Chapter 1

A portfolio optimization problem in a market with two prices generated by two information flow.

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We study a model in which the presence of different prices for the same asset or contingent claim is a consequence of different information settings. We consider a specific case involving defaults: here the two different levels of information concern the knowledge of the occurrence of a random time. Under no-short-sale restriction, we give conditions on the price dynamics in order to make the market arbitrage free despite the presence of price discrimination, and we construct the arbitrage portfolio if any. We then compute the optimal portfolio of an investor whose risk aversion is characterized by a logarithmic utility.

1. Introduction

In most financial models, an information set is given from the start and supposed to be common to all investors. Except models specifically dealing with the topic of insider trading (Grorud and Pontier, Imkeller et al., Hillairet, Elliott and Jeanblanc) or equilibrium models of the type of Kyle and Back (Cho, Kyle, Back), few models studied the impact of the information on the financial prices. Nevertheless, the amount and quality of the information affect the risk perception of investors when quantifying the financial risks or elaborating hedging strategies. For example, during the recent financial crisis, it was observed that defaults of important companies can have an important effect upon companies that might have seemed uncorrelated with the defaulted company and therefore, the default event represents itself a piece of information that needs to be explicitly taken into account in models.

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This paper proposes a toy financial model where the information upon which investors rely is explicitly modeled, as a key element. The information is modeled by a filtration, that is, an increasing sequence of sigma fields, reflecting the fact that one keeps track of the history. We first characterize the informational inefficiency of the market. In our broad definition of market, we can distinguish among two different market sectors (or two differently informed type of agents). The agents of the two market sectors have access to a different amount of information. Therefore the market sectors involve a different information settings, i.e., two filtrations $F$ and $G$. Furthermore, we assume that the agents of one market sector are better informed that the others, thus the following condition holds
\[ \forall t, F_t \subset G_t. \]

The main mathematical tool used is the theory of enlargements of filtrations. In each market sector, investors price the assets according to their information, and their pricing measure. We assume here that the pricing probability is the same for the whole market, i.e. the same for the two sectors. Even under a unique pricing measure $\mathbb{P}$, asymmetry of information gives rise to different pricing rules for the same product. We consider a contingent claim $\zeta \in F_T$ which may be not perfectly hedgeable (think of a weather derivative for example). The usual approach is to define the price of the claim as the discounted expected value of future cash flows, conditional to the information available to the agent. In the two market sectors, the time-$t$ price of the claim is calculated according to different information flows (assuming null interest rate, without loss of generality):
\[
\begin{align*}
\mathbb{E}[\zeta|F_t] & =: x_t \\
\mathbb{E}[\zeta|G_t] & =: X_t
\end{align*}
\]

Note that the information on $\zeta$ is fully known for both markets at time $T$ so that $X_T = x_T$, and that, if $F_0 = G_0$ are trivial (which is questionable, since, even at time 0, the market has information on the past) the prices of $\zeta$ are equal at time 0.

In general, the presence of multiple prices for the same asset is an indicator of an arbitrage opportunity, as a result of market inefficiency. In the case of a liquid market, arbitrage theory predicts that the price of the product must be unique. If it is not the case, arbitrageurs can make profit just by buying the asset from the market with a low price and in the same time short-selling it in the other market with a high price. However, analysis of real markets shows that discrepancies in prices for the same asset
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still arise. The presence of selling restrictions can prevent investors from exploiting arbitrages. Thus we introduce a no-short-sale restriction in our market; anyway we prove that this might be not sufficient to prevent market from arbitrage.

This paper studies the particular case where the additional information is synthesized in one event: the occurrence of a random time (which can be interpreted as a default time, or the time of switching to a new regime of the economy). The distribution of this time depends on the information available on a financial market, but its realization is not observable. The knowledge about its occurrence supposes some knowledge of the future evolution of the market prices, or of the states of the economy. Thus two financial entities or market sectors, having two different information sets (one entity being better informed than the other), will propose different bid prices (as for instance in the over the counter markets, where the prices are not transparent). In this framework, we study dynamically as information evolves the different prices evolution and their dependencies and we identify non arbitrage conditions. In the situation where the two prices converge to the same terminal value at some maturity date, there are trivial arbitrages in the market, which consist in short-selling the more expensive asset and buying the cheaper, then holding the portfolio until the maturity. We show that even with short-selling constraints, some non trivial arbitrages may exist, for instance based on the sign of the volatility (but these arbitrages may not be possible to realize). Since short-selling interdictions are natural on these markets where several prices may coexist, we provide a systematic study of the implications of these constraints in order to characterize the no-arbitrage conditions. Using results by Jouini and Kallal, and by Pulido, we are able to provide necessary and sufficient conditions for the parameters of the model such that the financial market is arbitrage free in absence of short-sales possibilities.

