A Modelisation of Public Private Parternships with failure time

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Abstract A commissioning of public works (Maitrise d'ouvrage publique, namely MOP in French) is a system where the community has commissioned equipment (hospital, prison ....) for its own needs and bear the cost, partly by self and partly by a loan from a bank. On another hand, Public-Private Partnership (PPP) means that community agrees on a period (15-25 years) with the contractor, and is billed rent. More or less it means "leasing" purchase, covering three parts: depreciation of equipment, maintenance costs, financial costs. This new formula is based on an "ordonnance" of June 17, 2004, amended by the law of 28 July 2008 (see legifrance.gouv.fr), justified by the emergency of requested equipment construction or its complexity. Our aim is to study the advantages and disadvantages of the new PPP formula. Here is a particular case of a risk-neutral consortium. We discuss of the advantages of outsourcing ('externalisation') in terms of model parameters and prove that externality is interesting only in case of large enough noise when we exclude the risk of bankruptcy. Indeed, this risk does not seem covered under current legislation. Finally, we study what could happen in case of failure penalities to be paid by the private consortium. In such a case, externality could be interesting in some context as high noise, high reference cost, short maturity, high enough penality.

1 Introduction

In the classic formula of a public project, a commissioning of public works (Maitrise d'ouvrage publique, namely MOP in French) the community realizes equipments (hospital, prison ....) for its own needs and bear the cost, partly by self and partly by a loan from a bank.

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In the formula “ partnership agreement ”(or public-private partnership, namely PPP) community agrees on a period (15 to 25 years) with the contractor, and is charged a rent. Somehow, it is a lease purchase, covering three parts: depreciation of equipment, maintenance costs, financial costs. This formula is based on an “ordonnance” of June 17, 2004, amended by the law of 28 July 2008 (see legifrance.gouv.fr). The justification for this device is mainly based on the urgency of requested equipment construction or its complexity. Here is studied the advantages and disadvantages of the PPP contract system. We have not included problems of taxation, (as the “ordonnance” and the law do). However, it should be noted that part of VAT (value added tax) is recoverable in case of MOP, whereas in the framework of a PPP, not only it is not, but it is added at the VAT payable on the loan. This particularity could have an influence, but this problem is not addressed in this paper. Here in particular, in the case of a risk-neutral consortium, we discuss the benefits of outsourcing in terms of model parameters. We show that when including the risk of bankruptcy, the externality can be interesting when a penalty is imposed on the consortium in case of bankruptcy and in a certain context: for instance when uncertainty is high enough, or the reference cost is important, or short maturity, or sufficient penalty. In fact, this corresponds to a risk transfer from public to private.

Section 2 sets the problem, following Iossa et al. model [2], and introduces the various parameters of the problem. In Section 3 we solve an optimization problem simultaneously for the consortium and the public community. Then we study in Section 4 the effects of introducing a bankruptcy time whose risk does not seem covered under the legislation above-named; this changes the model. If no penalty is required, the result of the optimization yields to choose a minimal externality in contrast to the result in case of absence of bankruptcy (Section 3). Finally, in Section 5, the consortium is obliged to pay penalties in case of bankruptcy: it is the only case discussed here where in a particular configuration of the game settings, outsourcing can be interesting for both parties. Section 6 gathers these results.

2 The problem setting

We follow here the framework of [2] by adding a stochastic view point.
To modelize the randomness of the model, we introduce a filtered probability space $(\Omega, \mathbf{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbf{P})$.

The **operational cost** $(C_s)_{s\in[0,T]}$ of the infrastructure maintenance is a nonnegative $\mathbf{F}$-adapted process. $(C_s)_{s\in[0,T]}$ is a rate (its unit is euros per unit time) and can be written as

$$C_s = \theta_0 - e_s - \delta a + \varepsilon_s, \quad s \in [0,T]. \tag{1}$$

where

- θ_0 = benchmark cost of the maintenance,

- e_s = effort on the maintenance done at date s to reduce the cost, it is a nonnegative $\mathbf{F}$-adapted process,
- a = effort on the construction to improve the infrastructure quality, it is a parameter in $\mathbf{R}^+$,
- $(\varepsilon_s)_{s\in[0,T]}$ is a centered bounded $\mathbf{F}$-adapted process that modelizes the random operational risk of the activity. We will assume that $\varepsilon \in [-m,M]$, $dt \otimes d\mathbf{P}\, a.s.$
- δ is the externality, it is a parameter in $\mathbf{R}^+$.

The externality represents the impact of the infrastructure on the maintenance cost. We assume that improving the infrastructure quality reduces the maintenance operational cost, thus the externality δ is non negative.
The maintenance cost is payable by the consortium until the maturity T or until a possible default of the consortium. We will assume the natural condition that the costs are nonegative a.s. This condition leads to some constraints detailed in section 3.2, using the expression of the optimal efforts.

The community pays to the consortium a rent $t(c)$ which is a function of the cost c : this rent permits both to pay the consortium for its work and to cover the maintenance costs that are in its charge. We assume that the community chooses a linear expression for the rent:

$$t(c) = \alpha - \beta c, \text{ with } \beta \geq -1, \text{ and } \alpha \text{ such that a.s. } t(C_s) \geq C_s \ \forall s \in [0,T].$$

$t(c) - c$ is a decreasing function of the costs C_s, and thus an increasing function of the efforts e_s. The larger β is, the greater is the incitement to the consortium to make effort on the infrastructure, but at the cost of a greater risk premium α.

Remark 1 *ε_s being in the interval $[-m,M]$ $\forall s \in [0,T]$ ($m>0$, $M>0$) the condition $t(C_s) \geq C_s$ $\forall s \in [0,T]$ is satisfied as soon as $\alpha \geq (\beta+1)(\theta_0+M)$.*

The consortium aims to maximize its terminal utility, discounted at the rate $r \geq 0$, its optimisation problem can be formulated as follows

$$\max_{(a,e)\in[0,+\infty[\times E} \left(\mathbf{E} \left(\int_0^T e^{-rs} \left(U(t(C_s) - C_s, s) - \phi(e_s) \right) ds \right) - \psi(a) \right) \tag{2}$$

with $E = \{(e_s)_{s\in[0,T]}\, \mathbf{F} \text{ adapted such that } \forall s \in [0,T]\, e_s \geq 0\, a.s.\}$.
The functions ϕ and ψ represent the effort cost, and following [2] we will choose $\phi(a) = \frac{a^2}{2}$ and $\psi(e) = \frac{e^2}{2}$. U is an utility function that modelizes the consortium risk aversion.

Definition 2. A function $U : (t,c) \to U(t,c)$ is called utility function if
(i) $U : [0,T] \times]0,+\infty[\to \mathbf{R}$ is continuous.
(ii) $\forall t \in [0,T]$, $U(t,\cdot)$ is strictly increasing and strictly concave.
(iii) The derivatives $\frac{\partial}{\partial t}U$, $\frac{\partial}{\partial c}U$, exist and are continuous on $[0,T] \times]0,+\infty[$.

This optimisation problem (2) will be reformulated in Section 4 in the case of a possible default of the consortium at a random time τ.

On the other hand, the community aims to maximize the social welfare defined as the social value of the project minus the rent payed at the consortium. The community optimization problem can be formulated as follows

$$\max_{(\alpha,\beta)\in\mathscr{A}} SW : (\alpha,\beta) \mapsto \left(\mathbf{E} \left[B_0 + \int_0^T e^{-rs} b(e_s) ds - \left(\int_0^T e^{-rs} t(C_s) ds - C_0 \right) \right] \right) \tag{3}$$

with $B_0 \geq 0$, $b : \mathbf{R}^+ \to \mathbf{R}^+$ C^1 and increasing, (for example $b(x) = bx,\ b > 0$), b represents the community utility.
$\mathscr{A} = \{(\alpha,\beta), \alpha \geq 0, \beta \geq -1$ such that $t(C_s) \geq C_s\ \forall s \in [0,T]\}$ in order that the consortium is refund of the maintenance costs ; C_0 is the initial cost payable by the consortium, B_0 is the initial social value of the project.

