Shape and Topology Optimization by the Level Set Method

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1 Introduction

Numerical methods of shape and topology optimization based on the level set representation and on shape differentiation make possible topology changes during the optimization process. But they do not solve the inherent problem of ill-posedness of shape optimization which manifests itself in the existence of many local minima, usually having different topologies. The reason is that the level set method can easily remove holes but can not create new holes in the middle of a shape. In practice, this effect can be checked by varying the initialization which yields different optimal shapes with different topologies. This absence of a nucleation mechanism is an inconvenient mostly in 2-d: in 3-d, it is less important since holes can appear by pinching two boundaries.

In [1] we have proposed, as a remedy, to couple our previous method with the topological gradient method (cf. [5][6][7][13]). Roughly speaking it amounts to decide whether or not it is favorable to nucleate a small hole in a given shape. Creating a hole changes the topology and is thus one way of escaping local minima. Our coupled method of topological and shape gradients in the level set framework is therefore much less prone to finding local, non global, optimal shapes. For most of our 2-d numerical examples of compliance minimization, the expected global minimum is attained from the trivial full domain initialization.

2 Setting of the problem

We restrict ourselves to linear elasticity. A shape is a bounded open set $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with a boundary made of two disjoint parts Γ_N and Γ_D, submitted to respectively Neumann and Dirichlet boundary conditions. All admissible shapes Ω are required to be a subset of a working domain $D \subset \mathbb{R}^d$. The shape Ω is occupied by a linear isotropic elastic material with Hooke's law A defined, for any symmetric matrix ξ, by $A\xi = 2\mu\xi + \lambda\big(\mathrm{Tr}\xi\big)\,\mathrm{Id}$, where μ and λ are the Lamé moduli. The displacement field u is the solution of the linearized elasticity

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system

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(A\,e(u)) & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma_D \\ \big(A\,e(u)\big)n & = & g & \text{on } \Gamma_N, \end{array}\right. \tag{1}$$

where $f \in L^2(D)^d$ and $g \in H^1(D)^d$ are the volume forces and the surface loads. If $\Gamma_D \neq \emptyset$, (1) admits a unique solution in $u \in H^1(\Omega)^d$. The objective function is denoted by $J(\Omega)$. In this paper, only the compliance will be considered:

$$J(\Omega) = \int_\Omega f \cdot u\,dx + \int_{\Gamma_N} g \cdot u\,ds = \int_\Omega A\,e(u) \cdot e(u)\,dx. \tag{2}$$

To avoid working on a problem with a volume constraint, we introduce a Lagrange multiplier ℓ and consider the minimization

$$\inf_{\Omega \subset D} \mathcal{L}(\Omega) = J(\Omega) + \ell|\Omega|. \tag{3}$$

3 Shape derivative

To apply a gradient method to the minimization of (3) we recall the classical notion of shape derivative (see e.g. [9][12]). Starting from a smooth open set Ω, we consider domains of the type $\Omega_\theta = \big(\mathrm{Id} + \theta\big)(\Omega)$, with Id the identity mapping of $\mathbb{R}^d$ and θ a vector field in $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

Definition: The shape derivative of J at Ω is defined as the Fréchet derivative at 0 of the application $\theta \to J\big((\mathrm{Id} + \theta)(\Omega)\big)$, i.e.

$$J\big((\mathrm{Id} + \theta)(\Omega)\big) = J(\Omega) + J'(\Omega)(\theta) + o(\theta) \quad \text{with} \quad \lim_{\theta \to 0} \frac{|o(\theta)|}{\|\theta\|} = 0\,,$$

where $J'(\Omega)$ is a continuous linear form on $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.
We recall the following classical result (see [3] and references therein).

Theorem 1 (shape derivative for the compliance): *Let Ω be a smooth bounded open set and $\theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$. If $f \in H^1(\Omega)^d$, $g \in H^2(\Omega)^d$, $u \in H^2(\Omega)^d$, then the shape derivative of (2) is*

$$\begin{aligned} J'(\Omega)(\theta) = & \int_{\Gamma_N} \left(2\left[\frac{\partial(g \cdot u)}{\partial n} + Hg \cdot u + f \cdot u \right] - Ae(u) \cdot e(u) \right) \theta \cdot n\,ds \\ & + \int_{\Gamma_D} Ae(u) \cdot e(u)\;\theta \cdot n\,ds, \end{aligned}$$

where H is the mean curvature defined by $H = \mathrm{div} n$.