The rest of the paper is organized as follow: Section 2 introduces the informational framework. The two information flows are modelized through a Brownian filtration and its enlargement with a random time. This leads to two different price processes whose dynamics are specified. Attempts to compare those prices are given. Section 3 studies arbitrage opportunities in this frictionless market and under short-selling constraints, and presents the computation of the optimal portfolio for a logarithmic utility function.
2. The informational framework and the prices dynamics

2.1. The default-free filtration and its progressive enlargement with a random time

We consider a default-free (sub-)market, whose information flow is conveyed by a filtration $\mathbb{F}$, where $\mathbb{F}$ is the natural filtration of a Brownian motion $B$. As the price $(x_t)_{0 \leq t \leq T}$ of the contingent claim $\zeta$, relatively to this information, is an $\mathbb{F}$-martingale, the Predictable Representation Property (PRP) in $\mathbb{F}$ implies the following dynamics

$$dx_t = \nu_t dB_t, \quad 0 \leq t \leq T$$

where $(\nu_t)_{0 \leq t \leq T}$ is an $\mathbb{F}$-predictable process. Besides, other investors or an other submarket is aware of the eventuality of a default that may affect the valuation of the contingent claim $\zeta$. Here the information flow is characterized by the filtration $\mathbb{G}$, that is the filtration $\mathbb{F}$ progressively enlarged with a random time $\tau$ (a strictly positive random variable),

$$\mathbb{G}_t = \cap_{s > t} \{ \mathcal{F}_s \vee \sigma(\tau \wedge s) \}.$$

For example the random time $\tau$ represents the default time. In order to make explicit the dynamics of the contingent claim price $(X_t)_{0 \leq t \leq T}$ in this sub-market, we assume the following hypotheses.

**Assumption 1.** We assume that $(\mathcal{H}')$ hypothesis is satisfied, that is any $\mathbb{F}$-martingale is a $\mathbb{G}$-semi-martingale. More precisely, we assume that there exists an integrable $\mathbb{G}$-adapted process $\mu$ such that $dB_t = dW_t + \mu_t dt$ where $W$ is a $\mathbb{G}$-Brownian Motion$^a$.

We also assume that there exists a $\mathbb{G}$-adapted non-negative process $\lambda^G$ such that $M_t = 1_{\tau \leq t} - \int_0^t \lambda^G_s ds$ is a $\mathbb{G}$-martingale. Note that $\lambda^G$ vanishes after $\tau$ and that there exists an $\mathbb{F}$-adapted process $\lambda^F$, such that $\lambda^G_t = 1_{t \leq \tau} \lambda^F_t$.

Under mild conditions, any $\mathbb{G}$-martingale $Y$ can be represented as a sum of two stochastic integrals (Predictable Representation Property (PRP) in $\mathbb{G}$)

$$Y_t = y + \int_0^t \hat{y}_s dW_s + \int_0^t \tilde{y}_s dM_s$$

where $\hat{y}$ and $\tilde{y}$ are $\mathbb{G}$-predictable processes (see, e.g., Jeanblanc and Le Cam$^8$ or Song$^{14}$).

$^a$It can be noted that the reverse does not hold, that is the existence of $\mu$ does not imply $(\mathcal{H}')$ hypothesis, due to integrability conditions (see Jeulin and Yor$^{10}$).
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The above conditions are satisfied in the case where the random time $\tau$ admits a conditional density with respect to $F$, that is if
\[
\mathbb{P}(\tau > u|F_t) = \int_u^\infty p_t(x)dx
\]
where, for any $x$, $p(x)$ is a positive $F$-martingale. Due to the PRP (in the filtration $F$) of the Brownian motion $B$, the martingale $p(x)$ admits a representation of the form
\[
dp_t(x) = p_t(x)\sigma_t(x)dB_t \quad t \geq 0
\]
where for any $x$, $\sigma(x)$ is an $F$-predictable process. Note that, since, by hypothesis, $\mathbb{P}(\tau > 0) = 1$, one has $\int_0^\infty p_t(x)dx = 1$, for any $t \geq 0$. Under some smoothness conditions allowing to differentiate this equality under the integral sign (with respect to the running time $t$), this implies that
\[
\int_0^\infty p_t(x)\sigma_t(x)dx = 0 = \left(\int_0^\infty p_t(x)\sigma_t(x)dx \right)dB_t
\]
In that case, using Itô-Ventcell formula, the conditional survival probability $G_t = \mathbb{P}(\tau > t|F_t)$ admits the Doob-Meyer decomposition
\[
G_t = 1 + \int_0^t \xi_s dB_s - \int_0^t p_s(s)ds
\]
with $\xi_t = \int_t^\infty p_t(x)\sigma_t(x)dx$.