3 Solution of the problem without default

3.1 Maximization of the consortium utility

Proposition 1. *The parameters of the rent (α,β) being fixed, there exists an unique solution $(\hat{a},\widehat{e_s})$ at the optimization problem (1), given by*

$$\begin{cases} \widehat{e_s} = (\beta+1)U'(\alpha - (\beta+1)(\theta_0 - e_s - \delta a + \varepsilon_s)) \\ \widehat{a} = \delta \mathbf{E}(\int_0^T e^{-rs} e_s ds). \end{cases}$$

Furthermore,

$$\widehat{a} \mapsto \widehat{e_s}(a)$$

is decreasing : the more effort the consortium makes for the construction, the less effort it has to do for the maintenance.

Proof: We have to optimize the function

$$(a,e) \mapsto= \mathbf{E}\left(\int_0^T e^{-rs} \left(U(\alpha - (\beta+1)(\theta_0 - e_s - \delta a + \varepsilon_s)) - \frac{1}{2}e_s^2) \right) ds \right) - \frac{1}{2}a^2,$$

which is concave in a and in e_s and thus which is maximum when its gradient is zero. This leads to the couple (a,e_s) solution of system in Proposition 1. We claim that for all a, there exists an unique nonnegative solution $e_s(a)$. Indeed, since U is strictly concave and increasing, we have

$$U'(\alpha - (\beta+1)(\theta_0 - e_s - \delta a + \varepsilon_s)) > 0,$$

and $g : x \mapsto (\beta+1)U'(\alpha - (\beta+1)(\theta_0 - x - \delta a + \varepsilon_s))$ is decreasing. Thus $e_s(a)$ is the abscissa of the intersection of the bisector and the function g graph, and it is solution of the implicit equation

$$F(x,a) = x - (\beta+1)U'(\alpha - (\beta+1)(\theta_0 - x - \delta a + \varepsilon_s)) = 0.$$

The relation between e_s and a is reflected by the derivative

$$\frac{de}{da} = -\frac{\partial_a F}{\partial_x F} = \frac{(\beta+1)^2 \delta U''(\alpha - (\beta+1)(\theta_0 - x - \delta a + \varepsilon_s))}{1 - (\beta+1)^2 U''(\alpha - (\beta+1)(\theta_0 - x - \delta a + \varepsilon_s))}.$$

Since $U'' < 0$, $\frac{de}{da} < 0$, $a \mapsto e_s(a)$ is decreasing. We do the same for the function $h : a \mapsto \delta \mathbf{E}[\int_0^T e^{-rs} e_s(a) ds]$ and we conclude by the existence of an unique optimal $\hat{a}$ solution of equation $a = h(a)$. •

Notation: We introduce the following notation, useful for the rest of the paper

$$A_t := \int_0^t e^{-rs} ds.$$

Example 1 : linear utility $U(x) = \eta + \gamma x$.
In this case, the consortium is risk neutral. The rent rule being fixed, the optimal efforts for the consortium are given by

$$\begin{cases} \widehat{e_s} = \gamma(\beta+1) \quad \forall s \in [0,T] \\ \widehat{a} = \delta \widehat{e_s}(\int_0^T e^{-rs} ds) = \delta\gamma(\beta+1)A_T. \end{cases}$$

Example 2 : quadratic utility
$U(x) = x - \frac{\gamma}{2}x^2$ with $\gamma > 0$ such that $t(C_s) < \frac{1}{\gamma}$ $\forall s \in [0,T]$.
In this case, the risk aversion of the consortium $\frac{\gamma}{1-\gamma x}$ is an increasing function of his wealth. The rent rule being fixed, the optimal efforts for the consortium are given by

$$\begin{cases} \widehat{a} = \delta \frac{(\beta+1)(1-\gamma(\alpha-(\beta+1)\theta_0))^+ A_T}{1+\gamma(\beta+1)^2(1+\delta^2 A_T)} \\ \widehat{e_s} = \frac{(\beta+1)}{1+\gamma(\beta+1)^2}(1 - \gamma(\alpha - (\beta+1)\theta_0 + (\beta+1)\delta\widehat{a} - (\beta+1)\varepsilon_s)) \quad \forall s \in [0,T] \end{cases}$$

The noise (ε_s) being centered, $\mathbf{E}(e_s)$ does not depend on s. Furthermore, ε_s taking values in $[-m,M]$ $\forall s \in [0,T]$ ($m > 0$, $M > 0$), the condition $t(C_s) < \frac{1}{\gamma}$ $\forall s \in [0,T]$ is satisfied as soon as $\alpha - (\beta+1)(\theta_0 - m) + \frac{(\beta+1)^2}{1+\gamma(\beta+1)^2}((1+\delta^2)(1-\gamma(\alpha-(\beta+1)\theta_0))^+ + \delta(\beta+1)M) < \frac{1}{\gamma}$.

Remark 3 *Comparison between these two examples: in the case of a quadratic utility function,* $\mathbf{E}(e_s) \leq (\beta+1)$ *and is a decreasing function of the risk aversion, whereas for a linear utility with slop* $\gamma > 1$, $\mathbf{E}(e_s) = e_s > (\beta+1)$. *Thus the more* *risk averse the consortium is, the less effort it will do both for the construction and for the maintenance of the infrastructure.*

Finding an explicit solution of the community optimization problem being tedious in a general setting, we will from now on focus on the framework of linear utility functions both for the community and the consortium

$$U(x) = \gamma x,\ \gamma > 0\ ;\ b(x) = b.x, b > 0.$$

This leads to the following constraints on the externality.

3.2 Constraints on the externality

It seems natural to assume the cost being nonnegative a.s. This leads to some constraints on the parameters that we will explicit, using the expression of the optimal efforts in the framework of a linear utility. Moreover, in practice, the community can not outsource more than a given level δ_{max}. We fix δ_{max} such that $C_s \geq 0$ almost surely:

$$C_s \geq 0 \Longleftrightarrow \theta_0 \geq e_s + \delta a - \varepsilon_s.$$

Using the expressions of a and e (see example 1) :

$$C_s \geq 0 \Longleftrightarrow \theta_0 - m \geq \gamma(\beta+1)(1+\delta^2 A_T) \tag{4}$$

This is a constraint linking δ and β. Note that this induces $\theta_0 \geq m$ since $\beta \geq -1$.

3.3 Maximization of the community social welfare

Our aim is to find explicit solutions in order to quantify the advantages of outsourcing, with the linear utilities $U(x) = \gamma x,\ \gamma > 0$ et $b(x) = b.x, b > 0$. The rent rule being fixed, the consortium optimal efforts are given by

$$\begin{cases} \widehat{e_s} = \gamma(\beta+1) \quad \forall s \in [0,T] \\ \widehat{a} = \delta\gamma(\beta+1)A_T. \end{cases}$$

Proposition 2. *We recall the social welfare (3) :*

$$SW(\alpha,\beta) = \left(\mathbf{E}\left[B_0 + \int_0^T e^{-rs} b(e_s) ds - \left(\int_0^T e^{-rs} t(C_s) ds - C_0\right)\right]\right)$$

with $C_s = \theta_0 - \widehat{e_s} - \delta\widehat{a} + \varepsilon_s$ *and* $t(C_s) = \alpha - \beta C_s$. *We assume that*

$$0 \leq \delta^2 \leq \delta_{max}^2 = \frac{\theta_0 - 2m - \gamma(b+1)}{\gamma A_T}. \tag{5}$$

Then the community optimal policy is given by

$$\widehat{\beta} = \frac{\gamma b + \theta_0 - \gamma(1+\delta^2 A_T)}{2\gamma(1+\delta^2 A_T)} \tag{6}$$

$$\widehat{\alpha} = \frac{\gamma b + \theta_0 + \gamma(1+\delta^2 A_T)}{2\gamma(1+\delta^2 A_T)}(M - b\gamma - \gamma(1+\delta^2 A_T)). \tag{7}$$

Remark here that assumption (5) implies that the benchmark cost θ_0 is bounded from below, otherwise negative costs can occur.

