Affine long term yield curves : an application of the Ramsey rule with progressive utility

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Abstract

The purpose of this paper relies on the study of long term affine yield curves modeling. It is inspired by the Ramsey rule of the economic literature, that links discount rate and marginal utility of aggregate optimal consumption. For such a long maturity modelization, the possibility of adjusting preferences to new economic information is crucial, justifying the use of progressive utility. This paper studies, in a framework with affine factors, the yield curve given from the Ramsey rule. It first characterizes consistent progressive utility of investment and consumption, given the optimal wealth and consumption processes. A special attention is paid to utilities associated with linear optimal processes with respect to their initial conditions, which is for example the case of power progressive utilities. Those utilities are the basis point to construct other progressive utilities generating non linear optimal processes but leading yet to still tractable computations. This is of particular interest to study the impact of initial wealth on yield curves.

Keywords: Progressive utility with consumption, market consistency, portfolio optimization, Ramsey rule, affine yields curves.

Introduction

This paper focuses on the modelization of long term affine yield curves. For the financing of ecological project, for the pricing of longevity-linked securities or any other investment with long term impact, modeling long term interest rates is crucial. The answer cannot be

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find in financial market since for longer maturities, the bond market is highly illiquid and standard financial interest rates models cannot be easily extended. Nevertheless, an abundant literature on the economic aspects of long-term policy-making has been developed. The Ramsey rule, introduced by Ramsey in his seminal work [26] and further discussed by numerous economists such as Gollier [4, 7, 11, 6, 8, 10, 5, 9] and Weitzman [28, 29], is the reference equation to compute discount rate, that allows to evaluate the future value of an investment by giving a current equivalent value. The Ramsey rule links the discount rate with the marginal utility of aggregate consumption at the economic equilibrium. Even if this rule is very simple, there is no consensus among economists about the parameters that should be considered, leading to very different discount rates. But economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions in the light of new knowledge and direct experiences: the utility criterion must be adaptative and adjusted to the information flow. In the classical optimization point of view, this adaptative criteria is called consistency. In that sense, market-consistent progressive utilities, studied in El Karoui and Mrad [15, 14, 13], are the appropriate tools to study long term yield curves.

Indeed, in a dynamic and stochastic environment, the classical notion of utility function is not flexible enough to help us to make good choices in the long run. M. Musiela and T. Zariphopoulou (2003-2008 [21, 22, 20, 19]) were the first to suggest to use instead of the classical criterion the concept of progressive dynamic utility, consistent with respect to a given investment universe in a sense specified in Section 1. The concept of progressive utility gives an adaptative way to model possible changes over the time of individual preferences of an agent. In continuation of the recent works of El Karoui and Mrad [15, 14, 13], and motived by the Ramsey rule (in which the consumption rate is a key process), [16] extends the notion of market-consistent progressive utility to the case with consumption: the agent invest in a financial market and consumes a part of her wealth at each instant. As an example, backward classical value function is a progressive utility, the way the classical optimization problem is posed is very different from the progressive utility problem. In the classical approach, the optimal processes are computed through a backward analysis, emphasizing their dependency to the horizon of the optimization problem, while the forward point of view makes clear the monotony of the optimal processes to their initial conditions. A special attention is paid to progressive utilities generating linear optimal processes with respect to their initial conditions, which is for example the case of power progressive utilities.

As the zero-coupon bond market is highly illiquid for long maturity, it is relevant, for small trades, to give utility indifference price (also called Davis price) for zero coupon, using progressive utility with consumption. We study then the dynamics of the marginal utility yield curve, in the framework of progressive and backward power utilities (since power utilities are the most commonly used in the economic literature) and in a model with affine factors, since this model has the advantage to lead to tractable computations while allowing for more stochasticity than the log normal model studied in [16]. Nevertheless, using power utilities implies that the impact of the initial economic wealth is avoided, since in this case the optimal processes are linear with respect to the initial conditions. We thus propose a way of constructing, from power utilities, progressive utilities generating non linear optimal processes but leading yet to still tractable computations. The impact of the initial wealth for yield curves is discussed.

The paper is organized as follows. After introducing the investment universe, Section 1 characterizes consistent progressive utility of investment and consumption, given the optimal wealth and consumption processes. Section 2 deals with the computation of the marginal utility yield curve, inspired by the Ramsey rule. Section 3 focuses on the yield curve with affine factors, in such a setting the yield curve does not depend on the initial wealth of the economy. Section 4 provides then a modelization for yield curves dynamics that are non-linear to initial conditions.

1 Progressive Utility and Investment Universe

1.1 The investment universe

We consider an incomplete Itô market, equipped with a n-standard Brownian motion, W and characterized by an adapted short rate (r_t) and an adapted n-dimensional risk premium vector (η_t). All these processes are defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ satisfying usual assumptions; they are progressively processes satisfying minimal integrability assumptions, as $\int_0^T (r_t + \|\eta_t\|^2 dt) < \infty, a.s.$.
The agent may invest in this financial market and is allowed to consume a part of his wealth at the rate $C_t \geq 0$. To be short, we give the mathematical definition of the class of admissible strategies (κ_t, C_t), without specifying the risky assets. Nevertheless, the incompleteness of the market is expressed by restrictions on the risky strategies constrained to live in a given progressive vector space $(\mathcal{R}_t)$, often obtained as the range of some progressive linear operator $\Im_t$.

Definition 1.1 (Test processes). (i) *The self-financing dynamics of a wealth process with risky portfolio κ and consumption rate C is given by*

$$dX_t^{\kappa,C} = X_t^{\kappa,C}[r_t dt + \kappa_t(dW_t + \eta_t dt)] - C_t\, dt, \qquad \kappa_t \in \mathcal{R}_t. \tag{1.1}$$

where C is a positive progressive process, κ is a progressive n-dimensional vector in $\mathcal{R}_t$, such that $\int_0^T C_t + \|\kappa_t\|^2 dt < \infty, a.s.$.
(ii) *A strategy (κ_t, C_t) is said to be admissible if it is stopped with the bankruptcy of the investor (when the wealth process reaches 0).*
(iii) *The set of the wealth processes with admissible (κ_t, C_t), also called test processes, is denoted by $\mathscr{X}^c$. When portfolios are starting from x at time t, we use the notation $\mathscr{X}_t^c(x)$.*

The following short notations will be used extensively. Let $\mathcal{R}$ be a vector subspace of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, $x^{\mathcal{R}}$ is the orthogonal projection of the vector x onto $\mathcal{R}$ and $x^{\perp}$ is the orthogonal projection onto $\mathcal{R}^{\perp}$.
The existence of a risk premium η is a possible formulation of the absence of arbitrage opportunity. From Equation (1.1), the minimal state price process Y_t^0, whose the dynamics is $dY_t^0 = Y_t^0[-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}).dW_t]$, belongs to the convex family $\mathscr{Y}$ of positive Itô's processes Y_t such that $(Y_t X_t^{\kappa,C} + \int_0^t Y_s C_s ds)$ is a local martingale for any admissible portfolio. The existence of equivalent martingale measure is obtained by the assumption that the exponential local martingale $L_t^{\eta^{\mathcal{R}}} = \exp(-\int_0^t \eta_s^{\mathcal{R}}.dW_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2\, ds)$ is a uniformly integrable martingale. Nevertheless, we are interested into the class of the so-called state price processes Y_t belonging to the family $\mathscr{Y}$ characterized below.

Definition 1.2 (State price process). (i) *An Itô semimartingale Y_t is called a state price process in $\mathscr{Y}$ if for any test process $X^{\kappa,C}$, $\kappa \in \mathcal{R}$,*

$$(Y_t X_t^{\kappa,C} + \int_0^t Y_s C_s ds) \text{ is a local martingale.}$$

(ii) *This property is equivalent to the existence of progressive process $\nu_t \in \mathcal{R}_t^{\perp}$, $(\int_0^T \|\nu_t\|^2 dt < \infty, a.s.)$ such that $Y = Y^\nu$ where Y^ν is the product of Y^0 $(\nu = 0)$ by the exponential local martingale $L_t^\nu = \exp\big(\int_0^t \nu_s.dW_s - 1/2\int_0^t ||\nu_s||^2 ds\big)$, and satisfies*

$$dY_t^\nu = Y_t^\nu[-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}).dW_t], \quad \nu_t \in \mathcal{R}_t^{\perp} \quad Y_0^\nu = y \tag{1.2}$$

From now on, to stress out the dependency on the initial condition, the solution of (1.2) with initial condition y will be denoted $(Y_t^\nu(y))$ and $Y_t^\nu := Y_t^\nu(1)$; the solution of (1.1) with initial condition x will be denoted $(X_t^{\kappa,C}(x))$ and $X_t^{\kappa,C} := X_t^{\kappa,C}(1)$.

