Compact Unstructured Representations for Evolutionary Topological Optimum Design

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Abstract

This paper proposes a few steps to escape structured extensive representations for evolutionary solving of Topological Optimum Design (TOD) problems: early results have shown the ability of Evolutionary methods to find numerical solutions to yet unsolved TOD problems, but those approaches were limited because the complexity of the representation was that of a fixed underlying mesh. Different compact unstructured representations are introduced, the complexity of which is self-adaptive, i.e. is evolved by the algorithm itself. The Voronoi-based representations are variable length lists of alleles that are directly decoded into shapes, while the IFS representation, based on fractal theory, involves a much more complex morphogenetic process. First results demonstrates that Voronoi-based representations allow one to push further the limits of Evolutionary Topological Optimum Design by actually removing the correlation between the complexity of the representations and that of the discretization. Further comparative results among all these representations on simple test problems indicate that the complex causality in the IFS representation disfavor it compared to the Voronoi-based representations.

1 Introduction

Evolutionary Algorithms (EAs) have been widely used in the framework of parametric optimization, i.e. when the search space is a structured space of fixed length vectors. In that context, EAs are just yet another optimization
method: They are indeed a powerful stochastic zero-th order global optimization method, and, as such, they have been successfully applied in many domains.

However, the most innovative and outstanding results have been recently obtained by taking advantage of the ability of EAs to deal with very unusual unstructured search spaces, such as spaces of unordered lists, of parse-trees, of graphs, ... The "only" prerequisites are an initialization procedure and variation operators that respect some minimal requirements [49]. Indeed, the larger the search space the better the solutions it contains, but the more difficult it might be to find them.

On the other hand, using unstructured search spaces is probably the only way to evolve complex solutions for which a description in extension will rapidly hit some scalability limits.

One first step away from structured representations is to use sparse variable-length lists instead of full extensive descriptions\(^1\): for instance, when searching in the space of polynomials, the extensive structured representation would be to look for the vector coefficients of all monomials up to a certain degree; an unstructured representation can be that of variable length lists of coefficients, describing only some particular monomials. The main problem of such compact unstructured representations is of course the design of meaningful variation operators (crossover, mutation) that will make evolution differ from random search.

Another further step toward scalability is to use list of groups of elementary items, also called component-based representations in [10]: in the context of polynomial identification, that would amount to manipulate some elementary polynomials not limited to simple monomials. But such compact unstructured representations can also be organized into structured spaces, to make their evolution easier: using Genetic Programming [33, 6] is an alternative representation for variable degree polynomials, with well-designed variation operators. Such search space also allows one to add useful features, such as modularity and recursion, to the representations [34], making another step toward the evolution of complex solutions: when the solution to a problem has some symmetries, is seems quite unlikely, and at least resource-wasting, that evolution will "discover" multiple instances of

\(^1\)Of course, there also exists extensive unstructured representation, such as the Messy GA representation [22], different representations for the TSP [43], or extensive description of variable-topology neural networks [5, 18] which will not be considered further in this paper in the light of the scalability issue.
the same mechanism.

But the to-date ultimate research direction toward the evolution of complex solutions seems to lie in the so-called *morphogenetic* approach: instead of evolving parts of solutions (simple item or more complex components), one evolves some programs that in turn give the solution when they are run. One of the early attempts of morphogenetic approach is the Cellular Encoding of F. Gruau [24] where a Neural Network is built from an embryo by a GP-like program – while many recent successes have been reported using GP in different domains [35, 36]. As they also can evolve modular solutions, morphogenetic approaches really are appealing to build very complex solutions to difficult problems whose components can hardly be designed directly. However, the increase in scalability goes together with a loss in causality: it is almost impossible for anyone to guess the influence of small parts of the genotype on the final solution.

In the framework of Topological Optimum Design, the plain direct extensive representation is the widely used bitarray approach based on a fixed mesh of the design domain. Though very successful to overcome the main limitations of deterministic methods for TOD [31, 29, 32], this representation does not scale up with the complexity of the mesh. Different compact unstructured representations based on Voronoi diagrams are introduced, that exhibit a self-adaptive complexity (i.e. the complexity of the solutions is adjusted by the algorithm). These representations do not involve exactly components, but do require some elementary alleles to be defined by the programmer; such alleles can be viewed as some sort of variable components: due to the high degree of epistasis of those representations, the phenotypic expression of each allele strongly depends on the other alleles. In an attempt to avoid the biases resulting from the manual choice of these alleles, the IFS representation, a morphogenetic approach based on fractal theory, is defined.