Proof: Since $\widehat{e_s} = \gamma(\beta+1)$ is constant :

$$\begin{aligned}\frac{SW(\alpha,\beta) - B_0 - C_0}{A_T} &= \mathbf{E}[be_s - \alpha + \beta C_s] = be_s - \alpha + \beta(\theta_0 - e_s(1+\delta^2 A_T)) \\ &= (b - \beta(1+\delta^2 A_T))e_s - \alpha + \beta\theta_0 \\ &= (b - \beta(1+\delta^2 A_T))\gamma(\beta+1) - \alpha + \beta\theta_0.\end{aligned}$$

SW is a polynomial function of degree 2 in β :

$$\frac{SW(\alpha,\beta) - B_0 - C_0}{A_T} = -\beta^2\gamma(1+\delta^2 A_T) + \beta(\gamma b + \theta_0 - \gamma(1+\delta^2 A_T)) - \alpha + b\gamma$$

The dominating coefficient is negative, thus there exists an unique maximum achieved for

$$\widehat{\beta} = \frac{\gamma b + \theta_0 - \gamma(1+\delta^2 A_T)}{2\gamma(1+\delta^2(A_T))} \tag{8}$$

that can be also written as

$$\gamma(1+\delta^2 A_T) = \frac{\gamma b + \theta_0}{2\widehat{\beta}+1}, \tag{9}$$

as soon as the constraint (4) is satisfied, that is as soon as

$$\theta_0 - m - \frac{1}{2}(b\gamma + \theta_0 + \gamma(1+\delta^2 A_T)) \geq 0,$$

which is indeed satisfied since the externality δ is bounded from above by $\delta^2_{max} = \frac{\theta_0 - 2m - \gamma(b+1)}{\gamma A_T}$.

The choice of α must satisfy the constraint $t(C_s) \geq C_s$, that is

$$\widehat{\alpha} \geq (\widehat{\beta}+1)(\theta_0 + \varepsilon_s - \widehat{e_s} - \delta\widehat{a}) \;\; ds \otimes d\mathbf{P} \; a.s.$$

The constraint must be satisfied in the linear case where $\widehat{e_s} = \gamma(\beta+1)$, $\widehat{a} = \delta\gamma(\beta+1)(A_T)$, and $\varepsilon_s \leq M$. We choose α saturating this constraint, $\widehat{\alpha}$ is given in terms of

$\widehat{\beta}$ and using relation (9)

$$\begin{aligned}\widehat{\alpha} &= (\widehat{\beta}+1)(\theta_0+M-\gamma(\widehat{\beta}+1)(1+\delta^2 A_T)) \\ &= (\widehat{\beta}+1)(\theta_0+M-(\widehat{\beta}+1)\frac{\gamma b+\theta_0}{2\widehat{\beta}+1}) \\ \widehat{\alpha} &= \frac{\widehat{\beta}+1}{2\widehat{\beta}+1}[\hat{\beta}(\theta_0+2M-b\gamma)+M-b\gamma].\end{aligned}$$

Since $\widehat{\alpha} \geq (1+\widehat{\beta})C_s$ and since the cost is nonnegative via (4), we necessarily have $\widehat{\alpha} \geq 0$. •

Remark that (9) implies that $\widehat{\beta}$ is a decreasing function of the externality δ, that satisfies

$$\frac{b\gamma+m}{\theta_0-2m-b\gamma} \leq \widehat{\beta} \leq \frac{b\gamma+\theta_0-\gamma}{2\gamma} \ :$$

The upper bound corresponds to $\delta=0$ (no outsourcing), the lower bound corresponds to δ maximum (5). Thus the study of the impact of the externality δ on the social welfare can be done through the study of the function $\beta \mapsto SW(\alpha(\beta),\beta)$ where we replace δ by its function of β using (9).

Proposition 3. *We assume (5). If $b\gamma - \frac{\gamma^2}{b\gamma+\theta_0} \leq M$ – that is if the noise level is high enough – the social welfare is optimal for the maximal externality $\delta = \sqrt{\frac{\theta_0-2m-\gamma(b+1)}{\gamma A_T}}$. Otherwise, is the noise level is lower, the social welfare is optimal for $\delta=\delta_{max}$ or for $\delta=0$ (depending on whether $SW(\beta_{max}) < SW(\beta_{min})$ or not).*
In conclusion, if we exclude the default risk, the externality is attractive only if the noise level is high enough.

Proof: We want to optimize the following function

$$\beta \mapsto \frac{SW(\alpha,\beta)-B_0-C_0}{A_T} = \beta^2\frac{\gamma b+\theta_0}{2\beta+1} - \alpha + b\gamma,$$

that is, replacing α by its optimal value function of β

$$\beta \mapsto \frac{1}{2\beta+1}[\beta^2 2(b\gamma-M) - \beta(3M+\theta_0-4b\gamma) - M + 2b\gamma].$$

We recall the constraint on β

$$\beta_{min} = \frac{\gamma b+m}{\theta_0-2m-b\gamma} \leq \beta \leq \beta_{max} = \frac{\gamma b+\theta_0-\gamma}{2\gamma}.$$

More precisely, we study on the interval $[\beta_{min},\beta_{max}]$ the function

$$f(\beta) = \frac{1}{2\beta+1}[\beta^2 2(b\gamma - M) - \beta(3M + \theta_0 - 4b\gamma) - M + 2b\gamma]. \qquad (10)$$

Differentiating with respect to β, we get

$$\frac{2\beta+1}{4(b\gamma-M)} f'(\beta) = \beta^2 + \beta - \frac{\theta_0 + M}{4(b\gamma - M)}$$

The discriminant of this polynomial function of degree 2 is $\Delta = 1 + \frac{\theta_0+M}{b\gamma-M} = \frac{\theta_0+b\gamma}{b\gamma-M}$. If $\Delta > 0$, the positive root is $\beta_r = \frac{-1+\sqrt{\frac{\theta_0+b\gamma}{b\gamma-M}}}{2}$. Remark that

$$\beta_r < \beta_{max} \Longleftrightarrow M < b\gamma - \frac{\gamma^2}{b\gamma + \theta_0}.$$

This can only happen if $b\gamma - \frac{\gamma^2}{b\gamma+\theta_0} > 0$, that is (by solving the second degree inequation $(b\gamma)^2 + b\gamma\theta_0 - \gamma^2 > 0$) if $b\gamma > \frac{\theta_0}{2}(\sqrt{1+\frac{4\gamma^2}{\theta_0^2}} - 1)$.

- First case : $b\gamma \le M$ (i.e. high level of noise), f (and thus SW) is a strictly decreasing function of β (and thus strictly increasing in δ). The social welfare is optimal for $\delta = \delta_{max}$ (maximal externality) for a high level of noise.

- Second case : $b\gamma - \frac{\gamma^2}{b\gamma+\theta_0} \le M \le b\gamma$ (i.e. medium level of noise). Since $\beta_r > \beta_{max}$, f (and thus SW) is again a strictly decreasing function of β (and thus strictly increasing in δ). The social welfare is optimal for $\delta = \delta_{max}$.

- Third case : $b\gamma - \frac{\gamma^2}{b\gamma+\theta_0} \ge M$ (i.e. low level of noise and $b\gamma$ large enough). The optimal externality depends on whether or not β_r is greater than β_{min}:
  - If $\beta_r > \beta_{min}$, then SW is a strictly decreasing function of β on $[\beta_{min}, \beta_r]$, and strictly increasing on $[\beta_r, \beta_{max}]$. Thus, the social welfare is optimal for $\delta = \delta_{max}$ or for $\delta = 0$ (whether $SW(\beta_{max}) < SW(\beta_{min})$ or not).
  - If $\beta_r \le \beta_{min}$, then SW is a strictly increasing function of β (and a strictly decreasing function of δ). Thus, the social welfare is optimal when there is no outsourcing ($\delta = 0$).

  To summarize this third case, the derivative $f'(\beta)$ being successively negative then positive, the optimum is achevied at one of the interval bound and is equal to $SW(\beta_{max}) \vee SW(\beta_{min})$.

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In conclusion, when we exclude the default risk of the consortium, outsourcing becomes attractive only if the noise level is high enough. This corresponds to a risk transfer from the community to the consortium.

We conclude this section with a toy numerical example in order to quantify numerically this benchmark noise level under which outsourcing is not attractive.