1.2 $\mathscr{X}^c$-consistent Utility and Portfolio optimization with consumption

In long term (wealth-consumption) optimization problems, it is useful to have the choice to adapt utility criteria to deep macro-evolution of economic environment. The concept of progressive utility is introduced in this sense. As we are interested in optimizing both the terminal wealth and the consumption rate, we introduce two progressive utilities $(\mathbf{U}, \mathbf{V})$, $\mathbf{U}$ for the terminal wealth and $\mathbf{V}$ for the consumption rate, often called utility system. For sake of completeness, we start refer the reader to [15] for a detailled study.

Definition 1.3 (Progressive Utility).
(i) *A* progressive utility *is a $\mathcal{C}^2$- progressive random field on $\mathbb{R}_+^*$, $\mathbf{U} = \{U(t,x); t \geq 0, x > 0\}$, starting from the deterministic utility function u at time 0, such that for every (t,ω), $x \mapsto U(\omega,t,x)$ is a strictly concave, strictly increasing, and non negative utility function, and satisfying thee Inada conditions:*

- *for every (t,ω), $U(t,\omega,x)$ goes to 0 when x goes to 0*
- *the derivative $U_x(t,\omega,x)$ (also called marginal utility) goes to ∞ when x goes to 0,*
- *the derivative $U_x(t,\omega,x)$ goes to 0 when x goes to ∞.*

For $t=0$, the deterministic utilities $U(0,.)$ and $V(0,.)$ are denoted $u(.)$ and $v(.)$ and in the following small letters u and v design deterministic utilities while capital letters refer to progressive utilities.

As in statistical learning, the utility criterium is dynamically adjusted to be the best given the market past information. So, market inputs may be viewed as a calibration universe through the test-class $\mathscr{X}^c$ of processes on which the utility is chosen to provide the best satisfaction. This motivates the following definition of $\mathscr{X}^c$-consistent utility system.

Definition 1.4. *A $\mathscr{X}^c$-consistent progressive utility system of investment and consumption is a pair of progressive utilities* $\mathbf{U}$ *and* $\mathbf{V}$ *on* $\Omega\times[0,+\infty)\times\mathbb{R}^+$ *such that,*
(i) CONSISTENCY WITH THE TEST-CLASS: *For any admissible wealth process* $X^{\kappa,C}\in\mathscr{X}^c$,

$$\mathbb{E}\big(U(t,X_t^{\kappa,C})+\int_s^t V(s,C_s)ds/\mathcal{F}_s\big)\leq U(s,X_s^{\kappa,C}),\ \ \forall s\leq t\ \ a.s.$$

In other words, the process $\big(U(t,X_t^{\kappa,C})+\int_0^t V(s,C_s)ds\big)$ *is a positive supermartingale, stopped at the first time of bankruptcy.*
(ii) EXISTENCE OF OPTIMAL STRATEGY: *For any initial wealth* $x>0$, *there exists an optimal strategy* (κ^*,C^*) *such that the associated non negative wealth process* $X^*=X^{\kappa^*,C^*}\in\mathscr{X}^c$ *issued from* x *satisfies* $\big(U(t,X_t^*)+\int_0^t V(s,C_s^*)ds\big)$ *is a local martingale.*
(iii) *To summarize,* $U(t,x)$ *is the value function of optimization problem with optimal strategies, that is for any maturity* $T\geq t$

$$U(t,x)=\operatorname{ess\,sup}_{X^{\kappa,C}\in\mathscr{X}_t^c(x)}\mathbb{E}\big(U(T,X_T^{\kappa,C})+\int_t^T V(s,C_s)\mathbf{1}_{\{X_s^{\kappa,C}(x)\geq 0\}}ds|\mathcal{F}_t\big)a.s. \quad (1.3)$$

The optimal strategy (X^*,C^*) *which is optimal for all these problems, independently of the time-horizon* T, *is called a myopic strategy.*
(iv) STRONGLY $\mathscr{X}^c$-CONSISTENCY *The system* $(\mathbf{U},\mathbf{V})$ *is said to be strongly* $\mathscr{X}^c$*-consistent if the optimal process* $X^*(x)$ *is strictly increasing with respect to the initial condition* x.

Convex analysis showed the interest to introduce the convex conjugate utilities $\tilde{U}$ and $\tilde{V}$ defined as the Fenchel-Legendre random field $\tilde{U}(t,y)=\sup_{c\geq 0,c\in\mathcal{Q}^+}(U(t,c)-cy)$ (similarly for $\tilde{V}$). Under mild regularity assumption, we have the following results (Karatzas-Shreve [12], Rogers [27]).

Proposition 1.5 (Duality)**.** *Let* (U,V) *be a pair of stochastic* $\mathscr{X}^c$*-consistent utilities with optimal strategy* (κ^*,C^*) *leading to the non negative wealth process* $X^*=X^{\kappa^*,C^*}$. *Then the convex conjugate system* $(\tilde{U},\tilde{V})$ *satisfies :*
(i) *For any admissible state price density process* $Y^\nu\in\mathscr{Y}$ *with* $\nu\in\mathcal{R}^\perp$, $\Big(\widetilde{U}(t,Y_t^\nu)+\int_0^t\widetilde{V}(s,Y_s^\nu)ds\Big)$ *is a submartingale, and there exists a unique optimal process* $Y^*:=Y^{\nu^*}$ *with* $\nu^*\in\mathcal{R}^\perp$ *such that* $\Big(\widetilde{U}(t,Y_t^*)+\int_0^t\widetilde{V}(s,Y_s^*)ds\Big)$ *is a local martingale.*
(ii) *To summarize,* $\widetilde{U}(t,y)$ *is the value function of optimization problem with myopic*

optimal strategy, that is for any maturity $T \geq t$

$$\widetilde{U}(t,y) = \operatorname{ess\,sup}_{Y^\nu \in \mathscr{Y}_t(y)} \mathbb{E}\Big(\widetilde{U}(T, Y_T^\nu(y)) + \int_t^T \widetilde{V}(s, Y_s^\nu)(y) ds / \mathcal{F}_t\Big), \ a.s. \tag{1.4}$$

(iii) Optimal Processes characterization *Under regularity assumption, first order conditions imply some links between optimal processes, including their initial conditions,*

$$Y_t^*(y) = \quad U_x(t, X_t^*(x)) = V_c(t, C_t^*(c)), \qquad y = u_x(x) = v_c(c) \tag{1.5}$$

The optimal consumption process $C_t^*(c)$ is related to the optimal portfolio $X_t^*(x)$ by the progressive monotonic process $\zeta_t^*(x)$ defined by

$$c = \zeta(x) = -\tilde{v}_y(u_x(x)), \quad \zeta_t^*(x) = -\widetilde{V}_y(U_x(x)), \quad C_t^*(c) = \zeta_t^*(X_t^*(x)) \tag{1.6}$$

(iv) *By Equation* (1.5)*, strong consistency of (U, V, X^*) implies the monotony of $y \mapsto Y_t^*(y)$. The system $(\tilde{U}, \tilde{V}, Y^*)$ is strongly $\mathscr{Y}$-consistent.*

The main consequence of the strong consistency is to provide a closed form for consumption consistent utility system.