2.2. Prices dynamics

Under Hypothesis 1, the two prices dynamics $x_t = \mathbb{E}(\zeta | F_t)$ and $X_t = \mathbb{E}(\zeta | G_t)$ of the contingent claim $\zeta$, with respect to the information flow $F$ and $G$, are the following : there exist processes $\nu, \alpha, \beta$ such that $\nu$ is $F$-predictable, $\alpha$ and $\beta$ are $G$-predictable, and for $t \in [0,T]$

\[
dx_t = \nu_t dW_t + \mu_t dt \quad (2)
\]
\[
x_t = \alpha_t dW_t + \beta_t dM_t \quad (3)
\]

We recall from Jeulin\textsuperscript{9} Lemma 4.4 and Song\textsuperscript{14} the decomposition of any $G$-predictable or optional process. Let $\varphi$ be a $G$-predictable (resp. optional) process, and for a family of processes $\{\varphi_t(\theta), \theta \leq t \leq T, \theta \in [0,T]\}$ where $\varphi_t(\theta)$ is $\mathcal{P}(F) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable (resp. $\mathcal{O}(F) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable), such that
\[
\varphi_t = \varphi_t^{F}1_{t \leq \tau} + \tilde{\varphi}_t(\tau)1_{\tau > t}, \quad 0 \leq t \leq T
\]
(respectively $\varphi_t = \varphi_t^{G}1_{t \leq \tau} + \tilde{\varphi}_t(\tau)1_{\tau > t}, \quad 0 \leq t \leq T$).
The question of comparison of this two prices naturally follows. Unfortunately, it is impossible to find a model where the information drift \((\mu_s)\) has a constant sign, as proved in the Lemma 1:

**Lemma 1.** If \(B\) is an \(\mathcal{F}\)-Brownian motion and a \(\mathcal{G}\)-semimartingale such that there exists a \(\mathcal{G}\)-Brownian motion \(W\) and a \(\mathcal{G}\)-adapted process \((\mu_s)\) satisfying \(dB_t = dW_t + \mu_t\,dt\), then for any \(s\), \(\mathbb{E}(\mu_s|\mathcal{F}_s) = 0\).

**Proof.** First recall that \(\mathbb{E}(\int_0^t \mu_s\,ds|\mathcal{F}_t) - \int_0^t \mathbb{E}(\mu_s|\mathcal{F}_s)\,ds\) is an \(\mathcal{F}\)-martingale. Then taking expectation of \(B_t = W_t + \int_0^t \mu_s\,ds\) implies that \(B_t = \mathbb{E}(W_t|\mathcal{F}_t) + \text{mart}\mathcal{F}_t + \int_0^t \mathbb{E}(\mu_s|\mathcal{F}_s)\,ds\) which implies that \(\int_0^t \mathbb{E}(\mu_s|\mathcal{F}_s)\,ds = 0\), hence \(\mathbb{E}(\mu_s|\mathcal{F}_s) = 0\) for any \(s\). Indeed, the finite variation process \(\int_0^t \mathbb{E}(\mu_s|\mathcal{F}_s)\,ds\) is a continuous \(\mathcal{F}\)-martingale, as sum of different martingales in the same Brownian filtration \(\mathcal{F}\), hence it is constant. Being null at \(t = 0\), this process is null, and its derivative (which exists) is null too. \(\square\)

### 2.3. Comparison of prices

One can ask the following question: can we compare the price of the contingent claim \(\zeta\) computed in the filtration \(\mathcal{F}\) and the price of \(\zeta\) computed in the filtration \(\mathcal{G}\)?

It is known that, \(\mathcal{G}\) being the progressive enlargement filtration of \(\mathcal{F}\),

\[
\mathbb{E}(\zeta|\mathcal{G}_t) = 1_{t<\tau} \frac{\mathbb{E}(1_{t<\tau}|\mathcal{F}_t)}{G_t} + 1_{\tau\leq t} \mathbb{E}(\zeta|\mathcal{G}_t), \quad \text{where} \quad G_t = \mathbb{P}(t < \tau|\mathcal{F}_t).
\]

Thus before \(\tau\), this prices comparison reduces to the comparison between \(\mathbb{E}(\zeta|\mathcal{F}_t)\) and \(\frac{\mathbb{E}(1_{t<\tau}|\mathcal{F}_t)}{G_t}\), or, equivalently to the comparison between \(G_t\mathbb{E}(\zeta|\mathcal{F}_t)\) and \(\mathbb{E}(1_{t<\tau}|\mathcal{F}_t)\).

This result is related with the sign of the conditional covariance

\[
\mathbb{E}(1_{t<\tau}|\mathcal{F}_t) - G_t \mathbb{E}(\zeta|\mathcal{F}_t) = \mathbb{E}(1_{t<\tau}|\mathcal{F}_t) - \mathbb{E}(1_{t<\tau}|\mathcal{F}_t) \mathbb{E}(\zeta|\mathcal{F}_t) = \text{Cov}_t(\zeta, 1_{t<\tau}).
\]