3.4 Numerical example

In this example we take $\theta_0 = 100$ euros per unit time. The noise represents the randomness of the cost around this value, that is $\Theta_0 = \theta_0 + \varepsilon$ is a random variable with values in $[\theta_0 - M, \theta_0 + M]$ (here we take $m = M$). In the case of a linear utility $U(x) = \gamma x$, we let $\gamma = 25$ euros per unit time and $b = 1$.

$$\theta_0 = 100 \; ; \; \gamma = 25 \; ; \; b = 1. \tag{11}$$

Proposition 4. *For $\theta_0 = 100$; $\gamma = 25$; $b = 1$, outsourcing is attractive if and only if noise level M is greater than $\frac{50}{3}$ (that is around 16,7 % of the benchmark cost θ_0).*

Proof: First, the level $b\gamma - \frac{\gamma^2}{b\gamma + \theta_0}$ given in Proposition 3 is equal to 20 in this example, thus if $M \geq 20$ the maximal externality is optimal. Remark that in this case

$$\beta_{max} = 2, \; \forall M \; ; \; f(\beta_{max}) = 50 - 3M,$$

where f (which has the same behavior as SW) was defined in (10):

$$f(\beta) = \frac{1}{2\beta + 1}[\beta^2 2(b\gamma - M) - \beta(3M + \theta_0 - 4b\gamma) - M + 2b\gamma].$$

Now we study the case where $M < 20$. Proposition 3 says that the optimum depends on the position of $f(\beta_{min})$ with respect to this value $50 - 3M$. We have

$$\beta_{min} = \frac{25 + M}{75 - 2M}.$$

We set $M = 5\mu$, then we obtain

$$f(\beta_{min}) = \frac{2500 - 20 \times 50\mu + 95\mu^2}{5(15 - 2\mu)},$$

to be compared to $f(\beta_{max}) = 5(10 - 3\mu)$. Then, $f(\beta_{min}) < f(\beta_{max})$ if and only if $M < \frac{50}{3} \sim 16.7$, leading β optimal equal to β_{max}, and $\widehat{\delta} = 0$. •

4 Introduction of a default time, without penalty

We extend here the previous model in a more dynamic point of view and by introducing a default time. We still consider linear utilities ($U(x) = \gamma x$ and $b(x) = bx$) and we consider the operational cost as a semimartingale :

$$dC_s = (\theta_0 - e_s - \delta a)ds + \sigma dW_s.$$

The community chooses then the following expression for the rent

$$dt(C_s) = \alpha ds - \beta dC_s.$$

We define the default time τ as the first time as the consortium can not refund its debt anymore. In a first step, we assume that no penalty is imposed to the consortium in case of default (the case of penalty will be studied in Section 5).

4.1 Utility maximisation for the consortium

The consortium refund the debt at a rate D ($dD_s = De^{-rs}ds$) that is deducted from its profit. Its aim is to optimize (with $U(x) = \gamma x$ being its utility function)

$$\begin{aligned}(e,a,\tau) \mapsto \mathbf{E}\left(\int_0^{\tau\wedge T} e^{-rs}(\gamma[dt(C_s) - dC_s - dD_s] - \frac{1}{2}e_s^2)\right) - \frac{1}{2}a^2 \\ = \mathbf{E}\left(\int_0^{\tau\wedge T} e^{-rs}[\gamma(\alpha - D) - \gamma(\beta+1)(\theta_0 - e_s - \delta a) - \frac{1}{2}e_s^2]ds\right) - \frac{1}{2}a^2\end{aligned}$$

since $\mathbf{E}\left(\int_0^{\tau\wedge T} e^{-rs}dW_s\right) = 0$.

Proposition 5. *We assume that the initial effort does not depend on the default time (which is unknown at date 0). Then the optimal policy of a risk neutral consortium is given by*

$$\begin{cases} \widehat{e_s} = \gamma(\beta+1)\mathbb{1}_{[0,\tau\wedge T]}(s) \\ \widehat{a} = \delta\gamma(\beta+1)A_T. \end{cases}$$

Proof: Since the default time τ is unknown at the initial date, we do the optimization only in (e,a) with the fact that the effort e is done on the interval $[0,\tau]$, thus

$$\begin{cases} \widehat{e_s} = \gamma(\beta+1)\mathbb{1}_{[0,\tau\wedge T]}(s) \\ \widehat{a} = \delta\gamma(\beta+1)\mathbf{E}(\int_0^{\tau\wedge T} e^{-rs}ds) \end{cases}$$

But rather than taking an initial effort a depending on τ, it is more relevant to take the optimal initial effort as in the case with no default (thus we may overevaluate it):

$$\widehat{a} = \delta\gamma(\beta+1)\mathbf{E}(\int_0^{T} e^{-rs}ds) = \delta\gamma(\beta+1)A_T. \qquad \bullet$$

4.2 Definition of the default time and of the community optimization problem

We introduce the initial fund financing the project : $DA_T = \int_0^T e^{-rs} D ds$.
The consortium must refund its debt, $dt \otimes d\mathbf{P}$ a.s.:

$$DA_T + t(C_t) - C_t - D_t \geq 0,$$

that is

$$Y_t = DA_T + \int_0^t e^{-rs}(\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))ds - \int_0^t e^{-rs}(\beta+1)\sigma dW_s \geq 0.$$

Thus the default occurs when this constraint is not satisfied anymore.

Definition 4. The default time τ is defined as

$$\tau = \inf\{t : \; Y_t < 0\}.$$

If $r = 0$ (then $A_t = t$) :

$$\tau = \inf\{t : \; \int_0^t e^{-rs}(\beta+1)\sigma dW_s > DA_T + (\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))A_t\}.$$

If $r > 0$:

$$\tau = \inf\{t : \; \int_0^t e^{-rs} dW_s > A_r - B_r e^{-rt}\},$$

where

$$A_r = \frac{rDA_T + (\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))}{r(\beta+1)\sigma},$$
$$B_r = \frac{\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))}{r(\beta+1)\sigma}.$$

We remark here that this defaut time τ is increasing in α: the greater the community rent is, the longer the consortium avoids the default (and this $\forall r$). Considering that it is optimal for the community to postpone the default as longer as possible, we will choose the maximum α satisfying the constraints detailed in the following.

We adapt the definition of SW because in case of default, the community should take over from the consortium to refund the debt

$$SW(\alpha,\beta) - B_0 - C_0 = \mathbf{E}\left(\int_0^T e^{-rs} b e_s ds - \int_0^\tau e^{-rs} dt(C_s) - \int_\tau^T e^{-rs} D ds\right)$$

$$= \mathbf{E}\left(\int_0^{\tau\wedge T} e^{-rs}[b\gamma(\beta+1) - \alpha + \beta(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))]ds - D\int_\tau^T e^{-rs} ds\right)$$

$= [D + b\gamma(\beta+1) - \alpha + \beta(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))]\mathbf{E}[A_{\tau\wedge T}] - DA_T.$

Introducing

$$H := D + b\gamma(\beta+1) + \beta(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T)),$$

$$SW(\alpha,\beta) - B_0 - C_0 + DA_T = (H-\alpha)\mathbf{E}[A_{\tau\wedge T}],$$

which is the product of a decreasing and an increasing function in α. If $\alpha \geq H$, then $\alpha \mapsto SW(\alpha,\beta) - B_0 - C_0 + DA_T$ is decreasing thus an optimal α must be less than H and we get

$$SW(\alpha,\beta) - B_0 - C_0 + DA_T = (H-\alpha)\mathbf{E}[A_{\tau\wedge T}] \geq 0.$$

Therefore, the optimum exists in the interval $[0,H]$ and we will study the following function of the parameters α,β,δ

$$E[A_{\tau\wedge T}] = \mathbf{E}[A_\tau \mathbf{I}_{\tau<T}] + A_T\mathbf{P}(\tau > T).$$

4.3 Solution in the case $r = 0$

If $r = 0$,

$$\tau = \inf\{t : W_t > A - Bt\},$$

where

$$A = \frac{DA_T}{(\beta+1)\sigma},$$
$$B = \frac{D + (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 T)) - \alpha}{(\beta+1)\sigma}.$$

We define

$$K := D + (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T)). \quad (12)$$

4.3.1 The law of the default time, case $r = 0$

Using 3.2.3. page 148 [3], the law of τ is given by

Proposition 6. *If $r = 0$,, the default time is defined by*

$$\tau = \inf\{t : W_t > \frac{DA_T}{(\beta+1)\sigma} + \frac{\alpha - K}{(\beta+1)\sigma}t\}$$

where K is defined in (12). Then the density of τ on $\mathbf{R}^+$ is

$$t \mapsto \frac{DA_T}{(\beta+1)\sigma\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T-(\alpha-K)t}{(\beta+1)\sigma}\right)^2].$$

If $r=0$, $\mathbf{P}\{\tau<\infty\}=\exp(A(K-\alpha)-|A(K-\alpha)|)$ (cf. [4] page 197). Thus, $\mathbf{E}[\tau]$ is finite if and only if $\alpha<K$. In order to postpone the default, we take $\alpha\geq K$.

