RECOVERY OF SMALL INHOMOGENEITIES FROM THE SCATTERING AMPLITUDE AT A FIXED FREQUENCY

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Abstract. We rigorously derive the leading order term in the asymptotic expansion of the scattering amplitude of a collection of a finite number of dielectric inhomogeneities of small diameter. We then apply this asymptotic formula for the purpose of identifying the location and certain properties of the shapes of the small inhomogeneities from scattering amplitude measurements at a fixed frequency. Our main idea is to reduce this reconstruction problem to the calculation of an inverse Fourier transform.

Key words. inverse scattering problem, scattering amplitude, Helmholtz equation, dielectric imperfections, reconstruction

AMS subject classifications. 35R30, 78A46

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1. Introduction. In this paper, we consider three-dimensional electromagnetic scattering from a collection of small dielectric inhomogeneities. We suppose that there is a finite number of dielectric imperfections in \( \mathbb{R}^3 \), each of the form \( z_j + \alpha B_j \), where \( B_j \subset \mathbb{R}^3 \) is a bounded, smooth \( (C^\infty) \) domain containing the origin. This regularity assumption could be considerably weakened. The total collection of imperfections thus takes the form

\[ I_\alpha = \bigcup_{j=1}^m (z_j + \alpha B_j). \]

The points \( z_j \in \mathbb{R}^3 \), \( j = 1, \ldots, m \), that determine the location of the imperfections are assumed to satisfy

\[ 0 < d_0 \leq |z_j - z_l| \quad \forall j \neq l. \tag{1} \]

We also assume that \( \alpha > 0 \), the common order of magnitude of the diameters of the imperfections, is small enough such that the imperfections are disjoint.

Our first goal is to provide a rigorous derivation of the asymptotic expansion of the scattering amplitude for such a collection of small dielectric imperfections. Our second goal is to use this expansion for efficiently determining the locations and/or shapes of the small inhomogeneities from scattering amplitude measurements at a fixed frequency by reducing the reconstruction problem of the small inhomogeneities to the calculation of an inverse Fourier transform. We expect that our asymptotic formulas will form the basis for very effective computational identification algorithms, aimed at determining information about the small inhomogeneities from scattering amplitude measurements.

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To the best of our knowledge, the present paper is the first attempt to design an effective and accurate method to determine the location and the size of small dielectric inhomogeneities with both different electric permittivities and magnetic permeabilities from scattering amplitude measurements. Our method is quite similar to the ideas used by Calderon [5] in his proof of uniqueness of the linearized conductivity problem and later by Sylvester and Uhlmann in their important work [20] on uniqueness of the three-dimensional inverse conductivity problem. Our technique for studying the scattering problem is to reduce the problem to a bounded domain with the aid of integral equation methods. On the bounded domain, the derivation of the asymptotic expansion of the solution relies heavily on the results of [24]. The current work is also a natural extension of the identification procedure that we have presented in [2], where we demonstrated numerically its accuracy and stability. For discussions on other closely related inverse scattering problems, the reader is referred, for example, to [7], [15], [10], [11], [12], [22], [23], [14], [17], [18], [19], and [8].

Let $\mu_{0} > 0$ and $\varepsilon_{0} > 0$ denote the permeability and the permittivity of the free space; we shall assume that these are positive constants. Let $\mu_{j} > 0$ and $\varepsilon_{j} > 0$ denote the permeability and the permittivity of the $j$th inhomogeneity, $z_{j} + \alpha B_{j}$; these are also assumed to be positive constants. Using this notation, we introduce the piecewise constant magnetic permeability

$$
\mu_{\alpha}(x) = \begin{cases} 
\mu_{0}, & x \in \mathbb{R}^3 \setminus \bar{I}_{\alpha}, \\
\mu_{j}, & x \in z_{j} + \alpha B_{j}, \; j = 1, \ldots, m.
\end{cases}
$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_{0}(x)$ equals the constant $\mu_{0}$. The piecewise constant electric permittivity $\varepsilon_{\alpha}(x)$ is defined analogously. We need to introduce some additional notation. Let $\gamma_{j}, 1 \leq j \leq m$, be a set of positive constants. In effect, $\{\gamma_{j}\}$ will be either the set $\{\varepsilon_{j}\}$ or the set $\{\mu_{j}\}$. For any fixed $1 \leq j_{0} \leq m$, let $\gamma$ denote the coefficient given by

$$
\gamma(x) = \begin{cases} 
\gamma_{0}, & x \in \mathbb{R}^3 \setminus \bar{B}_{j_{0}}, \\
\gamma_{j_{0}}, & x \in B_{j_{0}}.
\end{cases}
$$

By $\phi_{l}, 1 \leq l \leq 3$, we denote the solution to

$$
\begin{align*}
\nabla_{y} \cdot \gamma(y) \nabla_{y} \phi_{l} &= 0 \quad \text{in} \; \mathbb{R}^3, \\
\phi_{l} - y_{l} &\to 0 \quad \text{as} \; |y| \to \infty.
\end{align*}
$$

This problem may alternatively be written as

$$
\begin{cases} 
\Delta \phi_{l} = 0 & \text{in} \; B_{j_{0}}, \text{ and in} \; \mathbb{R}^3 \setminus \overline{B_{j_{0}}}, \\
\phi_{l} \text{ is continuous across} \; \partial B_{j_{0}}, \\
\frac{\nu \gamma_{j_{0}}}{\gamma_{j_{0}}} (\partial_{\nu} \phi_{l})^{+} - (\partial_{\nu} \phi_{l})^{-} = 0 & \text{on} \; \partial B_{j_{0}}, \\
\phi_{l}(y) - y_{l} &\to 0 \quad \text{as} \; |y| \to \infty.
\end{cases}
$$

Here $\nu$ denotes the outward unit normal to $\partial(z_{j} + \alpha B_{j})$; superscripts $+$ and $-$ indicate the limiting values as we approach $\partial(z_{j} + \alpha B_{j})$ from outside $z_{j} + \alpha B_{j}$ and from inside $z_{j} + \alpha B_{j}$. It is obvious that the function $\phi_{l}$ depends only on the coefficients $\gamma_{0}$ and $\gamma_{j_{0}}$ through the ratio $c = \frac{\gamma_{0}}{\gamma_{j_{0}}}$. The existence and uniqueness of this $\phi_{l}$ can be established using single layer potentials with suitably chosen densities. It is essential here that
the constant $c$, by assumption, cannot be 0 or a negative real number. We now define the polarization tensor $M^b_{kl}(c)$ of the inhomogeneity $B^b_{j0}$ (with aspect ratio $c$), by

$$(4) \quad M^b_{kl}(c) = c^{-1} \int_{B^b_{j0}} \partial_{y_k} \phi_l \, dy.$$ 

It is quite easy to see that the tensor $M^b_{kl}(c)$ is symmetric; since $c$ is a positive real number, it is furthermore positive definite (see [6], [13]).

