Reinforcement Learning
Reinforcement Learning: Advanced Techniques in the Tabular Setting

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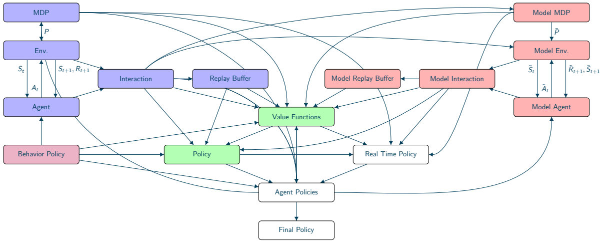
M2DS - Reinforcement Learning – Fall 2023

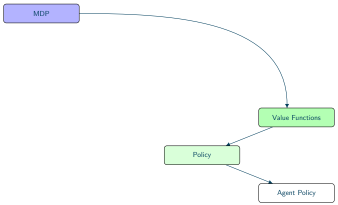
RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View

Operations Research and MDP

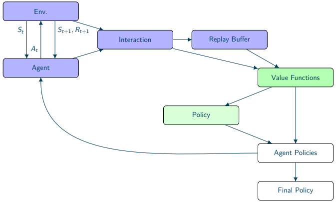


How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.

Reinforcement Learning and Interactions

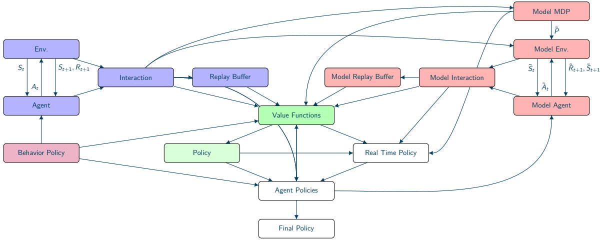


How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA. . .)
- Policy deduced by a statewise optimization of the value function over the actions.

More Tabular Reinforcement Learning



Can We Do Better?

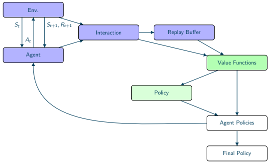
- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).

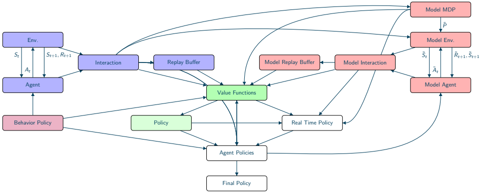
Outline

Advanced Tabular Reinforcemcent Learning

From



to



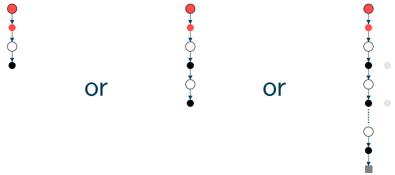
- Core idea: Approximate Bellman Operators with Stochastic Approximation...

Advanced Ideas?

- Between MC and TD?
- Off-policy vs on-policy?
- Exploration vs Exploitation?
- Model? Replay?
- Real Time Planning?

Outline

n-steps



How many steps before backup?

- One step: TD.
- As many steps as required to end the episod: MC.
- n-steps: n-steps TD.

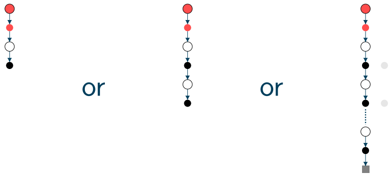
$$\left(\mathcal{T}^{\Pi}\right)^{n} v(s) = \mathbb{E}_{\Pi}\left[\underbrace{R_{t+1} + \gamma R_{t+2} + \gamma^{n-1} R_{t+n} + \gamma^{n} v(S_{t+n})}_{G_{t:t+n}} \middle| S_t = s\right]$$

- Family of stochastic approximation algorithms:

$$V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))\left(G_{t:t+n} - V(S_t)\right)$$

Discounted

n-steps TD



$$V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))\left(G_{t:t+n} - V(S_t)\right)$$

n-steps TD

- Convergence for prediction.
- Need to be combined with Policy Improvement for planning: n-steps SARSA.
- n-steps Q-learning could be an extension of API. . . but this means following the optimized policy Π. . . i.e. SARSA!

