Probab. Theory Relat. Fields
https://doi.org/10.1007/s00440-018-0838-9

Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$

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Received: 29 August 2017 / Revised: 6 February 2018

Abstract We prove that for Voronoi percolation on $\mathbb{R}^d$ ($d \geq 2$), there exists $p_c = p_c(d) \in (0, 1)$ such that

- for $p < p_c$, there exists $c_p > 0$ such that $\mathbb{P}_p[0 \text{ connected to distance } n] \leq \exp(-c_p n)$,
- there exists $c > 0$ such that for $p > p_c$, $\mathbb{P}_p[0 \text{ connected to infinity}] \geq c(p - p_c)$.

For dimension 2, this result offers a new way of showing that $p_c(2) = 1/2$. This paper belongs to a series of papers using the theory of algorithms to prove sharpness of the phase transition; see [10,11].

Mathematics Subject Classification 60K35 · 82B21 · 82B43

1 Introduction

Motivation Bernoulli percolation was introduced in [5] by Broadbent and Hammersley to model the diffusion of a liquid in a porous medium. Originally defined on a lattice, the model was later generalized to a number of other contexts. Of particular interest is the developments of percolation in continuum environment, see [15] for a book on the subject.

One of the most fundamental example of continuum models is Voronoi percolation (or Poisson Voronoi percolation), where the Voronoi cells associated to a Poisson point process in $\mathbb{R}^d$ are colored independently black or white with respective probability

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Published online: 24 March 2018

p and $1-p$. Voronoi percolation behaves very similarly to Bernoulli percolation, but is harder to study, due to local dependencies (the colors of two disjoint points are always correlated, since two points have always a positive probability to belong to the same cell). Because of these dependencies, several techniques for Bernoulli percolation do not apply, and the analysis of Voronoi percolation requires new and more robust methods to be developed.

In the celebrated work [6], Bollobás and Riordan proved that Voronoi percolation in the plane undergoes a sharp phase transition at the critical parameter $p = 1/2$. More precisely, for $p > 1/2$, the connected component of black cells containing origin is infinite with positive probability, while for $p < 1/2$, the connected component of black cells containing origin has radius larger than n with probability decaying exponentially fast in n. Since this result, several other results came to complement the picture on planar Voronoi percolation, including a fine description of the critical behavior [3,17]. Let us also mention an earlier result obtained by Benjamini and Schramm [8] which states asymptotic invariance of the crossing probabilities with respect to a conformal change of metric. We refer the reader to their article for more on the history of the model.

The recent advances in the understanding of Voronoi percolation were mostly restricted to the planar case, and several fundamental questions, including sharpness of the phase transition, remained widely open in higher dimension. This article provides a first proof of sharpness for Voronoi percolation in any dimension $d \geq 2$. As a consequence, it also offers an alternative computation of the critical point in the two-dimensional case.

Let $d \geq 2$ be a positive integer and let $\mathbb{R}^d$ be the d-dimensional Euclidean space with $\|\cdot\|$ denote the ℓ^2 norm. For $r > 0$, set $\mathsf{B}_r := \{y \in \mathbb{R}^d : \|y\| \leq r\}$ and $\mathsf{S}_r := \{y \in \mathbb{R}^d : \|y\| = r\}$ for the ball and sphere of radius r around the origin.

Let $\mathbb{P}_p$ denote the Voronoi percolation measure with parameter p on $\mathbb{R}^d$, that is $\mathbb{P}_p$ is the law of two independent point processes η^b and η^w with respective intensities p and $1-p$ (here, η^b and η^w are two locally finite subsets of $\mathbb{R}^d$). Define $\eta = \eta^b \cup \eta^w$. For a point $x \in \eta$, define the *Voronoi cell* of x

$$C(x) := \left\{ y \in \mathbb{R}^d : \|x - y\| = \min_{x' \in \eta} \|x' - y\| \right\}.$$

The measure $\mathbb{P}_p$ induces a coloring ω on the points of $\mathbb{R}^d$ defined as follows. Set $\omega(y) = 1$ for every y belonging to the Voronoi cell of some $x \in \eta^b$. Set $\omega(y) = 0$ for all the other points in $\mathbb{R}^d$. We say that y is *black* if $\omega(y) = 1$, and *white* otherwise.