The paper is organized the following way. The context of evolutionary TOD is recalled in section 2, from the mechanical background to the adaptive penalty method used within the fitness function. Section 3 introduces a series of three different representations based on the idea of Voronoi diagrams while section 4 presents original experimental results obtained with the simple Voronoi representations, assessing the power of the compact unstructured approach. Comparative results on cantilever benchmark problems are then presented, allowing one to discriminate among the different Voronoi-based representations. Section 5 introduces the IFS representation, based

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the fractal theory, together with preliminary experimental results assessing its possible advantages and setting its limits, at least for simple problems of TOD. Section 6 discusses the relevance of the different representations introduced in the paper and concludes on further directions of research.

2 Background

2.1 The mechanical problem

The general framework of this paper is the Topological Optimum Design (TOD) problem: find the optimal shape of a structure (i.e. a repartition of material in a given design domain) such that the mechanical behavior of that structure meets some requirements – here a bound on the maximal displacement under a prescribed loading, but it could also involve bounds on the eigenfrequencies, or any combination of stiffness and modal optimization. The optimality criterion is here the weight of the structure, but it could also involve other technological costs.

The mechanical model used in this paper is the standard two-dimensional (except in section 4.2) plane stress linear model, and only linear elastic materials will be considered (see e.g. [15]). All mechanical figures are adimensional (e.g. the Young modulus is set to 1) and the effects of gravity are neglected.

One of the most popular benchmark problem of Optimum Design is the optimization of a cantilever plate: the design domain is rectangular, the plate is fixed on the left vertical part of its boundary (displacement is forced to 0), and the loading is made of a single force applied on the middle of its right vertical boundary. Figure 1 shows the design domain for the $2 \times 1$ cantilever plate problem.

2.2 State of the art in Shape Optimization

The main trends in structural optimization can be sketched as follows. A first approach is that of domain variation [12] (also termed sensitivity analysis in Structural Mechanics). It consists in successive small variations of an initial design domain, and is based on the computation of the gradient of the objective function with respect to the domain. The original approach has two major defects: first, it requires a good initial guess, as it demonstrated
unstable for large variations of the domain; second, it does not allow to modify the topology of the initial domain (e.g. add or remove holes). However, the idea of topological gradient was recently proposed and successfully used in [19], allowing the modification of the topology of the solution. Nevertheless, this method is strictly limited to the linear elasticity framework.

The other method for topology optimization is the now standard approach of homogenization, introduced in [8], which deals with a continuous density of material in $[0, 1]$. This relaxed problem is known to have a solution in the case of linear elasticity [3] – and the corresponding numerical method does converge to a (non-physical) generalized solution made of fine composite material. It can then be post-processed to obtain an admissible solution with boolean density [2]. The homogenization method is also insofar limited to the linear-elasticity. The theoretical results about optimal microstructures only handle single-loading cases, though numerical solution to multi-loading cases have been proposed [1]. In addition, this method cannot address loadings that apply on the (unknown) actual boundary of the shape (e.g. uniform pressure).

A possible approach to overcome these difficulties of TOD is to use stochastic optimization methods.

Stochastic optimization methods have been successfully applied to other problems of structural optimization: in the framework of discrete truss structures, for cross-section sizing [38, 47] among others, as well as for topological optimization [23, 11] and for the optimization of composite materials [37].

TOD problems have also already been addressed by stochastic methods:
Simulated Annealing has been used to find the optimal shape of the cross-
section of a beam in [4]; and Evolutionary Algorithms have been used to
solve cantilever problems as the one presented in Section 2.1 in [26, 14, 31].

The above-mentioned limitations of the deterministic methods have been
successfully overcome by these works – in [31, 29, 32] for instance, results
of TOD in nonlinear elasticity, as well as the optimization of an underwater
dome (where the loading is applied on the unknown boundary) have been
proposed, both out of reach for the deterministic methods.

2.3 Fitness computation

The problem tackled in this paper is to find a structure of minimal weight
such that its maximal displacement stays within a prescribed limit \( D_{\text{lim}} \) when
some given pointwise force is applied on the loading point (see Figure 1). The
computation of the maximal displacement is made using a Finite Element
Analysis solver [27]).