We give an example. Recall that \(B\) is a Brownian motion with natural filtration \(\mathcal{F}\). Assume that \(\mathbb{P}(\tau > \theta|\mathcal{F}_t) = G_t(\theta) := \Phi\left(\frac{m_t - h(\theta)}{\sigma(t)}\right)\) where \(m_t = \int_0^t f(u)\,dB_u\), \(f\) is a deterministic function such that \(\int_0^\infty f^2(s)\,ds = 1\) and \(\Phi\) is the survival function for a standard Gaussian law. The positive function \(\sigma\) is defined as \(\sigma^2(t) = \int_t^\infty f^2(s)\,ds\) and \(h\) is a deterministic increasing function such that \(h(0) = -\infty\) (see El Karoui et al.\(^4\) for more details on
the construction of \( \tau \). Then, for a fixed \( \theta \), \( (G_t(\theta))_{t \geq 0} \) is an \( \mathbb{F} \)-martingale and

\[
d G_t(\theta) = -G_t(\theta) \frac{1}{\Phi \left( \frac{m_t - h(\theta)}{\sigma(t)} \right)} \varphi \left( \frac{m_t - h(\theta)}{\sigma(t)} \right) \frac{f(t)}{\sigma(t)} dB_t = -G_t(\theta) \Sigma_t(\theta) dB_t,
\]

where \( \Sigma_t(\theta) = \left( \Phi \left( \frac{m_t - h(\theta)}{\sigma(t)} \right) \right)^{-1} \varphi \left( \frac{m_t - h(\theta)}{\sigma(t)} \right) \frac{f(t)}{\sigma(t)} \). Assume now that the price \( x_t = \mathbb{E}(\zeta | \mathcal{F}_t) \), where \( \zeta \) is positive, has a positive volatility, i.e.

\[
dx_t = x_t \sigma_t dB_t
\]

with \( \sigma_t > 0 \) (this would be the case for a European call). If \( \zeta \in \mathcal{F}_T, \mathbb{E}(\mathbf{1}_{t < \tau} | \mathcal{F}_t) = \mathbb{E}(x_T G_T(t) | \mathcal{F}_t) \). Then, from integration by parts

\[
x_T G_T(t) = x_t G_t(t) + \int_t^T G_s(t) dx_s + \int_t^T x_s dG_s(t) - \int_t^T x_s G_s(t) \Sigma_s(t) ds,
\]

where the notation \( d_s G_s(t) \) makes precise that the stochastic differential is w.r.t. the running time \( s \). From our hypothesis on the sign of the volatility of \( x \), and assuming that \( f \) is non negative we obtain (since, obviously \( \Sigma_s(t) > 0 \))

\[
\mathbb{E}(x_T G_T(t) | \mathcal{F}_t) = x_t G_t(t) - \int_t^T \mathbb{E}(x_s \sigma_s(t) G_s(t) \Sigma_s(t) | \mathcal{F}_t) ds < x_t G_t(t) = x_t G_t
\]

therefore, on the set \( t < \tau \), one has \( X_t = \frac{1}{\mathbb{E}(x_T G_T(t) | \mathcal{F}_t)} < x_t \).

2.4. Link between the price dynamics coefficients

**Proposition 1.** Let \( x \) and \( X \) be two processes with dynamics given by (2) and (3). Then

\[
\nu_t = \mathbb{E}(\alpha_t + \mu_t X_t | \mathcal{F}_t).
\]

**Proof.** The equality \( x_t = \mathbb{E}(X_t | \mathcal{F}_t) \) (and \( x_0 = X_0 \)) implies \( \int_0^t \nu_s dB_s = \mathbb{E} \left( \int_0^t \alpha_s dW_s + \int_0^t \beta_s dM_s | \mathcal{F}_t \right) \) which is equivalent to

\[
\mathbb{E} \left( \int_0^t \nu_s dB_s \int_0^t z_s dB_s \right) = \mathbb{E} \left( \int_0^t \alpha_s dW_s + \int_0^t \beta_s dM_s \right) \int_0^t z_s dB_s
\]

for any \( \mathbb{F} \)-adapted bounded process \( z \). By integration by parts, the left-hand side is \( \mathbb{E}(\int_0^t \nu_s z_s ds) \).
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We now transform the right-hand side. On the one hand, from the relation between $W$ and $B$, using again integration by parts,

$$
E\left(\int_0^t \alpha_s dB_s \int_0^s z_s dW_s \right) = E\left(\int_0^t \alpha_s z_s ds + \int_0^t \alpha_s dW_s \int_0^s z_s \mu_s ds \right).
$$

Using integration by parts, one obtains

$$
E\left(\int_0^t \alpha_s dB_s \int_0^s z_s \mu_s ds \right) = E\left(\int_0^t \alpha_s \left(\int_0^s \alpha_u dW_u|\mathcal{F}_s\right) ds \right).
$$

On the other hand, using the fact that the martingales $W$ and $M$ are orthogonal,

$$
E\left(\int_0^t \beta_s dM_s \int_0^s z_s ds \right) = E\left(\int_0^t \beta_s dM_s \int_0^s z_s \mu_s ds \right) + E\left(\int_0^t \beta_s dM_s \int_0^t z_s dW_s \right)
$$

and by integration by parts once again

$$
E\left(\int_0^t \beta_s dM_s \int_0^s z_s ds \right) = E\left(\int_0^t \beta_s \int_0^s \beta_u dM_u|\mathcal{F}_s\right) ds \right).
$$