For $x, y \in \mathbb{R}^d$, let the event x connected to y (denoted by $\{x \longleftrightarrow y\}$) be the existence of a continuous path of black points connecting x to y. If $X, Y \subset \mathbb{R}^d$, the event $\{X \longleftrightarrow Y\}$ denotes existence of $x \in X$ and $y \in Y$ such that x is connected to y. Also, $\{0 \longleftrightarrow \infty\}$ is the event that 0 belongs to an unbounded connected component of black points. For $p \in [0, 1]$ and $n \geq 0$, define $\theta(p) := \mathbb{P}_p[0 \longleftrightarrow \infty]$ and $\theta_n(p) := \mathbb{P}_p[0 \longleftrightarrow \mathsf{S}_n]$. Finally, we set $p_c := \inf\{p \in [0, 1] : \theta(p) > 0\}$.

The main result of this paper is the following theorem.

Theorem 1 *Fix $d \geq 2$. For any $p < p_c$, there exists $c_p > 0$ such that for any $n \geq 1$,*

$$\theta_n(p) \leq \exp(-c_p n). \tag{1}$$

Furthermore, there exists $c > 0$ *such that* $\theta(p) \geq c(p - p_c)$ *for any* $p > p_c$.

This result has an immediate corollary, namely the result of Bollobás and Riordan [6] on planar Voronoi percolation.

Corollary 2 *The critical parameter of Voronoi percolation on* $\mathbb{R}^2$ *is equal to 1/2. Furthermore,* $\theta(1/2) = 0$.

Existing proofs of exponential decay for more standard models such as Bernoulli percolation [1,12,14] or the Ising model [2,12] do not extend to the context of Voronoi percolation. The reason is a lack of a BK-type inequality. In two dimensions, Bollobàs and Riordan use crossing probabilities and introduce tools from Boolean functions [13] to bypass this difficulty. This strategy was proved very fruitful in two dimension, since several results were proved for dependent percolation models using similar ideas; see e.g. [4,9]. Unfortunately, applying such arguments in higher dimension seemed to be very challenging, so that even Bernoulli-type percolation models remained out of reach of the previous method. Recently, a new technique based on randomized algorithms was introduced to prove sharpness of the phase transition for the random-cluster and Potts models on transitive graphs [10]. This method, based on an inequality connecting randomized algorithms and influences in a product space first proved in [16], seems applicable to a variety of continuum models including Voronoi percolation or Boolean percolation [11].

The strategy consists in proving a family of differential inequalities. More precisely, for every $\delta \in (0, \frac{1}{2})$, we will prove that there exists $c = c(\delta) > 0$ such that for all $n \geq 1$ and $p \in [\delta, 1 - \delta]$,

$$\theta_n'(p) \;\geq\; c\,\frac{n}{S_n(p)}\,\theta_n(p), \tag{2}$$

where $S_n := \sum_{k=0}^{n-1} \theta_k$.

The proof of Theorem 1 follows easily. Indeed, it is well-known that $0 < p_c < 1$ [7, p 270] so that the following lemma can be applied to $\alpha_0 := \delta < p_c < 1 - \delta =: \alpha_1$, and $f_n := \theta_n / c$.

Lemma 3 *Consider a converging sequence of increasing differentiable functions* $f_n : [\alpha_0, \alpha_1] \longrightarrow [0, M]$ *satisfying*

$$f_n' \geq \frac{n}{\Sigma_n} f_n \tag{3}$$

for all $n \geq 1$*, where* $\Sigma_n = \sum_{k=0}^{n-1} f_k$*. Then, there exists* $\beta_1 \in [\alpha_0, \alpha_1]$ *such that*

- *For any* $\beta < \beta_1$*, there exists* $c_\beta > 0$ *such that for any* n *large enough,* $f_n(\beta) \leq M \exp(-c_\beta n)$.
- *For any* $\beta > \beta_1$*,* $f = \lim\limits_{n\to\infty} f_n$ *satisfies* $f(\beta) \geq \beta - \beta_1$.

This lemma can be found in [10]. Also, note that Theorem 1 can be proved without the knowledge of $p_c < 1$. Indeed, if $p_c = 1$, Lemma 3 applied to $\alpha_0 = \delta$ and $\alpha_1 = 1 - \delta$ implies exponential decay for every $p < 1 - \delta$. Since δ is arbitrary, we obtain exponential decay for every $p < 1$. This comment could be relevant when working on spaces different from $\mathbb{R}^d$ for which $p_c < 1$ is not proved yet.