Theorem 1.6. *Let $\bar{\zeta}_t(x)$ be a positive progressive process, increasing in x and let $\bar{X}_t(x)$ be a strictly monotonic solution with inverse $\bar{\mathcal{X}}_t(z)$ of the SDE,*

$$d\bar{X}_t(x) = \bar{X}_t(x)[r_t dt + \kappa_t^*(\bar{X}_t(x))(dW_t + \eta_t^{\mathcal{R}} dt)] - \bar{\zeta}_t(\bar{X}_t(x))\, dt, \ \ \kappa_t^*(\bar{X}_t(x)) \in \mathcal{R}_t.$$

Let $\bar{Y}_t(y)$ be a strictly monotonic solution with inverse $(\bar{\mathcal{Y}}_t(z))$ of the SDE

$$d\bar{Y}_t(y) = \bar{Y}_t(y)\big[-r_t dt + (\nu_t^*(\bar{Y}_t(y)) - \eta_t^{\mathcal{R}}).dW_t\big], \quad \nu_t^*(\bar{Y}_t(y)) \in \mathcal{R}_t^{\perp}.$$

Given a deterministic utility system (u,v) such that $\bar{\zeta}_0(x) = \zeta(x) = -\tilde{v}_y(u_x(x))$, there exists a $\mathscr{X}^c$-consistent progressive utility system (U,V) such that $(\bar{X}_t(x), \bar{Y}_t(y))$ are the associated optimal processes, defined by:

$$U_x(t,x) = \bar{Y}_t(u_x(\bar{\mathcal{X}}_t(x)), \ V_c(t,c) = U_x(t, \bar{\zeta}_t^{-1}(c)) \text{ with } \bar{\zeta}^{-1}(c) = u_x^{-1}(v_c(c)) \tag{1.7}$$

Observe that the consumption optimization contributes only through the conjugate $\tilde{V}$ of the progressive utility $\mathbf{V}$. We refer to [16] for detailed proofs.

1.3 $\mathscr{X}^c$-consistent utilities with linear optimal processes

The simplest example of monotonic process is given by linear processes with positive (negative) stochastic coefficient. It is easy to characterize consumption consistent utility sytems (U,V) associated with linear optimal processes

$$X_t^*(x) = x X_t^* \ \text{ with } \ X_t^* := X_t^*(1), \qquad Y_t^*(y) = y Y_t^* \ \text{ with } \ Y_t^* := Y_t^*(1)$$

Proposition 1.7. (i) *A strongly $\mathscr{X}^c$-consistent progressive utility (U,V) generates linear optimal wealth and state price processes if and only if it is of the form*

$$U(t,x)=Y_t^*X_t^*\,u(\frac{x}{X_t^*}),\ V(t,c)=\zeta_t^*U(t,\frac{c}{\zeta_t^*})\quad \textit{with } \zeta_t(x)=x\,\zeta_t.$$

The optimal processes are then given by

$$X_t^*(x)=xX_t^*,\ Y_t^*(y)=yY_t^*,\ \textit{and}\quad C_t^*(x)=X_t^*(x)\zeta_t^*.$$

(ii) Power utilities *A consumption consistent progressive power utility (U^θ,V^θ) (with risk aversion coefficient θ) generates necessarily linear optimal processes and is, consequently, of the form $\big(U^{(\theta)}(t,x)=\frac{Y_t^*X_t^*}{1-\theta}\big(\frac{x}{X_t^*}\big)^{1-\theta},V^{(\theta)}(t,x)=(\hat{\psi}_t)^\theta\frac{Y_t^*X_t^*}{1-\theta}\big(\frac{x}{X_t^*}\big)^{1-\theta}\big)$.*

Proof. If $X_t^*(x)=xX_t^*$ and $Y_t^*(y)=yY_t^*$, their inverse flows are also linear and $\mathcal{X}(t,x)=\frac{x}{X_t^*},\mathcal{Y}(t,y)=\frac{y}{Y_t^*}$.

(i) a) The linearity with respect to its initial condition of the solution of one dimensional SDE with drift $b_t(x)$ and diffusion coefficient $\sigma_t(x)$ can be satisfied only when the coefficients $b_t(x)$ and $\sigma_t(x)$ are affine in x, that is $b_t(x)=xb_t$ and $\sigma_t^i(x)=x\sigma_t^i$, b and σ^i being one dimensional progressive processes. Since the only coefficient with some non linearity in the dynamics of $Y_t^*(y)$ is $y\nu_t^*(y)$, the previous condition implies that $\nu_t^*(y)$ does not depend on y. By the same argument, we see that $x\kappa_t^*(x)$ is linear and $\kappa_t^*(x)$ also does not depend on x. For the consumption process, $\zeta_t^*(x)$ the linear condition becomes $\zeta_t^*(x)=x\zeta_t^*$.

b) We are concerned by strongly consistent progressive utilities, since optimal processes are monotonic by definition. Then, since $Y_t^*(u_x(x))=U_x(t,X_t^*(x))$, we see that the marginal utility U_x is given by $U_x(t,x)=u_x(x/X_t^*)Y_t^*$. By taking the primitive with the condition $U(t,0)=0$, U is given by $U(t,x)=u(x/X_t^*)X_t^*Y_t^*$.

c) We know that $C_t^*(x)=X_t^*(x)\zeta_t^*$ and $\bar{\zeta}_t(t,X_t^*(x))=-\widetilde{V}_y(t,U_x(t,X_t^*(x)))$ (from optimality conditions). Thus $-\widetilde{V}_y(t,U_x(t,X_t^*(x)))=X_t^*(x){\zeta^*}_t$, a.s. $\forall t,x$. From monotonicity of X^* and U_x, we then conclude that $V_c(t,c)=U_x(t,\frac{c}{\zeta_t^*})$ a.s. $\forall t,c$. Integrating yields the desired formula.

(ii) Power-type utilities generate linear optimal processes. So, we only have to consider initial power utilities $u(x)=kx^{1-\theta}/(1-\theta),v(c)=k_vc^{1-\theta}/(1-\theta)$ with the same risk-aversion coefficient θ to characterize the system. □

Remark 1.1. *In order to separate the messages and as the risk aversion does not vary in this result, we have deliberately omitted the indexing of the optimal process by θ, especially in the explicit case of power utilities. Although, optimal process may reflect a part of this risk aversion, therefore in the last section of this work, we take care to make them also dependent on this parameter*

1.4 Value function of backward classical utility maximization problem

As for example in the Ramsey rule, utility maximization problems in the economic literature use classical utility functions. This subsection points out the similarities and the differences between consistent progressive utilities and backward classical value functions, and their corresponding portfolio/consumption optimization problems.

Classical portfolio/consumption optimization problem and its conjugate problem The classic problem of optimizing consumption and terminal wealth is determined by a fixed time-horizon T_H and two deterministic utility functions $u(.)$ and $v(t,.)$ defined up to this horizon. Using the same notations as previously, the classical optimization problem is formulated as the following maximization problem,

$$\sup_{(\kappa,c)\in\mathscr{X}^c} \mathbb{E}\Big(u(X^{\kappa,c}_{T_H}) + \int_0^{T_H} v(t,c_t)dt\Big). \tag{1.8}$$

For any $[0,T_H]$-valued $\mathbb{F}$-stopping τ and for any positive random variable $\mathcal{F}_\tau$-mesurable ξ_τ, $\mathscr{X}^c(\tau,\xi_\tau)$ denotes the set of admissible strategies starting at time τ with an initial positive wealth ξ_τ, stopped when the wealth process reaches 0. The corresponding value system (that is a family of random variables indexed by (τ,ξ_τ)) is defined as,

$$\mathcal{U}(\tau,\xi_\tau) = \operatorname{ess\,sup}_{(\kappa,c)\in\mathscr{X}^c(\tau,\xi_\tau)} \mathbb{E}\Big(u(X^{\kappa,c}_{T_H}(\tau,\xi_\tau)) + \int_\tau^{T_H} v(s,c_s)ds|\mathcal{F}_\tau\Big),\ a.s. \tag{1.9}$$

with terminal condition $\mathcal{U}(T_H,x) = u(x)$.
We assume the existence of a progressive utility still denoted $\mathcal{U}(t,x)$ that aggregates these system (that is more or less implicit in the literature). When the dynamic programming principle holds true, the utility system $(\mathcal{U}(t,x), v(t,.)$ is $\mathscr{X}^c$-consistent. Nevertheless, in the backward point of view, it is not easy to show the existence of optimal monotonic processes, or equivalently the strong consistency. Besides, the optimal strategy in the backward formulation is not myopic and depends on the time-horizon T_H. In the economic literature, T_H is often taken equal to $+\infty$ and the utility function is separable in time with exponential decay at a rate β interpreted as the pure time preference parameter: $v(t,c) = e^{-\beta t}v(c)$. It is implicitly assumed that such utility function are equal to zero when t tends to infinity.

