Numerical schemes of resolution of stochastic optimal control HJB equation

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Journée des doctorants

7 mars 2007

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CMAP

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

INRIA FUTURS

Équipe COMMANDS

Outline

1. Previous results
   - Model problem
   - Generalized Finite Differences (GFD)
2. A fast algorithm for the 2D HJB equation of stochastic control
   - Structure of 2D diffusion matrices
   - Stern-Brocot tree
   - Decomposition of the scaled diffusion matrix
   - Projection error
   - Numerical results
3. Other schemes : implicit and splitting

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Model problem

$$
(P_{\tau,x}) \qquad \begin{cases} \mathrm{Min}\, E \displaystyle\int_{\tau}^{T} \ell(t, y(t), u(t))\mathrm{d}t + \ell_F(y(T)); \\ \begin{cases} \mathrm{d}y(t) = f(t, y(t), u(t))\mathrm{d}t + \sigma(t, y(t), u(t))\mathrm{d}w(t), \\ y(\tau) = x, \end{cases} \\ u(t) \in U, \quad \tau \in [0, T], \quad t \in [\tau, T]. \end{cases}
$$

- $y(t) \in R^n$, $u(t) \in R^m$ state and control variable,
- ℓ and ℓ_F, real functions, running and final cost,
- $f : R \times R^n \times R^m \longrightarrow R^n$ the drift,
- σ mapping into the space of $n \times r$ matrices,
- w standard r dimensional Brownian motion,
- ℓ, ℓ_F, f and σ Lipschitz and bounded.

HJB Equation

V value function of problem $(P_{\tau,x})$, unique viscosity solution of

$$
\begin{aligned}
-v_t(t,x) = & \inf_{u \in U} \left\{ \ell(t,x,u) + f(t,x,u) \cdot v_x(t,x) + a(t,x,u) \circ v_{xx}(t,x) \right\}, \\
& \quad \text{for all } \; t,x \in [0,T] \times R^n. \\
v(T,x) = & \; \ell_F(x), \; \text{for all } \; x \in R^n.
\end{aligned} \tag{HJB}
$$

- $a(t,x,u) := \frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^\top$ covariance matrix,
- $A \circ B := \sum_{i,j=1}^{n} A_{ij}B_{ij}$ scalar product associated with the Frobenius norm,

Previous results

- Lions and Mercier (1980), Menaldi (1989) : Classical finite difference methods,
- Kushner (1977), Kushner and Dupuis (1992) : Markov chain approximation,
- Camilli and Falcone (1995) : methods based on a priori time discretization (and the related dynamic programming principle for discrete time problems),
- Krylov (2000) : an error estimate of a large class of discretization schemes.
- Barles and Jakobsen (2002, 2003) : improvements of the error estimates.

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Discretization

- Space and time discretization
  - $h_1, \ldots, h_n$ space discretization steps,
  - $x_k := \sum_{i=1}^{n} k_i e_i$ point of the state space, $k \in \mathbb{Z}^n$,
  - $Q \in \mathbb{N}$ number of time steps, $h_0 := T/Q$ time step, $t_q := qh_0$, $q = 0, \ldots, Q$.
  - v_k^q approximation of the value function V at $(t, x) = (t_q, x_k)$.
- Upwind finite difference operator

$$(D^{\pm}\varphi_k)_i = \frac{\varphi_{k+e_i} - \varphi_k}{h_i} \quad \text{if} \quad f(t_q, x_k, u)_i \geq 0, \quad \frac{\varphi_k - \varphi_{k-e_i}}{h_i} \quad \text{if not.}$$

$\varphi = \{\varphi_k\}$ real valued function over $\mathbb{Z}^n$.