The paper is organized as follows. The next section contains some preliminaries. In Sect. 3, we prove (2). Section 4 contains the proof of Corollary 2. For completeness, we include the proof of Lemma 3 in Sect. 5.

2 Preliminaries

2.1 Monotone events and the FKG inequality

An event A is said to be *increasing* if for every configurations (η^b, η^w), $(\bar{\eta}^b, \bar{\eta}^w)$,

$$\left.\begin{array}{c}(\eta^b, \eta^w) \in A \\ \eta^b \subset \bar{\eta}^b,\ \eta^w \supset \bar{\eta}^w\end{array}\right\} \implies (\bar{\eta}^b, \bar{\eta}^w) \in A.$$

An event is said to be *decreasing* if its complement is increasing. The FKG inequality for Voronoi percolation (see e.g.[7, p 278]) states that for any increasing events A and B,

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\mathbb{P}_p[B]. \tag{FKG}$$

Note that it implies that $\mathbb{P}_p[A \cap B] \leq \mathbb{P}_p[A]\mathbb{P}_p[B]$ whenever A is increasing and B is decreasing.

2.2 A Russo-type formula for Voronoi percolation

For an increasing event A, define the set of *pivotal points*

$$\mathsf{Piv}_A := \left\{x \in \eta : \mathbf{1}_A(\eta^b \backslash \{x\}, \eta^w \cup \{x\}) \neq \mathbf{1}_A(\eta^b \cup \{x\}, \eta^w \backslash \{x\})\right\}.$$

Call an increasing event A *local* if there exists $n \geq 0$ such that A is measurable with respect to the σ-algebra generated by $\{\omega(x)\}_{x \in \mathsf{B}_n}$.

Lemma 4 *Consider a local increasing event A. Then, $p \mapsto \mathbb{P}_p[A]$ is finitely differentiable and*

$$\frac{\mathrm{d}\mathbb{P}_p[A]}{\mathrm{d}p} = \mathbb{E}_p[|\mathsf{Piv}_A|].$$

Proof of Lemma 4 In this proof, $d\eta$ denotes the law of η (in particular it does not contain information on colors). Write

$$\mathbb{P}_{p+\delta}[A] - \mathbb{P}_p[A] = \int_\eta \mathbb{P}_{p+\delta}[A \mid \eta] - \mathbb{P}_p[A \mid \eta]\, d\eta.$$

Conditioned on η, the law of η^b is Bernoulli percolation with parameter p on points of η. Since A is measurable with respect to the σ-algebra generated by $\{\omega(x)\}_{x \in \mathsf{B}_n}$, apply Russo's formula (for Bernoulli percolation) and Fubini to get

$$\begin{aligned}\mathbb{P}_{p+\delta}\big[A\big]-\mathbb{P}_p\big[A\big] &= \int_\eta \Big(\int_{p\le s\le p+\delta} \mathbb{E}_s[|\mathsf{Piv}_A|\ |\eta]ds\Big)\,d\eta\\ &= \int_{p\le s\le p+\delta} \Big(\int_\eta \mathbb{E}_s[|\mathsf{Piv}_A|\ |\eta]d\eta\Big)\,ds\\ &= \int_{p\le s\le p+\delta} \mathbb{E}_s[|\mathsf{Piv}_A|]\,ds.\end{aligned}$$

The proof follows by continuity in s of $\mathbb{E}_s[|\mathsf{Piv}_A|]$, which is a direct consequence of the domination (4).

Note that even though the event A may depend only on the colors of the points in B_n, the set Piv_A can a priori contain points outside the ball. To check that $|\mathsf{Piv}_A|$ is integrable, we have

$$|\mathsf{Piv}_A| \le |D_n(\eta)| \tag{4}$$

where $D_n(\eta)$ is the set of points in η, whose cells intersect the ball B_n. The integrability of D_n follows from standard estimates of the Poisson–Voronoi tessellation. For example, observe that there exists $c > 0$ such that for every $t \ge n$,

$$\mathbb{P}_p[D_n \cap (\mathbb{R}^d \backslash \mathsf{B}_{4t}) \neq \emptyset] \le \mathbb{P}_p[\eta \cap \mathsf{B}_t = \emptyset] \le e^{-ct^d} \tag{5}$$

and

$$\mathbb{P}_p[|D_n| \cap \mathsf{B}_{4t} \ge t^{d+1}] \le \mathbb{P}_p[\eta \cap \mathsf{B}_{4t} \ge t^{d+1}] \le e^{-ct}. \tag{6}$$