Corollary 5 *If $r=0$, we choose $\alpha=K$, and the default time is defined as $\tau=\inf\{t:\ W_t>\frac{DA_T}{(\beta+1)\sigma}\}$, the density of τ on $\mathbf{R}^+$ is*

$$t \mapsto \frac{DA_T}{(\beta+1)\sigma\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T}{(\beta+1)\sigma}\right)^2].$$

4.3.2 Constraints on the parameters, case $r=0$.

A raisonnable constraint is to take a nonnegative instantaneous operational cost (in expectation) :

$$\mathbf{E}[C_s ds]=(\theta_0-\gamma(\beta+1)(1+\delta^2T))ds\geq 0.$$

that is

$$\theta_0\geq\gamma(\beta+1)(1+\delta^2T).$$

We have previously justified choosing $\alpha\geq K$ to move the default back, such that $\mathbf{E}[\tau]=+\infty$.

Proposition 7. *The expected instantaneaous cost being nonnegative, and $\alpha\geq K$ (such that $E[\tau]=+\infty$) induce the following constraints*

$$\theta_0-b\gamma(\beta+1)\leq\gamma(\beta+1)(1+\delta^2T)\leq\theta_0. \tag{13}$$

Furthermore $0\leq K\leq H$ and this prove the existence of an optimal α in the interval $[K,H]$.

Proof: The expected instantaneaous cost being nonnegative is equivalent to

$$\mathbf{E}[C_s ds]=(\theta_0-\gamma(\beta+1)(1+\delta^2T))ds\geq 0,$$

thus we get the right hand side inequality

$$\theta_0\geq\gamma(\beta+1)(1+\delta^2T).$$

This implies

$$K=D+(\beta+1)(\theta_0-\gamma(\beta+1)(1+\delta^2T))\geq D\geq 0.$$

The left hand size inequality follows from

$$H-K=(\beta+1)b\gamma-\theta_0+\gamma(\beta+1)(1+\delta^2T)\geq 0.$$

•

We now choose $\alpha = K$, that maximizes the first factor $SW - B_0 - C_0 + DA_T$. We remark that in this case, in expectation, the instantaneous rent is positive :

$$E[t(C_s) - C_s]ds = \alpha - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 T)) = D > 0.$$

4.3.3 Study of the social welfare, function of β, δ, case $r = 0$.

If $r = 0$, the law of τ is explicit, furthermore (Corollaire 5) *we choose* $\alpha = K = D + (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 T))$ and thus the factor of the expectation is

$$H - K = (\beta+1)b\gamma - \theta_0 + \gamma(\beta+1)(1+\delta^2 T).$$

Corollary 6 *If $r = 0$, we choose $\alpha = K < H$, and the function* $SW(\alpha,\beta) - B_0 - C_0 + DA_T = (H-K)\mathbf{E}[\tau \wedge T]$

$$= [(\beta+1)b\gamma - \theta_0 + \gamma(\beta+1)(1+\delta^2 T)] \left[\int_0^\infty (t \wedge T) \frac{DA_T}{(\beta+1)\sigma\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T}{(\beta+1)\sigma}\right)^2]dt \right].$$

With the choice $\alpha = K$, the default time is the hitting time of $\frac{DA_T}{(\beta+1)\sigma}$ by a Brownian motion, thus the density of τ is $\frac{DA_T}{(\beta+1)\sigma\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T}{(\beta+1)\sigma}\right)^2]$. Assuming that β, δ satisfy (13), Corollary 6 gives the function we want to optimize :

$$(\beta,\delta) \mapsto [(\beta+1)b\gamma - \theta_0 + \gamma(\beta+1)(1+\delta^2 T)] \int_{\mathbf{R}^+} t \wedge T \frac{DA_T}{\sigma(\beta+1)\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T}{(\beta+1)\sigma}\right)^2]dt.$$

Proposition 8. *Let $r = 0$. We assume that the default time is postponed as longer as possible and that, the consortium optimal policy $(\widehat{e_s}, \widehat{a})$ being established, the PPP contract requires nonnegative (in expectation) operational cost and rent. Then the optimal rent rule and the optimal externality are*

$$\begin{aligned} \alpha^* &= D, \\ -1 < \beta^* &= \frac{\theta_0}{\gamma} - 1, \\ \delta^* &= 0. \end{aligned} \tag{14}$$

Outsourcing is not optimal in this case.

Proof: The function

$$\delta \mapsto [(\beta+1)b\gamma - \theta_0 + \gamma(\beta+1)(1+\delta^2 T)] \int_{\mathbf{R}^+} t \wedge T \frac{DA_T}{\sigma(\beta+1)\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T}{(\beta+1)\sigma}\right)^2]dt$$

is increasing in δ and using (13), $(1+\delta^2 T)^* = \frac{\theta_0}{\gamma(\beta+1)}$. This optimum is greater than 1 thus we get the constraint for β :

$$\gamma(\beta+1) \leq \theta_0.$$

Replacing $1+\delta^2 T$ by its optimal value, we want to optimize the function

$$\beta \mapsto b\gamma \int_{\mathbf{R}^+} t \wedge T \frac{DA_T}{\sigma\sqrt{2\pi t^3}} \exp[-\frac{1}{2t}\left(\frac{DA_T}{(\beta+1)\sigma}\right)^2]dt.$$

This function is increasing, thus $\beta^* = \frac{\theta_0}{\gamma} - 1$, and using the expression of $(1+\delta^2 T)^*$ we get that $\delta^* = 0$. Finally,

$$\alpha^* = D + (\beta^*+1)(\theta_0 - \gamma(\beta^*+1)) = D. \qquad \bullet$$

The interpretation is the following: if there is no penalty in case of a default, the community optimal policy is to outsource the less possible (and MOP are better and more secure than PPP). Furthermore, we remark that at the optimum, $\mathbf{E}(C_s) = 0$, and the rent is $t(C_s) - C_s = D - \frac{\theta_0}{\gamma} C_s$, $\mathbf{E}(t(C_s) - C_s) = D$. Thus, the rent coincides, in expectation, to the refund of the consortium debt.