- Second order finite difference operator associated with $\xi \in \mathbb{Z}^n$

$$\Delta_\xi \varphi_k := \varphi_{k+\xi} - 2\varphi_k + \varphi_{k-\xi}$$

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Strong consistency

- Stencil $\mathcal{S}$: finite subset of $\mathbb{Z}^n \setminus \{0\}$,
- $a^h := \{a_{ij}/h_i h_j\}$: scaled covariance matrix,
- Approximation of the second-order term in the HJB equation

$$\sum_{\xi \in \mathcal{S}} \alpha^u_{q,k,\xi} \Delta_\xi v^q_k, \quad \text{where } \alpha^u_{q,k,\xi} \text{ are to be set}$$

- Strongly consistent[1] approximation of $a(t,x,u) \circ D^2_{xx}$ if

$$\sum_{\xi \in \mathcal{S}} \alpha^u_{q,k,\xi} \xi \xi^\top = a^h(t_q, x_k, u), \quad \text{for all} \quad k \in \mathbb{Z}^n.$$

(1) Consistency of generalized finite difference schemes for the stochastic HJB equation. Bonnans, Zidani (2003)

Explicit backward scheme

$$
\begin{aligned}
\frac{v_k^{q-1} - v_k^q}{h_0} &= \inf_{u \in U} \left\{ \ell(t_q, x_k, u) + f(t_q, x_k, u) \cdot D_{q,k}^u v_k^q + \sum_{\xi \in \mathcal{S}} \alpha_{q,k,\xi}^u \Delta_\xi v_k^q \right\} \\
v_k^Q &= \ell_F
\end{aligned}
$$

If the coefficients $\alpha_{q,k,\xi}^u$ are nonnegative and

$$
\sum_{i=1}^n \frac{\|f_i\|_\infty}{h_i} + 2\|\text{trace } a^h\|_\infty \leq \frac{1}{h_0}.
$$

the scheme is monotone.

When $\min_i h_i \downarrow 0$, we may take $h_0 = C \min_i \left(h_i^2\right)$, for $C > 0$ small enough (depending on f and a).

Scheme properties

- Strong consistency and monotonicity $\Rightarrow$ GFD are a particular case of consistent Markov chain approximations $\Rightarrow$ convergent in view of Kushner and Dupuis (1992),
- monotonicity, stability and consistence $\Rightarrow$ convergence[(2)],
- Krylov, Barles and Jacobsen hypotheses satisfied, the error estimates apply.

(2) Barles and Souganidis (1991)

Computation of coefficients $\alpha^u_{q,k,\xi}$

- To be done at each point of the grid, for each time step, possibly for each control,
- Solution of a linear program,
- Expensive if the stencil is large.

$$\text{size}(\mathcal{S}) := \max\{\|\xi\|_\infty;\ \xi \in \mathcal{S}\}$$

- Main result[(3)]
  - For 2D problems : an algorithm for computing the coefficients in $O(\text{size}(\mathcal{S}))$ operations,
  - For nonconsistent problems : computation of the closest consistent matrix in $O(\text{size}(\mathcal{S}))$ operations.
  - The closest consistent matrix for stencil of size p_{max} is computed in $O(1)$ operations after having obtained the closest consistent matrix for stencil of size $p_{max} - 1$.

(3) A fast algorithm for the two dimensional HJB equation of stochastic control,

Bonnans, Ottenwaelter, Zidani (2004)

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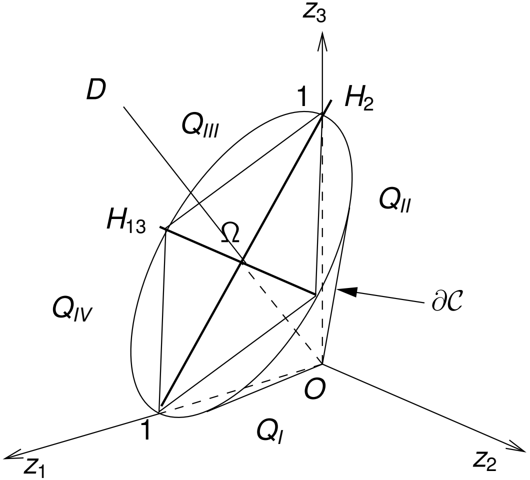
Cone of positive semidefinite matrices