From mechanical considerations, all structures that do not connect the
loading point and the fixed boundary are given an arbitrary high fitness
value. Moreover, the material in the design domain that is not connected
to the loading point – and has thus no effect on the mechanical behavior of
the structure – is discarded during the Finite Element Analysis, but slightly
penalizes the structure at hand (see [31, 29] for a detailed discussion on
both these issues). In summary, for connected structures, the problem is to
minimize the (connected) weight subject to one constraint for each loading
case, namely \( D_{\text{Max}}^i \leq D_{\text{lim}}^i \), where \( D_{\text{Max}}^i \) its maximal displacement computed
by the FEM under loading \( i \), and \( D_{\text{lim}}^i \) its prescribed limit.

Introducing the positive penalty parameters \( \alpha_i \), the fitness function to
minimize is

\[
\text{Weight} + \sum_i \alpha_i(D_{\text{Max}}^i - D_{\text{lim}}^i)^+
\]

However, adjusting \( \alpha_i \) is not an easy task, and many specific methods
exist in Evolutionary Computation [40].

The adaptive penalty method used here updates the penalty parameter
based upon global statistics of feasibility in the population. Its main goal is
to explore the neighborhood of the boundary of the feasible region by trying
to keep in the population individuals that are on both sides of that boundary:
in the context of stiffness optimization in TOD, the solution does lie on the
boundary, ... but for the continuous problem only! Once discretized, this is no longer true, and it can only be said that the solution lies close to the boundary.

The objective is to maintain in the population a minimum proportion of feasible individuals as well as a minimum proportion of infeasible individuals. Denote by \( \Theta_{\text{feasible}}^k \) the proportion of feasible individuals at generation \( k \), and by \( \Theta_{\text{inf}} \) and \( \Theta_{\text{sup}} \) two user-defined parameters. As small penalty parameters favor the infeasible individuals (and vice-versa), the following update rule for the \( \alpha_i \) parameters is proposed to try to keep \( \Theta_{\text{feasible}}^k \) in \([\Theta_{\text{inf}}, \Theta_{\text{sup}}]\):

\[
\alpha_{k+1} = \begin{cases} 
\beta \cdot \alpha_k & \text{if } \Theta_{\text{feasible}}^k < \Theta_{\text{inf}} \\
(1/\beta) \cdot \alpha_k & \text{if } \Theta_{\text{feasible}}^k > \Theta_{\text{sup}} \\
\alpha_k & \text{otherwise}
\end{cases}
\]

with \( \beta > 1 \). User-defined parameters of this method are \( \Theta_{\text{inf}}, \Theta_{\text{sup}}, \beta \) and the initial value \( \alpha_0 \). The robust values \( \beta = 1.1, \Theta_{\text{inf}} = 0.4, \) and \( \Theta_{\text{sup}} = 0.8 \) were used in all experiments presented this paper.

Note that the variations of \( \alpha \) are non monotonous, and hence there is no a priori guarantee that the best individual in the population is feasible. It can even happen that the population contains no feasible individual – though in that case the steady increase of \( \alpha \) should favor individuals with lower constraint violation, and rapidly result in the emergence of feasible individuals.

Some comparative results assessing the power of that population-based adaptive penalty method can be found in [9] for test problems, and in [25] in the context of TOD.

### 2.4 Representations of structures for TOD

All the works cited in section 2.2 that address TOD problems with EAs use the same ‘natural’ binary representation, termed bitarmug in [31]: it relies on a mesh of the design domain – the same mesh that is used to compute the mechanical behavior of the structure in order to give it a fitness (see section 2.3). Each element of the mesh is given value 1 if it contains material, 0 otherwise (see Figure 1). Note that this bit-based representation is not equivalent to the usual bitstring representation, and that some specific geometrical crossover operators had to be designed [30], similar to the crossover operator described below for the Voronoi-based representations.
In spite of its successes in solving TOD problems [31, 29, 32], bitarray representation suffers from a strong limitation due to the dependency of its complexity on that of the underlying mesh. Indeed, the size of the individual (the number of bits used to encode a structure) is the size of the mesh. Unfortunately, according to both the theoretical results in [13] and the empirical considerations in [21], the critical population size required for convergence should be increased at least linearly with the size of the individuals. Moreover, larger populations generally require a greater number of generations to converge. Hence it is clear that the bitarray approach will not scale up when using very fine meshes. This greatly limits the practical application of this approach to coarse (hence imprecise) 2D meshes, whereas Mechanical Engineers are interested in fine 3D meshes!

These considerations appeal for some more compact representations whose complexity does not depend on a fixed discretization. The ultimate step in the direction of complexity-free representation is to let the complexity itself evolve and be adjusted by the EA.