4.4 Solution of the problem in the case $r > 0$

If $r > 0$:

$$\tau = \inf\{t : \int_0^t e^{-rs} dW_s > A_r - B_r e^{-rt}\},$$

where

$$A_r = \frac{rDA_T + (\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))}{r(\beta+1)\sigma},$$

$$B_r = \frac{\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))}{r(\beta+1)\sigma}.$$

4.4.1 Constraints on the parameters

By continuity, we have almost surely

$$\int_0^\tau e^{-rs}(\beta+1)\sigma dW_s = DA_T + (\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))A_\tau$$

that implies in the case $r > 0$

$$0 = \mathbf{E}[\int_0^\tau e^{-rs}(\beta+1)dW_s] = DA_T + (\alpha - D - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 A_T))\mathbf{E}[A_\tau].$$

Thus

$$\mathbf{E}[A_\tau] = \frac{DA_T}{D-\alpha+(\beta+1)(\theta_0-\gamma(\beta+1)(1+\delta^2A_T))}, \tag{15}$$

this implies the constraint on the parameters (since $0 \leq A_\tau \leq 1/r$) :

$$0 \leq \mathbf{E}[A_\tau] = \frac{DA_T}{D-\alpha+(\beta+1)(\theta_0-\gamma(\beta+1)(1+\delta^2A_T))} \leq 1/r. \tag{16}$$

We recall

$$K = D+(\beta+1)(\theta_0-\gamma(\beta+1)(1+\delta^2A_T)),$$

thus we have the condition on α :

$$rDA_T \leq D-\alpha+(\beta+1)(\theta_0-\gamma(\beta+1)(1+\delta^2A_T)) = K-\alpha,\ \alpha \leq K-rDA_T. \tag{17}$$

Furthermore, the instantaneous cost being nonnegative (in expectation) :

$$\mathbf{E}[C_s ds] = (\theta_0-\gamma(\beta+1)(1+\delta^2T))ds \geq 0.$$

implies that

$$\theta_0 \geq \gamma(\beta+1)(1+\delta^2T).$$

4.4.2 Study of the social welfare, function of β,δ ; case $r \neq 0$.

Proposition 9. *Let $r>0$. We assume that the default time is postponed as longer as possible and that, the consortium optimal policy $(\widehat{e_s},\widehat{a})$ being established, the PPP contract requires nonnegative (in expectation) operational cost and rent. Then the optimal rent rule and the optimal externality are*

$$\begin{aligned} \widehat{\alpha} &= De^{-rT}, \\ -1 < \widehat{\beta} &= \frac{\theta_0}{\gamma}-1, \\ \widehat{\delta} &= 0. \end{aligned} \tag{18}$$

Proof: We summarize the constraints : the cost rate is nonnegative (in expectation) :

$$\theta_0 \geq \gamma(\beta+1)(1+\delta^2A_T)$$

as for the rent :

$$\alpha \geq (\beta+1)(\theta_0-\gamma(\beta+1)(1+\delta^2A_T)) = (K-D).$$

The optimal parameters must satisfy

$$\alpha \leq H \wedge (K-rDA_T).$$

As in the case $r=0$, it seems to be relevant to choose α such that to postpone the default as longer as possible, that is $\alpha = K - rDA_T$ (that satisfies the constraint $\alpha \geq K - D$ since $rA_T \leq 1$). We get

$$H - \alpha = (\beta+1)b\gamma - \theta_0 + \gamma(\beta+1)(1+\delta^2 A_T) + rDA_T$$

that is increasing in δ and must be nonnegative. This implies a constraint linking β and δ:

$$b\gamma(\beta+1) - \theta_0 + \gamma(\beta+1)(1+\delta^2 A_T) + rDA_T \geq 0.$$

Furthermore, with this choice of α, we get

$$A_r = 0,\ B_r = \frac{-DA_T}{(\beta+1)\sigma}.$$

Thus $\tau = \inf\{t / \int_0^t e^{-rs} dW_s > \frac{DA_T}{(\beta+1)\sigma} e^{-rt}\}$ does not dependent on δ.
For continuity reason,

$$\int_0^\tau e^{-rs} dW_s = \frac{DA_T}{(\beta+1)\sigma} e^{-r\tau}$$

thus $\mathbf{E}[e^{-r\tau}] = 0$, that is $\tau = +\infty$ a.s. and $\tau \wedge T = T$, $A_{\tau\wedge T} = A_T$. Therefore, for this choice of α,

$$SW(\alpha,\beta,\delta) = [(\beta+1)b\gamma - \theta_0 + \gamma(\beta+1)(1+\delta^2 A_T) + rDA_T]A_T.$$

SW is increasing in δ and the optimal δ is given by

$$(\widehat{1+\delta^2} A_T) = \frac{\theta_0}{\gamma(\beta+1)}$$

with the constraint $\frac{\theta_0}{\gamma(\beta+1)} \geq 1$, that is $\beta \leq \frac{\theta_0}{\gamma} - 1$. Finally,

$$\widehat{\alpha} = De^{-rT}$$

and we easily check that $H - \alpha = rDA_T + (\beta+1)b\gamma$ is positive.
The last step is to find the optimum $\beta+1$ for the function

$$\beta \mapsto f(\beta+1) = (rDA_T + (\beta+1)b\gamma)A_T.$$

This function is increasing, the optimal β is given such as in the case $r=0$:

$$\widehat{\beta} = \frac{\theta_0}{\gamma} - 1$$

and $\hat{\delta} = 0$. The community optimal policy is the same as in the case $r=0$. •

In conclusion, whatever the interest rate is, outsourcing is NEVER optimal if we consider the possibility that the consortium defaults and if no penalty is

administered in case of default.

Given the maturity of PPP contract, it is obvious that we have to take into account the possibility of default. We will now focus on finding some cases where outsourcing is attractive if a penalty is administered in case of default.

5 Penalty in case of default with $r = 0$

Here we add in Section 4 model a penalty $\rho V(T-t)^+$ that the consortium should pay in case of default, whereas the community receives the compensation $\rho' V(T-t)^+$. We assume the natural constraint $\rho V \leq D$ and we denote $\rho = \rho' + \varepsilon$ where εV is used to pay the liquidation cost. We summarize the constraints

$$\rho V \leq D \; ; \; \rho = \rho' + \varepsilon, \; \varepsilon > 0. \tag{19}$$

We consider the rent $dt(C_s) = \alpha ds - \beta dC_s$, thus the consortium optimal policy remains the following.

Proposition 10. *Considering the rent dynamic $dt(C_s) = \alpha ds - \beta dC_s$ and the operational cost dynamic $dC_s = (\theta_0 - e_s - \delta a)ds + \sigma dW_s$, the consortium optimal policy is*

$$\hat{e}_t = \gamma(\beta+1)\mathbb{1}_{[0,\tau]}(t), \; \hat{a} = \gamma(\beta+1)\delta T.$$

The default time is now defined as

$$\tau = \inf\{t : \; (\beta+1)\sigma W_t > DT + (\alpha - K)t - \rho V(T-t)^+\} \wedge T\}$$

where K is defined in (12). Thus $\tau = \tilde{\tau} \wedge T$ with

$$\tilde{\tau} := \inf\{t : \; (\beta+1)\sigma W_t > DT + (\alpha - K + \rho V)t - \rho VT\}.$$

5.1 Constraints on the parameters

As in the previous section 4, we choose to postponed the default as longer as possible, for both the consortium and the community interest :

$$\alpha \geq K - \rho V = D + (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 T)) - \rho V.$$

Using the fact that the operational cost and the rent are nonnegative, we precise the constraints on the parameters.

Proposition 11. *We assume that the operational cost and the rent are nonnegative (in expectation) and we choose the bigger externality satisfying this assumption.*

Then the optimal parameters α, β, δ satisfy the following constraints :

$$\gamma(\beta+1)(1+\delta^2 T) = \theta_0, \tag{20}$$
$$\gamma(\beta+1) \leq \theta_0, \tag{21}$$
$$0 \leq D-\rho V \leq \alpha < D+b\gamma(\beta+1)-\rho' V, \tag{22}$$

This last interval is not empty since $\rho > \rho'$ and $\gamma(\beta+1) \geq 0$.

Corollary 7 *With the choice of a maximal externality, the consortium optimal effort can be written with respect to (θ_0, δ, T) :*

$$\hat{e}_t = \frac{\theta_0}{1+\delta^2 T}\mathbb{1}_{[0,\tau]}(t),\ \hat{a} = \frac{\theta_0}{1+\delta^2 T}\delta T.$$

In this case $dC_s = \sigma dW_s$ on $[0,\tau]$.