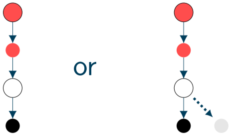
- Best convergence often for intermediate n.
- No proof beside TD for $n > 1$!

n-steps TD

Discounted: Prediction by n-steps TD

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    $Q(S_{t-n}, A_{t-n}) \leftarrow Q(S_{t-n}, A_{t-n}) + \alpha(N(S_t, A_t))\,(G_{t-n:t} - Q(S_t, A_t))$
    $t \leftarrow t + 1$
    $t' \leftarrow t' + 1$
  until *episod ends at time T' or $t' == T$*
until $t' == T$
output: State-Action value function Q

Expected SARSA



Expected SARSA

- The policy Π is known so that we can use it in a formula:
$$R_t + \gamma Q(S_t, A_t) \longrightarrow R_t + \gamma \sum_a \pi(a|S_t) Q(S_t, a)$$
- Make the update independent of the action chosen (and thus of the policy used to play).
- Reduce the variance for a computational cost.
- Amount to use the current estimate for $V(S_t)$...

Expected SARSA

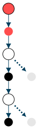
Discounted: Prediction by Expected SARSA

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(R_{t+1} + \gamma \sum_a \pi(a|S_t) Q(S_{t+1}, a) - Q(S_t, A_t)\right)$
  - $t \leftarrow t + 1$
  - $t' \leftarrow t' + 1$
- **until** *episod ends at time* T' *or* $t' == T$

until $t' == T$
output: State-Action value function Q

Discounted

n-steps Tree Backup



n-steps Tree Backup

- At each time step, use the expected SARSA average over the action while replacing the Q value for the picked action by a deeper estimate.
  - 1-step return (Expected Sarsa)
$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$
  - 2-step return:
$$\begin{aligned} G_{t:t+2} &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) \\ &\quad + \gamma \pi(A_{t+1}|S_{t+1}) \left(R_{t+2} + \gamma \sum_a \pi(a|S_{t+2}) Q(S_{t+2}, a) \right) \\ &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2} \end{aligned}$$

Discounted

n-steps Tree Backup

- 1-step return (Expected Sarsa)

$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$

- 2-step return:

$$\begin{aligned} G_{t:t+2} &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2} \\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) \left(G_{t+1:t+2} - Q(S_{t+1}, A_{t+1}) \right) \end{aligned}$$

- Recursive definition of n-step return:

$$\begin{aligned} G_{t:t+n} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \\ + \gamma \pi(A_{t+1}|S_{t+1}) \left(G_{t+1:t+n} - Q(S_{t+1}, A_{t+1}) \right) \end{aligned}$$

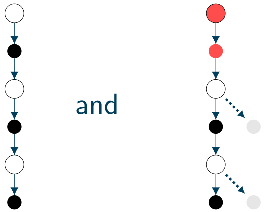
- TD update

$$Q(S_{t-n}, A_{t-n}) = Q(S_{t-n}, A_{t-n}) + \alpha(N(S_{t-n}, Q_{t-n})) \left(G_{t-n:t} - Q(S_{t-n}, A_{t-n}) \right)$$

Discounted

Q(σ)

Between and



Sampling or Averaging

- Unifying algorithm!
- Recursive definition of *n*-step return:

$$G_{t:t+n} = R_{t+1} + \sigma G_{t+1:t+n} + (1-\sigma)\Big(\gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma\pi(A_{t+1}|S_{t+1})\left(G_{t+1:t+n} - Q(S_{t+1}, A_{t+1})\right)\Big)$$

Discounted

Averaged n-steps return?

- n-step return:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- Averaged n-step return: (compound update)

$$G_t^\omega = \sum_{n=1}^{\infty} \omega_n G_{t:t+n} \quad \text{with} \sum_{i=1}^{\infty} \omega_n = 1$$

- TD(λ): specific averaging

$$\begin{aligned} G_t^\lambda &= (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n} \\ &= (1-\lambda) \sum_{n=1}^{T-t} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t} G_t \quad \text{(Episodic)} \end{aligned}$$

  interpolating between TD (a.k.a TD(0)) and MC for $\lambda = 1$.
- Can be mixed with tree backup strategies (TB(λ))

Discounted

λ-return and Temporality

True λ-return

- Require to wait until the end of an episode before we can update.
- Unusable in a non episodic setting!

Truncated λ-return

- Truncated λ-return:

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{H-t} \lambda^{n-1} G_{t:t+n} + \lambda^{H-t} G_{t:H}$$

- The virtual horizon H may vary during the algorithm.

λ-return and Temporality

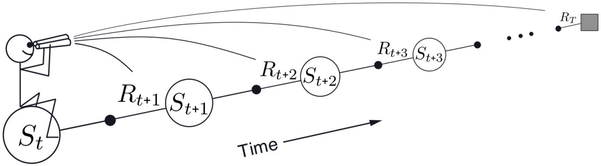
Temporality

- n-step return
$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$
depends on a current estimate V (or Q)!
- In G_λ should we use
  - an estimate available at time t?
  - an estimate available at time $t+n$?
  - an estimate available at time H?
- Off-Line vs On-Line!
  - Off-line: keep V constant during the episodes.
  - On-line: Used updated V when available.
  - True on-line (Sutton and Barto): restart algorithm with a growing horizon.