□

2.3 The OSSS inequality

Assume I is a countable set, and let $(\Omega^I, \pi^{\otimes I})$ be a product probability space, and $f : \Omega^I \to \{0, 1\}$. An *algorithm* T determining f takes a configuration $\omega = (\omega_i)_{i\in I} \in \Omega^I$ as an input, and reveals the value of ω in different coordinates one by one. At each step, which coordinate will be revealed next depends on the values of ω revealed so far. The algorithm stops as soon as the value of f is the same no matter the values of ω on the remaining coordinates. Here, we always assume that T determines f in finite steps almost surely. We will use the following inequality. For any function $f : \Omega^I \to \{0, 1\}$, and any algorithm T determining f,

$$\mathrm{Var}(f) \le \sum_{i\in I} \delta_i(\mathsf{T})\,\mathrm{Inf}_i(f), \tag{OSSS}$$

where $\delta_i(\mathsf{T})$ and $\mathrm{Inf}_i(f)$ are respectively the *revealment* and the *influence* of the i-th coordinate defined by

$$\delta_i(\mathsf{T}) := \pi^{\otimes I}[\mathsf{T} \text{reveals the value of } \omega_i],$$
$$\mathrm{Inf}_i(f) := \pi^{\otimes I}\left[f(\omega) \neq f(\tilde{\omega}) \right].$$

Above, $\tilde{\omega}$ denotes the random element in Ω^I which is the same as ω in every coordinate except the i-th coordinate which is resampled independently.

Remark 5 The (OSSS) inequality is originally stated for the case when the sets Ω and I are finite. However, the proof of [16] carries over to the case where (Ω, π) is a general probability space and I is countably infinite without any need for modification. The reader could also consult [10, Theorem 2.5].

2.4 Tensorization of Voronoi percolation

We will eventually apply (OSSS). In order to do so, we introduce a suitable finite product space to encode the measure of Voronoi percolation.

Fix $\varepsilon > 0$. For $x \in \varepsilon\mathbb{Z}^d$, introduce the box $\mathsf{R}_x^\varepsilon := x + [0, \varepsilon)^d$ as well as $\eta_x^b = \eta^b \cap \mathsf{R}_x^\varepsilon$, $\eta_x^w = \eta^w \cap \mathsf{R}_x^\varepsilon$ and $\eta_x = \eta_x^w \cup \eta_x^b$. Let (Ω_x, π_x) be the measured space associated to the random variable $\eta_x = (\eta_x^b, \eta_x^w)$, and consider the product space $(\prod_{x\in\varepsilon\mathbb{Z}^d} \Omega_x, \bigotimes_{x\in\varepsilon\mathbb{Z}^d} \pi_x)$. Since the random variables (η_x^b, η_x^w) are independent for different x, this space is in direct correspondence with the original space on which Voronoi percolation was defined.

For $x \in \varepsilon\mathbb{Z}^d$ and an increasing event A, define

$$\mathrm{Inf}_x^\varepsilon[A] := \mathbb{P}_p[\mathbf{1}_A(\eta) \neq \mathbf{1}_A(\tilde{\eta})], \tag{7}$$

where $\eta = (\eta_z)_{z\in\varepsilon\mathbb{Z}^d}$ has law $\bigotimes_{z\in\varepsilon\mathbb{Z}^d} \pi_z$ and $\tilde{\eta}$ is equal to η except on the x-coordinate which is resampled independently. Here and below, we use a slight abuse of notation by denoting the measure on the probability space in which η and $\tilde{\eta}$ are defined by $\mathbb{P}_p$.

