, $\hat{\mathcal{W}}$ and $\mathcal{W}$ the stored elastic potentials respectively in term of the gradient F or the right Cauchy-Green strain tensor $C = F^t \cdot F$. The cofactor matrix of the matrix F is denoted by $cof \, F = \partial_F \det F$.

2.2 Variational quasi incompressible formulation

We introduce variational spaces for displacements, velocities and pressures:

$$\begin{cases} \mathcal{U}_0 \subset \{u \in W^{1,s}(\Omega)^3; \quad u = 0 \text{ on } \Gamma_0\}, \\ \mathcal{V} \subset \{w \in L^2(\Omega)^3\}, \\ \mathcal{P} \subset \{p \in L^q(\Omega); \quad \frac{3}{s} + \frac{1}{q} \leq 1\}. \end{cases} \tag{3}$$

We assume that $\rho \in L^\infty(\Omega)$, with $\rho \geq \rho_0 > 0$ almost everywhere on Ω, that $f \in L^2(0,T; L^{s^*}(\Omega))$ with $\frac{1}{s} + \frac{1}{s^*} = 1$, and that $\varphi_d \in L^2\left(0,T; W^{1-1/s,s}(\Gamma_0)^3\right)$. The elastic potentials $\hat{\mathcal{W}}$ and $\mathcal{W}$ are assumed to be continuously differentiable with respect to their arguments. Then, the variational formulation of our model consists in finding:

$$\begin{cases} \varphi - \varphi_d \in L^2(0,T;\mathcal{U}_0), \\ \dot{\varphi} \in L^2(0,T;\mathcal{V}), \\ \rho \ddot{\varphi} \in L^2(0,T;\mathcal{U}_0'), \\ p \in L^2(0,T;\mathcal{P}), \end{cases} \tag{4}$$

such as for almost every $t \in]0,T[$, any $v \in \mathcal{U}_0$ and any $q \in \mathcal{P}$:

$$\begin{cases} \langle \rho \ddot{\varphi}(t), v \rangle_{\mathcal{U}_0', \mathcal{U}_0} + \displaystyle\int_\Omega \Pi(t) : \nabla v = \int_\Omega f(t) \cdot v, \\ \displaystyle\int_\Omega \left(\det \nabla \varphi(t) - 1 + \epsilon p(t) \right) q = 0. \end{cases} \tag{5}$$

Remark 1 • *We will assume henceforward that $g = 0$ to avoid a technical difficulty. Indeed, since the speed is in $L^2(\Omega)^3$, it has no trace on the boundary Γ_1 and the surfacic work cannot be defined so easily without additional regularity study.*

- *The spaces $\mathcal{U}_0$, $\mathcal{V}$ and $\mathcal{P}$ can be finite dimensional spaces in the framework of space discretized approximation.*
- *The compression term ϵp generalizes the problem to quasi incompressible situations. $1/\epsilon$ has the physical meaning of a bulk modulus.*
- *To obtain the uniform convergence towards the incompressible limit as $\epsilon \to 0$, it is necessary to have the following compatibility condition (see [Bab73, Bre74]):*

$$\exists \beta > 0, \quad \forall \varphi \in \varphi_d + \mathcal{U}_0,$$

$$\inf_{q \in \mathcal{P}, \|q\|_{\mathcal{P}}=1} \sup_{v \in \mathcal{U}_0, \|v\|_{\mathcal{U}_0}=1} \int_\Omega q\,((cof\ \nabla\varphi) : \nabla v) \geq \beta. \tag{6}$$

2.3 Conservation properties

With a minimal regularity analysis, one obtain the following expected physical properties of conservation. They hold for almost every $t \in]0, T[$:

- Energy conservation.

$$\mathcal{E}(t) - \mathcal{E}(0) = \int_0^t \int_\Omega f \cdot \dot{\varphi}, \tag{7}$$

  the total energy being defined by:

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \rho\dot{\varphi}(t,x)^2\,dx + \int_\Omega \hat{\mathcal{W}}(x, \nabla\varphi(t,x))\,dx + \frac{\epsilon}{2}\int_\Omega p(t)^2. \tag{8}$$

- Angular momentum conservation (for $\Gamma_0 = \emptyset$).

$$\mathcal{J}(t) - \mathcal{J}(0) = \int_0^t \int_\Omega \varphi \times f, \tag{9}$$

  with:

$$\mathcal{J}(t) = \int_\Omega \rho\varphi(t,x) \times \dot{\varphi}(t,x)\,dx. \tag{10}$$

- Linear momentum conservation (for $\Gamma_0 = \emptyset$).

$$\mathcal{I}(t) - \mathcal{I}(0) = \int_0^t \int_\Omega f, \tag{11}$$

  with:

$$\mathcal{I}(t) = \int_\Omega \rho\dot{\varphi}(t,x)\,dx. \tag{12}$$

Remark 2 *If $\varphi \in W^{2,2}(0,T;\mathcal{U}_0)$ and $p \in W^{1,2}(0,T;\mathcal{P})$, these properties are directly derived from the variational formulation by using respectively $v = \dot{\varphi}$, $v = a \times \varphi$ for all $a \in \mathbb{R}^3$, and any constant vector $v \in \mathbb{R}^3$.*

3 Efficiency and semi explicit strategies

When regarding industrial computations, the necessity to obtain low cost methods is obvious. The disadvantage of implicit methods is the introduction of a nonlinear problem at each time step.

At the opposite, an explicit time integration would be simple and economic to compute, but for stability reasons, time step has to satisfy a CFL condition, particularly restrictive for quasi-incompressible problems. A compromise could consist in impliciting the compression term in the scheme, which is equivalent to project an explicit time iteration of the compressible problem on the manifold of incompressible displacements. It is obvious that the CFL condition to be satisfied for the time step is then far less restrictive: it must only insure the stability of the explicit compressible scheme. Thus, we could propose to adopt as a time iteration, the following system:

$$\begin{cases} \displaystyle\int_\Omega \rho\ddot{\varphi}_n \cdot v + \int_\Omega \left(\frac{\partial \hat{\mathcal{W}}}{\partial F}(\nabla\varphi_n) + \frac{1}{2}\left(p_{n-1}\, cof\, F_{n-1} + p_{n+1}\, cof\, F_{n+1}\right)\right) : \nabla v = \int_\Omega f_n \cdot v, \\ \displaystyle\int_\Omega q\left(\det F_{n+1} - 1 - \epsilon p_{n+1}\right) = 0, \end{cases} \tag{13}$$

for all $(v, q) \in \mathcal{U}_0 \times \mathcal{P}$, with the centered finite difference approximation:

$$\ddot{\varphi}_n = \frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{\Delta t_n^2}. \tag{14}$$

Using a standard resolution by Newton method, increments $\delta\varphi_{n+1}^{(k)}$ and $\delta p_{n+1}^{(k)}$ are solutions of tangent problems of the following type:

$$\begin{cases} \displaystyle\frac{1}{\Delta t_n^2}\int_\Omega \rho\, \delta\varphi_{n+1}^{(k)} \cdot v + \frac{1}{2}\int_\Omega p_{n+1}\left(\frac{\partial\, cof\, F_{n+1}^{(k)}}{\partial F_{n+1}^{(k)}} : \nabla v\right) : \nabla\delta\varphi_{n+1}^{(k)} \\ \displaystyle + \frac{1}{2}\int_\Omega \delta p_{n+1}^{(k)}\, cof\, F_{n+1}^{(k)} : \nabla v = \left\langle R_{n+1}^{(k)}, v\right\rangle, \quad \forall v \in \mathcal{U}_0, \\ \\ \displaystyle\int_\Omega q\left(cof\, F_{n+1} : \nabla\delta\varphi_{n+1}^{(k)} - \epsilon\, \delta p_{n+1}^{(k)}\right) = -\left\langle S_{n+1}^{(k)}, v\right\rangle, \quad \forall q \in \mathcal{P}. \end{cases} \tag{15}$$

The only way to make it computable at low cost compared to an implicit scheme would be to invert only a pressure problem, which requires to adopt a lumped diagonal mass and to neglect the term in displacement:

$$\frac{1}{2}\int_\Omega p_{n+1}\left(\frac{\partial\, cof\, F_{n+1}^{(k)}}{\partial F_{n+1}^{(k)}} : \nabla v\right) : \nabla\delta\varphi_{n+1}^{(k)}. \tag{16}$$

This modification leads to a modified algorithm which we have observed not to converge in practice in nonlinear incompressible elasticity. Indeed, because the set of incompressible displacements is very nonlinear, the projection operator cannot be simplified. Thus, a displacement problem must be inverted and the associated cost is comparable to the cost of an implicit time step. Then, in our framework, totally implicit schemes seem to be the only way to explore.