2. Asymptotic formula for the solution. Consider in this section a homogeneous background medium in all of $\mathbb{R}^3$ with electric permittivity $\varepsilon^0$ and magnetic permeability $\mu^0$, and let $\varepsilon_\alpha$ and $\mu_\alpha$ be the corresponding dielectric functions in the presence of the small inhomogeneities described above. Let $u_\alpha$ be the solution to the Helmholtz equation

$$(5) \quad \left( \nabla \cdot \frac{1}{\mu_\alpha} \nabla + \omega^2 \varepsilon_\alpha \right) u_\alpha = 0 \quad \text{in} \quad \mathbb{R}^3,$$

with the radiation condition as $r \to \infty$,

$$(6) \quad |\partial_r (u_\alpha - e^{ik\eta \cdot x} - ik(u_\alpha - e^{ik\eta \cdot x})| = O \left( \frac{1}{r^2} \right),$$

where $\omega$ is the frequency, $k^2 = \omega^2 \varepsilon^0 \mu^0$, $\eta$ is a vector on the unit sphere $S^2$ in $\mathbb{R}^3$, $\eta \cdot \eta = 1$, and $u_0 = e^{ik\eta \cdot x}$ is an incident plane wave. Note that $u_0$ satisfies the homogeneous Helmholtz equation

$$(7) \quad \left( \nabla \cdot \frac{1}{\mu^0} \nabla + \omega^2 \varepsilon^0 \right) u_0 = 0 \quad \text{in} \quad \mathbb{R}^3.$$

In this section, we find and prove a formula, asymptotic with respect to the inhomogeneity size $\alpha$, for $u_\alpha$ in terms of $u_0$. We begin by defining the outgoing Green function $G(x, y)$ to satisfy

$$(8) \quad \left( \Delta_y + k^2 \right) G(x, y) = -\delta_x(y) \quad \text{in} \quad \mathbb{R}^3,$$

$$|\partial_r G - ikG| = O \left( \frac{1}{r^2} \right) \quad \text{as} \quad r \to \infty.$$

In fact, we know $G$ explicitly:

$$G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x - y|}.$$ 

Let $\Omega$ denote some fixed domain in $\mathbb{R}^3$ that contains the inhomogeneities. Without loss of generality, we can assume that $k^2$ is not an eigenvalue of $-\Delta$ in $\Omega$ corresponding to Dirichlet boundary conditions on $\partial \Omega$. We know that Proposition 1 in [24], which is based on properties of collectively compact operators, guarantees that, for $\alpha$ sufficiently small, the trivial solution is the unique solution to $(\nabla \cdot \frac{1}{\mu_\alpha} \nabla + \omega^2 \varepsilon_\alpha) v_\alpha = 0$ in $\Omega$, with the boundary condition $v_\alpha = 0$ on $\partial \Omega$.

If we consider the equation for $u_\alpha$ in the exterior of $\Omega$, multiply $G$, and integrate by parts, we get that, for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$,

$$u_\alpha(x) = u_0(x) + \int_{\partial \Omega} \partial_{\nu_y} G u_\alpha(y) \, d\sigma_y - \int_{\partial \Omega} G \partial_{\nu} u_\alpha(y) \, d\sigma_y,$$
where \( \nu \) is the unit outward normal to \( \partial \Omega \). Of course, this equation does not hold up to the boundary of \( \Omega \), but if we take the limit as \( x \rightarrow \partial \Omega \), we get (see, for example, [7] and [16])

\[
\frac{1}{2} u_\alpha|_{\partial \Omega} = u_0|_{\partial \Omega} + \int_{\partial \Omega} \partial_{\nu_y} G u_\alpha(y) \, d\sigma_y - \int_{\partial \Omega} G \partial_{\nu_x} u_\alpha(y) \, d\sigma_y
\]  

for \( x \in \partial \Omega \). Now define the Dirichlet to Neumann map

\[
N_\alpha : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega),
\]

\[
N_\alpha(f) = \partial_{\nu} v_\alpha,
\]

where \( v_\alpha \) is the solution to

\[
\left( \nabla \cdot \frac{1}{\mu_\alpha} \nabla + \omega^2 \varepsilon_\alpha \right) v_\alpha = 0 \quad \text{in} \quad \Omega,
\]

\[
v_\alpha = f \quad \text{on} \quad \partial \Omega.
\]

Hence

\[
N_\alpha(u_\alpha|_{\partial \Omega}) = \partial_{\nu} u_\alpha|_{\partial \Omega}.
\]

Similarly, let

\[
N_0 : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)
\]

be the Neumann to Dirichlet map for the limiting problem so that

\[
N_0(u_0|_{\partial \Omega}) = \partial_{\nu} u_0|_{\partial \Omega}.
\]

We also define the single and double layer potential operators

\[
S : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial \Omega)
\]

and

\[
D : H^{1/2}(\partial \Omega) \rightarrow H^{1/2}(\partial \Omega),
\]

where

\[
S : g \rightarrow \int_{\partial \Omega} G(x, y) g(y) \, d\sigma_y
\]

and

\[
D : f \rightarrow \int_{\partial \Omega} \partial_{\nu_y} G(x, y) f(y) \, d\sigma_y.
\]

Using this operator notation, we see that from (9) we have

\[
\left( \frac{I}{2} - D + SN_\alpha \right) (u_\alpha|_{\partial \Omega}) = u_0|_{\partial \Omega}.
\]
Similarly, \( u_0 \) satisfies
\[
\left( \frac{I}{2} - D + SN_0 \right) (u_0|_{\partial \Omega}) = u_0|_{\partial \Omega}.
\]
Define
\[
T_\alpha : H^{1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)
\]
by
\[
(11)\quad T_\alpha = \frac{I}{2} - D + SN_\alpha,
\]
and let
\[
(12)\quad T_0 = \frac{I}{2} - D + SN_0.
\]
By subtracting the two above equations, we have that
\[
T_\alpha(u_\alpha|_{\partial \Omega}) - T_0(u_0|_{\partial \Omega}) = 0,
\]
and hence
\[
T_\alpha((u_\alpha - u_0)|_{\partial \Omega}) = S(N_0 - N_\alpha)(u_0|_{\partial \Omega}).
\]

We will need the following proposition. The reader is referred to the appendix for its proof. In the following proposition and in the remainder of this paper, all asymptotic terms and constants may depend on the separation \( d_0 \) of the inhomogeneities.

**Proposition 1.** Let \( T_\alpha \) be defined by (11) and \( T_0 \) by (12). Then we have the following:

(a) \( T_\alpha \) converges to \( T_0 \) pointwise.

(b) \( T_\alpha - T_0 \) is collectively compact.

(c) There exists a constant \( C \) that is independent of \( \alpha \) and the set of points \( \{z_j\}_{j=1}^m \) such that, for any \( f \in H^{1/2}(\partial \Omega) \), \( T_\alpha^{-1} \) exists and
\[
\|T_\alpha^{-1}f\|_{H^{1/2}(\partial \Omega)} \leq C \|f\|_{H^{1/2}(\partial \Omega)}.
\]

(d) The following asymptotic formula holds:
\[
(13)\quad \frac{1}{\alpha^3} \sum_{j=1}^m \left( 1 - \frac{\mu^j}{\mu^0} \right) \nabla u_0(z_j) \cdot M^j \left( \frac{\mu^j}{\mu^0} \right) \nabla_y G(x, z_j) + k^2 \left( 1 - \frac{\epsilon^j}{\epsilon^0} \right) u_0(z_j) G(x, z_j) + o(\alpha^3),
\]
where the asymptotic term \( o(\alpha^3) \) is independent of \( x \in \partial \Omega \) and the set of points \( \{z_j\}_{j=1}^m \).