Lemma 6 *For a local increasing event A,*

$$\frac{\mathrm{d}\mathbb{P}_p[A]}{\mathrm{d}p} \geq \frac{1}{2} \limsup_{\varepsilon\to 0} \sum_{x\in\varepsilon\mathbb{Z}^d} \mathrm{Inf}_x^\varepsilon[A]. \tag{8}$$

Proof Assume A depends on the colors in B_n only. Let us start by proving that for any $m \geq 1$,

$$\frac{\mathrm{d}\mathbb{P}_p[A]}{\mathrm{d}p} \geq \frac{1}{2} \limsup_{\varepsilon\to 0} \sum_{x\in\varepsilon\mathbb{Z}^d\cap\mathsf{B}_m} \mathrm{Inf}_x^\varepsilon[A]. \tag{9}$$

Fix $x \in \varepsilon\mathbb{Z}^d \cap \mathsf{B}_m$ and use the notation for η and $\tilde{\eta}$ introduced above. Observe that with probability $1 - O(\varepsilon^{2d})$, the union $\eta_x \cup \tilde{\eta}_x$ contains at most one point. Then, using that $\eta = \tilde{\eta}$ when $\eta_x = \tilde{\eta}_x = \emptyset$ and that η_x and $\tilde{\eta}_x$ play symmetric roles, we obtain that

$$\mathrm{Inf}_x^\varepsilon[A] = 2\mathbb{P}_p[\mathbf{1}_A(\eta) \neq \mathbf{1}_A(\tilde{\eta}), |\eta_x| = 1, |\tilde{\eta}_x| = 0] + O(\varepsilon^{2d}).$$

Under the condition that $|\eta_x| = 1$ and $|\tilde{\eta}_x| = 0$, the configuration $\tilde{\eta}$ is simply obtained from η by removing the only point x of η in R_x^ε. Furthermore, by monotonicity, this point x must be pivotal in η when $\mathbf{1}_A(\eta) \neq \mathbf{1}_A(\tilde{\eta})$. Hence, writing Piv_A for the pivotal set corresponding to η, the equation above implies

$$\begin{aligned}\mathrm{Inf}_x^\varepsilon[A] &\leq 2\mathbb{P}_p[|\mathsf{Piv}_A \cap \mathsf{R}_x^\varepsilon| \geq 1, |\eta_x| = 1, |\tilde{\eta}_x| = 0] + O(\varepsilon^{2d})\\ &\leq 2\mathbb{E}_p[|\mathsf{Piv}_A \cap \mathsf{R}_x^\varepsilon|] + O(\varepsilon^{2d}).\end{aligned}$$

Summing this equation over the points $x \in \varepsilon\mathbb{Z}^d \cap \mathsf{B}_m$ gives

$$\sum_{x \in \varepsilon\mathbb{Z}^d \cap \mathsf{B}_m} \mathrm{Inf}_x^\varepsilon[A] \leq 2\mathbb{E}_p[|\mathsf{Piv}_A|] + O(\varepsilon^d).$$

Equation (9) follows by taking the lim sup and using the derivative formula of Lemma 4.

Obtaining (8) from (9) follows readily from the existence of $c > 0$ such that

$$\mathrm{Inf}_x^\varepsilon[A] \leq 2\varepsilon^d \exp(-c|x|^d) \tag{10}$$

uniformly in ε and $x \in \varepsilon\mathbb{Z}^d$ with $\|x\| \geq 4n$. To see this, assume that the value of $\mathbf{1}_A$ is changed when η is replaced by $\tilde{\eta}$. Then, $\eta \cup \tilde{\eta}$ must have at least one points in R_x^ε (which occurs with probability smaller than $2\varepsilon^d$), and the cell of one of these points must intersect $\mathsf{B}_{\|x\|/4}$, and therefore $\mathsf{B}_{\|x\|/4}$ cannot contain a point of η. Applying (5) concludes the proof. □

3 Proof of Theorem 1

As mentioned in the introduction, we only need to prove (2). For this, we fix $\delta > 0$. Fix $n > 0$ and $p \in [\delta, 1-\delta]$. Below, constants c_i ($i \leq 4$) are positive and depend on δ and d only. In particular, these constants are independent of n and p.

For $\varepsilon \in (0, 1)$, consider the product space $(\prod_{x\in\varepsilon\mathbb{Z}^d} \Omega_x, \bigotimes_{x\in\varepsilon\mathbb{Z}^d} \pi_x)$ introduced in Sect. 2.4. Applying (OSSS) to $f = \mathbf{1}_{0 \longleftrightarrow \mathsf{S}_n}$ and an algorithm T_k determining f gives that

$$\theta_n(p)(1-\theta_n(p)) \leq \sum_{x\in\varepsilon\mathbb{Z}^d} \delta_x(\mathsf{T}_k)\mathrm{Inf}_x^\varepsilon[0 \longleftrightarrow \mathsf{S}_n]. \tag{11}$$

The algorithm T_k will be provided by the following lemma, whose proof is postponed to the end of this section.