4 Conservation analysis for some usual schemes

4.1 General concepts

We write (5) under a first order form, useful for time integration concepts:
For all $v \in \mathcal{U}_0$, $w \in \mathcal{V}$, $q \in \mathcal{P}$,

$$\begin{cases} \partial_t \int_\Omega \rho \dot{\varphi} \cdot v + \int_\Omega \Pi : \nabla v = \int_\Omega f \cdot v, \quad \text{in } \mathcal{D}'(0,T), \\ \partial_t \int_\Omega \varphi \cdot w = \int_\Omega \dot{\varphi} \cdot w, \quad \text{in } \mathcal{D}'(0,T), \\ \int_\Omega (\det \nabla \varphi(t) - 1 + \epsilon p(t))\, q = 0, \quad \text{in } \mathcal{D}'(0,T). \end{cases} \tag{17}$$

The time interval $[0,T]$ is splitted into subintervals $[0,T] = \cup_{n=0}^N [t_n; t_{n+1}]$, with $\Delta t_n = t_{n+1} - t_n$. We will call numerical approximation of (17), any sequence $(\varphi_n, \dot{\varphi}_n, p_n)_{0 \leq n \leq N}$ of $(\mathcal{U}_0 \times \mathcal{V} \times \mathcal{P})^N$, given by an integration scheme of the type:

$$\begin{cases} \int_\Omega \rho \dot{\varphi}_{n+1} \cdot v = \int_\Omega \rho \dot{\varphi}_n + \Delta t_n \Big\langle P\left(\varphi_n, \varphi_{n+1}, \dot{\varphi}_n, \dot{\varphi}_{n+1}, p_n, p_{n+1}\right); v \Big\rangle, \quad \forall v \in \mathcal{U}_0, \\ \int_\Omega \varphi_{n+1} \cdot w = \int_\Omega \varphi_n \cdot w + \Delta t_n \Big\langle Q\left(\varphi_n, \varphi_{n+1}, \dot{\varphi}_n, \dot{\varphi}_{n+1}\right); w \Big\rangle, \quad \forall w \in \mathcal{V}, \\ \Delta t_n \int_\Omega q G(\varphi_n, \varphi_{n+1}) = 0, \quad \forall q \in \mathcal{P}. \end{cases} \tag{18}$$

Let us assume that the data and the solution of (17) are smooth in time. Then, the scheme (18) will said to be consistent and accurate at order r if when replacing $(\varphi_n, \dot{\varphi}_n, p_n)_{0 \leq n \leq N}$ by $(\varphi(t_n), \dot{\varphi}(t_n), p(t_n))_{0 \leq n \leq N}$ in (18), the relations are satisfied up to a $O((\Delta t_n)^{r+1})$ error term. For the schemes described in this paper, consistency and second order accuracy are quite obvious and will not be detailed further.

4.2 Midpoint based schemes

We analyze below some natural second order time integration schemes of the form:

$$\begin{cases} \int_\Omega \rho \dot{\varphi}_{n+1} \cdot v = \int_\Omega \rho \dot{\varphi}_n \cdot v - \Delta t_n \int_\Omega \Pi_{n+1/2} : \nabla v + \Delta t_n \int_\Omega \frac{f_n + f_{n+1}}{2} \cdot v, \\ \int_\Omega \varphi_{n+1} \cdot w = \int_\Omega \varphi_n \cdot w + \Delta t_n \int_\Omega \frac{\dot{\varphi}_n + \dot{\varphi}_{n+1}}{2} \cdot w, \\ \int_\Omega q \left(D_{n+1/2} - 1 + \epsilon \, \frac{p_n + p_{n+1}}{2} \right) = 0, \end{cases} \tag{19}$$

entirely determined by the expressions of $\Pi_{n+1/2}$ and $D_{n+1/2}$. The different choices analyzed in this paper are summarized on figure 1.

4.2.1 Trapezoidal rule

Proposition 1 *The trapezoidal rule achieves the following properties:*

Scheme	$\Pi_{n+1/2} =$	$D_{n+1/2} =$
Trapezoidal (stress averaging)	$\frac{1}{2}\left(\frac{\partial\hat{\mathcal{W}}}{\partial F}(F_n) + \frac{\partial\hat{\mathcal{W}}}{\partial F}(F_{n+1})\right)$ $-\frac{1}{2}\left(p_n \, cof \, F_n + p_{n+1} \, cof \, F_{n+1}\right)$	$\frac{1}{2}\left(\det F_n + \det F_{n+1}\right)$
Midpoint (strain averaging)	$\frac{\partial\hat{\mathcal{W}}}{\partial F}\left(\frac{F_n + F_{n+1}}{2}\right)$ $-\frac{p_n + p_{n+1}}{2} \, cof \left(\frac{F_n + F_{n+1}}{2}\right)$	$\frac{1}{2}\det\left(\frac{F_n + F_{n+1}}{2}\right)$
Conservative	$\left(\frac{F_n + F_{n+1}}{2}\right) \cdot \Sigma_{n+1/2}$ with: $\frac{1}{2}\int_\Omega \Sigma_{n+1/2} : (C_{n+1} - C_n) =$ $\int_\Omega \left(\mathcal{W}(C_{n+1}) + \frac{\epsilon}{2}p_{n+1}^2\right) - \int_\Omega \left(\mathcal{W}(C_n) + \frac{\epsilon}{2}p_n^2\right)$	$\frac{1}{2}\left(\det F_n + \det F_{n+1}\right)$

Figure 1: Some choices for $\Pi_{n+1/2}$ and $D_{n+1/2}$ in midpoint based schemes.

- **Discrete energy conservation.**

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \mathfrak{P}_n + c_n \Delta t_n^3, \tag{20}$$

where the discrete work between times t_n and t_{n+1} is given by:

$$\mathfrak{P}_n = \int_\Omega \frac{f_n + f_{n+1}}{2} \cdot (\varphi_{n+1} - \varphi_n),$$

and the discrete energy at discrete time n by:

$$\mathcal{E}_n = \frac{1}{2}\int_\Omega \rho\dot{\varphi}_n^2 + \int_\Omega \hat{\mathcal{W}}(\nabla\varphi_n) + \frac{\epsilon}{2}\int_\Omega p_n^2.$$

The scalar c_n only depends on φ_n, $\dot{\varphi}_n$, p_n, φ_{n+1}, $\dot{\varphi}_{n+1}$, p_{n+1}, and on the approximate time derivative of the pressure $\frac{p_{n+1}-p_n}{\Delta t_n}$. We will say that c_n only depends on the approximate solution at times n and $n+1$.

- **Discrete angular momentum.** *If $\Gamma_0 = \emptyset$,*

$$\mathcal{J}_{n+1} - \mathcal{J}_n = \mathfrak{M}_n + c_n \Delta t_n^3, \tag{21}$$

where the resultant moment between times t_n and t_{n+1} is given by:

$$\mathfrak{M}_n = \Delta t_n \int_\Omega \frac{\varphi_n + \varphi_{n+1}}{2} \times \frac{f_n + f_{n+1}}{2},$$

and the angular momentum at discrete time n by:

$$\mathcal{J}_n = \int_\Omega \rho \varphi_n \times \dot{\varphi}_n.$$

The constant c_n only depends on the approximate solution at times n and $n+1$.

- **Discrete linear momentum.** *If $\Gamma_0 = \emptyset$,*

$$\mathcal{I}_{n+1} - \mathcal{I}_n = \mathfrak{F}_n, \tag{22}$$

where the resultant force between times t_n and t_{n+1} is given by:

$$\mathfrak{F}_n = \Delta t_n \int_\Omega \frac{f_n + f_{n+1}}{2},$$

and the discrete linear momentum at time n by:

$$\mathcal{I}_n = \int_\Omega \rho \dot{\varphi}_n.$$

Proof :

1. **Energy conservation.** We take $v = (\varphi_{n+1} - \varphi_n)/\Delta t$ in (19). The inertial term gives the discrete increase of kinetic energy:

$$\int_\Omega \rho\left(\dot{\varphi}_{n+1} - \dot{\varphi}_n\right) \cdot v = \frac{1}{2}\int_\Omega \rho \dot{\varphi}_{n+1}^2 - \frac{1}{2}\int_\Omega \rho \dot{\varphi}_n^2.$$

The work $\mathfrak{P}_n$ is obtained as:

$$\Delta t_n \int_\Omega \frac{f_n + f_{n+1}}{2} \cdot v = \int_\Omega \frac{f_n + f_{n+1}}{2} \cdot \left(\varphi_{n+1} - \varphi_n\right).$$

The elastic term gives by a standard Taylor expansion:

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial \hat{\mathcal{W}}}{\partial F}(F_n) + \frac{\partial \hat{\mathcal{W}}}{\partial F}(F_{n+1})\right) : \nabla v \ &= \frac{1}{2\Delta \tau_n}\left(\frac{\partial \hat{\mathcal{W}}}{\partial F}(F_n) + \frac{\partial \hat{\mathcal{W}}}{\partial F}(F_{n+1})\right) : \left(F_{n+1} - F_n\right), \\
&= \frac{1}{\Delta \tau_n}\left(\hat{\mathcal{W}}_{n+1} - \hat{\mathcal{W}}_n\right) + \frac{c}{\Delta \tau_n}\frac{\partial^3 \mathcal{W}}{\partial F^3}(F_*)(F_{n+1} - F_n)^3, \\
&= \frac{1}{\Delta \tau_n}\left(\hat{\mathcal{W}}_{n+1} - \hat{\mathcal{W}}_n\right) + \frac{c}{8}\Delta t_n^2 \frac{\partial^3 \mathcal{W}}{\partial F^3}(F_*)(\nabla \dot{\varphi}_{n+1} + \nabla \dot{\varphi}_n)^3,
\end{aligned}$$

for a given constant $0 < c < 1/8$ and an unknown matrix F_*.