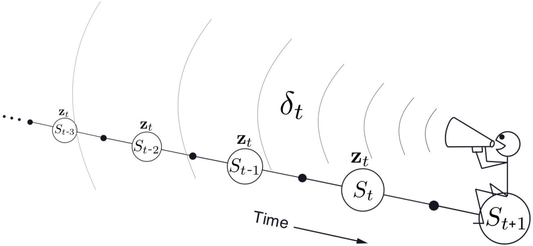
Outline

Forward and Backward Point of View

From a forward view



To a backward one:



Source: Sutton and Barto

Returns and Temporal Differencies

- n-step returns:

$$\begin{aligned} G_{t:t+n} - Q(S_t, A_t) &= R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} \\ &\quad + \gamma^n Q(S_{t+n}, A_{t+n}) - Q(S_t, A_t) \\ &= \sum_{l=1}^{n} \gamma^{l-1} (R_{t+l} + \gamma Q(S_{t+l}, A_{t+l}) - Q(S_{t+l-1}, A_{t+l-1})) \\ &= \sum_{l=0}^{n-1} \gamma^{l-1} \delta_{t+l} \end{aligned}$$

- λ return:

$$\begin{aligned} G_t^\lambda - Q(S_t, A_t) &= (1-\lambda) \sum_n \lambda^n (G_{t:t+n} - Q(S_t, A_t)) \\ &= \sum_{n=0} \lambda^n \gamma^n \delta_{t+n} \end{aligned}$$

Forward View and Backward View

Forward View

- Updates:

$$Q_t(s,a) = Q_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}\alpha_t(s,a)\left(\sum_{t''\geq t}\lambda^{t''-t}\gamma^{t''-t}\delta_{t''}\right)$$

- Cumulative updates:

$$Q_t(s,a) = Q_0(s,a) + \sum_{t'\leq t}\mathbf{1}_{(s,a)=(S_{t'},A_{t'})}\alpha_{t'}(s,a)\left(\sum_{t''\geq t'}\lambda^{t''-t'}\gamma^{t''-t'}\delta_{t''}\right)$$

- Limit:

$$Q_\infty(s,a) = Q_0(s,a) + \sum_{t'}\mathbf{1}_{(s,a)=(S_{t'},A_{t'})}\alpha_{t'}(s,a)\left(\sum_{t''\geq t'}\lambda^{t''-t'}\gamma^{t''-t'}\delta_{t''}\right)$$

- Focus on the update place.

Forward View and Backward View

Limit(s)

- Limit:

$$\begin{aligned} Q_\infty(s,a) &= Q_0(s,a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right) \\ &= Q_0(s,a) + \sum_{t''} \delta_{t''} \sum_{t' \leq t''} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t''-t'} \gamma^{t''-t'} \end{aligned}$$

- Focus on the update place or and the temporal differencies. . .

Forward View and Backward View

Backward View

- Same limit with cumulative udpates over temporal differencies

$$Q_t(s,a) = Q_0(s,a) + \sum_{t''\leq t} \delta_{t''} \sum_{t'\leq t''} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t''-t'} \gamma^{t''-t'}$$

- Updates

$$Q_t(s,a) = Q_{t-1}(s,a) + \delta_t \underbrace{\sum_{t'\leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t-t'} \gamma^{t-t'}}_{z_t(s,a)}$$

- Pseudo Eligibility trace:

$$\begin{aligned} z_t(s,a) &= \sum_{t'\leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t-t'} \gamma^{t-t'} \\ &= \lambda\gamma z_{t-1}(s,a) + \alpha_t(s,a) \mathbf{1}_{(s,a)=(S_t,A_t)} \end{aligned}$$

- Proof of convergence toward the same target.

Discounted

Eligibility Trace

$$Q_t(s,a) = Q_{t-1}(s,a) + \alpha_t \delta_t z_t(s,a)$$

Eligibility Trace

- Focus on temporal differencies with simultaneous update on all states.
- TD(λ) eligibility trace: $z_t(s,a) = \lambda\gamma z_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$
- Strictly equivalent to the previous scheme for constant stepsize
- Other eligibility trace:
  - Replacing trace:

$$z_t(s,a) = \begin{cases} 1 & \text{if } (s,a) = (S_t, A_t) \\ \lambda\gamma z_{t-1}(s,a) & \text{otherwise} \end{cases}$$

  - Time dependent trace:

$$z_t(s,a) = c_t \gamma z_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$$

where c_t is defined *in a appropriate way* to ensure the convergence of the algorithm.