$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \downarrow \begin{pmatrix} a_{11} \\ \sqrt{2}a_{12} \\ a_{22} \end{pmatrix}$$

Cone of positive semidefinite matrices

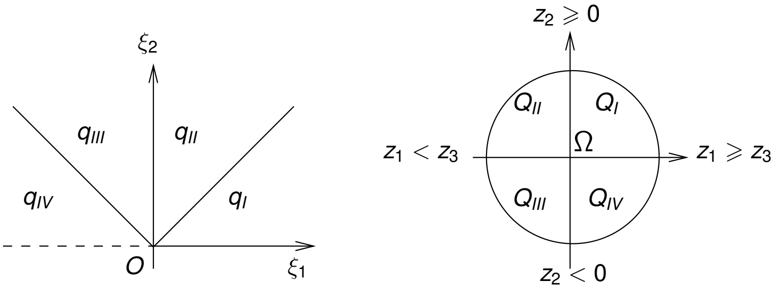


Figure: Correspondence of regions

$$\xi \in \mathcal{D} \subset R^2 \qquad \longrightarrow \qquad \xi\xi^\top \in \partial\mathcal{C}$$

Stencil

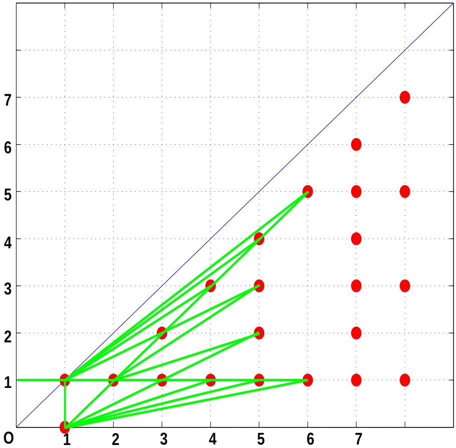


Figure: Family relations in regular grid

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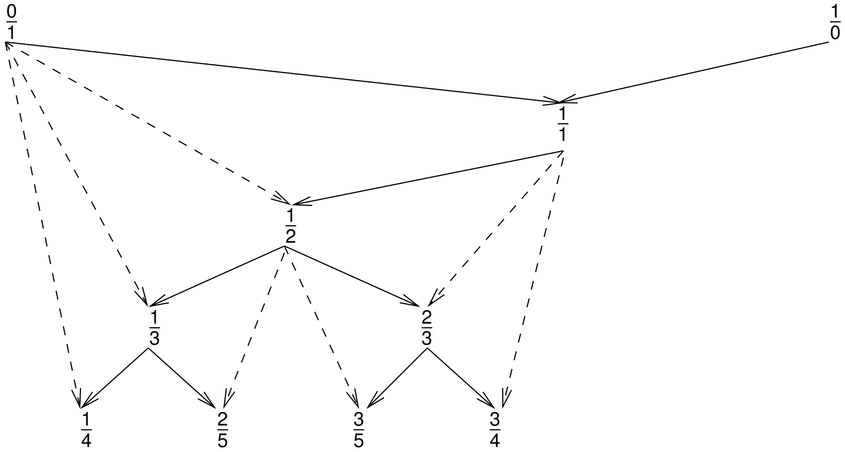
Stern-Brocot tree



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Algorithm DECOMP

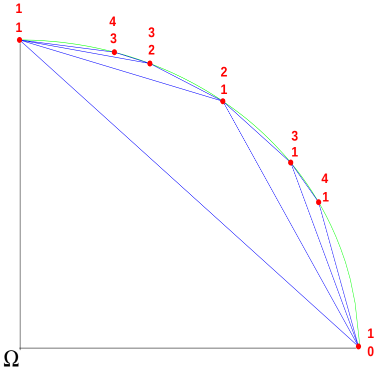

Figure: Correspondence of directions

$p_{max} \quad \longrightarrow \quad \mathcal{S}_{p_{max}} \quad \longrightarrow \quad$ Polyhedral cone $\mathcal{C}(\mathcal{S}_{p_{max}})$

Data : $p_{max} \in \mathbb{N}^*$, $\quad \varepsilon = \mathrm{dist}\left(\frac{a}{\|a\|}, \mathcal{C}(\mathcal{S}_{p_{max}})\right)$ relative error.