Concerning the compression term:

$$\begin{aligned}
\frac{1}{2}\left(p_n \ \mathit{cof}\ F_n + p_{n+1} \ \mathit{cof}\ F_{n+1}\right) : \nabla v \ &= \frac{1}{2\Delta t_n}\left(p_n \ \mathit{cof}\ F_n + p_{n+1} \ \mathit{cof}\ F_{n+1}\right) : (F_{n+1} - F_n), \\
&= \frac{1}{2\Delta t_n}\frac{p_n + p_{n+1}}{2}\left(\mathit{cof}\ F_n + \mathit{cof}\ F_{n+1}\right) : (F_{n+1} - F_n) \\
&\quad + \frac{1}{2\Delta t_n}\frac{p_{n+1} - p_n}{2}\left(\mathit{cof}\ F_{n+1} - \mathit{cof}\ F_n\right) : (F_{n+1} - F_n), \\
&= \frac{1}{\Delta t_n}\frac{p_n + p_{n+1}}{2}\left(\det F_{n+1} - \det F_n\right) \\
&\quad + c\frac{1}{\Delta t_n}\frac{p_n + p_{n+1}}{2}\frac{\partial^3 \det F}{\partial F^3}(F_{n+1} - F_n)^3 \\
&\quad + \frac{1}{2\Delta t_n}\frac{p_{n+1} - p_n}{2}\left(\mathit{cof}\ F_{n+1} - \mathit{cof}\ F_n\right) : (F_{n+1} - F_n),
\end{aligned}$$

with $0 < c < 1/8$. Moreover, if the initial kinematic constraint holds:

$$\int_\Omega q\,(\det \nabla\varphi_0 - 1 + \epsilon p_0) = 0, \quad \forall q \in \mathcal{P},$$

then it holds at every discrete time. Then, using (19):

$$\int_\Omega \frac{p_n + p_{n+1}}{2}\,(\det F_{n+1} - \det F_n) = \epsilon \int_\Omega \frac{p_n + p_{n+1}}{2}(p_{n+1} - p_n) = \frac{\epsilon}{2}\int_\Omega (p_{n+1}^2 - p_n^2).$$

Up to an integration over Ω, we have:

$$\begin{aligned}\frac{1}{2}\,(p_n\ \ cof\ F_n + p_{n+1}\ \ cof\ F_{n+1}) : \nabla v \quad &= \epsilon\frac{p_{n+1}^2 - p_n^2}{2\Delta t_n} \\ &+ \frac{c}{8}\Delta t_n^2 \frac{p_n + p_{n+1}}{2}\frac{\partial^3 \det F}{\partial F^3}\,(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})^3 \\ &+ \frac{\Delta t_n^2}{2}\frac{p_{n+1} - p_n}{2\Delta t_n}\frac{\partial^2 \det F}{\partial F^2}(F_*)(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})^2,\end{aligned}$$

and the announced result holds:

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \mathfrak{P}_n + c_n \Delta t_n^3,$$

with:

$$\begin{aligned}c_n &= \alpha\frac{\partial^3\mathcal{W}}{\partial F^3}(F_*)(\nabla\dot\varphi_{n+1} + \nabla\dot\varphi_n)^3 + \beta\frac{p_n + p_{n+1}}{2}\frac{\partial^3 \det F}{\partial F^3}\,(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})^3 \\ &- \frac{1}{4}\frac{p_{n+1} - p_n}{\Delta t_n}\frac{\partial^2 \det F}{\partial F^2}(F_*)(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})^2.\end{aligned}$$

2. **Angular momentum conservation.** If $\Gamma_0 = \emptyset$, taking $v = a \times \dfrac{\varphi_n + \varphi_{n+1}}{2} = \mathbb{J}_a \cdot \dfrac{\varphi_n + \varphi_{n+1}}{2}$ in (19), the elastic term becomes:

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial\hat{\mathcal{W}}}{\partial F}(F_n) + \frac{\partial\hat{\mathcal{W}}}{\partial F}(F_{n+1})\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a \\ &= \left(F_n \cdot \frac{\partial\mathcal{W}}{\partial C}(C_n) + F_{n+1}\cdot\frac{\partial\mathcal{W}}{\partial C}(C_{n+1})\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a, \\ &= \frac{1}{2}\,(F_n + F_{n+1})\cdot\left(\frac{\partial\mathcal{W}}{\partial C}(C_n) + \frac{\partial\mathcal{W}}{\partial C}(C_{n+1})\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a \\ &\quad + \frac{1}{2}\,(F_{n+1} - F_n)\cdot\left(\frac{\partial\mathcal{W}}{\partial C}(C_{n+1}) - \frac{\partial\mathcal{W}}{\partial C}(C_n)\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a.\end{aligned}$$

The first term vanishes because of the skew-symmetry of $\mathbb{J}_a$. Therefore, we get:

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial\hat{\mathcal{W}}}{\partial F}(F_n) + \frac{\partial\hat{\mathcal{W}}}{\partial F}(F_{n+1})\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a \\ &= \frac{1}{4}\Delta t_n\,(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})\cdot\left(\frac{\partial^2\mathcal{W}}{\partial C^2}(C_*) : (C_{n+1} - C_n)\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a, \\ &= \frac{1}{4}\Delta t_n\,(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})\cdot\left(\frac{\partial^2\mathcal{W}}{\partial C^2}(C_*) : (\nabla\varphi_*^t \cdot (F_{n+1} - F_n)\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a, \\ &= \frac{1}{8}\Delta t_n^2\,(\nabla\dot\varphi_n + \nabla\dot\varphi_{n+1})\cdot\left(\frac{\partial^2\mathcal{W}}{\partial C^2}(C_*) : \left(\nabla\varphi_*^t \cdot \nabla(\dot\varphi_n + \dot\varphi_{n+1})\right)\right)\cdot (F_n + F_{n+1})^t : \mathbb{J}_a.\end{aligned}$$

Concerning the momentum of compression terms, we have:

$$
\begin{aligned}
&\frac{1}{2}\left(p_n\, cof\ F_n + p_{n+1}\, cof\ F_{n+1}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a\\
&= \left(p_n F_n\cdot\frac{\partial \det C_n^{1/2}}{\partial C_n} + p_{n+1}F_{n+1}\cdot\frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a,\\
&= \frac{1}{2}\left(F_n+F_{n+1}\right)\cdot\left(p_n\frac{\partial \det C_n^{1/2}}{\partial C_n} + p_{n+1}\frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a\\
&\quad+\frac{1}{2}\left(F_{n+1}-F_n\right)\cdot\left(p_{n+1}\frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}} - p_n\frac{\partial \det C_n^{1/2}}{\partial C_n}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a,
\end{aligned}
$$

whose first term vanishes because of the skew-symmetry of $\mathbb{J}_a$. Therefore, we have simply:

$$
\begin{aligned}
&\frac{1}{2}\left(p_n\, cof\ F_n + p_{n+1}\, cof\ F_{n+1}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a\\
&= \frac{1}{4}\Delta t_n\left(\nabla\dot{\varphi}_n+\nabla\dot{\varphi}_{n+1}\right)\cdot\left(p_{n+1}\frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}} - p_n\frac{\partial \det C_n^{1/2}}{\partial C_n}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a.
\end{aligned}
$$

We detail the central factor:

$$
\begin{aligned}
&p_{n+1}\frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}} - p_n\frac{\partial \det C_n^{1/2}}{\partial C_n}\\
&= \frac{1}{2}(p_{n+1}-p_n)\left(\frac{\partial \det C_n^{1/2}}{\partial C_n} + \frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}}\right) + \frac{1}{2}(p_n+p_{n+1})\left(\frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}} - \frac{\partial \det C_n^{1/2}}{\partial C_n}\right),\\
&= \Delta t_n\ \frac{1}{2}\frac{p_{n+1}-p_n}{\Delta t_n}\left(\frac{\partial \det C_n^{1/2}}{\partial C_n} + \frac{\partial \det C_{n+1}^{1/2}}{\partial C_{n+1}}\right)\\
&\quad+\Delta t_n\ \frac{1}{4}(p_n+p_{n+1})\left(\frac{\partial^2 \det C^{1/2}}{\partial C^2}(C_*) : \left(\nabla\varphi_*^t\cdot\nabla(\dot{\varphi}_n+\dot{\varphi}_{n+1})\right)\right) = A_n\Delta t_n,
\end{aligned}
$$

where A_n is a fourth order symmetric tensor depending on the approximate solution at times n and $n+1$. Then, we have:

$$
\frac{1}{2}\left(p_n\, cof\ F_n + p_{n+1}\, cof\ F_{n+1}\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a = \frac{1}{4}\Delta t_n^2\left(\nabla\dot{\varphi}_n+\nabla\dot{\varphi}_{n+1}\right)\cdot A_n\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a.
$$

Let us introduce:

$$
c_n^a := \Delta t_n^2\left(\nabla\dot{\varphi}_n+\nabla\dot{\varphi}_{n+1}\right)\cdot\left(A_n+\frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial C^2}(C_*) : \left(\nabla\varphi_*^t\cdot\nabla(\dot{\varphi}_n+\dot{\varphi}_{n+1})\right)\right)\cdot\left(F_n+F_{n+1}\right)^t : \mathbb{J}_a,
$$

the discrete momentum of the tensions in the material between times t_n and t_{n+1} along the $a\in\mathbb{R}^3$ direction. By linearity $c_n^a = c_n\cdot a$, for a $c_n\in\mathbb{R}^3$. Plugging this in (19), and from the identity $u\cdot(a\times w) = (w\times u)\cdot a$, we obtain that:

$$
\int_\Omega \rho\frac{\varphi_n+\varphi_{n+1}}{2}\times\frac{\dot{\varphi}_{n+1}-\dot{\varphi}_n}{\Delta t_n} = \int_\Omega \frac{\varphi_n+\varphi_{n+1}}{2}\times\frac{f_n+f_{n+1}}{2}
$$

$$+\int_\Gamma \frac{\varphi_n+\varphi_{n+1}}{2}\times\frac{g_n+g_{n+1}}{2}-c_n. \tag{23}$$

From (19)-2, we have:

$$\int_\Omega \rho\frac{\varphi_{n+1}-\varphi_n}{\Delta t_n}\times\frac{\dot\varphi_n+\dot\varphi_{n+1}}{2}=0,$$

which from (23), implies:

$$\mathcal{J}_{n+1}-\mathcal{J}_n=\mathfrak{M}_n+c_n\Delta t_n^3.$$

3. **Linear momentum conservation.** If $\Gamma_0=\emptyset$, the result is straightforward by using any constant vector $v\in\mathbb{R}^3$ in (19).