Define the correction
\[
(14)\quad u^{(1)}(x) = \sum_{j=1}^m \left( 1 - \frac{\mu^j}{\mu^0} \right) \nabla u_0(z_j) \cdot M^j \left( \frac{\mu^j}{\mu^0} \right) \nabla_y G(x, z_j) + k^2 \left( 1 - \frac{\epsilon^j}{\epsilon^0} \right) u_0(z_j) G(x, z_j)
\]
for $x \neq z_j$, $j = 1, \ldots, m$. We have therefore shown that

$$T_\alpha((u_\alpha - u_0)|_{\partial\Omega}) = \alpha^3 u^{(1)}|_{\partial\Omega} + o(\alpha^3)$$

uniformly for $x \in \partial\Omega$. Note that, from the definition of $G$, $u^{(1)}$ satisfies

$$\nabla \cdot \left[ \nabla \left( \frac{\mu^j}{\mu^0} \nabla u_0(z_j) \cdot M^j \right) \nabla \delta_{z_j} \right] + k^2 \left( 1 - \frac{\varepsilon^j}{\varepsilon^0} \right) u_0(z_j) \delta_{z_j},$$

in the sense of distributions, where $\delta_{z_j}$ is the Dirac delta function at the point $z_j$.

**Lemma 1.** Let the correction term $u^{(1)}$ be defined by (14). Then we have

$$T_0(u^{(1)}|_{\partial\Omega}) = u^{(1)}|_{\partial\Omega}.$$

**Proof.** Multiplying (16) by $G$, integrating by parts over $\Omega$, and taking the limit as $x \to \partial\Omega$, we get

$$\frac{1}{2} u^{(1)}|_{\partial\Omega} - \int_{\partial\Omega} \partial_\nu G u^{(1)}(y) \, d\sigma_y + \int_{\partial\Omega} G \partial_\nu u^{(1)}(y) \, d\sigma_y = 0$$

for $x \in \partial\Omega$. Define $v^{(1)}$ as the unique solution to

$$\begin{cases}
\nabla v^{(1)} + k^2 v^{(1)} = 0 & \text{in } \Omega, \\
v^{(1)} = u^{(1)} & \text{on } \partial\Omega,
\end{cases}$$

that is,

$$\partial_\nu v^{(1)} = N_0(u^{(1)}|_{\partial\Omega}).$$

Green’s formula yields, for any $x \in \Omega$ away from the centers of the inhomogeneities,

$$\int_{\partial\Omega} G(x, y) \partial_\nu (u^{(1)} - v^{(1)})(y) \, d\sigma_y = \sum_{j=1}^m \left( 1 - \frac{\mu^j}{\mu^0} \right) \nabla u_0(z_j) \cdot M^j \nabla G(x, z_j)$$

$$+ k^2 \left( 1 - \frac{\varepsilon^j}{\varepsilon^0} \right) u_0(z_j) G(x, z_j) - u^{(1)}(x) + v^{(1)}(x)$$

$$= v^{(1)}(x).$$

Hence, for $x \in \partial\Omega$,

$$\int_{\partial\Omega} G(x, y) \partial_\nu (u^{(1)} - v^{(1)})(y) \, d\sigma_y = u^{(1)}(x).$$

Using this, we can rewrite

$$\int_{\partial\Omega} G \partial_\nu u^{(1)}(y) \, d\sigma_y = \int_{\partial\Omega} G N_0(u^{(1)})(y) \, d\sigma_y + \int_{\partial\Omega} G (\partial_\nu u^{(1)}(y) - N_0(u^{(1)})(y)) \, d\sigma_y$$

$$= \int_{\partial\Omega} G N_0(u^{(1)})(y) \, d\sigma_y + u^{(1)}(x),$$

from which it follows that

$$\frac{1}{2} u^{(1)}|_{\partial\Omega} - \int_{\partial\Omega} \partial_\nu G u^{(1)}(y) \, d\sigma_y + \int_{\partial\Omega} G N_0(u^{(1)})(y) \, d\sigma_y = u^{(1)}(x).$$
for $x \in \partial \Omega$. This just says exactly that $T_0(u^{(1)}|_{\partial \Omega}) = u^{(1)}|_{\partial \Omega}$. □

**Lemma 2.** The following estimate holds:

\[
\|u_\alpha - u_0 - \alpha^3 u^{(1)}\|_{H^{1/2}(\partial \Omega)} = o(\alpha^3),
\]

where the term $o(\alpha^3)$ goes to zero faster than $\alpha^3$ independent of the set of points $\{z_j\}_{j=1}^m$.

*Proof. From (15) it follows that*

\[
T_\alpha((u_\alpha - u_0 - \alpha^3 u^{(1)}|_{\partial \Omega}) = \alpha^3 u^{(1)}|_{\partial \Omega} - \alpha^3 T_\alpha(u^{(1)}|_{\partial \Omega}) + o(\alpha^3).
\]

Lemma 1 yields

\[
T_\alpha((u_\alpha - u_0 - \alpha^3 u^{(1)}|_{\partial \Omega}) = \alpha^3(T_0 - T_\alpha)(u^{(1)}|_{\partial \Omega}) + o(\alpha^3).
\]

Therefore, due to the pointwise convergence of $T_\alpha$ to $T_0$, we obtain

\[
T_\alpha((u_\alpha - u_0 - \alpha^3 u^{(1)}|_{\partial \Omega}) = o(\alpha^3),
\]

which leads, by using point (c) in Proposition 1, to the desired estimate (17). □

From this lemma, we obtain the following theorem.

**Theorem 1.** Let $u_\alpha$ be the solution to (5), and let $M^j(\mu^j, \nu^j)$ be the polarization tensors for the shapes $B_j$ defined by (4). Then, for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$ bounded away from $\partial \Omega$, we have the pointwise expansion

\[
u_\alpha(x) = e^{ik\gamma x} + \alpha^3 \sum_{j=1}^m e^{ik\gamma z_j} \left[ ik \left( 1 - \frac{\mu^j}{\mu^0} \right) \nabla_y G(x, z_j) \cdot M^j \left( \frac{\mu^j}{\mu^0} \right) \eta \right.
\]

\[
+ k^2 \left( 1 - \frac{\varepsilon^j}{\varepsilon^0} \right) |B_j| G(x, z_j) \]  

\[
+ o(\alpha^3).
\]

*Here the remainder term $o(\alpha^3)$ is independent of $x$ and the set of points $\{z_j\}_{j=1}^m$. 