Main result

Theorem

Algorithm **DECOMP** *provides a decomposition of* a^h *with a relative error lower than* ε*, and stops after at most* p_{max} *iterations. The cost of each iteration is* $O(1)$ *operations, and hence, its total cost is no more than* $O(p_{max})$.

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Projection error

Lemma

The distance from a PSD matrix a to $\mathcal{C}(\mathcal{S}_{p_{max}})$ *is at most* $\varepsilon_{p_{max}} \|a\|$, *and*

$$\varepsilon_{p_{max}} = \frac{\sqrt{p_{max}^2 + 1} - p_{max}}{\sqrt{2}\,\sqrt{2\,p_{max}^2 + 1}} \leq \frac{1}{4}\,p_{max}^{-2}.$$

Conversely, given $\varepsilon > 0$, *the distance from a to* $\mathcal{C}(\mathcal{S}_{p_{max}})$ *is at most* ε *when* $p_{max} \geq p_\varepsilon$, *with*

$$p_\varepsilon := \left\lceil \frac{\sqrt{1 - \varepsilon^2} - \varepsilon}{2\sqrt{\varepsilon\sqrt{1-\varepsilon^2}}} \right\rceil.$$

Solution error

Remark

If consistency does not hold, and $\varepsilon = 0$, then algorithm **DECOMP** computes the decomposition of the projection of $a(t, x, u)$ onto $\mathcal{C}(\mathcal{S}_{p_{max}})$. In that case, the numerical scheme can be interpreted as a consistent approximation for the perturbed HJB equation.

- v, v' : resp. the solution of the HJB equation and the (well-defined) solution of the perturbed HJB equation,
- When the step size vanishes, the limit of error between the two solutions is $\|v - v'\|_\infty$,
- Combining Jakobsen and Karlsen estimates of this error (2002) with the previous Lemma, we obtain

$$\|v - v'\|_\infty \leq C\|a - a'\|_\infty^{1/2} \leq C'\varepsilon_{p_{max}}^{1/2} \leq C''/p_{max}$$

where C and C' do not depend on p_{max}.

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An uncontrolled problem

Test function 1

$$\begin{cases} W(t, x_1, x_2) = (1+t)\sin x_1 \sin x_2 \\ 0 \leqslant x_1 \leqslant \pi; \quad 0 \leqslant x_2 \leqslant \pi; \quad 0 \leqslant t \leqslant 1. \end{cases}$$

$$\text{where} \quad \sigma(t, x_1, x_2) = \begin{pmatrix} \sin(x_1 + x_2) & \beta & 0 \\ \cos(x_1 + x_2) & 0 & \beta \end{pmatrix}$$

$$f(t, x_1, x_2) = 0$$

Ex 1 : $p_{max} = 5$, $\beta^2 = 0.1 \rightarrow a \in \mathring{\mathcal{C}}$
Ex 2 : $p_{max} = 5$, $\beta^2 = 0 \rightarrow a \in \partial\mathcal{C}$