- Need to store (and update) this information...

Temporal Differencies

$$\delta_t?$$

Temporal Differencies

- Basic temporal differencies:

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

- Expected temporal differencies:

$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$

- Average of both:

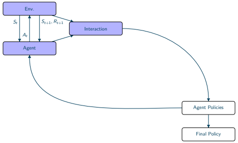
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma\sigma Q(S_{t+1}, A_{t+1}) + \gamma(1-\sigma) V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma\sigma \left(Q(S_{t+1}, A_{t+1}) - V(S_{t+1})\right) - Q(S_t, A_t)\end{aligned}$$

- Only expected temporal average is independent of the next action.
- No generic proof of convergence...

Outline

On-Policy vs Off-Policy

From



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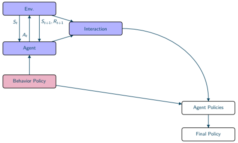


On-Policy vs Off-Policy

- On-Policy: the policy b used to interact is the same than the policy Π evaluated or optimized.
- Off-Policy: the policy b used to interact may be different from the policy Π evaluated or optimized.

- Off-Policy allows in particular to (re)use interactions from previous experiments.
- Q-learning was possible in off-policy setting.

Importance Sampling

$$\rho_{t:t'} = \frac{\mathbb{P}_\Pi(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}|S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}|S_t)} = \frac{\pi(A_t|S_t) \ldots \pi(A_{t'}|S_{t'})}{b(A_t|S_t) \ldots \pi(A_{t'}|S_{t'})}$$

Importance Sampling

- For any law p and q, and any function g

$$\mathbb{E}_p[g(x)] = \mathbb{E}_q\left[\frac{p(x)}{q(x)} g(x)\right]$$

provided $q(x) = 0$ implies $p(x) = 0$.

- $\mathbb{V}\text{ar}_q\left[\frac{p(x)}{q(x)} g(x)\right]$ may be large with respect to $\mathbb{V}\text{ar}_p\left[g(x)\right]$ if the ratio $p(x)/q(x)$ is large...

Importance Sampling for Trajectories

- For any trajectory $\tau_{t:t'} = S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1}),$,

$$\frac{\mathbb{P}_\Pi(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1})|S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1})|S_t)} = \frac{\pi(A_t|S_t) \ldots \pi(A_{t'}|S_{t'})}{b(A_t|S_t) \ldots b(A_{t'}|S_{t'})}$$

Importance Sampling and Returns

$$\mathbb{E}_\Pi[g(\tau_{t:t'})|S_t = s] = \mathbb{E}_b[\rho_{t:t'} g(\tau_{t:t'})|S_t = s] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t)\dots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\dots b(A_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}
\mathbb{E}_\pi[G_{t:t'}|S_t = s] &= \mathbb{E}_\pi\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} V(S_{t'}) \middle| S_t = s\right] \\
&= \mathbb{E}_b\left[\rho_{t:(t'-1)}\left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} V(S_{t'})\right) \middle| S_t = s\right] \\
&= \mathbb{E}_b\left[\sum_{t''=t+1}^{t'} \rho_{t:(t''-1)} \gamma^{t''-t-1} R_{t''} + \rho_{t:(t'-1)} \gamma^{t'-t} V(S_{t'}) \middle| S_t = s\right]
\end{aligned}$$

Importance Sampling and Returns

$$\mathbb{E}_\Pi[g(\tau_{t:t'})|S_t, A_t] = \mathbb{E}_b\left[\rho_{(t+1):t'} g(\tau_{t:t'})\middle|S_t, A_t\right] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t)\dots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\dots b(A_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}
\mathbb{E}_\pi[G_{t:t'}|S_t, A_t] &= \mathbb{E}_\pi\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\middle|S_t, A_t\right] \\
&= \mathbb{E}_b\left[\rho_{(t+1):(t'-1)} \left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\right)\middle|S_t, A_t\right] \\
&= \mathbb{E}_b\left[\rho_{(t+1):(t''-1)} \sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \rho_{(t+1):t'} \gamma^{t'-t} Q(S_{t'}, A_{t'})\middle|S_t, A_t\right]
\end{aligned}$$

