

STRUCTURAL OPTIMIZATION BY THE HOMOGENIZATION METHOD

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Abstract

We discuss a method of structural optimization in the context of linear elasticity. We seek the optimal shape of an elastic body which is both of minimum weight and maximal stiffness under specified loadings. Mathematically, a weighted sum of the elastic compliance and of the weight is minimized among all possible shapes. This problem is known to be "ill-posed", namely there is generically no optimal shape and the solutions computed by classical numerical algorithms are highly sensitive to the initial guess and mesh-dependent. Our method is based on the homogenization theory which makes this problem well-posed by allowing microperforated composites as admissible designs. A new numerical algorithm is thus obtained which allows to capture an optimal shape on a fixed mesh. Such a procedure is called topology optimization since it places no explicit or implicit restriction on the topology of the optimal shape, i.e. on its number of holes or members.

1. Introduction.

We consider the following structural optimization problem : find the optimal shape that minimizes a weighted sum of its elastic compliance and weight. As usual the compliance (i.e. the work done by the load) is a global measure of the design's rigidity. No explicit or implicit restrictions are made on the

shape's boundary. We work in a bounded reference domain Ω , occupied by a linearly elastic material with isotropic Hooke's law A_0 (with bulk and shear moduli κ and μ), and loaded on its boundary by some given force f . An admissible structure ω is a subset of the reference domain Ω obtained by removing one or more holes (the new boundaries created this way are traction-free). The equations of elasticity for the resulting design are

$$\begin{cases} \sigma = A_0 e(u) & e(u) = \frac{1}{2}(\nabla u + \nabla^t u) \\ \operatorname{div} \sigma = 0 & \text{in } \omega \\ \sigma \cdot n = f & \text{on } \partial\Omega \\ \sigma \cdot n = 0 & \text{on } \partial\omega \setminus \partial\Omega \end{cases} \quad (1)$$

where u is the displacement vector and σ the constraint matrix. The compliance of the structure ω is

$$c(\omega) = \int_{\partial\Omega} f \cdot u = \int_{\omega} A_0 e(u) \cdot e(u) = \int_{\omega} A_0^{-1} \sigma \cdot \sigma. \quad (2)$$

Introducing a positive Lagrange multiplier λ , our structural optimization problem is to minimize, over all subsets $\omega \subset \Omega$, the objective function $E(\omega)$ equal to the weighted sum of the compliance and weight of ω . In other words we want to compute minimizers of

$$\inf_{\omega \subset \Omega} (E(\omega) = c(\omega) + \lambda |\omega|). \quad (3)$$

The Lagrange multiplier λ has the effect of balancing the two contradictory objectives of rigidity and lightness of the shape (increasing its value decreases the weight).

As is well known, in absence of any supplementary constraints on the admissible designs ω (namely, without smoothness of the boundary), the objective function $E(\omega)$ may have no minimizer, i.e. there is no optimal shape (see e.g. (Kohn *et al.*, 1986), (Lurie *et al.*, 1982), (Murat *et al.*, 1985)). The physical reason for this non-existence is that it is often advantageous to cut infinitely many small holes (rather than just a few big ones) in a given design in order to decrease the objective function. Thus, achieving the minimum may require a limiting procedure leading to a "generalized" design consisting of composite materials made by microperforation of the original material.

To cope with this physical behavior of nearly optimal shapes, we have to enlarge the space of admissible designs by permitting perforated composites from the start : this process is called *relaxation*. Such composite structures are determined by two functions $\theta(x)$ and $A(x)$: θ is the local volume fraction of the original material, taking values between 0 and 1, and $A(x)$ is the effective Hooke's law determined by the microstructure of perforations.

Of course, we need to find an adequate definition of the relaxed objective function $\tilde{E}(\theta, A)$ which generalizes $E(\omega)$. This is done in the next section by using *homogenization theory* which gives optimal effective properties of composite materials. The ultimate goal is twofold : prove an existence theorem of relaxed or composite optimal design, and find a new numerical algorithm for computing optimal shapes.

The interested reader is referred to the pioneering works (Kohn *et al.*, 1986), (Lurie *et al.*, 1982), and (Murat *et al.*, 1985) for more details on the mathematical theory of relaxation by homogenization for structural design problems. The first numerical applications on meaningful problems have appeared more recently (see (Allaire *et al.*, 1997), (Allaire *et al.*, 1993), (Bendsoe *et al.*, 1988), (Jog *et al.*, 1994), (Suzuki *et al.*, 1991)).