□

Remark 3 *Some key ideas about the trapezoidal scheme:*

- *In the compressible case, exact energy conservation is only achieved if $\hat{\mathcal{W}}$ is a quadratic elastic potential as a function of F. It is easy to check that angular momentum is then also conserved. Nevertheless, this assumption is non realistic because incompatible with:*

$$\lim_{\det F\to 0}\hat{\mathcal{W}}(F)\to+\infty,$$

as shown in [Cia88].

- *A crucial argument in the proof of energy conservation is that if the initial kinematic constraint holds:*

$$\int_\Omega q\left(\det\nabla\varphi_0-1+\epsilon p_0\right)=0,\quad \forall q\in\mathcal{P},$$

then it holds at every discrete time.

- *Energy and angular momentum conservations are achieved with an error term $c_n\Delta t^3$, and the dependance of c_n with respect to the approximate solution is quite regular. Nevertheless, an accretive behaviour (local increase of energy or/and momentum) cannot be excluded for nonlinear problems with large time steps. It can entail numerical instability and a poor behaviour with respect to the group of rotations.*
- *The perfect behaviour of the scheme with respect to the group of translations is insured by the exact conservation of linear momentum.*

4.2.2 Midpoint scheme

With:

$$\begin{cases}\Pi_{n+1/2}=\dfrac{\partial\hat{\mathcal{W}}}{\partial F}\left(\dfrac{F_n+F_{n+1}}{2}\right)-\dfrac{p_n+p_{n+1}}{2}\,cof\left(\dfrac{F_n+F_{n+1}}{2}\right),\\ D_{n+1/2}=\det\left(\dfrac{F_n+F_{n+1}}{2}\right),\end{cases} \tag{24}$$

we prove:

Proposition 2 *The midpoint scheme achieves the following conservation properties:*

1. **Discrete energy conservation.** *For a constant time step* Δt,

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \mathfrak{P}_n + c_n \Delta t^3. \tag{25}$$

Here, c_n *depends on the approximate solution at times* n *and* $n+1$*, but also of the discrete third order time derivative of acceleration:* $\dddot{\varphi}_{n+1/2} = \frac{1}{2\Delta t^2}(\dot{\varphi}_{n+2} - \dot{\varphi}_{n+1} - \dot{\varphi}_n + \dot{\varphi}_{n+1})$.

2. **Discrete angular momentum conservation.** *If* $\Gamma_0 = \emptyset$,

$$\mathcal{J}_{n+1} - \mathcal{J}_n = \mathfrak{M}_n. \tag{26}$$

3. **Discrete linear momentum conservation.** *If* $\Gamma_0 = \emptyset$,

$$\mathcal{I}_{n+1} - \mathcal{I}_n = \mathfrak{F}_n. \tag{27}$$

Proof :

1. **Energy conservation.** We take $v = \frac{1}{2}(\dot{\varphi}_n + \dot{\varphi}_{n+1})$ in (19). The elastic and compression terms are the only one which differ from the trapezoidal case. For the elastic one, we have by a direct Taylor expansion:

$$\frac{\partial \hat{\mathcal{W}}}{\partial F}\left(\frac{F_n + F_{n+1}}{2}\right) : \nabla v = \frac{1}{\Delta t_n}\frac{\partial \hat{\mathcal{W}}}{\partial F}\left(\frac{F_n + F_{n+1}}{2}\right) : (F_{n+1} - F_n),$$

$$= \frac{\hat{\mathcal{W}}(F_{n+1}) - \hat{\mathcal{W}}(F_n)}{\Delta t_n} - \frac{c}{8}\Delta t_n^2 \frac{\partial^3 \hat{\mathcal{W}}}{\partial F^3}\left(\frac{F_n + F_{n+1}}{2}\right) \cdot (\nabla \dot{\varphi}_n + \nabla \dot{\varphi}_{n+1})^3.$$

The main difficulty comes from the kinematic constraint. We denote $\Box_{n+1/2} = \frac{1}{2}(\Box_n + \Box_{n+1})$ and $\frac{1}{2}(\dot{\Box}_n + \dot{\Box}_{n+1}) = \frac{1}{\Delta t_n}(\Box_{n+1} - \Box_n)$. We assume that the step time is a constant Δt, and introduce the interpolated displacement:

$$\overline{\varphi}_{n+1/2} = \frac{1}{2}(\varphi_{n+3/2} + \varphi_{n-1/2}) = \frac{1}{4}(\varphi_{n+2} + \varphi_{n+1} + \varphi_n + \varphi_{n-1}).$$

Then:

$$\begin{aligned} \overline{\varphi}_{n+1/2} - \varphi_{n+1/2} &= \frac{1}{4}(\varphi_{n+2} - \varphi_{n+1} - \varphi_n + \varphi_{n-1}) \\ &= \frac{\Delta t}{8}(\dot{\varphi}_{n+2} + \dot{\varphi}_{n+1} - \dot{\varphi}_n - \dot{\varphi}_{n-1}) \\ &= \frac{\Delta t^2}{4}(\ddot{\varphi}_{n+1} + \ddot{\varphi}_n), \end{aligned}$$

with $\ddot{\varphi}_n = \frac{\dot{\varphi}_{n+1} - \dot{\varphi}_{n-1}}{2\Delta t}$. The increase of displacement is defined by:

$$\begin{aligned} \delta &= \varphi_{n+3/2} - \varphi_{n-1/2} = \frac{1}{2}(\varphi_{n+2} + \varphi_{n+1} - \varphi_n - \varphi_{n-1}) \\ &= \frac{1}{2}(\dot{\varphi}_{n+3/2}\Delta t + \dot{\varphi}_{n-1/2}\Delta t + 2\varphi_{n+1} - 2\varphi_n) \\ &= 2\dot{\varphi}_{n+1/2}\Delta t + \frac{1}{2}\dddot{\varphi}_{n+1/2}\Delta t^3. \end{aligned} \tag{28}$$

From the kinematic constraint at half step time, we deduce:

$$\begin{aligned}\det F_{n+3/2} - \det F_{n-1/2} &= \left(cof\ \nabla\overline{\varphi}_{n+1/2}\right) : (\nabla\delta) + \frac{1}{24}\frac{\partial^2\, cof\ F}{\partial F^2}(F_*)(\nabla\delta)^3 \\ &= -\epsilon\left(p_{n+3/2} - p_{n-1/2}\right). \end{aligned} \tag{29}$$

The work of pressure forces is therefore:

$$\begin{aligned}&p_{n+1/2}\ cof\ F_{n+1/2} : \nabla\dot{\varphi}_{n+1/2} = \\ &p_{n+1/2}\left(cof\ \nabla\overline{\varphi}_{n+1/2} + \frac{\partial\, cof\ F}{\partial F}(F_{**}) : \left(F_{n+1/2} - \nabla\overline{\varphi}_{n+1/2}\right)\right) : \nabla\dot{\varphi}_{n+1/2} = \\ &p_{n+1/2}\left(cof\ \nabla\overline{\varphi}_{n+1/2} - \frac{\partial\, cof\ F}{\partial F}(F_{**}) : \left(\frac{1}{4}(\nabla\ddot{\varphi}_{n+1} + \nabla\ddot{\varphi}_n)\Delta t^2\right)\right) : \nabla\dot{\varphi}_{n+1/2}.\end{aligned}$$

Since $\dot{\varphi}_{n+1/2} = \dfrac{\delta}{2\Delta t} - \dfrac{\Delta t^2}{4}\dddot{\varphi}_{n+1/2}$, we have:

$$\begin{aligned}p_{n+1/2}\ cof\ F_{n+1/2} : \nabla\dot{\varphi}_{n+1/2} = \ & \frac{1}{2\Delta t}p_{n+1/2}\left(cof\ \nabla\overline{\varphi}_{n+1/2}\right) : (\nabla\delta) \\ & -\frac{\Delta t^2}{4}p_{n+1/2}\left(cof\ \nabla\overline{\varphi}_{n+1/2}\right) : \nabla\dddot{\varphi}_{n+1/2} \\ & -\frac{\Delta t^2}{4}\frac{\partial\, cof\ F}{\partial F}(F_{**}) : (\nabla\ddot{\varphi}_{n+1} + \nabla\ddot{\varphi}_n) : \nabla\dot{\varphi}_{n+1/2}.\end{aligned}$$

The two last terms are of order 2 in Δt. To tackle the first one, we use (29) and up to a second order term in Δt, we get:

$$\begin{aligned}p_{n+1/2}\ cof\ F_{n+1/2} : \nabla\dot{\varphi}_{n+1/2} = \ & -\frac{\Delta t^2}{48}p_{n+1/2}\frac{\partial^2\, cof\ F}{\partial F^2}(F_*)(\nabla\frac{\delta}{\Delta t})^3 \\ & -\frac{\epsilon}{2\Delta t}\ p_{n+1/2}\left(p_{n+3/2} - p_{n-1/2}\right).\end{aligned}$$

By rewriting (28) for the quantity $p_{n+3/2} - p_{n-1/2}$, we have:

$$p_{n+1/2}\ cof\ F_{n+1/2} : \nabla\dot{\varphi}_{n+1/2} = \frac{\epsilon}{2\Delta t}(p_{n+1}^2 - p_n^2),$$

up to a second order term in Δt.