*Proof. From Lemma 2, it follows that $u_\alpha - u_0$ satisfies in $\mathbb{R}^3 \setminus \overline{\Omega}$

\[
\begin{aligned}
\Delta(u_\alpha - u_0) + k^2(u_\alpha - u_0) &= 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega},
\end{aligned}
\]

\[
(u_\alpha - u_0) = \alpha^3 u^{(1)} + o(\alpha^3) \quad \text{on} \quad \partial \Omega,
\]

\[
|\partial_r(u_\alpha - u_0) - ik(u_\alpha - u_0)| = O(\frac{1}{\alpha^2}).
\]

Let $G$ denote the outgoing Dirichlet Green function that is defined by

\[
\begin{aligned}
\Delta G + k^2 G &= -\delta \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega},
G &= 0 \quad \text{on} \quad \partial \Omega,
|\partial_r G - ikG| &= O(\frac{1}{\alpha^2}).
\end{aligned}
\]

It is easy to see that $u_\alpha - u_0$ has the following integral representation in $\mathbb{R}^3 \setminus \overline{\Omega}$:

\[
(u_\alpha - u_0)(x) = \int_{\partial \Omega} \frac{\partial G}{\partial \nu}(x, y)(u_\alpha - u_0)(y) d\sigma(y) \quad \forall \ x \in \mathbb{R}^3 \setminus \overline{\Omega}.
\]
Moreover, for any \( x \in \mathbb{R}^3 \setminus \overline{\Omega} \) which is bounded away from \( \partial \Omega \), we obtain from the asymptotic expansion of the boundary condition in Lemma 2 that

\[
(u_\alpha - u_0)(x) = \alpha^3 \int_{\partial \Omega} \frac{\partial G}{\partial \nu_y}(x, y) u^{(1)}(y) \, d\sigma(y) + o(\alpha^3),
\]

where \( o(\alpha^3) \) is independent of \( x \) and the set \( \{z_j\}_{j=1}^m \). Since, for any \( x \in \mathbb{R}^3 \setminus \overline{\Omega} \) and \( z \in \Omega \), we have by standard integration by parts the identities

\[
\int_{\partial \Omega} \frac{\partial G}{\partial \nu}(x, y) G(y, z) \, d\sigma(y) = G(x, z)
\]

and

\[
\int_{\partial \Omega} \frac{\partial G}{\partial \nu}(x, y) \nabla_z G(y, z) \, d\sigma(y) = \nabla_z G(x, z),
\]

the expression of the correction term \( u^{(1)} \) immediately leads to the promised asymptotic expansion. \( \square \)

We also can obtain the next proposition on the norm convergence of the solutions.

**Proposition 2.** There exists a constant \( C \) that is independent of \( \alpha \) and the set of points \( \{z_j\}_{j=1}^m \) such that the following energy estimate holds:

\[
\|u_\alpha - u_0\|_{L^2(\Omega)} + \|\nabla u_\alpha - \nabla u_0\|_{L^2(\Omega)} \leq C\alpha^2.
\]

**Proof.** Let \( \tilde{u}_\alpha \) be defined as the unique solution to

\[
\begin{align*}
\Delta \tilde{u}_\alpha + k^2 \tilde{u}_\alpha &= 0 \quad \text{in } \Omega, \\
\tilde{u}_\alpha &= u_\alpha \quad \text{on } \partial \Omega.
\end{align*}
\]

We have

\[
\begin{align*}
\Delta (\tilde{u}_\alpha - u_0) + k^2 (\tilde{u}_\alpha - u_0) &= 0 \quad \text{in } \Omega, \\
(\tilde{u}_\alpha - u_0) &= u_\alpha - u_0 \quad \text{on } \partial \Omega,
\end{align*}
\]

which leads to

\[
\|\tilde{u}_\alpha - u_0\|_{H^1(\Omega)} \leq C\|u_\alpha - u_0\|_{H^{1/2}(\Omega)},
\]

where the constant \( C \) is independent of \( \alpha \). Using Lemma 2, we get that \( \|\tilde{u}_\alpha - u_0\|_{H^1(\Omega)} \) is of order \( \alpha^3 \). Now note that the function \( (u_\alpha - \tilde{u}_\alpha) \) is in \( H^1_0(\Omega) \), and for any \( v \in H^1_0(\Omega) \)

\[
\begin{align*}
\int_{\Omega} \frac{1}{\mu_\alpha} \nabla (u_\alpha - \tilde{u}_\alpha) \cdot \nabla v - \omega^2 \int_{\Omega} \varepsilon_\alpha (u_\alpha - \tilde{u}_\alpha) v &= \int_{\Omega} \frac{1}{\mu_\alpha} \nabla u_\alpha \cdot \nabla v - \omega^2 \int_{\Omega} \varepsilon_\alpha u_\alpha v \\
&\quad - \int_{\Omega} \frac{1}{\mu_0^0} \nabla \tilde{u}_\alpha \cdot \nabla v + \omega^2 \int_{\Omega} \varepsilon_0 \tilde{u}_\alpha v \\
&\quad + \sum_{j=1}^m \left( \frac{1}{\mu_0^j} - \frac{1}{\mu^j} \right) \int_{z_j + \alpha B_j} \nabla \tilde{u}_\alpha \cdot \nabla v \\
&\quad + k^2 \left( \frac{\varepsilon_j}{\varepsilon_0} - 1 \right) \int_{z_j + \alpha B_j} \tilde{u}_\alpha v.
\end{align*}
\]

Next we can bound

\[
\left| \int_{z_j + \alpha B_j} \nabla \tilde{u}_\alpha \cdot \nabla v \right| \leq \|\nabla \tilde{u}_\alpha\|_{L^2(z_j + \alpha B_j)} \|\nabla v\|_{L^2(\Omega)}
\]
and
\[ \left| \int_{z_j + \alpha B_j} \tilde{u}_\alpha v \right| \leq \| \tilde{u}_\alpha \|_{L^2(z_j + \alpha B_j)} \| v \|_{L^2(\Omega)}. \]

However, using the triangle inequality,
\[ \| \nabla \tilde{u}_\alpha \|_{L^2(z_j + \alpha B_j)} \leq \| \nabla (\tilde{u}_\alpha - u_0) \|_{L^2(\Omega)} + \| \nabla u_0 \|_{L^2(z_j + \alpha B_j)}, \]
and
\[ \| \tilde{u}_\alpha \|_{L^2(z_j + \alpha B_j)} \leq \| (\tilde{u}_\alpha - u_0) \|_{L^2(\Omega)} + \| u_0 \|_{L^2(z_j + \alpha B_j)}. \]

Therefore, since
\[ \| u_0 \|_{H^1(z_j + \alpha B_j)} = O(\alpha^2) \]
and
\[ \| (\tilde{u}_\alpha - u_0) \|_{H^1(\Omega)} = O(\alpha^3), \]
we obtain
\[ \left| \int_{\Omega} \frac{1}{\mu_\alpha} \nabla (u_\alpha - \tilde{u}_\alpha) \cdot \nabla v - \omega^2 \int_{\Omega} \varepsilon_\alpha (u_\alpha - \tilde{u}_\alpha) v \right| \leq C\alpha^2 \| v \|_{H^1(\Omega)} \]
for any \( v \in H^1_0(\Omega) \). From Proposition 1 in [24], it then follows that
\[ \| (u_\alpha - \tilde{u}_\alpha) \|_{H^1(\Omega)} = O(\alpha^2), \]
hence
\[ \| (u_\alpha - u_0) \|_{H^1(\Omega)} \leq \| (u_\alpha - \tilde{u}_\alpha) \|_{H^1(\Omega)} + \| (u_0 - \tilde{u}_\alpha) \|_{H^1(\Omega)} \leq C\alpha^2, \]
exactly as desired. \( \square \)

3. Asymptotic formula for the scattering amplitude. We now use the results derived in the previous section to prove an asymptotic formula for the scattering amplitude. The scattering amplitude, \( A_\alpha(x, z_j, \eta, k) \), is defined to be a function which satisfies

\[ u_\alpha(x) = e^{ik\cdot x} + A_\alpha \left( \frac{x}{|x|}, \eta, k \right) \frac{e^{ik|x|}}{|x|} + o \left( \frac{1}{|x|} \right) \]
as \( |x| \to \infty \). Recall that
\[ G(x, z_j) = \frac{e^{ik|x-z_j|}}{4\pi |x-z_j|}. \]

One can show from a simple calculation that, as \( |x| \to \infty \),

\[ G(x, z_j) = \frac{e^{ik|x|}}{|x|} \frac{e^{-ik\cdot z_j}}{4\pi} + o \left( \frac{1}{|x|} \right) \]
and
\begin{equation}
\nabla_y G(x, z_j) = \frac{e^{ik|x|}}{|x|} \frac{ikx}{4\pi|x|} e^{-ik \frac{x}{|x|} \cdot z_j} + o \left( \frac{1}{|x|} \right).
\end{equation}

The following asymptotic formula for the scattering amplitude holds.