To summarize, for any $z$

$$
E\left(\int_0^t \nu_s ds \right) = E\left(\int_0^t z_s \left(\alpha_s + \mu_s \int_0^s \alpha_u dW_u + \beta_u dM_u\right) d\mathcal{F}_s\right)
$$

which implies, since $\nu$ is $\mathbb{F}$-adapted,

$$
\nu_s = E\left(\alpha_s + \mu_s \int_0^s \alpha_u dW_u + \beta_u dM_u\right) d\mathcal{F}_s = E\left(\alpha_s + \mu_s \left(X_s - X_0\right)\right) d\mathcal{F}_s.
$$

Using the fact that $E(X_s|\mathcal{F}_s) = 0$, we get

$$
\nu_s = E(\alpha_s + \mu_s X_s|\mathcal{F}_s).
$$

\[\square\]

3. Portfolio Optimization problem

We now study the problem of maximizing the utility from terminal wealth of an investor, in a market where the two price processes $(x_t)_{0 \leq t \leq T}$ and $(X_t)_{0 \leq t \leq T}$ coexist. More precisely, we consider a market where the two risky assets correspond to the two price processes $(x_t)_{0 \leq t \leq T}$ and $(X_t)_{0 \leq t \leq T}$; this market consists of :

- a risk free asset with $r = 0$, 


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- a first risky asset with price dynamics $dx_s = \nu_s(dW_s + \mu_s ds)$,
- a second risky asset with price dynamics $dX_s = \alpha_s dW_s + \beta_s dM_s$.

Are there reasonable constraints (such that the short-selling constraints) that allow the existence of these two different prices for a single contingent claim, in an arbitrage free market? Even though it seems difficult to draw a general comparison between the two price processes $(x_t)$ and $(X_t)$, adding short selling constraints seems necessary to limit arbitrage opportunities in this market, as mentioned in the introduction. Indeed, for example, suppose that at time $t$, prices are such that $A_t = \{x_t > X_t\}$ on a set of positive probability. An arbitrage strategy is the following: on the set $A_t$, buy $\zeta$ at price $X_t$, short sale $\zeta$ at price $x_t$ and invest the difference $x_t - X_t$ in the riskless asset. On the complementary set $A_t^c$, do nothing. At maturity $T$, the asset $\zeta$ is received and immediately delivered; the amount invested in the riskless asset represents the gain for the arbitrageur.

3.1. No Arbitrage conditions

We prove that short-sale restriction might be sufficient to prevent market from arbitrages, if the market parameters satisfy the following inequalities.

**Proposition 2.** If there exist two $\mathcal{G}$-predictable processes $\psi$ and $\gamma$ such that, $dt \otimes dP$ almost surely,

\[
\begin{align*}
\nu_t(\mu_t + \psi_t) &\leq 0 \\
\psi_t \alpha_t + \lambda_t \beta_t \gamma_t &\leq 0 \\
\gamma_t &\geq -1
\end{align*}
\]

then excluding short-selling of the assets $x$ and $X$ (defined in (2) and (3)) precludes arbitrage opportunities, in the market with zero interest rate ($r = 0$).

**Proof.** The proof is based on Jouini and Kallal\textsuperscript{11} and Pulido\textsuperscript{13} result. The authors have established the equivalence between the existence of a supermartingale probability measure and the absence of arbitrage opportunities while excluding short-selling. A supermartingale probability measure $Q$ is a probability measure, equivalent to the historical one $P$, under which discounted prices are super-martingales. We note $L$ the Radon Nikodym
density of \( Q \) w.r.t. \( \mathbb{P} \),
\[
dL_t = L_t(-\mu_t dW_t + \gamma_t dM_t) \quad \text{with} \quad \gamma > -1.
\]

Applying Itô’s formula, it is straightforward to see that \( xL \) and \( XL \) satisfy
\[
\begin{align*}
qd(x_t L_t) &= L_t \nu_t (\mu_t + \psi_t) dt + x_t L_{t-} (\psi_t dW_t + \gamma_t dM_t) + L_t \nu_t dW_t \\
L_t (X_t L_t) &= L_t (\psi_t \alpha_t + \lambda_t \beta_t \gamma_t) dt + X_t L_{t-} (\psi_t dW_t + \gamma_t dM_t) + L_{t-} (\alpha_t dW_t + \beta_t dM_t)
\end{align*}
\]
hence the result holds. \( \square \)

