Dual weighted residual method for optimal control of hyperbolic equations of second order

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In this paper the dual weighted residual method for optimal control problems of hyperbolic equations of second order is considered. The state equation is written as a first order system in time and a posteriori error estimates separating the influences of time, space, and control discretization are derived to obtain a better accuracy of the discrete solution. A numerical example for optimal control of a nonlinear wave equation is presented.

1 Introduction

In this paper we consider the dual weighted residual method (DWR; cf. [1]) for finite element methods for optimal control problems of the following type

Minimize \( J(u, y) \), \( u \in U, \ y \in X \), s.t. \( y_{tt} - A(u, y) = f, \ y(0) = y_0(u), \ y_t(0) = y_1(u) \) (1.1)
governed by a (nonlinear) hyperbolic partial differential equation of second order. Thereby, \( U \) denotes the control space, \( X \) the state space, \( f \) a given force, \( A \) an operator depending on the control \( u \) and the state \( y \), and \( y_0 \) and \( y_1 \) denote the initial data, which may also depend on the control. The results presented here are an extension of [4, 5], where parabolic equations are the solution of the continuous problem and \((u_e, y_e)\) of the discretized one. Then, the aim is to estimate the error \( J(u, y) - J(u_e, y_e) \) by separating the influences of time, space, and control discretization to obtain an efficient algorithm to estimate the error in the cost functional. We present a numerical example for optimal control of a nonlinear wave equation. For a discussion of optimal control of hyperbolic equations, we refer the reader to [2, 3] and the references therein.

2 Optimal control problem

Let \( I = (0, T) \) with \( T > 0 \). Further, let \( V \subset H \subset V^* \) be a Gelfand triple, \((\cdot, \cdot)\) the inner product in \( H \), and \((\cdot, \cdot)_I = \int_0^T (\cdot, \cdot)_H dt\). We define the spaces \( X = L^2(I, V) \cap H^1(I, H) \cap H^2(I, V^*), \ Y = L^2(I, H) \cap H^1(I, V^*), \ X = X \times X, \ Y = X \times X \), and let \( U \subset L^2(I, Q) \) for a Hilbert space \( Q \). Using these spaces we write the state equation in a weak form as a first order system in time. For given \( u \in U, f \in L^2(I, H) \) and \( y_0(u) \in V, y_1(u) \in H \) we look for a unique solution \( y = (y^1, y^2) \in Y \) of

\[
\begin{align*}
(y^2, \xi^1) + a(u, y^1, \xi^1) + (y^2(0) - y_0(u), \xi^1(0))_H & = (f, \xi^1)_I, \quad \forall \xi^1 \in X, \\
(y^1, \xi^2) - (y^1, \xi^2)_I - (y_1(u), \xi^2(0))_H & = 0, \quad \forall \xi^2 \in X,
\end{align*}
\]

(2.1)

where \( a: U \times X \times X \to \mathbb{R} \) is a semilinear form associated to a given operator \( A: Q \times V \to V^* \) by the relations \( a(u, y)(\xi) = \int_0^T \tilde{a}(u(t), y(t))((\xi(t)))dt \) with \( \tilde{a}(\tilde{u}, \tilde{y})(\tilde{\xi}) = \langle A(\tilde{u}, \tilde{y}), \tilde{\xi} \rangle_{V^* \times V} \) for \( \tilde{u} \in Q \) and \( \tilde{y}, \tilde{\xi} \in V \). For existence and uniqueness results we refer the reader to [2]. Thus, the control problem reads as

Minimize \( J(u, y^1) \) s.t. (2.1), \( (u, y^1) \in U \times X \). (2.2)

We assume, that (2.2) obtains a locally unique solution.

3 Discretization

For the discretization of the state equation and the corresponding adjoint equation we apply a \( cG(r)cG(s) \) method. The reason for this discretization scheme is its property to conserve the energy in time on uniform meshes and thus, to reflect the natural behaviour of the continuous linear wave equations. At first the state equation is discretized in time by a Petrov Galerkin scheme, i.e. we use continuous ansatz functions and discontinuous test functions. Then we discretize the equation in space using conforming finite elements. Finally, we discretize the control space by choosing a finite dimensional subspace of \( U \). We assume, that the corresponding control problems on the different levels of discretization admit a locally unique solution and denote them by \( (u_k, y^1_k), (u_{kh}, y^1_{kh}), \) and \( (u_e, y^1_e) \). For a detailed description of the discretization we refer to [2].
4 A posteriori error estimate

To derive a posteriori error estimates we split the error in the cost functional as

\[
J(u, y^1) - J(u_\sigma, y_\sigma^1) = (J(u, y^1) - J(u_k, y_k^1)) + (J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1)) + (J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1))
\]

where the arguments of the functional are the solutions of the control problems on the continuous level as well as on the different levels of discretization; cf. Section 3. We define the Lagrangian \( L \) where the arguments of the functional are the solutions of the control problems on the continuous level as well as on the different levels of discretization; cf. Section 3. We define the Lagrangian \( L \)

\[
L(u, y, p) = J(u, y^1) + (f - y^2_1, p^1)_{L^2(\Omega)} - a(u, y^1)(p^1) - (y^1_1 - y^2_1, p^2)_{L^2(\Omega)} - (y^2_1(0) - y_1(u), p^1(0))_{H^1(\Omega)} + (y_0(u) - y^1_1(0), p^2(0))_{H^1(\Omega)}
\]

(4.1)

with \( y = (y^1, y^2) \in Y \) and \( p = (p^1, p^2) \in \mathbb{P}^2 \). Then the errors \( e_1, e_2, \) and \( e_3 \) in the cost functional can be estimated by first derivatives of the Lagrangian (4.1). For details of these techniques we refer the reader to [1, 2, 4].

5 Numerical example

In this numerical example we consider an optimal control problem for a nonlinear wave equation with distributed control. Let \( \Omega = [0, 1] \times [0, 1] \), \( V = H^1_0(\Omega) \), \( H = L^2(\Omega) \) and \( U = L^2(I, L^2(\Omega)) \). Then we consider the following control problem:

\[
\begin{align*}
\text{Minimize} & \quad J(u, y) = \frac{1}{2} \| \frac{\partial y}{\partial t} \|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \| u \|_{L^2(\Omega)}^2, \\
& \quad y_0 = 0, \\
& \quad y_{tt} - \Delta y + y^2 = u + f \text{ in } (0, T) \times \Omega, \\
& \quad y(0) = y_0 = 0 \text{ in } \Omega, \\
& \quad y_t(0) = y_1 = 0 \text{ in } \Omega, \\
& \quad y = 0 \text{ in } (0, T) \times \partial \Omega,
\end{align*}
\]

with

\[
f(t, x_1, x_2) = \begin{cases} 
100, & \text{if } x_1 < 0.125, \ t < 0.05, \\
0, & \text{else,}
\end{cases}
\]

\( y_0 = y_1 = 0, \ \alpha = 0.1, \)

for \( (t, x_1, x_2) \in [0, 1] \times \Omega \). The state and adjoint equation are discretized by a \( cG(1)cG(1) \) method and the control is discretized as the adjoint state. Figure 1 shows the error for adaptive in comparison to uniform refinement.

**Fig. 1** Error for adaptive and uniform refinement.

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**References**