2. **Angular momentum conservation.** Assuming that $\Gamma_0 = \emptyset$, we use $v = a \times \dfrac{\varphi_n + \varphi_{n+1}}{2} = \mathbb{J}_a \cdot \dfrac{\varphi_n + \varphi_{n+1}}{2}$ in (19). In the elastic part, we obtain the terms:

$$2\left(\frac{F_n + F_{n+1}}{2}\right)\cdot\frac{\partial\mathcal{W}}{\partial C}(C^*_{n+1/2})\cdot\left(\frac{F_n + F_{n+1}}{2}\right)^t : \mathbb{J}_a = 0,$$

and:

$$2\left(\frac{p_n + p_{n+1}}{2}\right)\left(\frac{F_n + F_{n+1}}{2}\right)\cdot\frac{\partial \det C^{1/2}}{\partial C}(C^*_{n+1/2})\cdot\left(\frac{F_n + F_{n+1}}{2}\right)^t : \mathbb{J}_a = 0,$$

that vanish because of the skew-symmetry of $\mathbb{J}_a$. The result proceeds then as in (23) but with $c_n = 0$.

3. **Linear momentum conservation.** Direct derivation as in the trapezoidal case.

□

Remark 4 *Some key points about the midpoint scheme:*

- *This scheme is known to be symplectic in the compressible framework (see [HLW02, SSC94, Gon96]). For small time steps, backward analysis shows the conservation of a discrete energy, close to the physical one up to a $O(\Delta t^2)$ term. Nevertheless, in practice, the desired time step may prove to be not sufficiently small to insure such a property. Moreover, symplecticity is hard to be obtained for constrained problems (see [SG93]); in particular, it is lost in the incompressible framework.*
- *For compressible materials with (irrealistic) quadratic elastic potential $\hat{\mathcal{W}}$, energy would be exactly conserved.*
- *The kinematic constraint at midpoint has bad consequences on energy conservation. In particular, it requires a very high regularity in time.*
- *Angular and linear momenta are exactly conserved, which is a proof of a perfect behaviour of the scheme with respect to rotations and translations groups.*

4.2.3 Exactly conservative schemes

At this stage, some remarks need to be done:

- The better conservation of energy achieved by the trapezoidal rule in comparison with the midpoint scheme, is due to the imposition of the kinematic constraint at time steps, and not at mid time steps. In (19), it is then natural to adopt:
$$D_{n+1/2} = \frac{1}{2}\left(\det F_n + \det F_{n+1}\right).$$
- The exact conservation of momenta performed by the midpoint scheme is due to the natural form of the first algorithmic stress tensor:
$$\Pi_{n+1/2} = \left(\frac{F_n + F_{n+1}}{2}\right) \cdot \Sigma_{n+1/2},$$
with a symmetric second stress tensor $\Sigma_{n+1/2}$.
- Then, it is straightforward to check that exact energy conservation is achieved iff we can satisfy:
$$\frac{1}{2}\int_\Omega \Sigma_{n+1/2} : (C_{n+1} - C_n) = \int_\Omega \left(\mathcal{W}(C_{n+1}) + \frac{\epsilon}{2}p_{n+1}^2\right) - \int_\Omega \left(\mathcal{W}(C_n) + \frac{\epsilon}{2}p_n^2\right). \tag{30}$$

Remark 5 *After finite element discretization, the integral over Ω in (30), could be replaced by the following relation:*
$$\frac{1}{2}\Sigma_{n+1/2} : (C_{n+1} - C_n) = \left(\mathcal{W}(C_{n+1}) + \frac{1}{2\epsilon}\mathbb{P}_{\mathcal{P}}\left(\det F_{n+1} - 1\right)^2\right)$$

$$
-\left(\mathcal{W}(C_n)+\frac{1}{2\epsilon}\mathbb{P}_{\mathcal{P}}\left(\det F_n-1\right)^2\right),
$$

where $\mathbb{P}_{\mathcal{P}}$ denotes the projection from $L^2(\Omega)$ to $\mathcal{P}$. It can be treated numerically by a subintegration technique.

A major goal is then to construct such a tensor $\Sigma_{n+1/2}$ satisfying (30). Following the idea of Simo and Tarnow in [ST92], extended to the quasi incompressible case, it is easy to prove by a mean value argument that there always exist scalar functions β_n and γ_n such that:

$$
\Sigma_{n+1/2} := \left(\frac{\partial\mathcal{W}}{\partial C}(\beta_n C_n+(1-\beta_n)C_{n+1})+\frac{\partial\mathcal{W}}{\partial C}((1-\beta_n)C_n+\beta_n C_{n+1})\right)
$$

$$
-\frac{p_n+p_{n+1}}{2}\left(\frac{\partial\det C^{1/2}}{\partial C}(\gamma_n C_n+(1-\gamma_n)C_{n+1})+\frac{\partial\det C^{1/2}}{\partial C}((1-\gamma_n)C_n+\gamma_n C_{n+1})\right),
$$

satisfies (30). The numerical difficulty of determining β_n and γ_n at each time step and at each Gauss point can be overcome by the proposal of Gonzalez in [Gon00], which proposes to compute the stress at mid interval by:

$$
\begin{aligned}
\Sigma_{n+1/2} \quad &:= 2\frac{\partial\mathcal{W}}{\partial C}(C_{n+1/2})+2\left(\mathcal{W}(C_{n+1})-\mathcal{W}(C_n)-\frac{\partial\mathcal{W}}{\partial C}(C_{n+1/2}):\delta C_n\right)\frac{\delta C_n}{\delta C_n:\delta C_n}\\
&-(p_n+p_{n+1})\left[\frac{\partial\det C^{1/2}}{\partial C}(C_{n+1/2})+\right.\\
&\left.+\left(\det C_{n+1}^{1/2}-\det C_n^{1/2}-\frac{\partial\det C^{1/2}}{\partial C}(C_{n+1/2}):\delta C_n\right)\frac{\delta C_n}{\delta C_n:\delta C_n}\right],
\end{aligned}
\tag{31}
$$

with $C_{n+1/2}=\frac{1}{2}(C_n+C_{n+1})$, and $\delta C_n=C_{n+1}-C_n$. By construction, the resulting scheme satisfies (30) and therefore conserves energy, angular and linear momentum.

Remark 6 *In the compressible framework with a quadratic elastic potential $\mathcal{W}$ (e.g. Saint Venant-Kirchhoff material), energy conservation is achieved with:*

$$
\Sigma_{n+1/2}=\frac{\partial\mathcal{W}}{\partial C}(C_n)+\frac{\partial\mathcal{W}}{\partial C}(C_{n+1}).
$$

Remark 7 *In a Newton method, nonlinear problems are solved by successive linearizations. Here, the linearized time integrator is non symmetric, which is a noticeable complication in numerical methods. An interesting idea, already mentionned in [AR01b], is to replace the linearized operator of the integration scheme by the linearized operator of the midpoint scheme, which is symmetric. Obviously, the disadvantage is a non quadratic convergence of the Newton method, but the practical overcost is negligeable.*

Remark 8 *When (5) is linearized around the undeformed configuration, the schemes presented in this section are reduced to the classical Newmark's trapezoidal rule, performing exact energy and linear momentum conservations. Because of the linearization, the angular momentum is no more conserved, neither for the continuous solution, nor for the approximate one. The spectral radius of the time integration operator is equal to one.*

4.3 Dissipative schemes

4.3.1 Preliminaries

In linear elastodynamics, it is necessary for stability reasons to use schemes whose spectral radius is $r \leq 1$. Moreover, when $r < 1$:

1. possible polynomial instabilities are avoided (arising when $r = 1$ in presence of a multiple unit eigenvalue),
2. information is dissipated at highest frequencies, that has no physical meaning,
3. the condition number of the linear systems to be solved is improved,
4. there exist a quadratic form whose value diminishes along the discrete evolution.

A good example is the popular second order Hilber-Hughes-Taylor (HHT) scheme. We present here a nonlinear analysis of this scheme, showing that the above advantages are no more conserved in a nonlinear framework.

4.3.2 HHT scheme

For a given $\alpha \geq 0$, the natural extension of the HHT scheme [HHT77, Hug87] to nonlinear elastodynamics is given by:

$$\begin{cases} \displaystyle\int_\Omega \rho\ddot{\varphi}_{n+1}\cdot v + \int_\Omega (\alpha\Pi_n + (1-\alpha)\Pi_{n+1}) : \nabla v = \int_\Omega f_{n+1-\alpha}\cdot v + \int_{\Gamma_1} g_{n+1-\alpha}\cdot v, \quad \forall v \in \mathcal{U}_0, \\ \displaystyle\int_\Omega q\,(\det F_{n+1} - 1) = 0, \quad \forall q \in \mathcal{P}, \end{cases} \tag{32}$$

with Newmark's relations:

$$\begin{cases} \varphi_{n+1} = \varphi_n + \Delta t_n\dot{\varphi}_n + \Delta t_n^2\left(\left(\frac{1}{2}-\beta\right)\ddot{\varphi}_n + \beta\ddot{\varphi}_{n+1}\right), \\ \dot{\varphi}_{n+1} = \dot{\varphi}_n + \Delta t_n\left((1-\gamma)\ddot{\varphi}_n + \gamma\ddot{\varphi}_{n+1}\right). \end{cases}$$

We have denoted $\gamma = \frac{1}{2}+\alpha$, and $\beta = \frac{(1+\alpha)^2}{4}$. The notation $\square_{n+1-\alpha}$ classically stands for $\alpha\square_n + (1-\alpha)\square_{n+1}$.

Remark 9 *In the linear case, we recall that [HHT77, Hug87]:*

- *HHT scheme is the natural modification of the Newmark's scheme combining spectral dissipation and second order accuracy.*
- *The choice of $\gamma = \frac{1}{2}+\alpha$ insures second order accuracy.*
- *For the Newmark's case (see [RT98]), the choice*

$$\frac{1+2\alpha}{4} \leq \beta \leq \frac{(1+\alpha)^2}{4},$$

corresponds to real eigenvalues for the integration operator, whereas:

$$\beta \geq \frac{(1+\alpha)^2}{4},$$

corresponds to complex conjugate eigenvalues.