2. Relaxed or homogenized formulation.

In this section we briefly recall the main results of (Allaire *et al.*, 1993), (Allaire *et al.*, 1997) concerning the relaxation procedure. We begin with a minimizing sequence for the objective function (3), i.e. a sequence of increasingly optimal shapes, $(\omega_\epsilon)_{\epsilon \rightarrow 0}$. This sequence can be regarded as finer and finer mixtures of the original material A_0 and void (holes) with lengthscale ϵ going to 0. Then, as a result of the homogenization theory, in the limit there exists an effective behavior of this fine mixture, i.e. a composite material of density $\theta(x) \in [0, 1]$ and Hooke's law $A(x)$ such that the sequence of solutions of (1) converges to the solution of

$$\begin{cases} \sigma = A(x)e(u) & e(u) = \frac{1}{2}(\nabla u + \nabla^t u) \\ \operatorname{div} \sigma = 0 & \text{in } \Omega \\ \sigma \cdot n = f & \text{on } \partial\Omega, \end{cases} \quad (4)$$

and the corresponding compliances also converge

$$\lim_{\epsilon \rightarrow 0} c(\omega_\epsilon) = \tilde{c}(\theta, A) = \int_{\partial\Omega} f \cdot u = \int_{\Omega} A(x)^{-1} \sigma \cdot \sigma,$$

where the stress σ is solution of the homogenized equation (4). Remark that, for a given value θ of the density, there are many different possible effective Hooke's law A in a set G_θ , the so-called G -closure set at volume fraction θ , which is the set of all possible homogenized Hooke's law with density θ . We can thus pass to the limit in the objective function and obtain the relaxed or homogenized functional

$$\lim_{\epsilon \rightarrow 0} E(\omega_\epsilon) = \min_{0 \leq \theta \leq 1, A \in G_\theta} \tilde{E}(\theta, A),$$

where

$$\tilde{E}(\theta, A) = \tilde{c}(\theta, A) + \lambda \int_{\Omega} \theta(x).$$

The relaxed functional $\tilde{E}(\theta, A)$ has to be minimized over all admissible composite designs, i.e. over all density θ and effective Hooke's law $A \in G_\theta$. This problem is not entirely explicit since the precise characterization of the G -closure set G_θ is unknown ! However, by using the principle of complementary energy, we can restrict the set G_θ of admissible composites to a smaller set of optimal composites, namely the so-called sequential laminates which are explicitly known. Indeed, we rewrite the compliance as

$$\tilde{c}(\theta, A) = \min_{\substack{\text{div} \sigma = 0 \text{ in } \Omega \\ \sigma \cdot n = f \text{ on } \partial\Omega}} \int_{\Omega} A(x)^{-1} \sigma \cdot \sigma. \quad (5)$$

Then, the two minimizations, in (θ, A) and in σ , can be switched. Furthermore, a classical result of homogenization implies that the microstructure can be optimized pointwise in the domain. Therefore, the relaxed formulation becomes

$$\min_{\substack{\text{div} \sigma = 0 \text{ in } \Omega \\ \sigma \cdot n = f \text{ on } \partial\Omega}} \int_{\Omega} \min_{0 \leq \theta \leq 1, A \in G_\theta} (A^{-1} \sigma \cdot \sigma + \lambda \theta). \quad (6)$$

For a fixed stress σ , the minimization of $A^{-1} \sigma \cdot \sigma$ on G_θ is a classical problem in the theory of homogenization and composite materials. It amounts to find the most rigid composite of given density θ under the stress σ . In two dimensions, the result is

$$\min_{A \in G_\theta} A^{-1} \sigma \cdot \sigma = A_0^{-1} \sigma \cdot \sigma + \frac{(\kappa + \mu)(1 - \theta)}{4\kappa\mu\theta} (|\sigma_1| + |\sigma_2|)^2 \quad (7)$$

where σ_1 and σ_2 are the eigenvalues of the 2 by 2 symmetric matrix σ . Furthermore, optimality in (7) is achieved for a so-called rank-2 sequential laminate aligned with the eigendirections of σ (see (Allaire *et al.*, 1993), (Gibianski *et al.*, 1984) for details). In three dimensions, the result is more complicated, and we give it in the special case of Poisson's ratio equal to zero, i.e. $3\kappa = 2\mu$ (the general case is not much different in essence)