2 Ramsey rule and Yield Curve Dynamics

As our aim is to study long term affine yields curves, we will focus in the following on affine optimal processes. But let us first recall some results on the Ramsey rule with progressive utility.

2.1 Ramsey rule

Financial market cannot give a satisfactory answer for the modeling of long term yield curves, since for longer maturities, the bond market is highly illiquid and standard financial interest rates models cannot be easily extended.

Economic point of view of Ramsey rule Nevertheless, an abundant literature on the economic aspects of long-term policy-making has been developed. The Ramsey rule is the reference equation in the macroeconomics literature for the computation of long term discount factor. The Ramsey rule comes back to the seminal paper of Ramsey [26] in 1928 where economic interest rates are linked with the marginal utility of the aggregate consumption at the economic equilibrium. More precisely, the economy is represented by the strategy of a risk-averse representative agent, whose utility function on consumption rate at date t is the deterministic function $v(t,c)$. Using an equilibrium point of view with infinite horizon, the Ramsey rule connects at time 0 the equilibrium rate for maturity T with the marginal utility $v_c(t,c)$ of the random exogenous optimal consumption rate (C_t^e) by

$$R_0^e(T) = -\frac{1}{T}\ln\frac{\mathbb{E}[v_c(T,C_T^e)]}{v_c(c)}. \tag{2.1}$$

Remark that the Ramsey rule in the economic literature relies on a backward formulation with infinite horizon, an usual setting is to assume separable in time utility function with exponential decay at rate $\beta > 0$ and constant risk aversion $\theta, (0 < \theta < 1)$, that is $v(t,c) = Ke^{-\beta t}\frac{c^{1-\theta}}{1-\theta}$. β is the pure time preference parameter, i.e. β quantifies the agent preference of immediate goods versus future ones. C^e is exogenous and is often modeled as a geometric Brownian motion.

In the financial point of view we adopt here the agent may invest in a financial market in addition to the money market. We consider an arbitrage approach with exogenously given interest rate, instead of an equilibrium approach that determines them endogenously. It seems also essential for such maturity to adopt a sequential decision scheme that allows to revise the first decisions in the light of new knowledge and direct experiences: the utility criterion must be adaptative and adjusted to the information flow. That is why we consider consistent progressive utility. The financial market is an incomplete Itô financial market: notations are the one described in Section 1.1, with a n standard Brownian motion W, a (exogenous) financial short term interest rate (r_t) and a n-dimensional risk premium $(\eta_t^{\mathcal{R}})$. In the following, we adopt a financial point of view and consider either the progressive or the backward formulation for the optimization problem.

Marginal utility of consumption and state price density process

(i) The forward dynamic utility problem

Proposition 1.5 gives a pathwise relation between the marginal utility of the optimal consumption and the optimal state price density process, where the parameterization is done through the initial wealth x, or equivalently c or y since $c = -\tilde{v}_y(u_x(x)) = -\tilde{v}_y(y)$,

$$V_c(t, C_t^*(c)) = Y_t^*(y), \quad t \geq 0 \quad \text{with} \quad v_c(c) = y. \tag{2.2}$$

The forward point of view emphasizes the key rule played by the monotony of Y with respect to the initial condition y, under regularity conditions of the progressive utilities (cf [15]). Then as function of y, c is decreasing, and $C_t^*(c)$ is an increasing function of c. This question of monotony is frequently avoided, maybe because with power utility functions (the example often used in the literature) $Y_t^*(y)$ is linear in y as ν^* does not dependent on y. We shall come back to that issue in Section 4.

(ii) THE BACKWARD CLASSICAL OPTIMIZATION PROBLEM

In the classical optimization problem, both utility functions for terminal wealth and consumption rate are deterministic, and a given horizon T_H is fixed. In this backward point of view, optimal processes are depending on the time horizon T_H : in particular the optimal consumption rate $C^{*,H}(y)$ depends on the time horizon T_H through the optimal state price density process $Y^{*,H}$, leading to the same pathwise relation (2.2) as in the forward case,

$$\frac{v_c(t, C_t^{*,H}(c))}{v_c(c)} = \frac{Y_t^{*,H}(y)}{y}, \quad 0 \leq t \leq T_H \quad \text{with} \quad v_c(c) = y. \tag{2.3}$$

So, in general the notation of the forward case are used, but with the additional symbol H $(Y^{*,H}, c^{*,H}, X^{*,H})$ to address the dependency on T_H in the classical backward problem.

Conclusion: Thanks to the pathwise relation (2.2), the Ramsey rule yields to a description of the equilibrium interest rate as a function of the optimal state price density process $Y^{*,e}$, $R_0^e(T)(y) = -\frac{1}{T}\ln \mathbb{E}[Y_T^{*,e}(y)/y]$, that allows to give a financial interpretation in terms of zero coupon bonds. More dynamically in time, $\forall t < T$,

$$R_t^e(T)(y) := -\frac{1}{T-t}\ln \mathbb{E}\left[\frac{V_c(T, C_T^{*,e}(c))}{V_c(t, C_t^{*,e}(c))}\big|\mathcal{F}_t\right] = -\frac{1}{T-t}\ln \mathbb{E}\left[\frac{Y_T^{*,e}(y)}{Y_t^{*,e}(y)}\big|\mathcal{F}_t\right]. \tag{2.4}$$

2.2 Financial yield curve dynamics

Based on the foregoing, it is now proposed to make the connection between the economic and the financial point of view through the state price densities processes and the pricing. Let $\big(B^m(t,T),\ t \leq T\big)$, ($m$ for market), be the market price at time t of a zero-coupon bond paying one unit of cash at maturity T. Then, the market yield curve is defined as usual by the actuarial relation, $B^m(t,T) = \exp(-R_t^m(T)(T-t))$. Thus our aim is to give a financial interpretation of $\mathbb{E}\Big[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\Big]$ for $t \leq T$ in terms of price of zero-coupon bonds. Remark that Y^* is solution of an optimization problem whose criteria depend on the utility functions, yet the utilities do not intervene in the dynamics of Y^*. In term of pricing, the terminal wealth at maturity T represents the payoff at maturity of the financial product, whereas the consumption may be interpreted as the dividend distributed by the financial product before T.

Replicable bond For admissible portfolio without consumption X^κ_t, it is straightforward that for any state price process $Y^\nu_t X^\kappa_t$ is a local martingale, and so under additional integrability assumption, $X^\kappa_t = \mathbb{E}\big[X^\kappa_T \frac{Y^\nu_T}{Y^\nu_t}|\mathcal{F}_t\big]$. So the price of X^κ_T does not depend on ν. This property holds true for any derivative whose the terminal value is replicable by an admissible portfolio without consumption, for example a replicable bond,

$$B^m(t,T) = \mathbb{E}\big[\frac{Y^0_T}{Y^0_t}|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds}|\mathcal{F}_t\big] = \mathbb{E}\Big[\frac{Y_T(y)}{Y_t(y)}|\mathcal{F}_t\Big]$$

Besides, $B^m(t,T) = \mathbb{E}\Big[\frac{Y_T(y)}{Y_t(y)}|\mathcal{F}_t\Big]$ for any state price density process Y with goods integrability property.