Proof: The expectation of the instantaneous cost being nonnegative

$$\mathbf{E}[C_s ds] = (\theta_0 - \gamma(\beta+1)(1+\delta^2 T))ds \geq 0.$$

that is $\theta_0 \geq \gamma(\beta+1)(1+\delta^2 T)$.
Our goal here is to find situations where outsourcing is attractive, thus we "a priori" choose δ maximum

$$\widehat{1+\delta^2} T = \frac{\theta_0}{\gamma(\beta+1)}.$$

This leads to the following constraint on β (since $1+\delta^2 T \geq 1$)

$$\beta+1 \leq \frac{\theta_0}{\gamma}.$$

With this choice of externality, the decision of postponing the default as longer as possible leads to the constraint on α

$$\alpha \geq K-\rho V = D-\rho V. \tag{23}$$

Furthermore, the instantaneous rent is nonnegative (in expectation)

$$\mathbf{E}[t(C_s)-C_s] = \alpha - (\beta+1)(\theta_0 - \gamma(\beta+1)(1+\delta^2 T)) \geq 0,$$

thus, with the choice of δ maximum, $\alpha \geq 0$. We compute the social welfare, with δ maximum and taking into account the compensation received in case of default :

$$SW(\alpha,\beta) - B_0 - C_0 = [D+b\gamma(\beta+1)-\alpha]\mathbf{E}[\tau\wedge T] - DT + \rho' V E[(T-\tau)^+],$$

that is, using $(T-\tau)^+ = T - \tilde{\tau}\wedge T$,

$$SW(\alpha,\beta) - B_0 - C_0 + (D-\rho' V)T = [D+b\gamma(\beta+1)-\alpha-\rho' V]\mathbf{E}[\tilde{\tau}\wedge T]. \tag{24}$$

This expression of SW requires the following constraint

$$D+b\gamma(\beta+1)-\alpha-\rho'V>0 \text{ i.e. } \alpha<D+b\gamma(\beta+1)-\rho'V.$$

Furthermore, the constraints on α ($\alpha\geq 0$ and (23)) lead to:

$$(D-\rho V)^+<\alpha<D+b\gamma(\beta+1)-\rho'V.$$

Using assumption (19), $(D-\rho V)^+=D-\rho V$ and

$$0\leq D-\rho V\leq\alpha<D+b\gamma(\beta+1)-\rho'V.$$

This interval is not empty since $\rho>\rho'$ and $b\gamma(\beta+1)\geq 0$. •

5.2 Maximisation of the social welfare

To emphasize the dependency on β, we now denotes $\tilde{\tau}$ by

$$\tau_\beta:=\inf\{t:\ (\beta+1)\sigma W_t>(D-\rho V)T+(\alpha-D+\rho V)t\}.$$

We remark that $\beta\mapsto\tau_\beta$ is decreasing. Using (24), we express the social welfare SW as a function of β.

Lemma 8 *Up to an additive constant, the social welfare is the sum of two functions of $\beta+1$:*

$$f(\beta+1)=b\gamma(\beta+1)\mathbf{E}[\tilde{\tau}\wedge T]=b\gamma\int_0^\infty t\wedge T\,\frac{(D-\rho V)T}{\sigma\sqrt{2\pi t^3}}\exp[-\frac{1}{2t}\left(\frac{(D-\rho V)T-(\alpha-D+\rho V)t}{(\beta+1)\sigma}\right)^2]dt$$

and

$$g(\beta+1)=[D-\alpha-\rho'V]\mathbf{E}[\tau_\beta\wedge T].$$

The following proposition gives the community optimal policy.

Proposition 12. *We assume (19). We assume that we postpone the default as longer as possible and that, the consortium optimal policy $(\widehat{e_s},\widehat{a})$ being established, the PPP contract requires nonnegative (in expectation) operational cost and rent. Then the optimal rent rule and the optimal externality are*

(i) if $D-\alpha-\rho'V\leq 0$, $\widehat{\beta}=\frac{\theta_0}{\gamma}$ and the same conclusions as in the case with no penalty hold (14).

(ii) if $D-\alpha-\rho'V>0$, we choose $\widehat{\alpha}=D-\rho V$ (this does not contradict (ii) since $\rho'<\rho$) and we denote $A=\gamma\frac{(D-\rho V)\sqrt{T}}{\theta_0\sigma}$. Then the sign $(f+g)'(\frac{\theta_0}{\gamma})$ is the one of the following expression

$$(b\theta_0+2(\rho-\rho')V)A(1-\Phi(A))+b\theta_0A^{-1}\left(\Phi(A)-\frac{1}{2}-A\phi(A)\right)-[2(\rho-\rho')V]\phi(A).$$

For a "small" A, $(f+g)'(\frac{\theta_0}{\gamma})<0$, and there exists an optimal β strictly less than $\frac{\theta_0}{\gamma}$, thus the optimal externality $\hat{\delta}$ is strictly positive.

Proof: On the one hand, the function f is increasing from $f(0)=0$ to

$$f(\infty)=b\gamma\int_{\mathbf{R}^+} t\wedge T\,\frac{(D-\rho V)T}{\sigma\sqrt{2\pi t^3}}dt=\frac{4b\gamma(D-\rho V)T\sqrt{T}}{\sigma\sqrt{2\pi}}.$$

On the other hand, concerning the function g, two cases may occur :
(i) if $D-\alpha-\rho' V\leq 0$, g is also increasing, the optimal β is $\frac{\theta_0}{\gamma}$ and the same conclusions as in the case without penalty hold (14).
(ii) if $D-\alpha-\rho' V>0$, g is decreasing and it is necessary to go into detail, using the constraints (21) and (22):

$$\gamma(\beta+1)\leq\theta_0,$$
$$0\leq D-\rho V\leq\alpha<D-\rho' V.$$

We will study the functions f and g in the interval $]0,\frac{\theta_0}{\gamma}]$. To do this, and in order to simplify the computations, we choose $\alpha=D-\rho V$ (thus τ_β is a.s. finite with an infinite expectation). We do the change of parameter :

$$\zeta=\frac{(D-\rho V)T}{(\beta+1)\sigma},\ \zeta\geq\frac{\gamma(D-\rho V)T}{\theta_0\sigma}.$$

Thus

$$\tilde{f}(\zeta)=b\gamma\frac{(D-\rho V)T}{\sigma}\int_{\mathbf{R}^+} t\wedge T\,\frac{1}{\sqrt{2\pi t^3}}\exp[-\frac{1}{2t}\zeta^2]dt, \tag{25}$$
$$\tilde{g}(\zeta)=[(\rho-\rho')V]\zeta\int_{\mathbf{R}^+} t\wedge T\,\frac{1}{\sqrt{2\pi t^3}}\exp[-\frac{1}{2t}(\zeta)^2]dt.$$

We will use the following technical lemma :

Lemma 9 *Let ϕ be the density of a standard centered Gaussian random variable, and Φ its cumulative function. Then for all positive A :*

$$\int_0^A u^2\phi(u)du=-A\phi(A)+\Phi(A)-\frac{1}{2}\ ;\ \int_A^\infty u^{-2}\phi(u)du=A^{-1}\phi(A)-1+\Phi(A).$$

We now compute the derivative function of $\tilde{g}$

Lemma 10

$$\tilde{g}'(\zeta)=[(\rho-\rho')V]4\zeta[\frac{\sqrt{T}}{\zeta}\phi(\frac{\zeta}{\sqrt{T}})-1+\Phi(\frac{\zeta}{\sqrt{T}})].$$

Proof: Before computing the derivative, we do the following change of variable in $\tilde{g}$: $u^2=\frac{\zeta^2}{t}$, $t=\frac{\zeta^2}{u^2}$, $dt=-2\frac{\zeta^2}{u^3}du$, and $\phi(u)=\frac{1}{\sqrt{2\pi}}\exp[-\frac{u^2}{2}]$:

$$\begin{aligned}\tilde{g}(\zeta) &= [(\rho-\rho')V]\zeta\int_{\mathbf{R}^+}\frac{\zeta^2}{u^2}\wedge T\frac{u^3}{\zeta^3}\frac{2\zeta^2}{u^3}\phi(u)du\\ &= [(\rho-\rho')V]2\int_{\mathbf{R}^+}\frac{\zeta^2}{u^2}\wedge T\phi(u)du\\ &= 2[(\rho-\rho')V]\left(\int_0^{\frac{\zeta}{\sqrt{T}}}T\phi(u)du+\int_{\frac{\zeta}{\sqrt{T}}}^{\infty}\frac{\zeta^2}{u^2}\phi(u)du\right).\end{aligned}$$