Numerical results 1

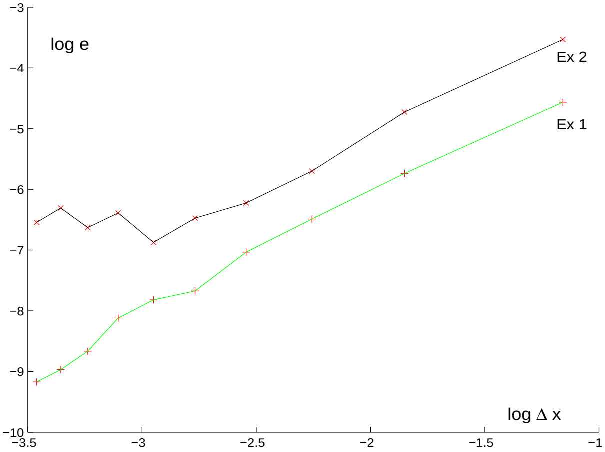

Figure: Error vs discretization step, $p_{max} = 5$

Optimal control

Test function 2

$$\begin{cases} W(t, x_1, x_2) = (1+t)\sin x_1 \sin x_2 \\ -1 \leqslant x_1 \leqslant 1; \quad -1 \leqslant x_2 \leqslant 1; \quad 0 \leqslant t \leqslant 0.5 \end{cases}$$

where

$\sigma(t, x_1, x_2) = \begin{pmatrix} \sqrt{2}\sin(x_1 + x_2) \\ \sqrt{2}\cos(x_1 + x_2) \end{pmatrix}$ does not depend on the control,

$f(t, x, u) = u, \; u_1^2 + u_2^2 \leq 1$ is the control.

Numerical results 2

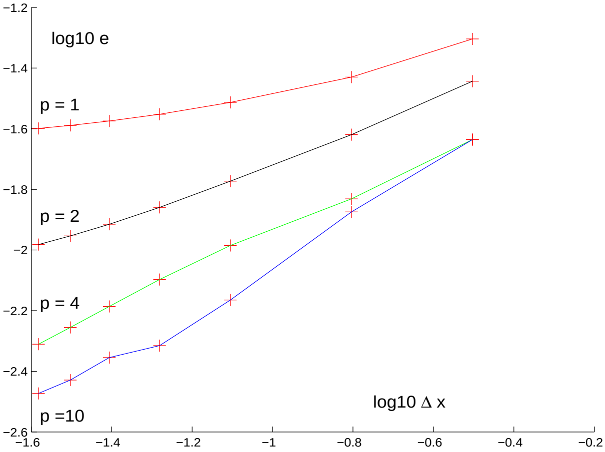

Figure: Error vs discretization step, optimal control, $p_{max} = 1,\ 2,\ 4,\ 10$

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Implicit and splitting schemes

- Explicit Euler scheme needs $\Delta t = O\left((\Delta x)^2\right)$.
- Implicit Euler scheme allows large time steps, but large scale linear systems to solve at each time step.

Since our scheme expresses the evolution operator as a sum of rank-one diffusion operators, whose directions are given by the stencil, a natural alternative is to use a splitting decomposition method, for which up to $|\mathcal{S}|$ (or $2|\mathcal{S}|$ for second-order schemes) tridiagonal linear systems have to be solved at each time step.

Cost

For 2D systems : on a square grid, in the case $h_1 = h_2$, $N_h = O(1/h_1)$ unknowns per column, the bandwidth is as much as $p_{max} N_h$, in the case of stencil $\mathcal{S}_{p_{max}}$.
The cost of factorization is, when $p_{max} \ll N_h$, of the order of

$$p_{max}^2 \times N_h^4$$

For the Laplace operator we have $p_{max} = 1$ and solving the implicit scheme is already expensive.