Non hedgeable bond For non hedgeable zero-coupon bond, the pricing by indifference is a way (among others) to evaluate the risk coming from the unhedgeable part.
The *utility indifference price* is the cash amount for which the investor is indifferent between selling (or buying) a given quantity of the claim or not. This pricing rule is non linear and provides a bid-ask spread. If the investor is aware of its sensitivity to the unhedgeable risk, they can try to transact for a little amount. In this case, the "fair price" is the marginal utility indifference price (also called Davis price [2]), it corresponds to the zero marginal rate of substitution. We denote by $B^u(t,T)$ (u for utility) the marginal utility price at time t of a zero-coupon bond paying one cash unit at maturity T, that is $B^u(t,T) = B^u_t(T,y) = \mathbb{E}\Big[\frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\Big]$. Based on the link between optimal state price density and optimal consumption, we see that

$$B^u_t(T,y) := B^u(t,T)(y) = \mathbb{E}\Big[\frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\Big] = \mathbb{E}\Big[\frac{V_c(T,C^*_T(c))}{V_c(t,C^*_t(c))}|\mathcal{F}_t\Big], \quad v_c(c) = y. \tag{2.5}$$

Remark that $B^u(t,T)(y)$ is also equal to $\mathbb{E}\Big[\frac{U_x(T,X^*_T(x))}{U_x(t,X^*_t(x))}|\mathcal{F}_t\Big]$. Nevertheless, besides the economic interpretation, the formulation through the optimal consumption is more relevant than the formulation through the optimal wealth : indeed the utility from consumption V is given, while the utility from wealth U is more constrained.
According to the Ramsey rule (2.4), equilibrium interest rates and marginal utility interest rates are the same. Nevertheless, for marginal utility price, this last curve is robust only for small trades.
The martingale property of $Y^*_t(y)B^u_t(T,y)$ yields to the following dynamics for the zero coupon bond maturing at time T with volatility vector $\Gamma_t(T,y)$

$$\frac{dB^u_t(T,y)}{B^u_t(T,y)} = r_t dt + \Gamma_t(T,y).(dW_t + (\eta^{\mathcal{R}}_t - \nu^*_t(y))dt). \tag{2.6}$$

Using the classical notation for exponential martingale, $\mathcal{E}_t(\theta) = \exp\big(\int_0^t \theta_s.dW_s - \frac{1}{2}\int_0^t \|\theta_s\|^2.ds\big)$, the martingale $Y^*_t(y)B^u_t(T,y)$ can written as an exponential martingale with volatility $\big(\nu^*_.(y) - \eta^{\mathcal{R}}_. + \Gamma_.(T,y)\big)$. Using that $B^u_T(T,y) = 1$, we have two characterisations of $Y^*_T(y)$,

$$Y^*_T(y) = B^u_0(T,y)\mathcal{E}_T\big(\nu^*_.(y) - \eta^{\mathcal{R}}_. + \Gamma_.(T,y)\big) = y\, e^{-\int_0^T r_s ds}\mathcal{E}_T\big(\nu^*_.(y) - \eta^{\mathcal{R}}_.\big).$$

Taking the logarithm gives

$$\int_0^T r_s ds = T R_0^u(T) - \int_0^T \Gamma_t(T,y).(dW_t + (\eta_t - \nu_t^*(y))dt) + \frac{1}{2}\|\Gamma_t(T,y)\|^2 dt. \qquad (2.7)$$

2.3 Yield curve for infinite maturity and progressive utilities

The computation of the marginal utility price of zero coupon bond is then straightforward using (2.5) leading to the yield curve dynamics ($R_t^u(T,y) = -\frac{1}{T-t}\ln B_t^u(T,y)$)

$$\begin{aligned} R_t^u(T,y) &= \frac{T}{T-t}R_0^u(T,y) - \frac{1}{T-t}\int_0^t r_s ds - \int_0^t \frac{\Gamma_s(T,y)}{T-t}dW_s \\ &+ \int_0^t \frac{||\Gamma_s(T,y)||^2}{2(T-t)}ds + \int_0^t < \frac{\Gamma_s(T,y)}{T-t}, \nu_s^* - \eta_s^{\mathcal{R}} > ds. \end{aligned}$$

for finite maturity, and $l_t^u(y) := \lim_{T\to+\infty} R_t^u(T,y)$ for infinite maturity.
As showed in Dybvig [24] and in El Karoui and alii. [23] the long maturity rate $l_t^u(y)$ behaves differently according to the long term behavior of the volatility when $T \to \infty$,

- If $\lim_{T\to\infty} \frac{\Gamma_t(T,y)}{T-t} \neq 0, a.s.$, then $\frac{||\Gamma_t(T,y)||^2}{T-t} \to \infty$ a.s and $l_t(y)$ is infinite.
- Otherwise, $l_t = l_0 + \int_0^t \lim_{T\to\infty}\left(\frac{||\Gamma_s(T,y)||^2}{2(T-s)}\right) ds$, and l_t is a non decreasing process, constant in t and ω if $\lim_{T\to\infty} \frac{||\Gamma_t(T,y)||^2}{T-t} = 0$.

In this last case, which is the situation considered by the economists, all past, present or future yield curves have the same asymptote.

3 Progressive utilities and yield curves in affine factor model

Recently, affine factor models have been intensively developed with some success to capture under the physical probability measure both financial and macroeconomics effects, from the seminal paper of Ang and Piazzesi (2003). As explained in Bolder&Liu (2007) [1], *Affine term-structure models have a number of theoretical and practical advantages. One of the principal advantages is the explicit description of market participants aggregate attitude towards risk. This concept, captured by the market price of risk in particular, provides a clean and intuitive way to understand deviations from the expectations hypothesis and simultaneously ensure the absence of arbitrage.*

3.1 Definition of affine market

The affine factor model makes it possible to compute tractable pricing formulas, it extends the log-normal model (studied in [16]) to a more stochastic model. Affine model, which generalizes the CIR one, was first introduced by D. Duffie and R. Kan (1996) [3], where the authors assume that the yields are affine function of stochastic factors, which implies an affine structure of the factors. Among many others, M. Piazzesi reports in [25] some recent successes in the study of affine term structure models. Several constraints must be fulfilled to define an affine model in mutidimentional framework, but we will not discuss the details here and refer to the works of Teichmann and coauthors [17], [18].

Properties of affine processes and their exponential We adopt the framework of the example in Piazzesi ([25], p 704). The factor is a N-dimensionnal vector process denoted by ξ and is assumed to be an affine diffusion process, that is the drift coefficient and the variance-covariance matrix are affine function of ξ :

$$d\xi_t = \delta_t(\xi_t)dt + \sigma_t(\xi_t)dW_t \tag{3.1}$$

The affine constraint is expressed as:

- $\delta_t(\xi_t) = \varrho_t^\delta \xi_t + \delta_t^0$, where $\varrho_t^\delta \in \mathbb{R}^{N\times N}$ and $\delta_t^0 \in \mathbb{R}^N$ are deterministic.
- $\sigma_t(\xi_t) = \Theta_t s_t(\xi_t)$, where $\Theta_t \in \mathbb{R}^{N\times N}$ is deterministic, and the matrix $s_t(\xi_t)$ is a diagonal $N \times N$ matrix, with eigenvalues $s_{ii,t}(\xi_t)$. The affine property concerns the variance covariance matrix $\Theta_t s_t(\xi_t)\tilde{s}_t(\xi_t)\tilde{\Theta}_t$ or equivalently (since $s_t(\xi_t)$ is diagonal) the positive eigenvalues $\lambda_{i,t} = s^2_{ii,t}(\xi_t)$ of $s_t(\xi_t)\tilde{s}_t(\xi_t)$: $\lambda_{i,t} = \tilde{\rho}^\lambda_{i,t}\xi_t + \lambda^0_{i,t}$ that must be positive [1] with deterministic $(\lambda^0_{i,t}, \tilde{\rho}^\lambda_{i,t}) \in \mathbb{R}\times\mathbb{R}^N$, where $\tilde{.}$ denotes the transposition of a vector or a matrix.

Characterization of market with affine optimal processes To be coherent with the previous market model (Section 1.1), we have to define the set of admissible strategies $\mathcal{R}_t$, at date t, and its orthogonal $\mathcal{R}_t^\perp$. Let us first to point out that the $(N\times 1)$ volatility vector of any process $\tilde{a}_t\xi_t$ (a is deterministic) is given by $s_t(\xi_t)\tilde{\Theta}_t.a_t$. Thus if $\mathcal{R}$ (linear space) is the set of admissible strategies, then at time t it depends on ξ_t and is necessarily given by,

$$\mathcal{R}_t(\xi_t) = \{\tilde{a}_t\Theta_t s_t(\xi_t),\ a_t \text{ progressive vector in some linear and deterministic space } E_t \subset \mathbb{R}^N\}$$

The deterministic space E_t and its orthogonal are assumed to be stable by $\tilde{\Theta}_t$, or equivalently the matrix $\tilde{\Theta}_t$ is commutative with the orthogonal projection on E_t. A block matrix Θ_t (up to an orthogonal transformation) satisfies this property. Furthermore, E_t and $E_t^\perp$ are stable by ϱ_t^δ. The set of admissibles strategies being well defined, we denote by $a_t^\mathcal{R}$ and by $a_t^\perp$ the elements of the linear spaces E_t and $E_t^\perp$.