Lemma 7 *There exists $c_0 > 0$ such that for any $k \in [0, n]$, there exists an algorithm T_k determining $\mathbf{1}_{0\longleftrightarrow \mathsf{S}_n}$ with the property that*

$$\delta_x(\mathsf{T}_k) \leq c_0\, \mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k]. \tag{12}$$

Now, using (12) in (11) gives

$$\theta_n(p) \le c_0 c_1 \sum_{x\in\varepsilon\mathbb{Z}^d} \mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k]\,\mathrm{Inf}_x^\varepsilon[0 \longleftrightarrow \mathsf{S}_n] \tag{13}$$

where $c_1 := (1-\theta_1(1-\delta))^{-1}$. Averaging (13) over integer $1 \le k \le n$ gives

$$\theta_n(p) \le \frac{c_0 c_1}{n} \sum_{x\in\varepsilon\mathbb{Z}^d} \left(\sum_{k=1}^n \mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k]\right) \mathrm{Inf}_x^\varepsilon[0 \longleftrightarrow \mathsf{S}_n].$$

A simple geometric observation using the invariance under translation of Voronoi percolation implies that

$$\sum_{k=1}^n \mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k] \le \sum_{k=1}^n \theta_{d(x,\mathsf{S}_k)}(p) \le 2\sum_{k=0}^{n-1}\theta_k = 2S_n(p)$$

(above, $d(x,\mathsf{S}_k)$ denotes the distance between x and S_k) so that

$$\theta_n(p) \le 2c_0c_1\frac{S_n(p)}{n} \sum_{x\in\varepsilon\mathbb{Z}^d} \mathrm{Inf}_x^\varepsilon[0 \longleftrightarrow \mathsf{S}_n].$$

Lemma 6 implies (2) by letting ε tend to 0. Overall, the proof of the theorem boils down to the proof of Lemma 7.

Proof of Lemma 7 Fix $0 \le k \le n$. We start by defining the algorithm.

For each $y \in \varepsilon\mathbb{Z}^d$, define an auxiliary algorithm $\mathsf{Discover}(y)$ revealing the random variables η_x around the point y until the color of each point in R_y^ε is determined. We define this algorithm more formally inductively. Set a parameter $s = 0$. When $s = t$ (for some integer t), if the color of all the points inside the box R_y^ε are determined by all the revealed coordinates so far, the algorithm stops and returns the colors of points as the output. If not, the algorithm reveals the value of η_x for $x \in \varepsilon\mathbb{Z}^d$ satisfying $\|x-y\| \le t$ and sets $s = t+1$. We write $x \in D(y)$ if x is revealed by $\mathsf{Discover}(y)$. We are now in a position to define the algorithm T_k. □

Definition 8 The algorithm T_k runs as follows: Initialize the algorithm by setting $X_0 = \emptyset$ and $Z_0 = \mathsf{S}_k$. At every step t, assume that $X_t \subset \varepsilon\mathbb{Z}^d$ and $Z_t \subset \mathbb{R}^d$ have been constructed. If there is no $y \in \varepsilon\mathbb{Z}^d\setminus X_t$ with $\mathsf{R}_y^\varepsilon \cap Z_t \neq \emptyset$, the algorithm stops. If such a y exists (if more than one exists, pick the smallest for an ordering of $\varepsilon\mathbb{Z}^d$ fixed before running the algorithm), then the algorithm does the following:

- run $\mathsf{Discover}(y)$.
- Set $X_{t+1} = X_t \cup \{y\}$.
- Set $Z_{t+1} = Z_t \cup \{$all the black points in$\omega \cap \mathsf{R}_y^\varepsilon\}$.

Note that this algorithm discovers the union of all the black connected components of S_k in $\omega \cap \mathsf{B}_n$. In particular, it clearly determines $\mathbf{1}_{0 \longleftrightarrow \mathsf{S}_n}$. We now bound the revealment of T_k.