The previous lemma leads to

$$\tilde{g}(\zeta)=2[(\rho-\rho')V]\left(T(\Phi(\frac{\zeta}{\sqrt{T}})-\frac{1}{2})+\zeta^2(\frac{\sqrt{T}}{\zeta}\phi(\frac{\zeta}{\sqrt{T}})-1+\Phi(\frac{\zeta}{\sqrt{T}}))\right).$$

Up to the multiplicative constant $2(\rho-\rho')V$, the derivative of the first term is $\sqrt{T}\phi(\frac{\zeta}{\sqrt{T}})$ and the second term is

$$\sqrt{T}\zeta\phi(\frac{\zeta}{\sqrt{T}})-\zeta^2+\zeta^2\Phi(\frac{\zeta}{\sqrt{T}})$$

whose derivative is ($\phi'(u)=-u\phi(u)$) :

$$\sqrt{T}\phi(\frac{\zeta}{\sqrt{T}})-\frac{\zeta^2}{\sqrt{T}}\phi(\frac{\zeta}{\sqrt{T}})-2\zeta(1-\Phi(\frac{\zeta}{\sqrt{T}}))+\frac{\zeta^2}{\sqrt{T}}\phi(\frac{\zeta}{\sqrt{T}})$$

and reduction leads to the result. •

The derivative of $\tilde{f}$ is

Lemma 11

$$\tilde{f}'(\zeta)=-2b\gamma\frac{(D-\rho V)T}{\sigma}\left(1-\Phi(\frac{\zeta}{\sqrt{T}})+T\zeta^{-2}[\Phi(\frac{\zeta}{\sqrt{T}})-\frac{1}{2}-\frac{\zeta}{\sqrt{T}}\phi(\frac{\zeta}{\sqrt{T}})]\right).$$

Proof: We deduce from (25):

$$\tilde{f}'(\zeta)=-b\gamma\frac{(D-\rho V)T}{\sigma}\zeta\int_{\mathbf{R}^+}t\wedge T\frac{1}{t\sqrt{2\pi t^3}}\exp[-\frac{1}{2t}\zeta^2]dt.$$

Doing a change of variable in $\tilde{f}'$:

$$\tilde{f}'(\zeta)=-2b\gamma\frac{(D-\rho V)T}{\sigma}\zeta\int_{\mathbf{R}^+}(\frac{\zeta^2}{u^2})\wedge T\frac{1}{\frac{\zeta^2}{u^2}\sqrt{2\pi(\frac{\zeta^2}{u^2})^3}}\exp[-\frac{u^2}{2}]\frac{\zeta^2}{u^3}du=$$

$$=-2b\gamma\frac{(D-\rho V)T}{\sigma}\int_{\mathbf{R}^+}(\frac{\zeta^2}{u^2})\wedge T\frac{1}{\sqrt{2\pi}}\frac{u^2}{\zeta^2}\exp[-\frac{u^2}{2}]du=$$

$$-2b\gamma\frac{(D-\rho V)T}{\sigma}\left(\int_0^{\frac{\zeta}{\sqrt{T}}} T\frac{u^2}{\zeta^2}\phi(u)du+\int_{\frac{\zeta}{\sqrt{T}}}^{\infty}\phi(u)du\right).$$

Lemma 9 yields :

$$\tilde{f}'(\zeta)=-2b\gamma\frac{(D-\rho V)T}{\sigma}\left(T\zeta^{-2}[-\frac{\zeta}{\sqrt{T}}\phi(\frac{\zeta}{\sqrt{T}})+\Phi(\frac{\zeta}{\sqrt{T}})-\frac{1}{2}]+1-\Phi(\frac{\zeta}{\sqrt{T}})\right).\ \bullet$$

Proof: of the proposition 12, case (ii) : We are looking at the sign of $(f+g)'(\frac{\theta_0}{\gamma})$ which is the sign of $-(\tilde{f}+\tilde{g})'(\zeta)$ (in $\zeta=\frac{\gamma(D-\rho V)T}{\theta_0\sigma}$). Using the two lemmas, $-(\tilde{f}+\tilde{g})'(\zeta)=$

$$2b\gamma\frac{(D-\rho V)T}{\sigma}\left(1-\Phi(\frac{\zeta}{\sqrt{T}})+\frac{T}{\zeta^2}[\Phi(\frac{\zeta}{\sqrt{T}})-\frac{1}{2}-\frac{\zeta}{\sqrt{T}}\phi(\frac{\zeta}{\sqrt{T}})]\right)$$

$$-4\zeta(\rho-\rho')V[\frac{\sqrt{T}}{\zeta}\phi(\frac{\zeta}{\sqrt{T}})-1+\Phi(\frac{\zeta}{\sqrt{T}})].$$

Thus the sign of $(f+g)'$ is the one of

$$b\gamma\frac{(D-\rho V)\sqrt{T}}{\sigma}\left(1-\Phi(\frac{\zeta}{\sqrt{T}})+T\zeta^{-2}[\Phi(\frac{\zeta}{\sqrt{T}})-\frac{1}{2}-\frac{\zeta}{\sqrt{T}}\phi(\frac{\zeta}{\sqrt{T}})]\right)$$

$$-2\frac{\zeta}{\sqrt{T}}[(\rho-\rho')V][\frac{\sqrt{T}}{\zeta}\phi(\frac{\zeta}{\sqrt{T}})-1+\Phi(\frac{\zeta}{\sqrt{T}})].$$

For $\zeta=\gamma\frac{(D-\rho V)}{\theta_0\sigma}$, we set $A:=\gamma\frac{(D-\rho V)\sqrt{T}}{\theta_0\sigma}$, and the sign of $(f+g)'(\frac{\theta_0}{\gamma})$ is the one of

$$b\theta_0A\left(1-\Phi(A)+A^{-2}[\Phi(A)-\frac{1}{2}-A\phi(A)]\right)-2A[(\rho-\rho')V][A^{-1}\phi(A)-1+\Phi(A)]$$

which is the expected expression of Proposition 12 (ii)

$$(b\theta_0+2(\rho-\rho')V)A(1-\Phi(A))+b\theta_0A^{-1}\left(\Phi(A)-\frac{1}{2}-A\phi(A)\right)-[2(\rho-\rho')V]\phi(A).$$

The asymptotics around zero of the two first terms are :

$$[2(\rho-\rho')V+b\theta_0]A[1-\Phi(A)]\sim[2(\rho-\rho')V+b\theta_0]\frac{A}{2},$$
$$b\theta_0A^{-1}\left([\Phi(A)-\frac{1}{2}-A\phi(A)]\right)\sim b\theta_0\frac{5A^2}{6\sqrt{2\pi}},$$

and the third term is equal for $A=0$ to $-2(\rho-\rho')V\phi(0)=-\frac{2(\rho-\rho')V}{\sqrt{2\pi}}<0$. Thus, for A small enough, $(f+g)'(\frac{\theta_0}{\gamma})<0$. $\bullet$

Remark 12 *This condition "$A = \gamma\frac{(D-\rho V)\sqrt{T}}{\theta_0\sigma}$ small enough" is satisfied if*
- the noise level is high (large σ),
- the benchmark cost θ_0 is high,
- the maturity T is short,
- $D-\rho V$ is small, that is the penalty ρ is large enough.

In this section that modelizes the better the reality, we show that outsourcing is attractive for the community in case of high uncertainty or high noise, short maturity, high benchmark operational cost or a sufficently high penalty in case of default.

6 Conclusion

Three models of PPP contracts have been studied in this paper :
- The first one assumes that there is no default risk and that the contract does not end before maturity.
- The second one introduces the default risk of the consortium, without any compensation for the community in case of an unreciprocated contract breaking-off.
- The third one also considers the default risk of the consortium, and the consortium has to pay penalty in case of default, the community receiving a part of this penalty as a compensation.

In the second model, whatever is the discount rate (positive or zero), the community optimal policy is to give up for outsourcing. In the first model, outsourcing is optimal if the noise level around the maintenance benchmark cost is higher than a threshold: this corresponds to a risk transfer from the community to the consortium. Remark that this threshold is an increasing function of the benchmark cost and of the coefficient of the consortium utility. Similarly, in the third model with penalty in case of default, outsourcing is optimal is the randomness is high enough, or if the contract maturity is short, if the benchmark cost or the penalty are high enough.

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