We consider two types of assumptions:

(i) The spot rate (r_t) and the consumption rate (ζ_t) are affine positive processes
$$r_t = a_t^r\xi_t + b_t^r \quad \text{ and } \quad \zeta_t = a_t^\zeta\xi_t + b_t^\zeta.$$

(ii) The volatilities of the optimal processes X^* and Y^* have affine structure,

$$\kappa_t^* = \tilde{a}_t^{X,\mathcal{R}}\Theta_t s_t(\xi), \quad \eta_t^\mathcal{R} = \tilde{a}_t^{Y,\mathcal{R}}\Theta_t s_t(\xi), \quad \nu_t^* = \tilde{a}_t^{Y,\perp}\Theta_t s_t(\xi), (a^{\cdot,\mathcal{R}} \in E_t \text{ and } a^{\cdot,\perp} \in E_t^\perp). \tag{3.2}$$

[1]This constraint is restrictive, but we do not go into the details of this hypothesis as this is not the goal of this work, for more details see the literature cited above concerning affine models.

Forward utility and marginal utility yields curve In the forward utility framework with linear portfolios, the marginal utility price of zero-coupon bond with maturity T is given by Equation (2.5), where $Y_t^*(y)$ is linear in y. The price of zero-coupon does not depend on y. Taking into account the specificities of the affine market, we have that

$$\begin{aligned} B_t^u(T) &= \mathbb{E}\big[\frac{Y_T^*}{Y_t^*}|\mathcal{F}_t\big] \qquad (3.3)\\ &= \mathbb{E}\Big[\exp\big(-\int_t^T (a_s^r\xi_s + b_s^r)ds + \int_t^T \tilde{a}_u^Y\Theta_u s_u(\xi)dW_u - \frac{1}{2}\int_t^T \|\tilde{a}_u^Y\Theta_u s_u(\xi)\|^2 du\big)|\mathcal{F}_t\Big] \end{aligned}$$

Thanks to the Markovian structure of the affine diffusion ξ, the price of zero-coupon bond is an exponential affine fonction of ξ, with the terminal constraint $B_T^u(T) = 1$,

$$\ln(B_t^u(T)) = \tilde{A}_t^T\xi_t + B_t^T, \quad A_T^T = 0,\ B_T^T = 0.$$

Ricatti equations To justify the Ricatti equations, we fix some notations and develop useful calculation for study exponential affine process. Moreover, when working with affine function $f(t,\xi) = \tilde{a}_t\xi + b_t$, it is sometimes useful to write $b_t = f_t(0) = f_t^0$ and $a_t = \nabla_\xi f_t(0) = \nabla f_t^0$.

Lemma 3.1. *Let $a_t \in \mathbb{R}^N$ and $b_t \in \mathbb{R}$ be two deterministic functions.*
(i) *The affine process $\tilde{a}_t\xi_t + b_t$ is a semimartingale with decomposition,*

$$d(\tilde{a}_t\xi_t + b_t) = \tilde{a}_t\Theta_t s_t(\xi)dW_t - \frac{1}{2}||\tilde{a}_t\Theta_t s_t(\xi)||^2 dt + f_t(a_t,\xi_t)dt$$

a) *The quadratic variation $||\tilde{a}_t\Theta_t s_t(\xi)||^2 = q_t(a,\xi_t)$ is quadratic in a and affine in ξ. More precisely, if Θ_t^i is the i-th column of the matrix Θ, we have*

$$q_t(a,\xi) = \nabla q_t^0(a)\xi + q_t^0(a),\ \text{with}\ \nabla q_t^0(a) = \Sigma_{i=1}^{i=N}\big(\tilde{\Theta}_t^i a_t\big)^2\rho_{i,t}^\lambda,\ \text{and}\ q_t^0(a) = \Sigma_{i=1}^{i=N}\big(\tilde{\Theta}_t^i a_t\big)^2\lambda_{i,t}^0$$

b) *The drift term $f_t(a,\xi_t)$ is an affine quadratic form in a and affine in ξ given by*

$$f_t(a,\xi_t) = (\partial_t\tilde{a}_t + \tilde{a}_t\varrho_t^\delta + \frac{1}{2}\nabla q_t^0(a)\xi_t + \partial_t b_t + \tilde{a}_t\delta_t^0 + \frac{1}{2}q_t^0(a) \qquad (3.4)$$

(ii) *A process $X_t = \tilde{a}_t\xi_t + b_t + \int_0^t \delta_s^X(\xi_s)ds$ ($\delta_t^X(\xi) = \nabla\delta_t^{0,X}\xi + \delta_t^{0,X}$) is the log of an exponential martingale if and only if the coefficients satisfy the Ricatti equation*

$$\partial_t\tilde{a}_t + \tilde{a}_t\varrho_t^\delta + \frac{1}{2}\nabla q_t^0(a) + \nabla\delta_t^{0,X} = 0, \quad \partial_t b_t + \tilde{a}_t\delta_t^0 + \frac{1}{2}q_t^0(a) + \delta_t^{0,X} = 0 \qquad (3.5)$$

Proof. (i) From Itô's formula, $d(\tilde{a}_t\xi_t + b_t) = \partial_t(\tilde{a}_t)\xi_t dt + \tilde{a}_t d\xi_t + \partial_t b_t dt$,
which implies from the dynamics (3.1) of ξ,

$$\begin{aligned} d(\tilde{a}_t\xi_t + b_t) &= \big(\partial_t(\tilde{a}_t)\xi_t + \tilde{a}_t\delta_t(\xi_t) + \partial_t b_t\big)dt + \tilde{a}_t\Theta_t s_t(\xi)dW_t\\ &= \Big(\partial_t(\tilde{a}_t)\xi_t + \tilde{a}_t\delta_t(\xi_t) + \partial_t b_t + \frac{1}{2}||\tilde{a}_t\Theta_t s_t(\xi)||^2\Big)dt + \Big(\tilde{a}_t\Theta_t s_t(\xi)dW_t - \frac{1}{2}||\tilde{a}_t\Theta_t s_t(\xi)||^2 dt\Big). \end{aligned}$$

a) The sequel is based on the decomposition of the quadratic variation of $\tilde{a}_t\xi_t + b_t$ as affine form in ξ_t and quadratic in a.
b) Then, the affine decomposition of the drift term may be rewritten in the same way.

(ii) $\exp(X_t)$ will be an exponential martingale, if and only if $dX_t = \tilde{a}_t\Theta_t s_t(\xi)dW_t - \frac{1}{2}||\tilde{a}_t\Theta_t s_t(\xi)||^2dt$, or equivalently if and only if $f_t(a,\xi)+\delta^X(t,\xi) \equiv 0$. Given that $f_t(a,\xi)+\delta^X(t,\xi)$ is affine in ξ, the condition is satisfied if both, the coefficient of ξ and the constant term are null functions. □

Application to bond pricing Solving Ricatti Equation (3.5) with any given initial condition $\tilde{a}_0\xi_0+b_0$ implies that the exponential affine process $\exp(\tilde{a}_t\xi_t+b_t+\int_0^t\delta_s^X(\xi_s)ds)$ is a local martingale.
In a backward formulation, solving Ricatti Equation (3.5) with terminal constraint $X_T = \tilde{a}_T\xi_T+b_T+\int_0^T\delta_s^X(\xi_s)ds$ implies under additional integrability assumption, that, if (a_t^T, b_t^T) are solutions of the Riccatti system with terminal condition (a_T, b_T).

$$\mathbb{E}\big[\exp(X_T - X_t)|\mathcal{F}_t\big] = \exp\big(\tilde{a}_t^T\xi_t + b_t^T\big). \tag{3.6}$$

In the theory of bond pricing, from Equation (3.3), we are looking for an affine process $\tilde{A}_t^T\xi_t + B_t^T$ such that, since $r_t = a_t^r\xi_t + b_t^r$,

$$\exp\big(\tilde{A}_t^T\xi_t+B_t^T\big) = \mathbb{E}\Big[\exp\big(\int_t^T -(a_t^r\xi_t+b_t^r)du+\tilde{a}_u^Y\Theta_u s_u(\xi)dW_u-\frac{1}{2}\|\tilde{a}_u^Y\Theta_u s_u(\xi)\|^2du\big)|\mathcal{F}_t\Big]$$

Given that the martingale part is the one associated with the affine process $\tilde{a}_t^Y\xi_t$, it is possible to define an affine process X^Y with affine drift such that,

$$X_t^Y = \tilde{a}_t^Y\xi_t - \int_0^t f_s(a_s^Y,\xi_s)\,ds = X_0^Y + \int_0^t \tilde{a}_u^Y\Theta_u s_u(\xi)dW_u - \frac{1}{2}\|\tilde{a}_u^Y\Theta_u s_u(\xi)\|^2du \tag{3.7}$$

where $f_t(a,\xi)$ is defined in Equation (3.4) with $b=0$. Since $\exp(X_t^Y)$ is a local martingale, $\tilde{a}_t^Y$ is solution of the Ricatti equation with $b_t \equiv 0$. In the new formulation, we are looking for some process $X_t^T = \tilde{A}_t^T\xi_t + B_t^T$ such that the exponential of the process $Z_t = X_t^T + X_t^Y - \int_0^t r_s ds$ is a local martingale. The problem belongs to the family of previous problem applied to the process Z. We summarize these results below.