$$\begin{aligned} \min_{A \in G_\theta} A^{-1} \sigma \cdot \sigma &= A_0^{-1} \sigma \cdot \sigma + \\ &+ \begin{cases} \frac{(1-\theta)}{4\mu\theta} (|\sigma_1| + |\sigma_2| + |\sigma_3|)^2 & \text{if } |\sigma_3| \leq |\sigma_1| + |\sigma_2| \\ \frac{(1-\theta)}{2\mu\theta} ((|\sigma_1| + |\sigma_2|)^2 + |\sigma_3|^2) & \text{if } |\sigma_3| \geq |\sigma_1| + |\sigma_2| \end{cases} \end{aligned} \quad (8)$$

where the eigenvalues of σ are labeled in such a way that $|\sigma_1| \leq |\sigma_2| \leq |\sigma_3|$. Furthermore, optimality in the first regime of (8) is achieved by a rank-3 sequential laminate aligned with the eigendirections of σ , while in the

second regime it is achieved by a rank-2 sequential laminate aligned with the two first eigendirections of σ .

After this crucial step, the minimization in θ can easily be done by hand, which completes the explicit calculation of the relaxed formulation. From a mathematical point of view, one can prove that the relaxed formulation (6) admits a minimizer, that any minimizing sequence of the original problem (3) converges (in the sense of homogenization) to a minimizer of (6), and that the two infimum values of (6) and (3) are equal. However, in general there is no uniqueness of the minimizer.

3. Numerical algorithm.

At this point, using homogenization theory and introducing a relaxed formulation might appear to be just a trick for proving existence theorems. In truth its importance goes much further, and it is at the root of new numerical algorithms for computing optimal shapes. Indeed, it permits to separate the original minimization (3) in two different tasks : first, optimize locally the microstructure (the effective Hooke's law A) with explicit formula, second, minimize globally on the density $\theta(x)$. This has the effect of transforming a difficult "free-boundary" problem into a much easier "sizing" optimization problem in a fixed domain. It has many advantages : on the one hand, the computational cost is very low compared to traditional algorithms since the mesh is fixed (shapes are captured rather than tracked), on the other hand, it behaves as a topology optimizer (the final optimal shape may have a topology completely different from that of the initial guess). The key features of this type of algorithms have been first recognized by M. Bendsoe and N. Kikuchi (Bendsoe *et al.*, 1988).

Let us describe more precisely our algorithm. It is an alternate direction algorithm : we start with an initial design (usually full material everywhere), then, at each iteration, we compute the stress σ solution of a linear elasticity problem with a Hooke's law corresponding to the previous design, and we update the design variables θ and A in terms of σ by using the explicit formula for the optimal laminated composite material in (7) or (8). We iterate this process until convergence which is detected when the density variation becomes smaller than some threshold. The elasticity problem is solved by finite elements (piecewise linear for the displacement and piecewise constant for the density and Hooke's law in (Allaire *et al.*, 1997)). There are some numerical difficulties associated to this choice of finite elements (checkerboard instabilities), but they can be easily cured (see (Allaire *et al.*, 1997), (Jog *et al.*, 1994), (Sigmund, 1994)).

This algorithm converges smoothly in a relatively small number of iterations (between 10 and 100, depending on the desired accuracy). Further-

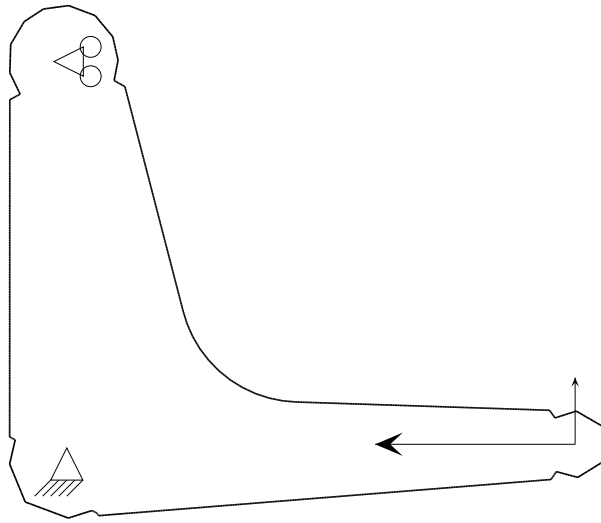


Figure 1. Boundary conditions for a suspension triangle.