4 Topological derivative

One drawback of the method of shape derivative is that no nucleation of holes inside the domain are allowed. Numerical methods based on the shape derivative may therefore fall into a local minimum. A remedy to this inconvenience has been proposed as the bubble method, or topological asymptotic method, [6], [7],

[13]. The main idea is to test the optimality of a domain to topology variations by removing a small hole with appropriate boundary conditions.

Consider an open set $\Omega \subset \mathbb{R}^d$ and a point $x_0 \in \Omega$. Introduce a fixed model hole $\omega \subset \mathbb{R}^d$, a smooth open bounded subset containing the origin. For $\rho > 0$ we define the translated and rescaled hole $\omega_\rho = x_0 + \rho\omega$ and the perforated domain $\Omega_\rho = \Omega \setminus \bar{\omega}_\rho$. The goal is to study the variations of the objective function $J(\Omega_\rho)$ as $\rho \to 0$.

Definition: *If the objective function admits the following so-called topological asymptotic expansion for small $\rho > 0$*

$$J(\Omega_\rho) = J(\Omega) + \rho^d D_T J(x_0) + o(\rho^d),$$

then $D_T J(x_0)$ is called the topological derivative at point x_0.

If the model hole ω is the unit ball, the following result gives the expression of the topological derivative for the compliance $J(\Omega)$ with Neumann boundary conditions on the hole in 2d (see [7], [13]).

Theorem 2: *Let ω be the unit ball of $\mathbb{R}^2$. If $f = 0$, $g \in H^2(\Omega)^2$ and $u \in H^2(\Omega)^2$, then $\forall x \in \Omega \subset \mathbb{R}^2$, if $C_2 = \pi(\lambda + 2\mu)/(2\mu(\lambda + \mu))$,*

$$D_T J(x) = C_2 \big\{ 4\mu A e(u) \cdot e(u) + (\lambda - \mu) tr(Ae(u)) tr(e(u)) \big\}(x).$$

The above expression is nonnegative. This means that, for compliance minimization, there is no interest in nucleating holes if there is no volume constraint. However, if a volume constraint is imposed, the topological derivative may have negative values due to the addition of the term $-\ell|\omega|$. For the minimization problem (3), the corresponding topological gradient is $D_T\mathcal{L}(x) = D_T J(x) - \ell|\omega|$. At the points where $D_T\mathcal{L}(x) < 0$, holes are introduced into the current domain.

5 Level set method for shape optimization

Consider $D \subset \mathbb{R}^d$ a bounded domain in which all admissible shapes Ω are included, i.e. $\Omega \subset D$. Following the idea of Osher and Sethian [10], the boundary of Ω is represented by means of a level set function ψ such that $\psi(x) < 0 \Leftrightarrow x \in \Omega$. The normal n to the shape Ω is recovered as $\nabla\psi/|\nabla\psi|$ and the mean curvature H is given by $\operatorname{div}(\nabla\psi/|\nabla\psi|)$.

During the optimization process, the shape $\Omega(t)$ is going to evolve according to a fictitious time parameter $t > 0$ which corresponds to descent stepping. The evolution of the level set function is governed by the following Hamilton-Jacobi transport equation [10]

$$\frac{\partial\psi}{\partial t} + V|\nabla\psi| = 0 \quad \text{in} \quad D, \tag{4}$$

where $V(t, x)$ is the normal velocity of the shape's boundary. The choice V is based on the shape derivative computed in Theorem 1

$$\mathcal{L}'(\Omega)(\theta) = \int_{\partial\Omega} v\,\theta \cdot n\, ds, \tag{5}$$

where the integrand $v(u, n, H)$ depends on the state u, the normal n and the mean curvature H. The simplest choice is to take the steepest descent $\theta = -vn$. This yields a normal velocity for the shape's boundary $V = -v$. Another choice consists in smoothing the velocity field vn by applying the Neumann-to-Dirichlet map to $-vn$. The method described in details in [8] is used in the numerical computations.

The main point is that the Lagrangian evolution of the boundary $\partial\Omega$ is replaced by the Eulerian solution of a transport equation in the whole fixed domain D. Likewise the elasticity equations for the state u are extended to the whole domain D by using the so-called "ersatz material" approach. The Hamilton-Jacobi equation (4) is solved by an explicit upwind scheme (see e.g. [11]) on a Cartesian grid with a time stepping satisfying a CFL condition. To regularize the level set function (which may become too flat or too steep), it is periodically reinitialized by solving another Hamilton-Jacobi equation which admits, as a stationary solution, the signed distance to the initial interface [11].