Theorem 3.2. *Assume an affine optimization framework, where the optimal state price has an affine volatility $\tilde{a}_t^Y\Theta_t s_t(\xi)$ with $\tilde{a}_t^Y$ solution of some Ricatti equation.*
(i) *Any zero-coupon bond is an exponential function $\exp\big(\tilde{A}_t^T\xi_t + B_t^T\big)$ such that $\tilde{a}_t^Z = \tilde{A}_t^T + \tilde{a}_t^Y$ is solution of a Ricatti function with terminal condition $\tilde{a}_T^Y$, and δ^Z function $\delta_t^Z(\xi) = -f_t(a_t^Y\xi) + a_t^r\xi_t + b_t^r$.*
(ii) *The volatility of a bond with maturity T is $\Gamma_t(T) = \tilde{A}_t^T\Theta_t s_t(\xi)$.*

Proof. The process $Z_t = X_t^T + X_t^Y - \int_0^t r_s ds$ is an affine process with affine integral term. The decomposition of Z is of the type $Z_t = \tilde{a}_t^Z\xi_t + b_t^Z + \int_0^t\delta_s^Z(\xi_s)ds$ with $\tilde{a}_t^Z = \tilde{A}_t^T + \tilde{a}_t^Y$, $b_t^Z = B_t^T$, $\delta_t^Z(\xi) = -f_t(a_t^Y,\xi) + a_t^r\xi_t + b_t^r$. So $\tilde{a}_t^Z$ satisfies a Ricatti equation with terminal value $\tilde{a}_T^Y$. □

3.2 Affine model and power utilities

As power utilities is the classical most important example for economics, we now study the marginal utility yield curve in affine model with progressive and backward power utilities. The backward case differs significantly of the forward case, since constraints appear on the optimal processes at maturity T_H.

Backward formulation In the classical backward problem with classical power utility function x^θ, $(0 < \theta < 1)$ and horizon T_H, we have the terminal constraint on the optimal processes: the terminal values of the optimal wealth process $X^{*,H}_{T_H}$ and of the state price process $Y^{*,H}_{T_H}$ satisfy (from optimality)

$$(Y^{*,H}_{T_H})^{1/\theta} X^{*,H}_{T_H} = \mathrm{Cst} = \exp k$$

We recall here by the index H the dependency on the horizon T_H; moreover for simplicity we make $k = 0$.

The question is then how this constraint is propagated at any time in an affine framework, with affine consumption rate ζ^*_t. For notational simplicity, we denote $X^{*,\zeta,H}$ (the optimal wealth process capitalized at rate ζ^*_t by the process $S^{0,\zeta^*}_t = \exp \int_0^t \zeta^*_u du$.
Using that $S^{0,\zeta^*} X^{*,H} Y^{*,H}$ is a martingale with terminal value $S^{0,\zeta^*}_{T_H}(Y^{*,H}_{T_H})^{1-1/\theta}$, we study the martingale $M^\theta_t = \mathbb{E}\big(S^{0,\zeta^*}_{T_H}(Y^{*,H}_{T_H})^{1-1/\theta}|\mathcal{F}_t\big)$ in two ways:
(i) the first one is based on the process $X^{*,H}_t$, since $M^\theta_t = S^{0,\zeta^*}_t X^{*,H}_t Y^{*,H}_t$
(ii) the second one is very similar to the study of zero-coupon bond, by observing that by the Markov property $\mathbb{E}\big[(S^{0,\zeta^*}_{T_H}/S^{0,\zeta^*}_t)(Y^{*,H}_{T_H}/Y^{*,H}_t)^{1-1/\theta}|\mathcal{F}_t\big]$ is an exponential affine, whose coefficients (A^θ, B^θ) are solutions of a Ricatti equation, that is

$$\mathbb{E}\big((S^{0,\zeta^*}_{T_H}/S^{0,\zeta^*}_t)(Y^{*,H}_{T_H}/Y^{*,H}_t)^{1-1/\theta}|\mathcal{F}_t\big) = \exp(\tilde{A}^\theta_t \xi_t + B^\theta_t)$$

The backward constraint is then equivalent to the stochastic equality

$$S^{0,\zeta^*}_t X^{*,H}_t Y^{*,H}_t = S^{0,\zeta^*}_t (Y^{*,H}_t)^{1-1/\theta} \exp(\tilde{A}^\theta_t \xi_t + B^\theta_t).$$

Proposition 3.3. (i) *The terminal optimal constraint $X^{*,H}_{T_H} = S^{0,\zeta^*}_{T_H}(Y^{*,H}_{T_H})^{-1/\theta}$ is propagated through the time into a closed relation with the state price process,*

$$X^{*,H}_t = (Y^{*,H}_t)^{-1/\theta} \exp(\tilde{A}^\theta_t \xi_t + B^\theta_t), \tag{3.8}$$

where $\exp(\tilde{A}^\theta_t \xi_t + B^\theta_t) = \mathbb{E}\big[(S^{0,\zeta^*}_{T_H}/S^{0,\zeta^*}_t)(Y^{*,H}_{T_H}/Y^{*,H}_t)^{1-1/\theta}|\mathcal{F}_t\big]$.
(ii) *In particular, the constraint that the volatility κ^*_t has an affine structure belonging to the space E, implies that $(\tilde{A}^\theta_t)^\perp = 1/\theta(\tilde{a}^Y_t)^\perp$.*
(iii) *The links with the zero-coupon bond is given by the relation,*

$$B(t,T) = \exp(\tilde{A}^T_t \xi_t + B^T_t) = \mathbb{E}\big(\frac{Y^{*,H}_T}{Y^{*,H}_t}|\mathcal{F}_t\big). \tag{3.9}$$

4 Yield curve dynamics non-linear on initial conditions

4.1 From linear optimal processes to more general progressive utilities

Until then, we have omitted the dependence of optimal processes with respect to risk aversion. In what follows, risk aversion plays an important role, therefore, an agent that has as risk aversion denoted θ, his utility process will be denoted U^θ, his optimal wealth is denoted $X^{*\theta}$ and finally his optimal dual process $Y^{*,\theta}$. For simplicity we are concerned in the following only by utility processes which are of power type. As we have already mentioned above, utilities of power type generate optimal processes $X^{*,\theta}$ and $Y^{*,\theta}$ which are linear with respect to their initial conditions, i.e, $X^{*,\theta}(x) = xX^{*,\theta}$ and $Y^{*,\theta}(y) = yY^{*,\theta}$ (with $X^{*,\theta} = X^{*,\theta}(1)$ and $Y^{*,\theta} = Y^{*,\theta}(1)$). Thus the marginal utility price at time t of a zero coupon with maturity T, given in (2.5) by

$$B^\theta(t,T)(y) := B^u(t,T)(y) = \mathbb{E}\Big[\frac{Y_T^{*,\theta}(y)}{Y_t^{*,\theta}(y)}|\mathcal{F}_t\Big] = \mathbb{E}\Big[Y_{t,T}^{*,\theta}|\mathcal{F}_t\Big], \text{ with } Y_{t,t}^{*,\theta} = 1$$

does not depend on y nor on the consumption of the market at time t. This is not surprising given that power utilities, although they are useful to compute explicit optimal strategies, are somehow restrictive because they generates only linear optimal processes. Besides, the economic litterature emphasizes the dependence of the equilibrium rate $R_0^e(T)$ on the initial consumption. To study this dependence, we have to give a nontrivial example of stochastic utility that generates a nonlinear state price density process and then calculate the price of zero coupon. This is not obvious, especially since our goal is to give an explicit formula for the optimal dual process. The idea is to first generate, from optimal process $X^{*,\theta}$ and $Y^{*,\theta}$ associated with progressive power utilities U^θ, a new processes $\bar{X}$ and $\bar{Y}$ which are both admissible, monotone and especially nonlinear with respect to their initial conditions. In a second step, we use the characterization (1.7) of Theorem 1.6 to thereby construct non-trivial stochastic utilities with $\bar{X}$ and $\bar{Y}$ as optimal processes. The method that we will develop in the following is the starting point of the work of El Karoui and Mrad [13], in which many other ideas and extensions can then be found.