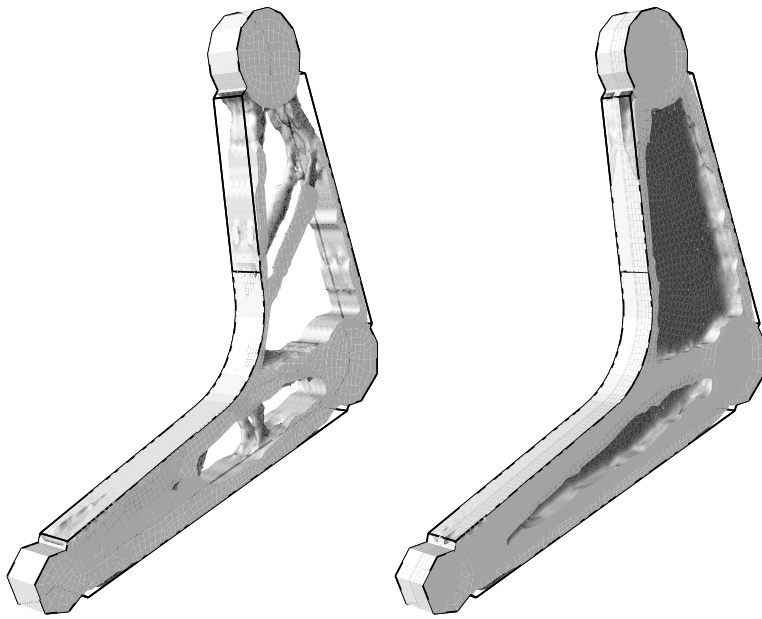


Figure 2. Optimal suspension triangle; full 3-D solution (left) and plate reinforced solution (right).

more, in practice it is insensitive to the choice of initial guess and convergent under mesh refinement, suggesting that the numerical algorithm always

picks up the same global minimum. However, as expected, it usually produces homogenized optimal designs that include large region of composite materials with intermediate density. From a practical point of view, this is an undesirable feature since the primal goal is to find an optimal shape, i.e. a density taking only the values 0 or 1 ! The remedy is to introduce a *penalization* technique that will get rid of composite materials. The strategy is the following : after convergence has been reached on a homogenized optimal design, we run a few more iterations (around 10) of our algorithm during which we force the density to take values close to 0 or 1. More specifically, denoting by θ_{opt} the true optimal density, the penalization procedure amounts to update the density at the value $\theta_{pen} = \frac{1 - \cos(\pi\theta_{opt})}{2}$. There is no specific reason to choose a cosinus-shape function for the penalized density, except that it works fine and yields surprisingly nice shapes featuring fine patterns instead of composite regions.

The success of this method is due to the fact that the relaxed design is characterized not only by a density θ but also by a microstructure A which is hidden at the sub-mesh level. The penalization has the effect of reproducing this microstructure at the mesh level (see figure 1). Of course it is strongly mesh-dependent in the sense that the finer the mesh the more complicated the resulting "almost optimal" structure.

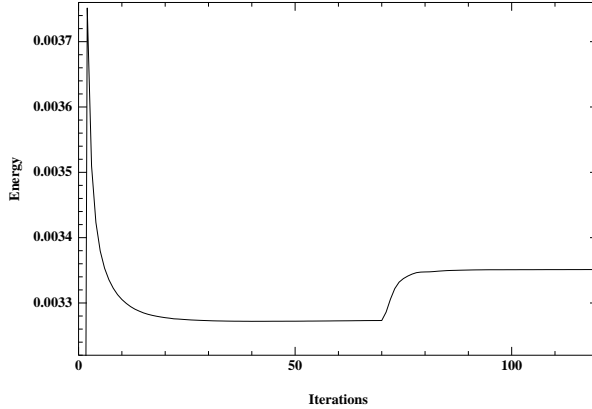


Figure 3. Objective function versus iteration number.

Figure 2 depicts a three-dimensional numerical example, namely a suspension triangle. The boundary conditions are indicated in a 2-D section of the 3-D mesh (see Figure 1). It is a multiple loads optimization, which means that the two arrows in Figure 1 correspond to two forces separately exerted. A full 3-D optimal structure has been computed and compared with a so-called "plate reinforced" optimal shape obtained by enforcing a

middle section of the structure full of material. The full 3-D shape has 40% of material left from the original box (50% for the plate reinforced shape). The process for computing such optimal shapes has been to compute first composite optimal shapes and then to apply the penalization procedure. The convergence is smooth and monotone, except when starting the penalization (see Figure 3).

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