Let us study the implications of the non arbitrage equations (4). It is easy to see that
(a) On the set \( \{ (t, \omega) \text{ such that } \alpha_t(\omega) \mu_t(\omega) > 0 \} \), defining \( \psi_t(\omega) = -\mu_t(\omega) \) and \( \gamma_t(\omega) = 0 \) satisfy (4).
(b) On the set \( \{ (t, \omega) \text{ such that } \mu_t(\omega) \nu_t(\omega) < 0 \} \), defining \( \psi_t(\omega) = \gamma_t(\omega) = 0 \) satisfy (4).
(c) On the remaining set \( \{ (t, \omega) \text{ such that } \mu_t(\omega) \nu_t(\omega) < 0 \text{ and } \mu_t(\omega) \nu_t(\omega) > 0 \} = \{ (t, \omega) \text{ such that } \alpha_t(\omega) \nu_t(\omega) < 0 \text{ and } \mu_t(\omega) \nu_t(\omega) > 0 \} \), the non arbitrage equations (4) are studied as follow:

(1) On the set \( \{ (t, \omega) \text{ such that } \nu_t(\omega) < 0 < \alpha_t(\omega) \text{ and } \mu_t(\omega) \nu_t(\omega) > 0 \} \) then necessarily \( \mu_t(\omega) < 0 \). The first condition of (4) is \( \psi_t(\omega) \nu_t(\omega) \leq \gamma_t(\omega) > -\mu_t(\omega) > 0 \).

The non arbitrage condition, after \( \tau \) is \( \psi_t(\omega) \alpha_t(\omega) \leq 0 \), which is impossible. Before \( \tau \), the NA condition is \( \psi_t(\omega) \alpha_t(\omega) + \lambda_t(\omega) \beta_t(\omega) \gamma_t(\omega) < 0 \).

- On the subset \( \{ (t, \omega), \beta_t(\omega) < 0 \} \), the condition is satisfied by taking \( \gamma_t(\omega) = \frac{-\mu_t(\omega) \alpha_t(\omega) - \psi_t(\omega) \alpha_t(\omega)}{\lambda_t(\omega) \beta_t(\omega)} \).
- On the subset \( \{ (t, \omega), \beta_t(\omega) > 0 \} \) the NA condition requires that \( -\mu_t(\omega) \alpha_t(\omega) + \lambda_t(\omega) \beta_t(\omega) \gamma_t(\omega) \leq 0 \), for a \( \gamma_t(\omega) > -1 \) hence there are arbitrages on the subset \( \{ (t, \omega), \mu_t(\omega) \alpha_t(\omega) + \lambda_t(\omega) \beta_t(\omega) < 0 \text{ and } \beta_t(\omega) > 0 \} \).

(2) On the set \( \{ (t, \omega) \text{ such that } \alpha_t(\omega) < \nu_t(\omega) \text{ and } \mu_t(\omega) \nu_t(\omega) > 0 \} \) then necessarily \( \mu_t(\omega) > 0 \). The first condition of (4) is \( \psi_t(\omega) \leq -\mu_t(\omega) < 0 \).

The non arbitrage condition, after \( \tau \) is \( \psi_t(\omega) \alpha_t(\omega) \leq 0 \), which is impossible. Before \( \tau \), the NA condition is \( \psi_t(\omega) \alpha_t(\omega) + \lambda_t(\omega) \beta_t(\omega) \gamma_t(\omega) \leq 0 \).

- On the subset \( \{ (t, \omega), \beta_t(\omega) < 0 \} \), the condition is satisfied by taking \( \gamma_t(\omega) = \frac{-\mu_t(\omega) \alpha_t(\omega) - \psi_t(\omega) \alpha_t(\omega)}{\lambda_t(\omega) \beta_t(\omega)} \).
- On the subset \( \{ (t, \omega), \beta_t(\omega) > 0 \} \) the NA condition requires that \( -\mu_t(\omega) \alpha_t(\omega) + \lambda_t(\omega) \beta_t(\omega) \gamma_t(\omega) \leq 0 \), for a \( \gamma_t(\omega) > -1 \).
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Hence there are arbitrages on the subset \( \{(t, \omega), \mu_t(\omega)\alpha_t(\omega) + \lambda_t(\omega)\beta_t(\omega) < 0 \text{ and } \beta_t(\omega) > 0 \} \).

**Proposition 3.** Under the constraint of no short selling, the market is arbitrage free

- after \( \tau \) iff, \( dt \otimes d\mathbb{P} \) almost surely, \( \alpha \mu > 0 \) or \( \alpha \nu < 0 \),
- before \( \tau \) iff, \( dt \otimes d\mathbb{P} \) almost surely, \( \alpha \mu > 0 \) or \( \alpha \nu < 0 \) or \( \beta < 0 \) or \( \mu \alpha + \lambda \beta < 0 \).

In other words, arbitrages are possible (after \( \tau \)) if and only if \( \alpha > 0 > \nu, \mu < 0 \) or \( \alpha < 0 < \nu, \mu > 0 \) on a \( dt \otimes d\mathbb{P} \) non null set.

Arbitrages are possible (before \( \tau \)) if and only if \( \alpha > 0 > \nu, \mu < 0, \mu \alpha + \lambda \beta < 0, \beta > 0 \) or \( \alpha < 0 < \nu, \mu > 0, \mu \alpha + \lambda \beta < 0, \beta > 0 \) on a \( dt \otimes d\mathbb{P} \) non null set.