- No correction if $t' = t + 1$

λ-return

λ-return

- Recursive definition of the λ-return:

$$G_t^\lambda | S_t = R_{t+1} + \gamma \left((1-\lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \Big((1-\lambda) (\sigma Q(S_{t+1}, A_{t+1}) + (1-\sigma) (\sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \pi(A_{t+1}|S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right))) + \lambda G_{t+1}^\lambda \Big)$$

- Off-line correction

$$G_t^\lambda | S_t = \rho_{t:t} \left(R_{t+1} + \gamma \left((1-\lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right) \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \Big((1-\lambda) (\sigma Q(S_{t+1}, A'_{t+1}) + (1-\sigma) (\sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \pi(A_{t+1}|S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right))) + \lambda \rho_{t+1:t+1} G_{t+1}^\lambda \Big)$$

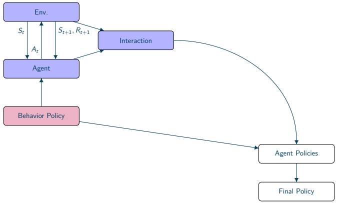
where A'_{t+1} is drawn following π (or multiply by $\rho_{t+1:t+1}$ to use A_{t+1}).

$$\delta_t?$$

Temporal Differencies

- Basic temporal differencies:
$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A'_{t+1}) - Q(S_t, A_t)$$
with A'_{t+1} drawn using π.
- Expected temporal differencies:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$
without any correction.
- Average of both:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma\sigma Q(S_{t+1}, A_{t+1}) + \gamma(1-\sigma) V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma\sigma \left(Q(S_{t+1}, A'_{t+1}) - V(S_{t+1})\right) - Q(S_t, A_t)\end{aligned}$$
with A'_{t+1} drawn using π.

Off-Policy Algorithm



Off-Policy Correction

- Replace any estimate of the gain by an importance-sampling corrected one.
- Works well for prediction.
- Can be combined with policy improvement (a la SARSA) but less (no?) theoretical guarantees.

Retrace(λ)

$$\widetilde{\mathcal{T}}Q(s,a) = Q(s,a) + \mathbb{E}_b\left[\sum_{t\geq 0}\gamma^t\left(\prod_{t'=1}^{t} c_{t'}\right)\delta_t \middle| S_0 = s, A_0 = a\right]$$

$$c_t = c(A_t, S_t, A_{t-1}, S_{t-1}, \cdots, A_0, S_0)$$

$$\mathbb{E}_b[\delta_t|S_t, A_t] = \mathbb{E}[R_{t+1} + \gamma\mathbb{E}_\pi[Q(S_{t+1},\cdot)] - Q(S_t, A_t)|S_t, A_t]$$

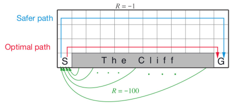
Generic Off-Policy Algorithm

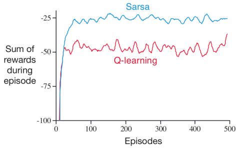
- Generic off-line algorithm including
  - Importance sampling: $c_t = \rho_{t:t} = \pi(A_t|S_t)/b(A_t|S_t)$
  - TB(λ): $c_t = \lambda\pi(A_t|S_t)$
  - Retrace(λ): $c_t = \lambda\min(1, \pi(A_t|S_t)/b(A_t/S_t))$
- **Prop:** Q_π is a fixed point as $\mathbb{E}_b[\delta_t|S_t, A_t] = \mathbb{E}[\mathcal{T}^\pi Q(S_t, A_t) - Q(S_t, A_t)|S_t, A_t]$.
- **Prop:** $\widetilde{\mathcal{T}}$ is a contraction provided $c_t \leq \rho_t = \pi(A_t|S_t)/b(A_t|S_t)$.
- Convergence for Importance sampling, TB(λ) and Retrace(λ) for any b.
- Partial results for policy improvement under more assumptions.

- For Q(λ), $c_t = \lambda$, convergence if $\|\pi(|s) - b(|s)\|_1 \leq \epsilon$ and $\lambda \leq (1-\gamma)/(\gamma\epsilon)$.

Outline

Q-Learning vs SARSA





How different are they?

- In Q-learning, the exploratory policy used is decoupled from the optimized policy.
- This exploratory policy may yield low rewards on average.
- In SARSA, the two policies are linked with the hope on having higher rewards during the learning step.
- Subtle different behavior even if we modify the exploratory policy in Q-Learning.

Exploration vs Exploitation

Exploration vs Exploitation

- Exploration: explore new policies to be able to discover the best ones.
- Exploitation: use good policies to obtain a good return.
- Exploration is a requirement.