3 Voronoi-based representations

The Voronoi representation is a first attempt toward unstructured representations for TOD. It has first been proposed in [44], but has been used since then mainly in the context of identification problems [46, 45]. This section recalls the definition of Voronoi representation, and proposes two other representations that also derive from the same ideas.

3.1 Voronoi representation

Voronoi diagrams: Consider a finite number of points $V_0, \ldots, V_N$ (the Voronoi sites) of a given subset of $\mathbb{R}^n$ (the design domain). To each site $V_i$ is associated the set of all points of the design domain for which the closest Voronoi site is $V_i$, termed Voronoi cell. The Voronoi diagram is the partition of the design domain defined by the Voronoi cells. Each cell is a polyhedral subset of the design domain, and any partition of a domain of $\mathbb{R}^n$ into polyhedral subsets is the Voronoi diagram of at least one set of Voronoi sites (see [42] for a detailed introduction to Voronoi diagrams, and a general presentation of algorithmic geometry).
**The genotype:** Consider now a (variable length) list of Voronoi sites, each site being labeled 0 or 1. The corresponding Voronoi diagram represents a partition of the design domain into two subsets, if each Voronoi cell is labeled as its associated site (see Figure 2).

![Voronoi Diagram](image)

**Figure 2:** The 2x1 cantilever plate test problem, and a Voronoi representation of a structure.

**Decoding:** Of course, as some FE analysis is required during the computation of the fitness function, and as re-meshing is a source of numerical noise that could ultimately take over the actual difference in mechanical behavior between two very similar structures, it is mandatory to use the very same mesh for all structures at the same generation. A partition described by Voronoi sites is easily mapped on any mesh: the subset (void or material) an element belongs to is determined from the label of the Voronoi cell in which the gravity center of that element lies.

However, the complexity of the individuals (i.e. the number of Voronoi sites in their representation) is totally independent of the choice of the mesh used for fitness computation, and will evolve according to the Darwinian principles underwining the whole evolutionary process.

**Initialization:** the initialization procedure for the Voronoi representation is a uniform choice of the number of Voronoi sites between 1 and a user-supplied maximum number, a uniform choice of the Voronoi sites in the structure, and a uniform choice of the boolean void/material label.

**Variation operators:** The variation operators for the Voronoi representation are problem-driven:
The crossover operator exchanges Voronoi sites on a geometrical basis. In this respect it is similar to the specific bitarray crossover described in [30]. Figure 3 is an example of application of this operator.

The mutation operator is chosen by a roulette wheel selection based on user-defined weights among the following operators:

- the displacement mutation performs a Gaussian mutation on the coordinates of the sites. As in Evolution Strategies [48], adaptive mutation is used: one standard deviations is attached to each coordinate of each Voronoi site, undergoes log-normal mutation before being used for the Gaussian mutation of the corresponding coordinate.

- the label mutation randomly flips the boolean attribute of one site.

- the add and delete mutations are specific variable-length operators that respectively randomly add or remove one Voronoi site on the list.

**Boundary control:** One crucial problem in TOD is the fine tuning of the boundary of the solution. The optimal shape can only be reached in reasonable time if the algorithm is able to precisely control the boundaries of the individuals in the population. Unfortunately, the Voronoi representation only offers indirect control of the boundary of the structure it represents. Moreover, the high epistasis of that representation makes it difficult to modify a single boundary without disturbing the adjacent ones. The idea behind the dipole representation presented in next section is to try to overcome that difficulty.
Figure 4: *The dipole representation. A single dipole (a) and the Voronoi diagram built using three dipoles (b): some unwanted corners appear at median meetings.*

### 3.2 Dipole representation

**Dipoles:** A dipole is a set of two Voronoi sites, one labeled 0 and the other labeled 1, standing almost at the same point in the design domain, but whose median has a prescribed angle in the plane. A dipole is hence defined by three real-valued variables, its coordinates $(x, y)$ and the angle of its median with the $x$-axis $\theta$. Figure 4-a is an example of a dipole. The direct control over $\theta$ allows a precise control over that part of the boundary that goes through the $(x, y)$ point.

**The genotype:** One individual in the dipole representation is a (variable length) list of dipoles. As in the Voronoi representation, the corresponding Voronoi diagram represents a partition of the design domain into two subsets.

**Decoding:** As for the Voronoi representation, the fitness of all structures is evaluated using a fixed mesh, and the projection on that fixed mesh is performed as in section 3.1.

However, as can be seen on Figure 4-b, the decoding of adjacent dipoles shows that the resulting structure has two kinds of boundaries: the median of the dipoles, which can hopefully be controlled by the evolutionary algorithm, and the medians between dipoles, whose fine tuning will be as difficult as in the Voronoi representation – and maybe even more, as some weird configurations will often arise, as the one shown in Figure 4-b.