step 1: To fix the idea, $(X^{*,\theta}, Y^{*,\theta}, \zeta^{*,\theta})$ denotes the triplet of optimal primal, dual and consumption processes associated with stochastic utility (U^θ, V^θ) of power type and relative risk aversion $U_x^\theta / xU_{xx}^\theta = \theta$. We consider also two strictly decreasing probability density functions f and g, $\int_0^{+\infty} f(\theta')d\theta' = \int_0^{+\infty} g(\theta')d\theta' = 1$, use the change of variable $\theta = z\theta'$ and define the strictly increasing functions $x^\theta(x) := f(\theta/x),\ x > 0$, $y^\theta(y) := g(\theta/y),\ y > 0$ satisfying the following identities

$$\int_0^{+\infty} x^\theta(x)d\theta = x,\ \forall x > 0$$

$$\int_0^{+\infty} y^\theta(y)d\theta = y, \ \forall y > 0$$

<u>**step 2:**</u> We are now concerned with the following processes $\bar{X}$, $\bar{Y}$ and $\bar{\zeta}$ defined by

$$\bar{X}_t(x) := \int_0^{+\infty} x^\theta(x) X_t^{*,\theta} d\theta, \ x > 0 \tag{4.1}$$

$$\bar{Y}_t(y) := \int_0^{+\infty} y^\theta(y) Y_t^{*,\theta} d\theta, \ y > 0 \tag{4.2}$$

where we recall that $X_t^{*,\theta}$ (resp. $Y_t^{*,\theta}$) denotes, in this integral, the optimal process starting from the initial condition equal to 1. As seen previously, these two processes are an admissible wealth and a state density process which are strictly increasing with respect to their initial conditions of which they depend on non-trivial way far from being linear. The consumption $\bar{\zeta}$ intuitively associated with $\bar{X}$ is given by

$$\bar{\zeta}_t(\bar{X}_t(x)) = \int_0^{+\infty} \zeta_t^{*,\theta}(X_t^{*,\theta}(x^\theta(x)))d\theta = \int_0^{+\infty} x^\theta(x)\zeta_t^{*,\theta}(X_t^{*,\theta})d\theta \tag{4.3}$$

where the last equality comes from the linearity, for a fixed θ, of $\zeta^{*,\theta}$ and $X^{*,\theta}$. To complete the construction of the progressive utility for which $(\bar{X}, \bar{Y}, \bar{\zeta})$ will be the optimal processes, a martingale property on the process $(e^{-\int_0^t \bar{\zeta}_s ds}\bar{X}_t\bar{Y}_t)$ is necessary. We, then, make the following assumption

Assumption 4.1. *The optimal policies $\kappa^{*,\theta}$ and $\nu^{*,\theta}$ and the market risk premium η are uniformly bounded.*

Assumption 4.1 implies that $(e^{-\int_0^{\cdot} \zeta_s^{*,\theta} ds}X^{*,\theta}Y^{*,\theta'})$ are martingales for all θ, θ' and consequently $(e^{-\int_0^t \bar{\zeta}_s ds}\bar{X}_t\bar{Y}_t)$ is also a martingale.

<u>**step 3:**</u> The last step is then to consider any classical utility functions u and v (not necessarily of power type nor generating linear optimal processes) and only impose that their derivatives u_x and v_c have good integrability conditions close to zero. All the ingredients were met, from (1.7) of Theorem 1.6, by considering the monotonic process $\bar{C}$ defined by

$$\bar{C}(v_c^{-1}(u_x(x))) := \bar{\zeta}_t(\bar{X}_t(x)) = \int_0^{+\infty} \zeta_t^{*,\theta}(X_t^{*,\theta}(x^\theta(x)))d\theta,$$

the pair of random fields defined by

$$\begin{cases} U(t,x) = \int_0^x \bar{Y}_t(u_x(\bar{\mathcal{X}}(t,z))dz, \\ V(t,c) = \int_0^c \bar{Y}_t(v_c(\bar{\mathcal{C}}_t(\theta))d\theta \end{cases} \tag{4.4}$$

is a consistent progressive utility of investment and consumption generating $(\bar{X}, \bar{Y}, \bar{C})$ as optimal wealth, dual and consumption processes, with dual $(\widetilde{U}, \widetilde{V})$:

$$\begin{cases} \widetilde{U}(t,y) = \int_y^\infty \bar{X}_t(-\tilde{u}_y(\bar{\mathcal{Y}}(t,z))dz, \\ \widetilde{V}(t,c) = \int_c^\infty \bar{C}_t(-\tilde{v}_c(\bar{\mathcal{Y}}(t,\alpha))d\alpha \end{cases} \tag{4.5}$$

where $(\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{C}})$ denotes the inverse flows of $\bar{X}$, $\bar{Y}$ and $\bar{C}$.

Example Suppose that for any θ, θ' we have $X^{*,\theta} = X^{*,\theta'} = X^*, \forall\theta$ a.s., then in this case $\bar{X}(x) = xX^*$ with inverse $\bar{\mathcal{X}}(x) = x/X^*$, consequently the progressive utility U is given by:

$$U(t,x) = \int_0^x \int_0^{+\infty} y^\theta(u_x(x/X_t^*))Y_t^{*,\theta} d\theta dz.$$

4.2 Application to Ramsey rule evaluation

Let us now, turn to the Ramsey rule, and study the price of zero coupon. We recall at first that the price in our new framework is then given by

$$B(t,T)(y) = \mathbb{E}\Big[\frac{\bar{Y}_T(y)}{\bar{Y}_t(y)}|\mathcal{F}_t\Big] \tag{4.6}$$

From the formula of $\bar{Y}$ (4.2), the price $B(t,T)$ becomes

$$B(t,T)(y) = \frac{1}{\int_0^{+\infty} y^\theta(y)Y_t^{*,\theta} d\theta}\mathbb{E}\Big[\int_0^{+\infty} y^\theta(y)Y_T^{*,\theta} d\theta \big|\mathcal{F}_t\Big] \tag{4.7}$$

Now, let us introduce $B^\theta(t,T)$ the zero coupon bond (independent on y because $Y^{*,\theta}$ is linear on y) associated with risk aversion θ defined by

$$B^\theta(t,T) = \mathbb{E}\Big[\frac{Y_T^{*,\theta}(y)}{Y_t^{*,\theta}(y)}|\mathcal{F}_t\Big] \tag{4.8}$$

it follows that

$$B(t,T)(y) = \frac{1}{\int_0^{+\infty} y^\theta(y)Y_t^{*,\theta} d\theta}\int_0^{+\infty} y^\theta(y)Y_t^{*,\theta} B^\theta(t,T) d\theta \tag{4.9}$$

It is clear from this formula, that the zero coupon is a mixture of prices $B^\theta(t,T)$ weighted by $\frac{y^\theta(y)Y_t^{*,\theta}}{\int_0^{+\infty} y^\theta(y)Y_t^{*,\theta} d\theta}$ which is strongly dependent on y of non-trivial way.
At this level, several questions naturally arise: What is the sensitivity of the bond with respect to y? It is monotone, concave, convex? What about its asymptotic behavior? Give complete and satisfactory answers to these questions is beyond the scope of this work but will be addressed in a future paper.

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