- No tradeoff if we optimize only the final result!
- Tradeoff between the two if we consider that the returns during training matters!
- Q-learning use the first approach and SARSA try to tackle the second.
- Tradeoff if we study a regret:
$$\sum_t \mathbb{E}_{\Pi_\star}[R_t] - \mathbb{E}_{\Pi_t}[R_t]$$
which forces us to be good as fast as possible.
- No natural definition in the discounted setting.

Bandits

$$\mathcal{S} = \{0\} \quad \text{and} \quad A = \{1, \dots k\} \quad \text{and} \quad r(s,a) = r_a$$

Bandits

- Very simple toy model where there is only one state!
- Optimal policy: pick $a_\star \in \operatorname{argmax} r_a$.
- Q estimation: estimate r_a by playing action a.
- Strategy:
  - Every arm has to be played until we are sure they are bad.
  - Best arm should be played as often as possible to maximime the rewards during the learnig phase.
- Simple enough setting to obtain result on the regret

$$r_T = \sum_{t \leq T} (r_{a_\star} - R_t)$$

- We will use $\Delta_a = r_{a_\star} - r_a$ and assume that $R|a$ is 1-subgaussian.

Explore Then Commit

Explore Then Commit (Random Exploration)

- Play the arm successively during Km steps and then play the optimal one during $T - Km$ steps.
- **Prop:**

$$r_T \leq \min(m, T/K) \sum_{a=1}^{k} \Delta(a) + \max(T - mK, 0) \sum_{a=1}^{k} \Delta(a) \exp(-m\Delta(a)^2/4)$$

Furthermore,

$$\mathbb{P}(a_T = a_*) \geq 1 - \sum_{a \neq a_*} \exp(-m\Delta(a)^2/4)$$

ϵ-greedy Strategy

- Estimate $Q(a) = r_a$ by MC:

$$Q_t(a) = \frac{\sum_{t'=1}^{t-1} \mathbf{1}_{A_{t'}=a} R_{t'}}{\sum_{i=1}^{t-1} \mathbf{1}_{A_{t'}=a}}$$

- Pick arm a at time t using

$$\pi(a) = \begin{cases} \epsilon_t/k + (1-\epsilon) & \text{if } a = \operatorname{argmax}_{a'} Q_t(a') \text{ (only the smallest if necessary)} \\ \epsilon_t/k & \text{otherwise} \end{cases}$$

- **Prop:**

$$r_T \geq \sum_{t=1}^{T} \frac{\epsilon_t}{k} \sum_{a=1}^{k} \Delta(a)$$

ϵ-greedy Strategy

ϵ-greedy Strategy

- **Prop:**

$$\mathbb{P}(A_T = a_*) \geq 1 - \epsilon_T - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

with $\Sigma_T = \sum_{s=1}^{T} \epsilon_s$.

Furthermore,

$$\mathbb{P}(a_* = \operatorname{argmax} Q_{T,a}) \geq 1 - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

If $\epsilon_t = c/t$,

$$r_T \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T) + 1}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

as soon as $c/(6k) > 1$ and $c \min_{a \neq a_*} \Delta(a)/4k < 1$.

If $\epsilon_t = c \log(t)/t$ then

$$r_T \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T)(\log(T) + 1)}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

UCB Strategy

Upper Confidence Bound

- Use an optimistic strategy to pick the best arm

$$A_t = \operatorname{argmax} Q_t(a) + \sqrt{\frac{c \log t}{N_t(a)}}$$

- **Prop:**

$$r_n(t) \leq C_c \sum_a \Delta(a) + \sum_a \frac{4c \ln t}{\Delta(a)}.$$

with $C_c < +\infty$ as soon as $c > 3/2$
Furthermore

$$\mathbb{P}(A_t = a_*) \geq 1 - 2kt^{-2c+2}$$

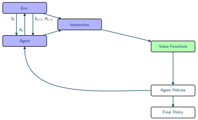
as soon as $t \geq \max_a \frac{4c \ln t}{\Delta(a)^2}$.

- Optimal regret!
- Hard to extend to RL setting but shows that ϵ-greedy may not be optimal.

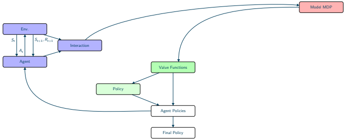
Outline

Model Based Approach

From



to



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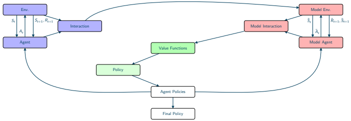
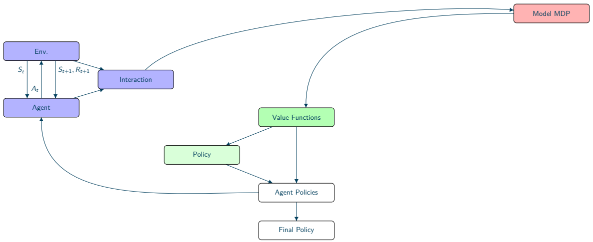


Model Based Approach

- Use the interactions to learn a model...
- that can be used to learn a good policy.
- This model can be:
  - a MDP,
  - a simulator.