- *The stability imposes $0 \leq \alpha \leq \frac{1}{2}$, with $\alpha = 0$ corresponding to the trapezoidal rule. The spectral radius decreases for $0 \leq \alpha \leq \frac{1}{3}$ and increases for $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$.*

In the nonlinear framework, we prove:

Proposition 3 *We assume the time step to be constant. Then, the nonlinear HHT scheme (32) achieves the following conservation properties:*

1. **Discrete energy conservation.** *Up to higher order terms depending only of time variations in force, we have:*

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \mathfrak{P}_n - \mathfrak{D}_n^\alpha \Delta t^3 + c_n \Delta t^3, \tag{33}$$

where c_n is defined as for the trapezoidal rule and depends on displacements and pressures at times n and $n+1$, on the approximate velocity $V_{n+1/2} = \frac{1}{\Delta t}(\varphi_{n+1} - \varphi_n)$, and on the approximate pressure time derivative $\pi_{n+1/2} = \frac{1}{\Delta t}(p_{n+1} - p_n)$. The coefficient $\mathfrak{D}_n^\alpha$ has the following expression:

$$\begin{aligned} \mathfrak{D}_n^\alpha &= \frac{\alpha^2}{8} \int_\Omega \rho \left(\ddot{\varphi}_{n+1}^2 - \ddot{\varphi}_n^2\right) + \frac{\alpha^3}{4} \int_\Omega \left(\ddot{\varphi}_{n+1} - \ddot{\varphi}_n\right)^2 \\ &\quad + k \int_\Omega \ddot{\Pi}_n : (\nabla V_{n+1/2}) + k \int_\Omega \frac{\partial^2 f}{\partial t^2}(t_n) \cdot V_{n+1/2}, \end{aligned}$$

where $\ddot{\Pi}_n$ is the centered second order finite difference:

$$\ddot{\Pi}_n = \frac{1}{\Delta t^2} \left(\Pi_{n-1} - 2\Pi_n + \Pi_{n+1}\right).$$

2. **Discrete angular momentum.** *Up to higher order terms, we have:*

$$\mathcal{J}_{n+1} - \mathcal{J}_n = \mathfrak{M}_n + c_n \Delta t^3, \tag{34}$$

where c_n depends on the approximate solution at times $n-1$, n and $n+1$, on the accelerations $\ddot{\varphi}_{n-1}$, $\ddot{\varphi}_n$ et $\ddot{\varphi}_{n+1}$, and on the approximate second order time derivatives $\frac{1}{\Delta t}(\pi_{n+1/2} - \pi_{n-1/2})$.

3. **Discrete linear momentum.** *Up to higher order terms depending only of the time variations in force, we have:*

$$\mathcal{I}_{n+1} - \mathcal{I}_n = \mathfrak{F}_n + c_n \Delta t^3, \tag{35}$$

where c_n only depends on the second order time derivative of f.

Proof : A linear combination of the discrete systems at times n and $n+1$, with respective coefficients $(1-\gamma) = \frac{1}{2} - \alpha$ and $\gamma = \frac{1}{2} + \alpha$ gives:

$$\begin{aligned} &\int_\Omega \rho \frac{\dot{\varphi}_{n+1} - \dot{\varphi}_n}{\Delta t_n} \cdot v + \int_\Omega \underbrace{k \left(\Pi_{n-1} - 2\Pi_n + \Pi_{n+1}\right)}_{\aleph_n} : \nabla v + \int_\Omega \frac{\Pi_n + \Pi_{n+1}}{2} : \nabla v = \\ &\int_\Omega \frac{f_n + f_{n+1}}{2} \cdot v + \int_\Omega \underbrace{k \left(f_{n-1} - 2f_n + f_{n+1}\right)}_{\mathfrak{f}_n} \cdot v, \end{aligned} \tag{36}$$

with coefficient $k = \alpha(\frac{1}{2} - \alpha) > 0$ for $0 \leq \alpha \leq \frac{1}{2}$. With $\gamma = \frac{1}{2} + \alpha$ and $\beta = \frac{(1+\alpha)^2}{4}$, Newmark's relations are:

$$\begin{cases} \dot{\varphi}_{n+1} - \dot{\varphi}_n = \Delta t_n \left((\frac{1}{2} - \alpha)\ddot{\varphi}_n + (\frac{1}{2} + \alpha)\ddot{\varphi}_{n+1} \right), \\ \varphi_{n+1} - \varphi_n = \Delta t_n \dfrac{\dot{\varphi}_n + \dot{\varphi}_{n+1}}{2} + \dfrac{\alpha^2 \Delta t_n^2}{4} \left(\ddot{\varphi}_{n+1} - \ddot{\varphi}_n \right). \end{cases} \tag{37}$$

The form (36),(37) of the scheme adds to the trapezoidal rule some "correction terms". We only detail these additional contributions.

1. **Energy conservation.** By using $v = (\varphi_{n+1} - \varphi_n)/\Delta t_n$ in (36), and the relations (37), the inertial term takes the form:

$$\begin{aligned} \int_\Omega \rho \frac{\dot{\varphi}_{n+1} - \dot{\varphi}_n}{\Delta t_n} \cdot v \quad &= \int_\Omega \rho \frac{\dot{\varphi}_{n+1} - \dot{\varphi}_n}{\Delta t_n} \cdot \frac{\dot{\varphi}_n + \dot{\varphi}_{n+1}}{2} \\ &\quad + \Delta t_n \frac{\alpha^2}{4} \int_\Omega \rho \left((\frac{1}{2} - \alpha)\ddot{\varphi}_n + (\frac{1}{2} + \alpha)\ddot{\varphi}_{n+1} \right) \cdot \left(\ddot{\varphi}_{n+1} - \ddot{\varphi}_n \right), \\ &= \frac{1}{2\Delta t_n} \int_\Omega \rho \dot{\varphi}_{n+1}^2 - \frac{1}{2\Delta t_n} \int_\Omega \rho \dot{\varphi}_n^2 \\ &\quad + \Delta t_n \frac{\alpha^2}{8} \int_\Omega \rho \left(\ddot{\varphi}_{n+1}^2 - \ddot{\varphi}_n^2 \right) + \Delta t_n \frac{\alpha^3}{4} \int_\Omega \left(\ddot{\varphi}_{n+1} - \ddot{\varphi}_n \right)^2. \end{aligned}$$

The non trapezoidal contribution to the stress terms is:

$$\int_\Omega \aleph_n : \nabla v = k \Delta t^2 \int_\Omega \ddot{\Pi}_n : (\nabla V_{n+1/2}),$$

defining the second order accurate finite difference approximations of the stress acceleration $\ddot{\Pi}_n$, and of the velocity $V_{n+1/2}$ by

$$\ddot{\Pi}_n = \frac{1}{\Delta t^2} \left(\Pi_{n-1} - 2\Pi_n + \Pi_{n+1} \right),$$

$$V_{n+1/2} = \frac{1}{\Delta t} (\varphi_{n+1} - \varphi_n).$$

Concerning corrections on the force term, we have

$$\int_\Omega \mathfrak{f}_n \cdot v = k \Delta t^2 \int_\Omega \frac{\partial^2 f}{\partial t^2}(t_n) \cdot V_{n+1/2} + O(\Delta t^3).$$

As a consequence, up to higher orders in Δt concerning only the variations of f, we obtain:

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \mathfrak{P}_n - \mathfrak{D}_n^\alpha \Delta t_n^3 + c_n \Delta t_n^3,$$

with:

$$\begin{aligned} \mathfrak{D}_n^\alpha \quad &= \frac{\alpha^2}{8} \int_\Omega \rho \left(\ddot{\varphi}_{n+1}^2 - \ddot{\varphi}_n^2 \right) + \frac{\alpha^3}{4} \int_\Omega \left(\ddot{\varphi}_{n+1} - \ddot{\varphi}_n \right)^2 \\ &\quad + k \int_\Omega \ddot{\Pi}_n : (\nabla V_{n+1/2}) + k \int_\Omega \frac{\partial^2 f}{\partial t^2}(t_n) \cdot V_{n+1/2}. \end{aligned}$$

2. Angular momentum conservation. Assuming that $\Gamma_0 = \emptyset$, we use $v = \mathbb{J}_a \cdot \frac{\varphi_n + \varphi_{n+1}}{2}$. The non trapezoidal contribution of stresses is:

$$\int_\Omega \aleph_n \cdot \left(\frac{F_n + F_{n+1}}{2}\right)^t : \mathbb{J}_a = \underbrace{\int_\Omega \aleph_n \cdot \left(\frac{F_{n+1} + F_{n-1}}{2}\right)^t : \mathbb{J}_a}_{I_1} + \underbrace{\frac{\Delta t}{2} \int_\Omega \aleph_n \cdot \left(\nabla V_{n-1/2}\right)^t : \mathbb{J}_a}_{I_2}.$$

We decompose $\aleph_n$ by writing:

$$\begin{aligned}
\aleph_n &= \left(\frac{F_{n+1} + F_{n-1}}{2}\right) \cdot (\Sigma_{n-1} - 2\Sigma_n + \Sigma_{n+1}) \\
&\quad + \left(\frac{F_{n+1} - F_{n-1}}{2}\right) \cdot (\Sigma_{n+1} - \Sigma_{n-1}) + (F_{n+1} - 2F_n + F_{n-1}) \cdot \Sigma_n, \\
&= \left(\frac{F_{n+1} + F_{n-1}}{2}\right) \cdot (\Sigma_{n-1} - 2\Sigma_n + \Sigma_{n+1}) \\
&\quad + \frac{\Delta t}{2} \left(\nabla V_{n+1/2} + \nabla V_{n-1/2}\right) \cdot (\Sigma_{n+1} - \Sigma_{n-1}) + \Delta t \left(\nabla V_{n+1/2} - \nabla V_{n-1/2}\right) \cdot \Sigma_n.
\end{aligned} \tag{38}$$