**THEOREM 2.** The scattering amplitude
\begin{equation}
A_\alpha \left( \frac{x}{|x|}, \eta, k \right) = \frac{\alpha^3 k^2}{4\pi} \sum_{j=1}^{m} e^{ik(\eta - \frac{x}{|x|}) \cdot z_j} \left[ \left( \frac{\mu_j}{\mu_0} - 1 \right) \frac{x}{|x|} \cdot M^j \eta - \left( \frac{\varepsilon_j}{\varepsilon_0} - 1 \right) |B_j| \right]
\end{equation}

for any \( \frac{x}{|x|} \) and \( \eta \in S^2 \), where \( o(\alpha^3) \) is independent of the set of points \( \{ z_j \}_{j=1}^{m} \).

**Proof.** This follows from (21), (22), and the expansion in Theorem 1.

4. Method for reconstruction of inhomogeneities at a fixed frequency.

In this section, we present a linear method to determine the locations and the polarization tensors of the small inhomogeneities from scattering amplitude measurements for a fixed frequency. Based on the asymptotic expansion (23), we reduce the reconstruction of the small dielectric inhomogeneities from the scattering amplitude to the calculation of an inverse Fourier transform. For convenience, we are going to assume that \( B_j \), for \( j = 1, \ldots, m \), are balls. In this case, the polarization tensors \( M^j \) have the following explicit forms (see, for example, [25]):
\begin{equation}
M^j \left( \frac{\mu_j}{\mu_0} \right) = m^j I_3,
\end{equation}

where \( I_3 \) is the \( 3 \times 3 \) identity matrix and the scalars \( m^j \) are given by
\begin{equation}
m^j = 8\pi |B_j| \frac{\mu_j}{\mu_j + \mu_0}.
\end{equation}

We assume that we are in possession of the scattering amplitude \( A_\alpha \left( \frac{x_l}{|x_l|}, \eta_l, k \right) \) for a collection of pairs \( \left( \frac{x_l}{|x_l|}, \eta_l \right) \), where \( l = 1, \ldots, L \) and \( l' = 1, \ldots, L' \). Introduce
\begin{equation}
g \left( \frac{x}{|x|}, \eta \right) = \sum_{j=1}^{m} e^{ik(\eta - \frac{x}{|x|}) \cdot z_j} \left[ \left( \frac{\mu_j}{\mu_0} - 1 \right) m^j \frac{x}{|x|} \cdot \eta - \left( \frac{\varepsilon_j}{\varepsilon_0} - 1 \right) |B_j| \right], \quad \frac{x}{|x|}, \eta \in S^2.
\end{equation}

We first observe that
\begin{equation}
g \left( \frac{x}{|x|}, \eta \right) = g \left( -\eta, -\frac{x}{|x|} \right) \quad \forall \frac{x}{|x|}, \eta \in S^2.
\end{equation}

Define, for \( l = 1, \ldots, L \) and \( l' = 1, \ldots, L' \), the coefficients \( a_{l,l'} \) by
\begin{equation}
a_{l,l'} = \frac{4\pi}{k^2 \alpha^3} A_\alpha \left( \frac{x_l}{|x_l|}, \eta_{l'}, k \right).
\end{equation}

Our reconstruction procedure is divided into three steps.

**Step 1.** Given that
\begin{equation}
g \left( \frac{x_l}{|x_l|}, \eta_{l'} \right) \approx a_{l,l'},
\end{equation}
we can compute using the fast Fourier transform (FFT) an accurate approximation of \( g(\frac{\vec{x}}{|\vec{x}|}, \eta) \) on \( S^2 \times S^2 \).

**Step 2.** Let \( M \) denote the following complex variety:

\[ M = \{ \xi \in \mathbb{C}^3, \; \xi \cdot \xi = 1 \}. \]

It is easy to see that \( g(\frac{\vec{x}}{|\vec{x}|}, \eta) \) has an analytic continuation to \( M \times M \). Let \( (Y_{p,q})_{-p \leq q \leq p, \; p=0,1,\ldots} \) denote the normalized (in \( L^2(S^2) \)) spherical harmonics. Denote by \( g_{p,q} \) the Fourier coefficients of \( g \):

\[ g \left( \frac{x}{|x|}, \eta \right) = \sum_{p,q} g_{p,q} \left( \frac{x}{|x|} \right) Y_{p,q}(\eta) \quad \forall \; \frac{x}{|x|}, \eta \in S^2. \]

Recall that, from Step 1, we are in fact in possession of an accurate approximation of \( g_{p,q}(\frac{\vec{x}}{|\vec{x}|}) \) on \( S^2 \) for \(-p \leq q \leq p\) and \( p \leq P \). In view of (26), the analytic continuation of the truncated Fourier series

\[ \sum_{p,q; p \leq P} g_{p,q} \left( \frac{x}{|x|} \right) Y_{p,q}(\eta) \]

of \( g(\frac{\vec{x}}{|\vec{x}|}, \eta) \) on \( M \times M \) can be obtained by using the standard analytic continuation of the spherical harmonics \( (Y_{p,q}(\eta))_{p,q} \) on the complex variety \( M \) followed by another analytic continuation of the Fourier expansion in \( \frac{x}{|x|} \). We know that the analytic continuation of \( g \) from \( S^2 \times S^2 \) to \( M \times M \) is unique.

**Step 3.** Recall that, given \( a_{l,l'} \) for \( l = 1, \ldots, L \) and \( l' = 1, \ldots, L' \), we have constructed by Steps 1 and 2 an accurate approximation of the function \( g(\frac{\vec{x}}{|\vec{x}|}, \eta) \) that is analytic on \( M \times M \) and is such that

\[ g \left( \frac{x_l}{|x_l|}, \eta_{l'} \right) \approx a_{l,l'} \quad \forall \; l = 1, \ldots, L \; \text{and} \; l' = 1, \ldots, L'. \]

However, for any \( \xi \in \mathbb{R}^3 \), we know that there exist \( \xi_1 \) and \( \xi_2 \) in \( M \) such that \( \xi = k(\xi_1 - \xi_2) \); see, for example, [5] and [20]. Let us now view \( (a_{l,l'}) \) as a function of \( \xi \in \mathbb{R}^3 \). We have

\[ g(\xi_1, \xi_2) = \sum_{j=1}^m e^{-i\xi \cdot z_j} \left[ \left( \frac{\mu_j}{\mu_0} - 1 \right) m_j \xi_1 \cdot \xi_2 - \left( \frac{z_j}{z_0} - 1 \right) |B_j| \right], \]

and, since

\[ \xi_1 \cdot \xi_2 = 1 - \frac{1}{2} k^2 |\xi|^2, \]

we can rewrite \( g \) as follows:

\[ g(\xi_1, \xi_2) = \sum_{j=1}^m e^{-i\xi \cdot z_j} \left[ \left( \frac{\mu_j}{\mu_0} - 1 \right) m_j \left( 1 - \frac{1}{2} k^2 |\xi|^2 \right) - \left( \frac{z_j}{z_0} - 1 \right) |B_j| \right]. \]

Define

\[ \tilde{g}(\xi) = g(\xi_1, \xi_2), \]
and note that we are now in possession of an approximation to \( \hat{g}(\xi) \) for any \( \xi \in \mathbb{R}^3 \). Here we rely on the fact that the analytic continuation is unique.