**Evolution operators:** these operators for the dipole representation are derived from the ones of the Voronoi representation: the initialization pro-
cEDURE chooses a number of dipoles, and initializes their coordinates uniformly in the design domain and their angle in \([0, 2\pi]\). The crossover operator exchanges dipoles exactly as its counterpart for Voronoi representation exchanged Voronoi sites (see Figure 3). The mutation operators include the displacement mutation, the Gaussian mutation of the angle of a dipole, and of course the addition and destruction of dipoles in the list.

**Truss-like structures** For cantilever problems, it is well-known that the best structures are in fact truss structures. Obtaining truss structures using Voronoi diagrams or dipoles requires the emergence of coupled subsets of either sites or dipole and thus might take some time to evolve.

Moreover, the defects of the dipole representation pointed out in Figure 4-b (together with experimental results as the ones of sections 4.3) demonstrate its inability to achieve the fine tuning of the boundary that was the main reason why it was designed.

The Voronoi-bar representation, introduced in next section, aims at both achieve the fine tuning of the boundary, and favor the evolution of truss structures by providing alleles that already are truss elements.

### 3.3 Bar representation

**Voronoi-Bars:** A Voronoi-bar is hence defined by four real-valued variables, its coordinates \((x, y)\), the angle of the bar with the \(x\)-axis \(\theta\) and its width. Figure 3.3-a is an example of a single Voronoi-bar.

**The genotype:** One individual in the Voronoi-bar representation is a (variable length) list of Voronoi-bars. When all Voronoi-bars are simply considered as Voronoi sites, the corresponding Voronoi diagram represents a partition of the design domain into convex polygons. Each such polygon is then separated into two subdomains, namely the central part, made of material, and the outer part, “filled” with void (see Figure 3.3). Whenever the width is large enough, the whole cell is \(1\), whereas a null value for the width turns the cell into a \(0\) cell; these extreme cases of the Voronoi-bar representation are nothing else than the Voronoi representation itself.

**Decoding:** As for the Voronoi representation, the fitness of all structures will be evaluated using a fixed mesh, and the projection on that fixed mesh is
performed as in section 3.1: an element is considered made of material if and only if its center of gravity falls within the material part of a Voronoi-bar.

As can be seen on Figure 3.3-b, the decoding of adjacent Voronoi-bars allows to directly control almost the whole boundary of the resulting structure, apart from some limited portions at the junction of two “bars”.

**Evolution operators**: these operators for the Voronoi-bar representation are once again derived from the ones of the Voronoi representation: the initialization procedure chooses a number of bars, and initializes their coordinates, angles and width uniformly. The crossover operator exchanges bars exactly as its counterpart for Voronoi representation exchanged Voronoi sites (see Figure 3). The mutation operators include the displacement mutation, the Gaussian mutation of the angle and width of a bar, and of course the addition and destruction of bars in the list.

4 Experimental results for Voronoi-based representations

This section introduces some results obtained using the Voronoi-based representations. Mesh-dependency experiments were run on the Voronoi representation to ensure the idea of compact unstructured representation was indeed playing its role: this was shown to be the case up to the error in discretization [25]. Some original results on some 3D cantilever problem further demonstrate that using unstructured representations did indeed allow innovative results in Evolutionary Topological Optimum Design. But the most impor-
tant part of this section deals with comparative results on the benchmark cantilever problems to try to assess the usefulness of the introduction of the other Voronoi-based representations.

4.1 Evolutionary experimental conditions

Unless otherwise stated, the experiments presented further on have been performed using the following settings: Standard GA-like evolution (linear rank-based selection and generational replacement of all parents by all offspring) with populations size 80; At most 40 Voronoi sites (or dipoles or bars) per individual; Crossover rate is 0.6 and mutation rate per individual is 0.3; Weights among the different mutations are 0.5 for the displacement mutation, the remaining mutations equally sharing the remaining 0.5; All runs are allowed at most 2000 generations, and the algorithm stops after 300 generations without improvement; All plots are the result of 21 independent runs; All CPU times are given related to a Pentium III processor running at 300MHz under Linux. For instance, the cost of one generation for the $1 \times 2$ or the $2 \times 1$ cantilever problems discretized with 200 elements is 2s.

4.2 Three-dimensional problem

This section introduces the first results of 3D TOD obtained using Evolutionary Computation (as far as we are aware of). They have been obtained using the Voronoi representation.