We denote the centered finite differente approximations of stress acceleration by:

$$\ddot{\Sigma}_n = \frac{1}{\Delta t^2}(\Sigma_{n-1} - 2\Sigma_n + \Sigma_{n+1}),$$

and of stress speed by:

$$\dot{\Sigma}_n = \frac{1}{2\Delta t}(\Sigma_{n+1} - \Sigma_{n-1}).$$

Considering Newmark's relations, we have:

$$\begin{aligned}
V_{n+1/2} - V_{n-1/2} = \quad & \Delta t \left((\frac{1}{2} - \alpha)\ddot{\varphi}_{n-1} + \ddot{\varphi}_n + (\frac{1}{2} + \alpha)\ddot{\varphi}_{n+1} \right) \\
& + \Delta t^2 \frac{\alpha^2}{4} \left(\ddot{\varphi}_{n+1} - 2\ddot{\varphi}_n + \ddot{\varphi}_{n-1}\right).
\end{aligned}$$

Then, we can rewrite $\aleph_n$ as:

$$\aleph_n = \overbrace{k\Delta t^2 \left(\frac{F_{n+1} + F_{n-1}}{2}\right) \cdot \ddot{\Sigma}_n}^{\aleph_n^1}$$

$$\underbrace{+ k\Delta t^2 \left(\nabla V_{n+1/2} + \nabla V_{n-1/2}\right) \cdot \dot{\Sigma}_n + k\Delta t^2 \left(\nabla \left(V_{n+1/2} - V_{n-1/2}\right) / \Delta t\right) \cdot \Sigma_n}_{\aleph_n^2}.$$

By skew-symmetry of $\mathbb{J}_a$, $\aleph_n^1$ has a vanishing contribution in I_1, whereas $\aleph_n^2$ has a second order contribution in I_1. Contributions of $\aleph_n$ in I_2 is at third order in Δt.

Concerning the additional resultant moment, up to a third order term:

$$\int_\Omega \mathfrak{f}_n \cdot v = k\Delta t^2 \int_\Omega \frac{\varphi_n + \varphi_{n+1}}{2} \times \frac{\partial^2 f}{\partial t^2}(t_n).$$

3. **Linear momentum conservation.** Assuming that $\Gamma_0 = \emptyset$, the HHT scheme entails a second order error in the discrete resultant force, given by:
$$\int_\Omega \mathfrak{f}_n \simeq \Delta t^2 \int_\Omega \frac{\partial^2 f}{\partial t^2}(t_n),$$
up to a third order term.

□

Remark 10 *Considering the linearized problem, we assume that $\hat{\mathcal{W}}$ is quadratic as a function of F, and that the incompressibility constraint is linearized. Then:*
$$\int_\Omega \Pi_n : \nabla v = \int_\Omega \frac{\partial \hat{\mathcal{W}}}{\partial F} : \nabla v - p\, div\, v, \quad \forall v \in \mathcal{U}_0.$$
If $f = 0$, we have $c_n = 0$ (i.e. the trapezoidal rule is energy conserving). Moreover, since:
$$\begin{aligned} \varphi_{n+1} - \varphi_n = \quad & \tfrac{1}{2}(\varphi_{n+1} - \varphi_n) + \tfrac{1}{2}(\varphi_n - \varphi_{n-1}) \\ & + \tfrac{1}{2}(\varphi_{n+1} - \varphi_n) - \tfrac{1}{2}(\varphi_n - \varphi_{n-1}), \end{aligned} \tag{39}$$
we have up to an integration over Ω:
$$\begin{aligned} \Delta t^3\, \ddot{\Pi}_n : \nabla V_{n+1/2} \quad & = \hat{\mathcal{W}}(F_{n+1} - F_n) - \hat{\mathcal{W}}(F_n - F_{n-1}) \\ & + \frac{\epsilon}{2}(p_{n+1} - p_n)^2 - \frac{\epsilon}{2}(p_n - p_{n-1})^2 \\ & + \hat{\mathcal{W}}(F_{n+1} - 2F_n + F_{n-1}) \\ & + \frac{\epsilon}{2}(p_{n+1} - 2p_n + p_{n-1})^2. \end{aligned}$$
Then, the following quadratic form:
$$\begin{aligned} \mathcal{E}_n^{HHT} & = \frac{1}{2}\int_\Omega \rho \dot{\varphi}_n^2 + \int_\Omega \hat{\mathcal{W}}(F_n) + \frac{\epsilon}{2}\int_\Omega p_n^2 \\ & + \frac{\alpha^2 \Delta t^3}{8}\int_\Omega \rho \ddot{\varphi}_{n+1}^2 + k\Delta t^2 \hat{\mathcal{W}}\left(\dot{F}_{n-1/2}\right) + \frac{k\epsilon}{2}\Delta t^2 (\dot{p}_{n-1/2})^2, \end{aligned}$$
with $\dot{F}_{n-1/2} = (F_n - F_{n-1})/\Delta t$ and $\dot{p}_{n-1/2} = (p_n - p_{n-1})/\Delta t$, diminishes along the dynamics. More precisely:
$$\begin{aligned} \mathcal{E}_{n+1}^{HHT} - \mathcal{E}_n^{HHT} & = -\frac{\alpha^3 \Delta t^3}{4}\int_\Omega (\ddot{\varphi}_{n+1} - \ddot{\varphi}_n)^2 \\ & - k\Delta t^4 \int_\Omega \hat{\mathcal{W}}\left(\ddot{F}_n\right) - \frac{k\epsilon}{2}\Delta t^4 (\ddot{p}_n)^2 \leq 0. \end{aligned}$$
Therefore, for linear elastodynamics, there exist a quadratic form $\mathcal{E}_n^{HHT}$ diminishing along the HHT discrete evolution. This comes from the fact that the spectral radius of the time integrator is less than one. Nevertheless, this quadratic form does not coincide with the usual mechanical energy. It introduces acceleration terms in the energy and high order terms in time in the dissipation, which are larger for larger frequencies.

Remark 11
- *It is straightforward to check in the expression of $\mathfrak{D}_n^\alpha$ that dissipation is not unconditional. In particular, energy is dissipated by the non trapezoidal terms during the acceleration phases of the dynamics when $0 < \alpha \leq \frac{1}{2}$. Nevertheless, the scheme becomes accretive when the dynamics slows down. This remark will be highlighted numerically in the following section.*
- *If $\beta = \frac{1+2\alpha}{4}$, it is straightforward to check in the proof that the two first inertial terms in $\mathfrak{D}_n^\alpha$ disappear. Then, $\mathfrak{D}_n^\alpha$ is proportional to $\alpha(\frac{1}{2} - \alpha)$ and maximal absolute values of $\mathfrak{D}_n^\alpha$ are obtained for $\alpha = \frac{1}{4}$.*
- *Groups of symmetry are not well preserved by the discrete dynamics as momenta conservation is not well insured. This remark confirms the work of Armero and Romero in [AR01a]; they prove the non existence of relative equilibria for the HHT discrete dynamics in the case of a nonlinear spring-mass system.*

4.3.3 Energy dissipation

Performing energy dissipation in the nonlinear framework is not as easy as in the linear case. The example of the HHT scheme is quite relevant from this point of view. The work of Armero and Romero in [AR01a, AR01b] about energy dissipation in the nonlinear case is a good example of such a difficulty. One major comment is that they introduce dissipation in the Gonzalez scheme [Gon00], resulting in a very complex nonlinear scheme. It tends to show that dissipating energy is more complicate than conserving it.

5 Numerical experiment

In this section, we illustrate numerically the previous analysis on the case of a bidimensional quasi incompressible cantilever beam in plane displacements. The elastic potential is given by the Mooney-Rivlin constitutive law:

$$\mathcal{W}(C) = c_1\,(tr\,C - 3) + c_2\,(tr\,cof\;C - 3)\,.$$

Data is presented on figure 3. On the mesh presented on figure 4, the discrete spaces Q_1 and Q_0 for displacements and pressures respectively are adopted. They are proved to be compatible in the selected case, as shown in [Tal81].

Figure 2: A cantilever beam with constant force F.

length	1 m
width	0.1 m
F	1000 N
ρ	1000 kg/m3
c_1	2 MPa
c_2	0.2 MPa
$1/\epsilon$	2.E12 Pa
T	10 s
Δt	0.02 s
Newton's tolerance	1E-7

Figure 3: Physical data and numerical choices.

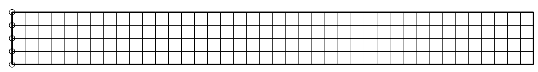

Figure 4: A 250 elements mesh of the beam.

As F is a constant force, we ideally expect to observe the conservation of the following discrete quantity:

$$H_n = \int_\Omega f \cdot \varphi_n - \mathcal{E}_n.$$

The constant time step Δt is chosen so that to have approximatively 20 time steps per oscillation of the cantilever beam. With this value of Δt, 4 or 5 Newton's iterations per time step are necessary to solve the problem with the required accuracy (10^{-7} m). Due to numerical instabilities, the number of Newton's iterations per time step can grow up; the simulation is stopped when it exceeds 20.

Our first observation is that when H_n decreases (global increase of the energy of the system), the number of Newton's iterations per time step grows up until the method does not converge any more. As a consequence, trapezoidal, midpoint and HHT [HHT77] schemes cannot complete the simulation on the whole time interval $[0, T]$ for the specified parameters. Only the conservative Gonzalez scheme can achieve long term time integration without such an overcost.

Energy evolution is presented on figure 6. The worst energy conservation holds for the midpoint scheme, as shown in the previous analysis. With the selected parameters, HHT is globally energy growing. Gonzalez scheme is quasi exactly conservative up to a very small error term depending on Newton's tolerance.