When $x \in \varepsilon\mathbb{Z}^d$ is revealed, there exist $y \in \varepsilon\mathbb{Z}^d$ and $y' \in \mathsf{R}_y^\varepsilon$ such that $x \in D(y)$ and $y' \longleftrightarrow \mathsf{S}_k$. Take $z \in \mathbb{Z}^d$ such that $y' \in \mathsf{R}_z^1 = z + [0,1)^d$. Define E_z to be the decreasing event that η^b does not intersect the Euclidean ball of radius $\|x - z\| - 3\sqrt{d}$ around z. The fact that $x \in D(y)$ implies that E_z occurs, since otherwise the colors of the points in R_z^1 are independent of the colors of the points in R_x^ε. We deduce that

$$\delta_x(\mathsf{T}_k) \;\le\; \sum_{z\in\mathbb{Z}^d} \mathbb{P}_p[\mathsf{R}_z^1 \longleftrightarrow \mathsf{S}_k \text{ and } E_z] \overset{\text{FKG}}{\le} \sum_{z\in\mathbb{Z}^d} \mathbb{P}_p[\mathsf{R}_z^1 \longleftrightarrow \mathsf{S}_k]\cdot \mathbb{P}_p[E_z].$$

A standard estimate on Poisson Point Processes in $\mathbb{R}^d$ implies that

$$\mathbb{P}_p[E_z] \le \tfrac{1}{c_2}\exp(-c_2\|z-x\|^d). \tag{14}$$

Furthermore, when $z \in \mathsf{B}_m$, by choosing a path $y_0, \dots, y_k = z$ in $\mathbb{Z}^d$ with $x \in \mathsf{R}_{y_0}^1$ and $k \le c_3\|z - x\|$, we deduce that

$$\begin{aligned}
\mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k | \mathsf{R}_z^1 \longleftrightarrow \mathsf{S}_k] \;&\ge\; \mathbb{P}_p[x \longleftrightarrow \mathsf{R}_z^1 \text{ and } \mathsf{R}_z^1 \text{ all black } | \mathsf{R}_z^1 \longleftrightarrow \mathsf{S}_k]\\
&\overset{\text{FKG}}{\ge} \mathbb{P}_p[x \longleftrightarrow \mathsf{R}_z^1 \text{ and } \mathsf{R}_z^1 \text{ all black}]\\
&\overset{\text{FKG}}{\ge} \prod_{i=1}^{k} \mathbb{P}_p[\mathsf{R}_{y_i}^1 \text{ all black}] \ge \exp(-c_4\|z-x\|).
\end{aligned} \tag{15}$$

(In the last inequality we used that $p \ge \delta$.) The bounds (14) and (15) imply that

$$\begin{aligned}
\delta_x(\mathsf{T}_k) &\le \mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k] \sum_{z\in\mathbb{Z}^d} \exp(c_4\|z-x\|)\cdot \tfrac{1}{c_2}\exp(-c_2\|z-x\|^d)\\
&\le c_0 \mathbb{P}_p[x \longleftrightarrow \mathsf{S}_k],
\end{aligned}$$

which concludes the proof (note the competition between a term growing exponentially fast in the distance and a term decaying exponential fast in the distance to the power d). □

4 Proof of Corollary 2

Let A_n be the event that $\Lambda_n := [-n,n]^2$ is crossed by a continuous path of black points going from left to right. Since the complement of A_n is the event that there is a continuous path of white vertices from top to bottom, which has the same probability, we deduce that

$$\mathbb{P}_{1/2}[A_n] = 1/2. \tag{16}$$

In particular, (16) implies that $\mathbb{P}_{1/2}[\mathsf{B}_1 \longleftrightarrow \mathsf{S}_n] \geq \frac{1}{2n}$ so that

$$\mathbb{P}_{1/2}[0 \longleftrightarrow \mathsf{S}_n] \overset{\text{FKG}}{\geq} \mathbb{P}_{1/2}[\mathsf{B}_1 \longleftrightarrow \mathsf{S}_n]\mathbb{P}_{1/2}[\mathsf{B}_1 \text{ all black}] \geq \tfrac{1}{n}\mathbb{P}_{1/2}[\mathsf{B}_1 \text{ all black}].$$

Since this quantity does not decay exponentially fast, we deduce that $p_c \leq 1/2$.