One can easily explicit arbitrage strategies in each situation where constraints (4) are not satisfied. On the event \( \Omega_t(\omega) := \{(t, \omega), \nu(t)(\omega) < 0 < \alpha_t(\omega)\} \) the portfolio that consists of investing \( \pi_t^1(\omega) = \alpha_t(\omega)1_{\Omega_t^1}(\omega) \) in the asset \( x \) and \( \pi_t(\omega) = -\nu_t(\omega)1_{\Omega_t^1}(\omega) \) in the asset \( X \) is admissible. The value of this portfolio is \( Y_t \) such that, for \( t > \tau \), \( dY_t = \alpha_t \mu_t \nu_t \ dt \). This is a risk-free asset, with return greater than the one of the savings account. For \( t < \tau \), \( dY_t = \nu_t(\alpha_t \mu_t + \beta_t \lambda_t)dt \) with \( \nu_t(\alpha_t \mu_t + \beta_t \lambda_t) > 0 \) if the conditions (4) are violated.

Similarly on the event \( \Omega_t(\omega) := \{(t, \omega), \alpha_t(\omega) < 0 < \nu_t(\omega)\} \) the portfolio that consists of investing \( \pi_t^1(\omega) = -\alpha_t(\omega)1_{\Omega_t^1}(\omega) \) in the asset \( x \) and \( \pi_t(\omega) = \nu_t(\omega)1_{\Omega_t^1}(\omega) \) in the asset \( X \) is admissible. The value of this portfolio is \( Y_t \) such that, for \( t > \tau \), \( dY_t = -\alpha_t \mu_t \nu_t \ dt \). This is a risk-free asset, with return greater than the one of the savings account. For \( t < \tau \), \( dY_t = -\nu_t(\alpha_t \mu_t + \beta_t \lambda_t)dt \) with \( -\nu_t(\alpha_t \mu_t + \beta_t \lambda_t) > 0 \) if the conditions (4) are violated.

**Remark:** We have found conditions under which there does not exist arbitrage before AND after \( \tau \). The arbitrages we exhibit (if these conditions are not satisfied) are based on the knowledge of the sign of the volatility, and that sign is not adapted to the filtration of the prices. Indeed, if the diffusion term can vanish, \( \mathbb{F}^x \) is strictly smaller that \( \mathbb{F}^B \) and does not contain the sign of \( \nu \). Note that the quantity \( \alpha \nu \) is the covariation of \( x \) and \( X \), hence is observable. Since these arbitrages can not be realized by investors having only informations on prices, we are in a well known situation in Economic: investors know that something exists, but they can not realize it.
3.2. Optimal portfolio for a logarithmic utility

We solve here the optimization problem for a logarithmic utility, as all computations can be done explicitly. Let \((\delta^1, \delta)\) be a self financing strategy, with \((\delta^1, \delta)\) the number of shares invested in the assets \((x, X)\) respectively. The corresponding wealth \(Y^{\delta^1, \delta}\) satisfies the following dynamics (we recall that the interest rate is zero)

\[
dY^{\delta^1, \delta}_t = \delta^1_t \nu_t \mu_t dt + (\delta^1_t \nu_t + \delta_t \alpha_t) dW_t + \delta_t \beta_t dM_t.
\]

The short selling strategies are excluded (otherwise, arbitrage strategies always exist): \(\delta^1 \geq 0\) and \(\delta \geq 0\). The set of admissible strategies is restricted to the no-short selling strategies leading to a positive wealth process. We will see that the no arbitrage constraints (4) naturally appear to exclude infinite value functions.

For a logarithmic utility \(U(x) = \ln(x)\), the optimization of \(\mathbb{E}(\ln(Y_T^{\delta^1, \delta}))\) can be computed explicitly. We normalize the portfolio process \((\delta^1, \delta)\) by the corresponding wealth, by considering the proportion of wealth \((\pi^1, \pi)\) invested in each asset: for \(t \in [0, T]\), \((\pi^1_t, \pi_t) = \frac{1}{\sqrt{1+\pi^1_t}}(\delta^1_t, \delta_t)\), leading to the exponential form of the wealth:

\[
\pi^1 \geq 0, \pi \geq 0 \quad \text{and} \quad dY^{\pi^1, \pi}_t = Y^{\pi^1, \pi}_t \pi^1_t \nu_t \mu_t dt + (\pi^1_t \nu_t + \pi_t \alpha_t) dW_t + \pi_t \beta_t dM_t.
\]

\[
Y^{\pi^1, \pi}_T = Y_0 \exp \left( \int_0^T (\pi^1_t \nu_t + \pi_t \alpha_t) dW_t - \frac{1}{2} (\pi^1_t \nu_t + \pi_t \alpha_t)^2 dt + \pi^1_t \nu_t \mu_t dt - \pi_t \beta_t \lambda^G dt + \ln(1 + \pi_t \beta_t)(dM_t + \lambda_t^G dt) \right)
\]

\[
\mathbb{E}(\ln(Y_T^{\pi^1, \pi})) = \ln(Y_0) + \mathbb{E} \left( \int_0^T \left( \pi^1_t \nu_t \mu_t - \frac{1}{2} (\pi^1_t \nu_t + \pi_t \alpha_t)^2 - \pi_t \beta_t \lambda^G + \lambda_t^G \ln(1 + \pi_t \beta_t) \right) dt \right)
\]