Implicit scheme

$$\frac{V(t+h_0,x)-V(t,x)}{h_0} = \inf_{u\in U}\{F(t+h_0,x,u,V)\}\,,\quad \forall\, t\in h_0\{1,\dots,N_t\}\,,\quad x\in R^N,$$
$$V(0,x) = \Phi(x),\quad \forall\, x\in R^N,$$

where

$$F(t,x,u,V)=\ell(t,x,u)+\sum_{1\le i\le|\mathcal{S}|}F_i(t,x,u,V)$$

and

$$\begin{cases} F_0(t,x,u,V) := & \ell(t,x,u),\\ F_i(t,x,u,V) := & UW_i(f,V)(t,x,u)+\alpha_{\xi^i}(t,x,u)\,\Delta_{\xi^i}V(t,x),\\ & 1\le i\le N\\ F_i(t,x,u,V) := & \alpha_{\xi^i}(t,x,u)\,\Delta_{\xi^i}V(t,x),\\ & N<i\le|\mathcal{S}| \end{cases}$$

$UW_i(f,V)$: upwind finite difference first order operator associated to f.

Implicit splitting scheme

- Minimization w.r.t. the control u : Howard algorithm.
- Loop on the stencil directions inside a time step.

$$
\begin{aligned}
V(0,x) &= \Phi(x), \quad \forall\, x \in R^N, \ \text{ and } \forall\, t \in h_0\,\{1,\dots,N_t\} \ : \\
\widehat{u}(0,x) &\in \operatorname*{argmin}_{u\in U} F(0,x,u,V) \\
V_0(x) &= V(t,x) \\
V_{i+1}(x) &= V_i(x) + h_0 F_i(t+h_0, x, \widehat{u}(t,x), V_{i+1}), \quad i = 0,\dots,|\mathcal{S}|, \\
V(t+h_0,x) &= V_{|\mathcal{S}|+1}(x) \\
\widehat{u}(t+h_0,x) &\in \operatorname*{argmin}_{u\in U} \{F(t+h_0,x,u,V)\}\,.
\end{aligned}
$$

Scheme properties

Since monotonicity and consistency $\Rightarrow$ convergence (Barles and Souganidis, 1991), we study the conditions of monotonicity and consistency for the splitting scheme inside a time step, for a loop on the directions.

- Monotonicity
  - For each direction the implicit scheme is monotone,
  - The splitting scheme is monotone as a composition of monotone schemes.
- Consistency : open problem.

Numerical test

Test function : 2D Gaussian function

$$W(t,x_1,x_2)=\frac{1}{2\pi(2t+1)}\,e^{-\frac{1}{2(2t+1)}\frac{(a_{22}x_1^2-2a_{12}x_1x_2+a_{11}x_2^2)}{(a_{11}a_{22}-a_{12}^2)}}$$

Solution of the PDE $W_t(t,x_1,x_2)=a\circ W_{xx}(t,x_1,x_2)$

where $a=\frac{\xi\xi^\top}{\|\xi\|^2}+\frac{\xi'\xi'^\top}{\|\xi'\|^2}$, $t_0=0.5$.

Tests with $\xi=\begin{pmatrix}1\\0\end{pmatrix}$, $\xi'=\begin{pmatrix}2\\1\end{pmatrix}$.

Numerical results : simple splitting

N_x	Simple splitting			N_t
	Error	Order	Time	
22	$2.482283\ 10^{-3}$	-	0 sec	22
42	$9.370919\ 10^{-4}$	1.41	0 sec	42
82	$3.529801\ 10^{-4}$	1.41	1 sec	82
162	$1.443794\ 10^{-4}$	1.29	10 sec	162
322	$6.383762\ 10^{-5}$	1.18	80 sec	322
642	$2.979488\ 10^{-5}$	1.10	651 sec	642

Remark

$$\Delta t = O(\Delta x)$$

Numerical results : Strang splitting

N_x	Strang splitting			N_t
	Error	Order	Time	
22	$2.107620\ 10^{-3}$	-	0 sec	22
42	$7.234636\ 10^{-4}$	1.54	0 sec	42
82	$2.432465\ 10^{-4}$	1.57	2 sec	82
162	$8.914378\ 10^{-5}$	1.45	20 sec	162
322	$3.617066\ 10^{-5}$	1.30	160 sec	322
642	$1.595454\ 10^{-5}$	1.18	1302 sec	642

Remark

Strang splitting divides the error by 2.