Recall that \( e^{-i\xi \cdot z_j} \) (up to a multiplicative constant) is exactly the Fourier transform of the Dirac function \( \delta_{z_j} \) (a point mass located at \( z_j \)). Multiplication by powers of \( \xi \) in Fourier space corresponds to differentiation of the Dirac function. Therefore, using the inverse Fourier transform, we obtain

\[
\mathcal{F}^{-1}(\hat{g}(\xi)) = \sum_{j=1}^{m} L_j(\delta_{z_j}),
\]

where \( L_j \) are, in view of (27), second order constant coefficient differential operators.

Hence \( \hat{g}(\xi) \) is the inverse Fourier transform of a distribution with its support at the locations of the centers of inhomogeneities \( z_j \). Therefore, we think that a numerical Fourier inversion of a sample of \( \hat{g}(\xi) \) will efficiently pin down the \( z_j \)'s. The method of location of the points \( z_j \) is then similar to that proposed for the conductivity problem [2] from boundary measurements. The number of data (sampling) points needed for an accurate discrete Fourier inversion of \( \hat{g}(\xi) \) follows from the Shannon theorem [9]. We need (conservatively), of order \((h/\delta)^3\), sampled values of \( \xi \) to reconstruct, with resolution \( \delta \), a collection of inhomogeneities that lie inside a square of side \( h \). Note, however, that real measurements are taken only in Step 1. It remains to be seen how many such measurements are needed. Once the locations \( \{z_j\}_{j=1}^{m} \) are known, we may calculate \( |B_j| \) by solving the appropriate linear system arising from (27). If \( B_j \) are general domains, our calculations become more complex, and eventually we have to deal with pseudodifferential operators (independent of the space variable \( x \)) applied to the same Dirac functions. Numerical experiments examining the feasibility of this approach will be presented in a forthcoming publication.

**Appendix. Proof of Proposition 1.** Recall that \( \Omega \) is some fixed domain in \( \mathbb{R}^3 \) containing the inhomogeneities. Define \( \hat{G}(x,z) \) to be the Dirichlet Green function for \( \Omega \),

\[
\Delta_x \hat{G}(x,z) + k^2 \hat{G}(x,z) = -\delta_x \quad \text{in } \Omega,
\]

\[
\hat{G}(x,z) = 0 \quad \text{on } \partial \Omega.
\]

Recall that

\[
N_\alpha f - N_0 f = \frac{\partial v_\alpha}{\partial \nu} - \frac{\partial v_0}{\partial \nu},
\]

where

\[
\nabla \cdot \frac{1}{\mu_\alpha} \nabla v_\alpha + \omega^2 \varepsilon_\alpha v_\alpha = 0 \quad \text{in } \Omega,
\]

\[
v_\alpha = f \quad \text{on } \partial \Omega,
\]

and

\[
\nabla \cdot \frac{1}{\mu_0} \nabla v_0 + \omega^2 \varepsilon_0 v_0 = 0 \quad \text{in } \Omega,
\]

\[
v_0 = f \quad \text{on } \partial \Omega.
\]
Integration by parts gives
\[
v_\alpha(x) = -\int_\Omega v_\alpha(z)(\Delta_z \hat{G} + k^2 \hat{G}) \, dz
\]
\[
= \int_{\partial \Omega} \frac{\partial \hat{G}}{\partial \nu_z} \, d\sigma_z + \int_\Omega \nabla v_\alpha \cdot \nabla_z \hat{G} \, dz - \int_\Omega k^2 v_\alpha \hat{G} \, dz
\]
\[
= v_0(x) + \sum_{j=1}^{m} \int_{z_j + \alpha B_j} \left[ \left( 1 - \frac{\mu_j}{\mu^0} \right) \nabla v_\alpha \nabla \hat{G} \, dz + k^2 \left( 1 - \frac{\epsilon_j}{\epsilon^0} \right) v_\alpha \hat{G} \right]
\]
(31)
since by (29) and (30)
\[
\int_\Omega \frac{1}{\mu_\alpha} \nabla v_\alpha \cdot \nabla_z \hat{G} \, dz - \omega^2 \int_\Omega \epsilon_\alpha v_\alpha \hat{G} \, dz = 0
\]
and
\[
v_0(x) = \int_{\partial \Omega} \frac{\partial \hat{G}}{\partial \nu_z} \, d\sigma_z.
\]

We first derive a uniform asymptotic expansion for \( \frac{\partial \nu_\alpha}{\partial \nu} \) on \( \partial \Omega \). We note that this is similar to Theorem 1 in [24], where the authors derived an expansion when \( n = 2 \) using the free space Green function. We use the Dirichlet Green function because it is more convenient for our purposes.

**Lemma 3.** Let \( v_\alpha \) and \( v_0 \) be defined as above. Then we have the pointwise expansion
\[
(N_\alpha - N_0)(f)
\]
\[
= \frac{\partial v_\alpha}{\partial \nu}(x) - \frac{\partial v_0}{\partial \nu}(x)
\]
\[
= \alpha^3 \sum_{j=1}^{m} \left[ \left( \frac{1}{\mu^j} - \frac{1}{\mu_0} \right) \nabla v_\alpha(z_j) \cdot M^j \left( \frac{\mu^j}{\mu_0} \right) \nabla \hat{G}(x, z_j) \right.
\]
\[
+ k^2 \left( 1 - \frac{\epsilon_j}{\epsilon^0} \right) v_\alpha(z_j) \frac{\partial}{\partial \nu_x} \hat{G}(x, z_j) \right] + o(\alpha^3),
\]
(32)

where the term \( o(\alpha^3) \) is uniform for \( x \in \partial \Omega \).