- Often easier to obtain a simulator.

Model based and MDP

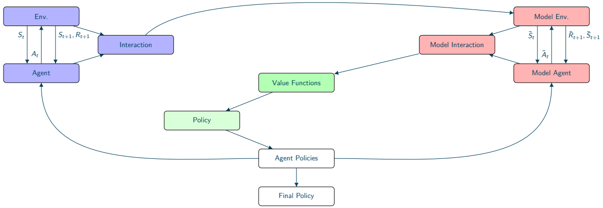


Estimated MDP: back to OR

- MDP can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated MDP, prediction and planning can be done using OR.

- Implicitely done by TD(0) when doing several passes.
- Model should be checked/improved as much as possible when new trajectories arrive.

Model based and RL

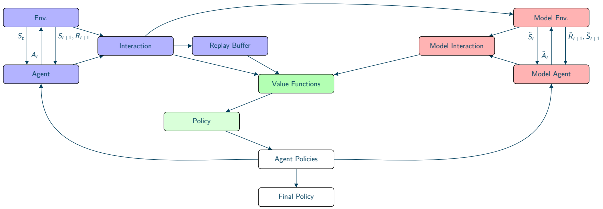


Estimated Simulator: back to RL

- Simulator can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated simulator, prediction and planning can be done using RL.

- Model should be checked/improved as much as possible when new trajectories arrive.

Model Free and Model Based Approach



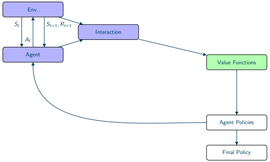
Dyna

- Combine true interactions with simulated ones.
- Simultaneous acting, model learning, OR learning and RL learning.
- Search for a tradeoff between the (slow) learning RL algorithm and the (wrong) model OR algorithm.
- Need to deal with schedule!

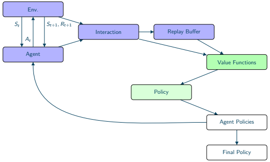
Outline

Replay Buffer and Prioritized Sweeping

From



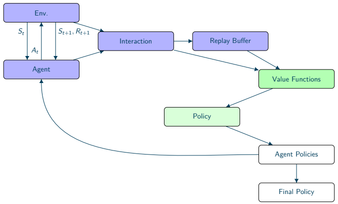
to



Replay Buffer and Prioritized Sweeping

- Can we reuse previous interactions?
- In which order?

Replay Buffer

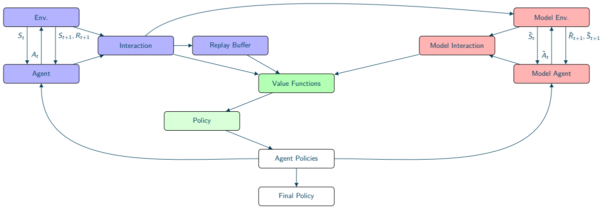


Replay Buffer

- Store previous interactions (trajectories) in a first-in first-out buffer.
- Draw a subsequence from those interactions (trajectories) and use it in a RL algorithm:
  - On-line: if the trajectory comes from the same policy.
  - Off-line: if the trajectory comes from a different policy.

- Similar to a simulator but no arbitrary choice of state or action.
- Often use with on-line algorithm if the policy has only mildy evolved...

Prioritized Sweeping



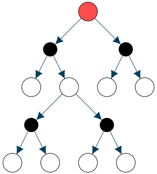
Prioritized Sweeping

- Plain Replay Buffer: subsequence drawn uniformly.
- Prioritized Sweeping: subsequence drawn favoring states with large temporal differencies.

- Can be combined with a model approach.

Outline

Real Time Planning

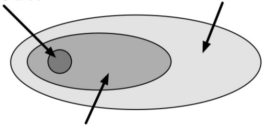


Real Time Planning

- Can we optimize the policy at the current state?
- Do we need to optimize it everywhere?
- What is required?

- Planning at decision time...

Real Time Dynamic Programming



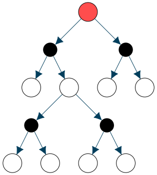
- Warmup in Dynamic Programming...