The design domain is a quadrangle subset of $\mathbb{R}^3$, and the problem is symmetrical: only half of the domain is discretized, according to a $16 \times 7 \times 10$ mesh. Its left face is fixed, and the loading is applied on the middle of the right face.

Here again the higher complexity of the problem lead to modify the settings: the population size is again set to 120 and the maximum number of Voronoi sites is also increased to 120.

Figure 6 demonstrates that the algorithm was able to find some good solutions in... a few days of CPU time (3D FEM analyses are far more costly than 2D for the same mesh size). Moreover, it also stresses the ability of EAs to find multiple quasi-optimal solutions to the same problem, some of them quite original indeed when compared to the results of the homogenization method on the same problem.
Figure 6: Two results for the symmetrical three-dimensional problem using a 16 × 7 × 10 mesh for half of the structure, with same constraint (CPU time = 6mn/gen).

4.3 Comparative results of Voronoi-based representations

This section presents comparative benchmark results on the three Voronoi-based representations. Two benchmark problems are considered: the 1 × 2 and 2 × 1 cantilever plates with respective limits on the maximal displacement of 20 and 220. In both cases, the vertical left boundary is fixed, and the point-wise force is applied at half-height of the right vertical boundary. The experimental conditions for all representations are those described in section 4.1.

Figures 7, 8 and 9 show typical best structures obtained with respectively the Voronoi, the dipole and the Voronoi-bar representations, while Figures 10 and 11 show statistics over 21 runs for both test cases.

The first conclusion of these experiments is that all three representations find almost equally good solutions among the 21 runs. However, the best representation according to the quality criterion is the Voronoi-bar representation: almost all solutions were similar to the ones of Figure 9, whereas many solutions found by the dipole representation were much worse, and the solutions found by the Voronoi representation were consistently slightly worse. These trends are reflected on the comparative runs shown in Figures 10 and 11. Note that both Voronoi and dipole representations sometimes showed re-
(a) : weight=0.215, 35 sites  (b) : weight=0.35, 32 sites

Figure 7: The two best benchmark results for the Voronoi representation

(a) : weight=0.215, 15 dipoles  (b) : weight=0.325, 36 dipoles

Figure 8: The two best benchmark results for the dipole representation
results similar to the Voronoi-bar representation, but the latter really appeared more robust.

Another criterion is the complexity of the solutions. The test cases are here very simple, and the solutions should reflect this simplicity. Here again the Voronoi-bar representation is a clear winner: In all runs, the Voronoi-bar representation found very compact solutions, compared with those found by the other representations. The perfect 2-bars V-shape was even found once for the 1 × 2 cantilever problem.

Hence it seems that the additional complexity in the elementary alleles of the Voronoi-bar representation does pay off, at least on these benchmark problems.

4.4 The 10 × 1 cantilever

The problem of the 10 × 1 cantilever (discretized using a 100 × 10 regular mesh) proved to be difficult for the bitarray representation as it raises an additional difficulty: most of the solutions do not connect the fixed boundary and the point where the loading is applied. Hence an alternate initialization procedure is used, where the average weight of random structures can be tuned (see [28] for details). Furthermore, the maximal number of solutions for each individual is increased to 120, and the best results were obtained with a population size of 120.

Nevertheless, the dipole representation was unable to find satisfactory
Evaluations

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Figure 10: Comparative on-line results for the Voronoi-based representations on the $1 \times 2$ cantilever for $D_{lim} = 20$

Evaluations

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Figure 11: Comparative on-line results for the Voronoi-based representations on the $2 \times 1$ cantilever for $D_{lim} = 220$
solutions – in most cases, it simply could never find a connected solution, similarly to the bitarray representation.

Figure 12 and 13 shows the most significant results obtained using respectively the Voronoi and the Voronoi-bar representations.

Again, a slight advantage can be seen for the Voronoi-bar representation in the quality of the best solution. However, the advantage in solution complexity is not so clear than it was on the 1 × 2 benchmark. But a very interesting feature is the quasi-regularity of the Voronoi-bar solution: indeed, any mechanical engineer would build such a structure by using the same part structure four or five times before ending with some specific part at the further end (think of how cranes are designed). But as the Voronoi-based representations do not have the ability to evolve modularity, such partial solutions have to be evolved six times.

5 IFS representation

The Voronoi-based representations were some attempts to escape the direct encoding of discretized structures using a predefined mesh. However, the basic blocks that build the structure had to be designed by the programmer, and wrong choices can bias the search in a wrong direction.