Concerning midpoint and trapezoidal schemes, the energy growth goes with numerical instabilities on velocities, as shown on figure 7.

To highlight the theoretical analysis done for the HHT scheme, figure 8 shows a zoom of energy evolution during one second of the dynamics, with displacements. It is worth noticing that in conformity with the analysis of (33), energy is dissipated

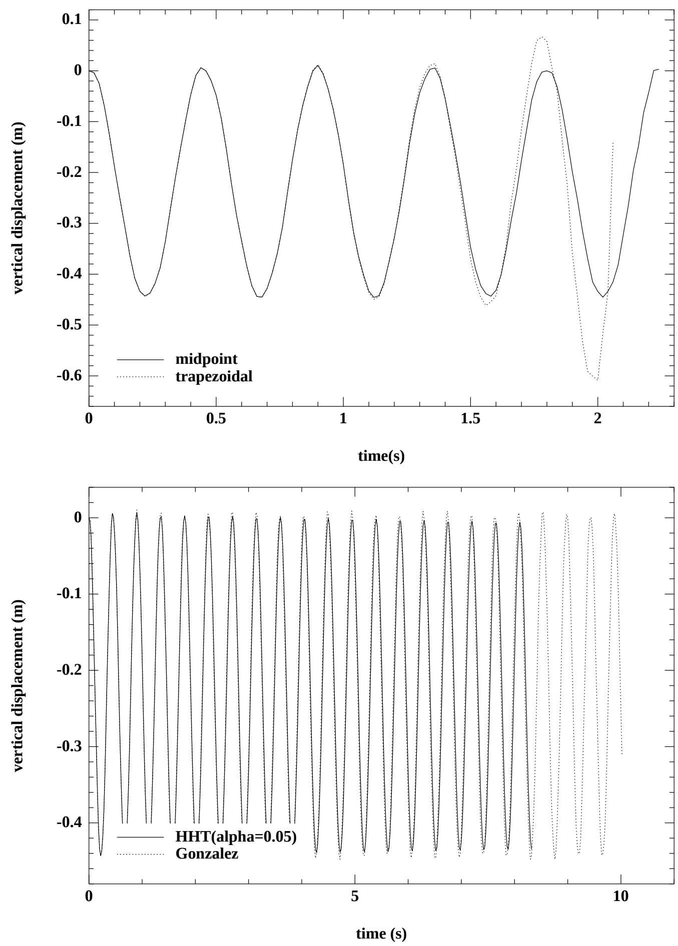

Figure 5: Vertical displacement of the tip of the cantilever beam. Simulation stops when the time step calculation exceeds 20 Newton's iterations.

when the beam goes up or down (acceleration of the dynamics) and grows when the deformation is in a neighbourhood of the minimum or the maximum.

Moreover, concerning HHT scheme, it is difficult to adopt the right value of the dissipation parameter α. We have seen that the usual value $\alpha = 0.05$ proposed in [Cri97, HHT77] is not sufficient to insure long term time integration in the nonlinear framework. At the opposite $\alpha = 0,2$ entails overdissipation, as shown on figure 9.

Remark 12 *The present numerical analysis holds for quasi incompressible nonlinear elastodynamics, but the same phenomena can be observed in large displacements for compressible systems.*

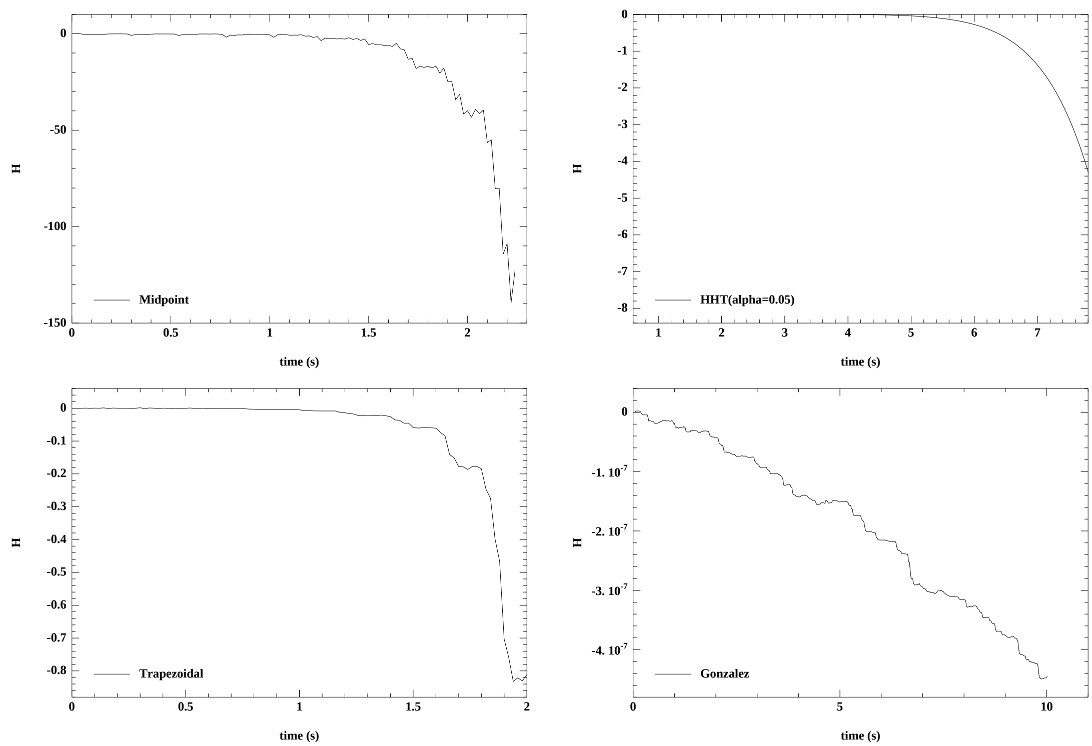

Figure 6: Evolution of the discrete total potential H (in Joules) as a function of time. As an indication, the maximal value of the deformation potential $\int_\Omega \mathcal{W}(C)$ is about 0.5 Joules.

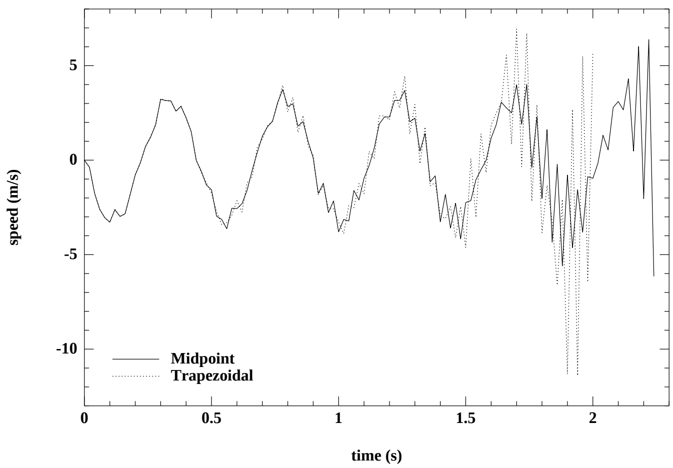

Figure 7: Instability of the vertical velocity at the tip of the cantilever beam, for midpoint and trapezoidal schemes.

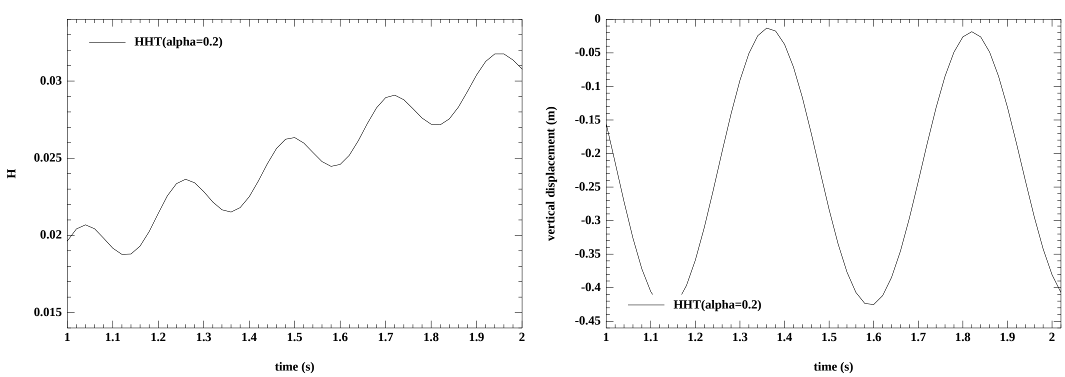

Figure 8: Zoom on total discrete potential and displacement for HHT scheme ($\alpha = 0.2$).

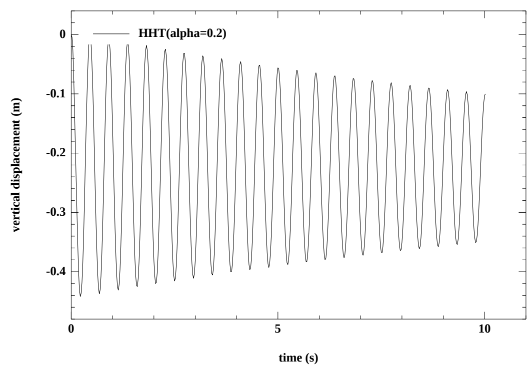

Figure 9: Overdissipation for vertical displacement at the tip of the beam for HHT scheme ($\alpha = 0.2$).

6 Conclusion

We have mainly proposed an energy conservation analysis for usual time integration schemes in nonlinear elastodynamics. We have shown numerically the great advantage of energy conserving schemes in terms of unconditional stability. Moreover, we have underlined that in general, linearly dissipative schemes do not lead to nonlinear energy dissipative generalizations (e.g. HHT).

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