The square-root trick (using the FKG inequality) implies that for any $n \geq k \geq 1$,

$$\mathbb{P}_{1/2}[\mathsf{B}_k \text{ is connected in } \Lambda_n \text{ to the top of } \Lambda_n] \geq 1 - \mathbb{P}_{1/2}[\mathsf{B}_k \not\longleftrightarrow \infty]^{1/4}$$

so that

$$\mathbb{P}_{1/2}[\mathsf{B}_k \text{ is connected in } \Lambda_n \text{ to the top and bottom of } \Lambda_n] \geq 1 - 2\mathbb{P}_{1/2}[\mathsf{B}_k \not\longleftrightarrow \infty]^{1/4}.$$

Now, the uniqueness of the infinite connected component [7, p 278] when it exists implies that

$$\liminf_{n\to\infty} \mathbb{P}_{1/2}[A_n] \geq 1 - 2\mathbb{P}_{1/2}[\mathsf{B}_k \not\longleftrightarrow \infty]^{1/4}.$$

Assume for a moment that $\theta(1/2) > 0$. Letting k tend to infinity, we would deduce that $\mathbb{P}_{1/2}[A_n]$ tends to 1 which would contradict (16). This implies $\theta(1/2) = 0$ and $p_c \geq 1/2$.

Remark 9 The argument that $\theta(1/2) > 0$ implies $\mathbb{P}_{1/2}[A_n]$ tends to 1 could have been replaced by Zhang's argument which is adopted to the setting of Voronoi percolation in [18].

Remark 10 Since the trace on $\mathbb{R}^2$ of Voronoi percolation on $\mathbb{R}^d$ also has the property of crossing squares with probability 1/2 when the parameter p is equal to 1/2, the previous reasoning readily implies that $p_c(d) \leq 1/2$.

5 Proof of Lemma 3

Define $\beta_1 := \inf\left\{\beta \,:\, \limsup_{n\to\infty} \dfrac{\log \Sigma_n(\beta)}{\log n} \geq 1\right\}$.

Assume $\beta < \beta_1$. Fix $\delta > 0$ and set $\beta' = \beta - \delta$ and $\beta'' = \beta - 2\delta$. We will prove that there is exponential decay at β'' in two steps.

First, there exists an integer N and $\alpha > 0$ such that $\Sigma_n(\beta) \leq n^{1-\alpha}$ for all $n \geq N$. For such an integer n, integrating $f_n' \geq n^\alpha f_n$ between β' and β – this differential inequality follows from (3), the monotonicity of the functions f_n (and therefore Σ_n) and the previous bound on $\Sigma_n(\beta)$ – implies that

$$f_n(\beta') \leq M \exp(-\delta\, n^\alpha), \quad \forall n \geq N.$$

Second, this implies that there exists $\Sigma < \infty$ such that $\Sigma_n(\beta') \leq \Sigma$ for all n. Integrating $f_n' \geq \frac{n}{\Sigma} f_n$ for all n between β'' and β'—this differential inequality is again due to (3), the monotonicity of Σ_n, and the bound on $\Sigma_n(\beta')$—leads to

$$f_n(\beta'') \le M \exp\left(-\frac{\delta}{\Sigma} n\right), \quad \forall n \ge 0.$$

Assume $\beta > \beta_1$. For $n \ge 1$, define the function $T_n := \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i}{i}$. Differentiating T_n with respect to β and using (3), we obtain

$$T_n' = \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i'}{i} \overset{(3)}{\ge} \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i}{\Sigma_i} \ge \frac{\log \Sigma_{n+1} - \log \Sigma_1}{\log n},$$

where in the last inequality we used that for every $i \ge 1$,

$$\frac{f_i}{\Sigma_i} \ge \int_{\Sigma_i}^{\Sigma_{i+1}} \frac{dt}{t} = \log \Sigma_{i+1} - \log \Sigma_i.$$

For $\beta' \in (\beta_1, \beta)$, using that Σ_{n+1} is increasing in β and integrating the previous differential inequality between β' and β gives

$$T_n(\beta) - T_n(\beta') \ge (\beta - \beta') \frac{\log \Sigma_n(\beta') - \log M}{\log n}.$$

Hence, the fact that $T_n(\beta)$ converges to $f(\beta)$ as n tends to infinity implies

$$f(\beta) - f(\beta') \ge (\beta - \beta') \left[\limsup_{n\to\infty} \frac{\log \Sigma_n(\beta')}{\log n}\right] \ge \beta - \beta'.$$

Letting β' tend to β_1 from above, we obtain $f(\beta) \ge \beta - \beta_1$.

Acknowledgements The authors are thankful to Asaf Nachmias for reading the manuscript and his helpful comments. This research was supported by the IDEX grant from Paris-Saclay, a grant from the Swiss FNS, and the NCCR SwissMAP.

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