The following fractal-based representation is an attempt to go further in
the morphogenetic direction: no assumption is made about what the building blocks of a structure could be – but the search space for the genotype is hopefully rich enough so that a large number of different structures can be evolved.

5.1 IFS Theory

An IFS $\Omega = \{F, (w_n)_{n=1, \ldots, N}\}$ is a collection of $N$ functions defined on a complete metric space $(F, d)$. Let $W$ be the Hutchinson operator, defined on the space of subsets of $F$:

$$\forall K \subset F, \ W(K) = \bigcup_{n \in [0, N]} w_n(K)$$

If all $w_n$ functions are contractive (i.e. there exists a positive real number $s < 1$ such that $d(w(x), w(y)) \leq s d(x, y)$ for all $(x, y) \in F^2$), the IFS is called hyperbolic, and there exists a unique set $A$, called the attractor of the IFS, such that $W(A) = A$.

The uniqueness of the attractor is a result of the contractive mapping fixed-point theorem for $W$, which is contractive according to the Hausdorff distance defined by

$$d_H(A, B) = \max_{x \in A} \max_{y \in B} (\min_{y \in B} d(x, y)), \max_{y \in B} (\min_{x \in A} d(x, y))]$$

From a computational viewpoint, there are two known ways to compute the attractor of an IFS:

- **Stochastic method (toss-coin):** Let $x_0$ be the fixed point of one of the $w_i$ functions. Build the sequence $x_n$ by $x_{n+1} = w_i(x_n)$, $i$ being randomly chosen in $\{1, \ldots, N\}$. Then $\bigcup_n x_n$ is an approximation of the attractor of $\Omega$ (the larger $n$, the more precise the approximation).

- **Deterministic method:** From any kernel $S_0$, build the sequence $\{S_n\}$ of subsets by $S_{n+1} = W(S_n)$. When $n$ goes to $\infty$, $S_n$ is an approximation of the real attractor of $\Omega$.

5.2 Evolutionary IFS identification

The first attempts to evolve IFS using EAs dealt with the inverse problem: given a target shape $A \subset F$, find the IFS whose attractor is $A$. 

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This problem can be formulated as an optimization problem: find the IFS whose attractor minimizes the distance to the target shape $A$. As the function to be optimized is extremely complex, some \textit{a priori} restrictive hypotheses are necessary. Usually, the search space is that of affine IFS, with a fixed number of functions: see \cite{7, 50} for early computational methods. More recently, solutions based on Evolutionary Algorithms have been presented for affine IFS, i.e. IFS in which all functions are affine functions \cite{51, 20, 41}.

But affine IFS are a small subset of possible IFS, and some previous work of one of the authors \cite{39} dealt with general non-affine IFS (called \textit{Mixed IFS}) using GP, that allows to evolve any type of function. However, whereas assessing the contractivity of affine functions is straightforward, the contractivity of general functions defined as GP trees could only be numerically checked a posteriori – and at a heavy computational cost. This drawback motivated the very recent introduction of Polar IFS \cite{16} in which the functions are sought (still using GP) in polar form around their fixed points: a simple condition on the $\rho$ functions ensures the local contractivity of the function around its fixed point. While this does not ensure the global contractivity, the proportion of contractive functions among that class of polar functions is much larger than that of contractive general GP trees - and the inverse problem can be solved more rapidly and accurately.

Unfortunately, when the present work started, only the GP program to identify mixed IFS was operational. Hence the first results presented in next sections using IFS representation for the TOD problem have been obtained using the mixed IFS GP-based program described in detail in \cite{39}.

5.3 IFS representation for TOD: first results

The idea of shape representation using IFS is now straightforward: The attractor of an IFS is a shape defined in the design domain. Hence the fitness of the IFS can be computed using that shape as a structure, potential solution of the TOD.

The attractor of a given IFS is computed on the mesh that is used for the FE analyses, and the fitness is computed as stated in section 2.3. The same $1 \times 2$ and $2 \times 1$ benchmark cantilever problems than in section 4 are used, and Figure 14 shows the best results obtained in 5 runs.

First, the good news is that reasonable structures were obtained. Moreover, their shapes are indeed more "lace-like" than when using a Voronoi-
Figure 14: The two benchmark results for the IFS representation.

Based representation – and without the cost of describing all small holes as in the bitarray representations.