For reasons of brevity, we restrict a significant part of the derivation of the asymptotic expansion (32) to the case of one inhomogeneity \( (m = 1) \). We suppose that this inhomogeneity is centered at the origin, so it is of the form \( \alpha B \). The general case may be verified by a fairly direct iteration of the argument we will present here, adding one inhomogeneity at a time. We will as usual make the change of variables
\[
y = x/\alpha,
\]
where
\[
\hat{\Omega} = \frac{1}{\alpha} \Omega
\]
and
\[
B = \frac{1}{\alpha} B_\alpha.
\]
Define the correction $w_\alpha(y)$ to be the unique solution to

\begin{align}
\Delta y w_\alpha + \alpha^2 \omega^2 \varepsilon^1 \mu_1^{1} w_\alpha &= 0 \quad \text{in } B, \\
\Delta y w_\alpha + \alpha^2 \omega^2 \varepsilon^0 \mu_0^{0} w_\alpha &= 0 \quad \text{in } \tilde{\Omega} \setminus \bar{B}, \\
\frac{1}{\mu_0^{0}} \frac{\partial w_\alpha^+}{\partial y} - \frac{1}{\mu_1^{1}} \frac{\partial w_\alpha^-}{\partial y} &= -\left( \frac{1}{\mu_0^{0}} - \frac{1}{\mu_1^{1}} \right) \nabla_x v_0(0) \cdot \nu \quad \text{on } \partial B, \\
\frac{\partial w_\alpha}{\partial \nu} &= 0 \quad \text{on } \partial \tilde{\Omega},
\end{align}

with $w_\alpha$ continuous across $\partial B$.

Also, define $w(y)$, which is independent of $\alpha$ and a sort of limit of $w_\alpha$, as the unique solution to

\begin{align}
\Delta y w = 0 \quad \text{in } B, \\
\Delta y w = 0 \quad \text{in } \mathbb{R}^n \setminus \bar{B}, \\
\frac{1}{\mu_0^{0}} \frac{\partial w^+}{\partial y} - \frac{1}{\mu_1^{1}} \frac{\partial w^-}{\partial y} &= -\left( \frac{1}{\mu_0^{0}} - \frac{1}{\mu_1^{1}} \right) \nabla_x v_0(0) \cdot \nu \quad \text{on } \partial B, \\
\lim_{|y| \to \infty} |w(y)| &= 0,
\end{align}

with $w$ continuous across $\partial B$.

Recall that $|w(y)| = O\left(\frac{1}{|y|}\right)$ as $|y| \to +\infty$. We now need to prove two lemmas before we can proceed with the derivation of the asymptotic formula (13).

**Lemma 4.** Let $\nu_\alpha$, $v_0$, and $w_\alpha$ be given by (29), (30), and (33), respectively. Let

$$z_\alpha(y) = v_\alpha(\alpha y) - v(\alpha y) - \alpha w_\alpha(y).$$

Then there exists a constant $C$ independent of $\alpha$ such that

$$\|z_\alpha\|_{L^2(\Omega)} \leq C$$

and

$$\|\nabla_y z_\alpha\|_{L^2(\Omega)} \leq C\alpha.$$

**Proof.** Note that $z_\alpha(x/\varepsilon) \in H_0^1(\Omega)$. For any $\phi \in H_0^1(\Omega)$, integration by parts gives us that

\[
\int_{\Omega} \nabla_y z_\alpha \cdot \nabla_y \phi(\alpha y) \, dy - \alpha^2 \omega^2 \int_{\Omega} \varepsilon_\alpha(\alpha y) z_\alpha \phi(\alpha y) \, dy
\]

\[
= \left( \frac{1}{\mu_0^{0}} - \frac{1}{\mu_1^{1}} \right) \int_{\partial B} \nabla_x (v_0(\alpha y) - v_0(0)) \cdot \nu \phi(\alpha y) \, d\sigma_y - \alpha^2 \omega^2 (\varepsilon^0 - \varepsilon^1) \int_{B} v_0(\alpha y) \phi(\alpha y) \, dy
\]

\[
= \left( \frac{1}{\mu_0^{0}} - \frac{1}{\mu_1^{1}} \right) \int_{B} \alpha \Delta_x (v_0(\alpha y) - v_0(0)) \phi(\alpha y) \, d\sigma_y - \alpha^2 \omega^2 (\varepsilon^0 - \varepsilon^1) \int_{B} v_0(\alpha y) \phi(\alpha y) \, dy
\]

\[
+ \left( \frac{1}{\mu_0^{0}} - \frac{1}{\mu_1^{1}} \right) \int_{B} \nabla_x (v_0(\alpha y) - v_0(0)) \cdot \nabla_y \phi(\alpha y) \, d\sigma_y.
\]
Next we change variables back to the small domain on the left-hand side and multiply by $\alpha$ to obtain

\[
\int_{\Omega} \frac{1}{\mu_\alpha} \nabla_x z_\alpha \cdot \nabla_x \phi \, dx - \omega^2 \int_{\Omega} \varepsilon_\alpha z_\alpha \phi \, dx
= \alpha^2 \left( \frac{1}{\mu^0} - \frac{1}{\mu^1} \right) \int_B \Delta_x (v_0(\alpha y) - v_0(0)) \phi(\alpha y) \, dy - \alpha^3 \omega^2 (\varepsilon^0 - \varepsilon^1) \int_B v_0(\alpha y) \phi(\alpha y) \, dy
+ \alpha \left( \frac{1}{\mu^0} - \frac{1}{\mu^1} \right) \int_B \nabla_x (v_0(\alpha y) - v_0(0)) \cdot \nabla_y \phi(\alpha y) \, dy.
\]

Using a Taylor expansion of $v_0$, we find that there exists $C$, depending on $v_0$ but independent of $\alpha$, such that

\[
\left| \int_{\Omega} \frac{1}{\mu_\alpha} \nabla_x z_\alpha \cdot \nabla_x \phi \, dx - \omega^2 \int_{\Omega} \varepsilon_\alpha z_\alpha \phi \, dx \right| \leq C \alpha^3 \| \phi(\alpha y) \|_{L^2(B)} + C \alpha^2 \| \nabla_y \phi(\alpha y) \|_{L^2(B)}.
\]

By rescaling, we see that

\[
\| \phi(\alpha y) \|_{L^2(B)} = \alpha^{-3/2} \| \phi \|_{L^2(\alpha B)}
\]

and

\[
\| \nabla_y \phi(\alpha y) \|_{L^2(B)} = \alpha^{-1/2} \| \nabla_x \phi \|_{L^2(\alpha B)}
\]

so that

\[
\left| \int_{\Omega} \frac{1}{\mu_\alpha} \nabla_x z_\alpha \cdot \nabla_x \phi \, dx - \omega^2 \int_{\Omega} \varepsilon_\alpha z_\alpha \phi \, dx \right| \leq C \alpha^{3/2} \| \phi \|_{H^1(\Omega)}.
\]

By Proposition 1 of [24], it follows that

\[
\| z_\alpha \|_{H^1(\Omega)} \leq C \alpha^{3/2}.
\]

The result then follows from another scaling. \qed

**Lemma 5.** Let $w_\alpha$ and $w$ be defined by (33) and (34), respectively. Then there exists $C$ independent of $\alpha$ such that

\[
\| \nabla_y (w_\alpha - w) \|_{L^2(\tilde\Omega)} \leq \frac{C}{\alpha^{1/2}}.
\]

**Proof.** Consider $w_\alpha(x/\alpha) - w(x/\alpha)$. Since $w_\alpha$ and $w$ share the same jump condition on the boundary of the ball, their difference satisfies an equation across this boundary. It is not hard to see that in fact we have

\[
\nabla_x \cdot \frac{1}{\mu_\alpha} \nabla_x (w_\alpha - w) + \omega^2 \varepsilon_\alpha (w_\alpha - w) = -\omega^2 \varepsilon_\alpha w \quad \text{in} \quad \Omega,
\]

\[
w_\alpha - w = -w \quad \text{on} \quad \partial\Omega.
\]