6 Optimization algorithm

For the minimization problem (3) we propose an iterative coupling of the level set method and of the topological gradient method. Both methods are gradient-type algorithms, so our coupled method can be thought of as an alternate directions descent algorithm.

The level set method relies on the shape derivative $\mathcal{L}'(\Omega)(\theta)$ of Section 3, while the topological gradient method is based on the topological derivative $D_T\mathcal{L}(x)$ of Section 4. These two types of derivative define independent descent directions that we simply alternate as follows.

In a first step, the level set function ψ is advected according to the velocity $-v$. Then, holes are introduced into the current domain Ω where the topological derivative $D_T\mathcal{L}(x)$ is minimum and negative.

Our proposed algorithm is structured as follows:

1. Initialization of the level set function ψ_0 corresponding to an initial guess Ω_0 (usually the full working domain D).

2. Iteration until convergence, for $k \geq 0$:

   (a) **Elasticity analysis.** Computation of the state u_k solving a problem of linear elasticity on Ω_k. This yields the shape derivative, the velocity v_k and the topological gradient.

   (b) **Shape gradient.** If $\mod(k, n_{top}) < n_{top}$, the current shape Ω_k, characterized by the level set function ψ_k, is deformed into a new shape Ω_{k+1}, characterized by ψ_{k+1} which is the solution of the Hamilton-Jacobi equation (4) after a time interval Δt_k with the initial condition ψ_k and a velocity $-v_k$. Δt_k is chosen such that $\mathcal{L}(\Omega_{k+1}) \leq \mathcal{L}(\Omega_k)$.

   (c) **Topological gradient.** If $\mod(k, n_{top}) = 0$, nucleation step: Ω_{k+1} is obtained by inserting new holes into Ω_k according to the topological gradient.

For details about the shape gradient step and the topological gradient step, we refer to our previous works [1][2][3].

7 A numerical example in 2-d

It is a variation of the classical cantilever, but its optimal solution have a more complex topology. It consists in a rectangular domain of dimensions 10×8 with a square hole whose boundaries are submitted to an homogeneous Dirichlet boundary condition. The domain is meshed with a regular 150×120 grid. Figure 1 shows the solution obtained by the algorithm coupling shape and topological sensitivity, starting from the full domain, with 1 step of topological gradient every 10 iterations.

The convergence history of Figure 2, for different numbers of initial holes, ranging from 0 to 160, gives some hints on the efficiency of the level set method without topological gradient: first, it confirms that a "topologically poor" initialization cannot convergence to a good solution; second, it shows that initializing with "many holes" is not a good idea too. The good strategy lies in between, but it is generally not easy to find. The topological gradient allows the convergence to a good solution, starting from the full domain, without the need of adjusting any tricky numerical parameters. Remark that the solution computed from initialization 3 (22 holes) is also good, but it has been reached after an history where it had to escape from many local minima, using the tolerance of the algorithm to small increases of the objective function.

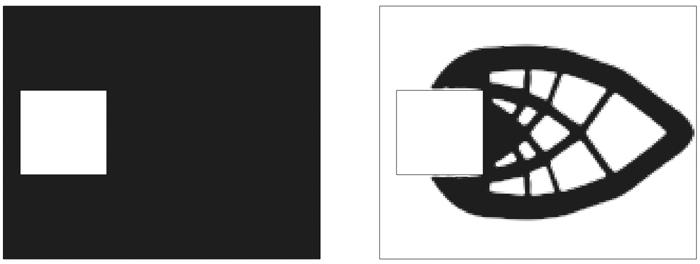

Figure 1: The initial configuration (full domain) and the solution obtained by the level set method with topological gradient.

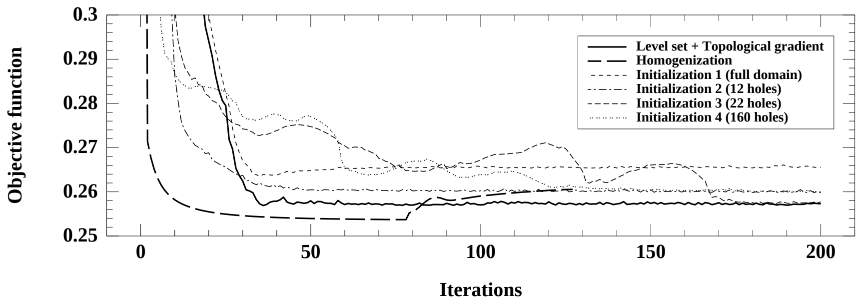

Figure 2: Convergence history of the homogenization method, the level set method with topological gradient (full domain initialization), and the plain level set method with 4 different initial states.

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