But some drawbacks can also be derived from these preliminary experiments:

- The variance of the results was very high – some results were really not good at all;
- the same adaptive penalty strategy was used here than for the Voronoi-based representations (see section 2.3). However, whereas all runs of Voronoi-based representations found feasible solutions, most runs using the IFS representation found slightly infeasible solutions;
- the computational time for decoding is much larger for the IFS representation than for the Voronoi representation;
- the influence of mesh refinement on the actual shape obtained by decoding an IFS is not easy to guess. However, first experiments suggest that different meshes might result in quite different shapes up to very fine meshes.

6 Discussion and conclusion

This paper has introduced new representations for evolutionary TOD. Departing from the raw bitarray representation based on a fixed mesh of the design domain, representations based on the theory of Voronoi diagrams have
been proposed, from the simple Voronoi representation to the more complex
dipole and Voronoi-bar representations. These three representations are un-
structured and compact, i.e. an individual is a variable-length unordered
list of alleles. The complexity of the structure of a single allele increases
when going from the Voronoi representation to the Voronoi-bar representa-
tion. However, all three representation implement self-adaptive complexity
of the solutions, i.e. the actual complexity of the individuals is evolved by
the algorithm and does not have to be pre-defined by the programmer.

These representations have been tested on simple test problems of TOD.
The results seem to show that all three representations can solve such prob-
lems, and require roughly the same computational effort for the same qual-
ity of solution, with a slight advantage for the Voronoi-bar representation.
However, when examining the complexity of the solution, there is a clear
advantage in using the Voronoi-bar representation, whose solutions consist-
tently involve less alleles than the two other. Note that this probably also
explains the observed slight improvement in quality vs computation effort, as
it is easier to fine tune the solution when only few alleles are to be adjusted.
However, it should be kept in mind that all 2D cantilever problems have
truss-like optimal solutions constructed from ... bar-like elements. Further
experiments on problems for which the optimal solutions does not exhibit
such characteristics should be carried on.

Finally, the IFS representation was presented, a morphogenetic represen-
tation in which the structure is defined indirectly as the attractor of a set of
contractive mappings on the design domain. Such representation does not
make any a priori supposition on the shape of building blocks for the solution
of the TOD at hand. This should allow more complex solutions to be evolved
without designing specific alleles.

However, though reasonable results were obtained, the IFS representa-
tion was unable to find very good solutions on the same simple test problems
of TOD. Of course, it can be argued that this increase of complexity of the
morphogenetic process might only prove beneficial for problems where the
solution is also complex – and further work will try to apply this representa-
tion to more difficult problems to try to validate that hypothesis. However,
it also might be the case that the lack of causality (direct feed-back from the
mechanical structure on the IFS) forbids any useful evolutionary process, at
least with so few individuals and generations. Some experiments on highly
parallel systems with distributed populations of hundreds of thousands of
individuals might help answering that question.

Another problematic issue is the dependency of the morphogenetic process on the mesh, that seems to be much higher than for all Voronoi representations. Two possible answers will be investigated: by using different unstructured meshes during evolution, or by making the decoding process smoother. First, by changing the mesh at every generation, or by averaging the fitness over a few meshes, it is hoped that only solutions that are robust with respect to the mesh will survive successive selections. Second, the numerical computation of the attractor of an IFS fills an element with material as soon as it is hit once by the toss-coin algorithm, whereas smoother decoding would be to consider only the hard core of the attractor requiring a minimal number of such hits before filling it.

In the present state of this research, however, the Voronoi-bar representation seems a good choice for evolutionary TOD, achieving a good compromise between compactness of the solutions and ease of search for good solutions. However, whereas the extension of the Voronoi and dipole representations to three dimensions is straightforward (and so is that of the IFS representation), that of the Voronoi-bar representation requires some more work: one will probably need plate and bars with different cross-section shapes to be included in the elementary alleles.

Nevertheless, none of the representations proposed here does include high level features such as the possibility to evolve symmetric, or re-usable sub-solutions, that seem to be the key to evolving highly complex solutions: for instance, it is clear satisfactory solutions of the $N \times 1$ cantilever problem for large $N$ can only be obtained by using many almost identical small truss-structures. Some hierarchical representations for shapes have been proposed already, such as the Quad-tree representation [17]. However quad-tree representation is not easy to evolve, as for instance standard tree crossover does not preserve the locality of quad-tree discretization. But some coupling between a hierarchical approach and one of the unstructured representations presented here might allow to reach the Graal of Evolutionary Design, the automatic design of highly complex structures. It is hoped that the work in this paper actually brings some building blocks to such higher level morphogenetic representation – while already allowing the direct computation of solutions to simple problems out of reach for deterministic methods.
References


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