By Proposition 1 and Corollary 1 in [24], there exists a constant $C$ independent of $\alpha$ such that

\[
\| w_\alpha - w \|_{H^1(\Omega)} \leq C \left( \| w \|_{L^2(\Omega)} + \| w \|_{H^1(\partial\Omega)} \right).
\]
Since $\Omega$ is a bounded domain and $w(y)$ is bounded, we clearly have $\|w\|_{L^2(\Omega)}$ bounded. Also, since $w(x/\alpha)$ decays as $\alpha \to 0$, we also have $\|w\|_{H^{1/2}(\partial \Omega)}$ bounded independently of $\alpha$. Hence

$$\|w_\alpha - w\|_{H^1(\Omega)} \leq C,$$

which by rescaling proves the lemma. □

Now define

$$r_\alpha(y) = v_\alpha(\alpha y) - v_0(\alpha y) - \alpha w - c_\alpha,$$

where the constant $c_\alpha$ is defined so that $r_\alpha$ satisfies

$$\int_{\partial B} r_\alpha \, d\sigma_y = 0.$$

The previous two lemmas together imply that

$$\|\nabla_y r_\alpha\|_{L^2(\tilde{\Omega})} \leq C \alpha^{1/2}.$$

Then, from (31),

$$v_\alpha(x) - v_0(x) = \int_{\alpha B} \left[ \left( 1 - \frac{\mu^0}{\mu^1} \right) \nabla_z v_\alpha(z) \nabla_z \hat{G}(x, z) + k^2 \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) v_\alpha(z) \hat{G}(x, z) \right] \, dz$$

$$= \alpha^3 \int_B \left[ \left( 1 - \frac{\mu^0}{\mu^1} \right) \nabla_z v_\alpha(\alpha y) \nabla_z \hat{G}(x, \alpha y) + k^2 \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) v_\alpha(\alpha y) \hat{G}(x, \alpha y) \right] \, dy$$

$$= \alpha^2 \int_B \left( 1 - \frac{\mu^0}{\mu^1} \right) \nabla_y (v_0 + \alpha w) \nabla_z \hat{G}(x, \alpha y)$$

$$+ k^2 \alpha^3 \int_B \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) v_\alpha(\alpha y) \hat{G}(x, \alpha y) \, dy$$

(35)

$$+ \alpha^2 \int_B \left( 1 - \frac{\mu^0}{\mu^1} \right) \nabla_y (r_\alpha) \nabla_z \hat{G}(x, \alpha y) \, dy.$$

By expanding $\hat{G}$ in a Taylor series and using the above estimate for $r_\alpha$, we have that

(36) $$\int_B \nabla_y r_\alpha \cdot \nabla_z \hat{G}(x, \alpha y) \, dy = \int_B \nabla_y r_\alpha \cdot \nabla_z \hat{G}(x, 0) \, dy + O(\alpha^{3/2}),$$

and since we have chosen $r_\alpha$ to have integral zero around the boundary of $B$, the first term on the right-hand side above is zero by integration by parts. Hence

(37) $$\int_B \nabla_y r_\alpha \cdot \nabla_z \hat{G}(x, \alpha y) \, dy = O(\alpha^{3/2}).$$

Inserting this into (35), we have shown that

$$v_\alpha(x) - v_0(x) = \alpha^2 \int_B \left( 1 - \frac{\mu^0}{\mu^1} \right) \nabla_y (v_0 + \alpha w) \nabla_z \hat{G}(x, \alpha y)$$

$$+ \alpha^3 k^2 \int_B \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) v_\alpha(\alpha y) \hat{G}(x, \alpha y) \, dy + o(\alpha^3).$$

(38)
From this expression, we now derive the formulae with the polarization tensor:

\[
v_{\alpha}(x) - v_0(x) = \alpha^3 \left( 1 - \frac{\mu^0}{\mu^1} \right) \left[ \int_B \nabla_x v_0(\alpha y) \cdot \nabla_z \hat{G}(x, \alpha y) \, dy \right. \\
+ \int_B \nabla_y w \cdot \nabla_z \hat{G}(x, \alpha y) \, dy \right] \\
+ k^2 \alpha^3 \int_B \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) v_{\alpha}(\alpha y) \hat{G}(x, \alpha y) \, dy + o(\alpha^3) \\
= \alpha^3 \left( 1 - \frac{\mu^0}{\mu^1} \right) |B| \nabla_x v_0(0) \cdot \nabla_z \hat{G}(x, 0) \\
+ \alpha^3 \left( 1 - \frac{\mu^0}{\mu^1} \right) \int_B \nabla_y w \cdot \nabla_z \hat{G}(x, 0) \, dy \\
+ k^2 \alpha^3 \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) |B| v_0(0) \hat{G}(x, 0) + o(\alpha^3)
\] (39)

by Taylor expansions for \(v_0\) and \(\hat{G}\). Note that

\[
\int_B \nabla_y w \, dy = \int_{\partial B} \frac{\partial w}{\partial \nu_y} \, y \, d\sigma_y
\]

and

\[
\psi = w + \nabla_x v_0(0) \cdot y = \frac{\partial v_0}{\partial \nu_x}(0) \phi_l,
\]

where the \(\phi_l\) are defined by (4). Hence

\[
|B| \nabla_x v_0(0) + \int_B \nabla_y w \, dy = \int_B \nabla_y \psi \, dy,
\]

from which we may rewrite (41) as

\[
v_{\alpha}(x) - v_0(x) = \alpha^3 \left( \frac{1}{\mu^1} - \frac{1}{\mu^0} \right) \nabla u_0(0) \cdot M \left( \frac{\mu^1}{\mu^0} \right) \nabla_z \hat{G}(x, 0) \\
+ k^2 \left( 1 - \frac{\varepsilon^1}{\varepsilon^0} \right) u_0(0) \hat{G}(x, 0) + o(\alpha^3)
\] (42)

for \(M\) defined by (4). By standard elliptic regularity, we obtain (32), where the term \(o(\alpha^3)\) is uniform for \(x \in \partial \Omega\).

We are now ready to prove Proposition 1. Integration by parts yields

\[
\int_{\partial \Omega} G(x, y) \frac{\partial}{\partial \nu_y} (\nabla_z \hat{G}(y, 0)) \, d\sigma_y = \nabla_z G(x, 0) \quad \text{and}
\int_{\partial \Omega} G(x, y) \frac{\partial}{\partial \nu_y} (\hat{G}(y, 0)) \, d\sigma_y = G(x, 0).
\] (43)

By applying the operator \(S\) to (32) and using (43), we arrive at the promised asymptotic expansion (13), which, along with the boundedness of the operator \(S\), implies that \(T_\alpha\) converges to \(T_0\) pointwise, which is the claim in point (a). Furthermore,
since the points $z_j$ are away from the boundary $\partial \Omega$, it follows from (13) that the family of operators $T_\alpha - T_0$ is collectively compact, and so point (b) holds. Rewriting $T_\alpha = T_0 + (T_\alpha - T_0)$ and recalling that the operator $T_0$ is invertible, it follows immediately from [4] that $T_\alpha^{-1}$ is well defined, and point (c) in Proposition 1 holds.

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REFERENCES