Thus the optimization problem relies on maximizing for every \(t \in [0, T]\)

\[
\pi^1_t \nu_t \mu_t - \frac{1}{2} (\pi^1_t \nu_t + \pi_t \alpha_t)^2 - \pi_t \beta_t \lambda^G + \lambda_t^G \ln(1 + \pi_t \beta_t), \quad \text{under the constraint} \quad \pi^1_t \geq 0, \pi_t \geq 0.
\]

(5)

We first exclude the cases for which \(\mathbb{E}(\ln(Y_T^{\pi^1, \pi}))\) is not bounded by above. This can be achieved only for \(\pi^1\) or \(\pi\) going to \(+\infty\) AND conditions on the market parameters such that the corresponding value tends to \(+\infty\). Explicitly, those conditions (in particular, we have to exclude the possibility of having \(\pi^1_t \nu_t + \pi_t \alpha_t = 0\) with the rest of the term in \(dt\) being positive), we again find the conditions of no arbitrage (4) and the arbitrage strategies.

Those cases being excluded, the Lagrangian of this optimization problem...
under constraints is

\[ \mathcal{L}(\pi^1_t, \pi_t) = \pi^1_t \nu_t \mu_t - \frac{1}{2} (\pi^1_t \nu_t + \pi_t \alpha_t)^2 - \pi_t \beta_t \lambda^G_t + \lambda^G_t \ln(1 + \pi_t \beta_t) + k^1_t \pi^1_t + k_t \pi_t \]

with \( k^1_t \pi^1_t = 0, k_t \pi_t = 0, k^1_t \geq 0, k_t \geq 0. \)

Due to the convexity of (5), the Kuhn Tucker first order conditions are sufficient (under no arbitrage conditions (4)):

\[
\begin{align*}
- (\pi^1_t \nu_t + \pi_t \alpha_t) \alpha_t - \beta_t \lambda^G_t + \frac{\lambda^G_t \beta_t}{1 + \pi_t \beta_t} + k_t &= 0 \\
\nu_t \mu_t - (\pi^1_t \nu_t + \pi_t \alpha_t) \nu_t + k^1_t &= 0 \\
k^1_t \pi^1_t &= 0, k_t \pi_t = 0, k^1_t \geq 0, k_t \geq 0.
\end{align*}
\]

Using the condition \( \pi_t \beta_t > -1 \) for \( t \leq \tau \) (so that the wealth remains positive after \( \tau \)), a systematic (but tedious) study of the Kuhn-Tucker conditions before \( \tau \) and after \( \tau \) (recall that \( \lambda^G = 0 \) after \( \tau \)) gives

- after \( \tau \), the optimal strategy is \( \pi_t = 0 \) and \( \pi^1_t = \frac{\mu_t}{\nu_t} \) if \( \nu_t \mu_t \) and \( \alpha_t \) have the same signs, and \( \pi_t = \pi^1_t = 0 \) otherwise
- before \( \tau \), the optimal strategy is \( \pi_t = \frac{-\mu_t \alpha_t}{\beta_t (\lambda^G_t \beta_t + \mu_t \alpha_t)} \) and \( \pi^1_t = \frac{\mu_t}{\nu_t} (1 + \frac{\alpha_t^2}{\beta_t (\lambda^G_t \beta_t + \mu_t \alpha_t)}) \) if \( \nu_t \mu_t > 0, \mu_t \alpha_t < 0 \) and \( \beta_t (\lambda^G_t \beta_t + \mu_t \alpha_t) > 0 \). Otherwise, \( \pi_t = \pi^1_t = 0. \)

Remark that the optimal portfolio often stands at the limit of the constrained domain, that is \( \pi^1 = 0 \) or \( \pi = 0. \)

Remark: For a general utility function \( U \), solving

\[
\sup_{\delta^1_t \geq 0, \delta_t \geq 0} E(U(Y^\delta^1_t, \delta_t)),
\]

leads to the problem of the replication with no short selling portfolios. Classical technics (duality or BSDE approaches) of portfolio optimization in incomplete market do not apply here. The particularity of our framework is that there does not exist any probability measure under which the prices are local martingales. In the literature in incomplete markets, there always exists at least one martingale measure and the existence of such measure is crucial in the replication methodology.

This problem of replication in an incomplete market with no martingale measure is a challenging problem that exceeds the aim of this paper.
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