RT DP

- Use trajectories to sample the states to update.
- Convergence holds with exploratory policy.
- Optimal policy does not require to specify the action in irrelevant states.
- Convergence holds even without full exploration in some specific cases!

- In practice, seems to be computationaly efficient.

Planning At Decision Time



Planning At Decision Time

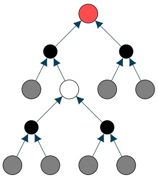
- Can we find a good action A_t at S_t... without having it precomputed?
- Policy Improvement

$$A_t = \text{argmax}\, Q_t(S_t, \cdot)$$

  can be seen as a first step.
- How to go deeper?

- **A model or a simulator will be required!**

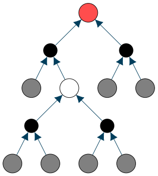
Heuristic Search



Heuristic Search

- Requires the knowledge of the MDP and of a heuristic based value function V.
- Strategy:
  - Build a limited depth tree by stopping after a few steps and at some specific states.
  - Backup the heuristic based value function using Dynamic Programming (Optimal Bellman operator).
  - Pick the action having the hight value.
- The deeper the better... but the more expensive due to branching!
- Requires a *suitable* heuristic...

Rollout Algorithm

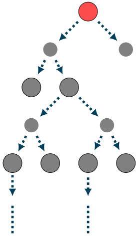


Rollout Policy

- Use a MC estimate with a default policy instead of a heuristic.
- Backup those estimates using Dynamic Programming.
- Simulation can even start after the first action (as in Policy Improvement).

- The values are (most of the time) discarded for the next state.

Monte Carlo Tree Search



Monte Carlo Tree Search

- Simultaneour tree growing, rollout and backup by DP.
- Repeat 4 steps:
  - Selection of a sequence of actions according to the current values with a tree policy.
  - Expansion of the tree at the last node without values.
  - Simulation with a rollout policy to estimate the values at this node.
  - Backup of the value by relaxed Dynamic Programming.
- MCTS focuses on promising paths using a UCB approach.

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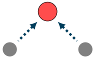

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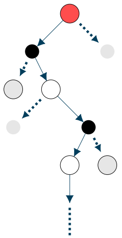

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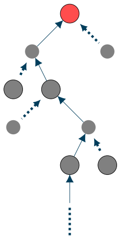

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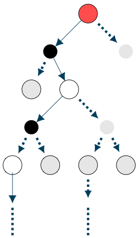

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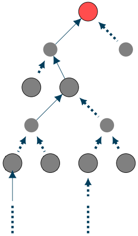

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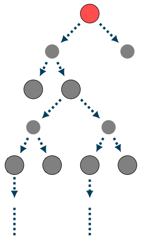

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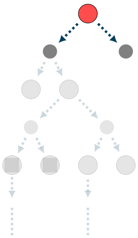

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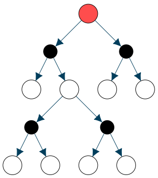
Model Predictive Control?



Model Predictive Control

- Open loop optimization:

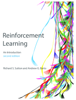
$$\max_{a_t, a_{t+1}, \ldots, a_{t+h}} \mathbb{E}\left[\sum_{t'=t}^{t+h} R_t\right]$$

  using a predictive model (simulator).
- Do not take into account state uncertainties in the control choice...
- But much simpler optimization...
- and equivalence for a linear Gaussian model.

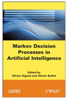
- Extensively used for short-term planning in Control.

Outline

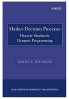
References



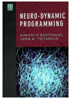
R. Sutton and A. Barto.
Reinforcement Learning, an Introduction (2nd ed.)
MIT Press, 2018



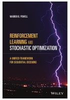
O. Sigaud and O. Buffet.
Markov Decision Processes in Artificial Intelligence.
Wiley, 2010



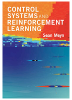
M. Puterman.
Markov Decision Processes. Discrete Stochastic Dynamic Programming.
Wiley, 2005



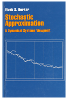
D. Bertsekas and J. Tsitsiklis.
Neuro-Dynamic Programming.
Athena Scientific, 1996



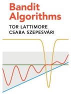
W. Powell.
Reinforcement Learning and Stochastic Optimization: A Unified Framework for Sequential Decisions.
Wiley, 2022



S. Meyn.
Control Systems and Reinforcement Learning.
Cambridge University Press, 2022



V. Borkar.
Stochastic Approximation: A Dynamical Systems Viewpoint.
Springer, 2008



T. Lattimore and Cs. Szepesvári.
Bandit Algorithms.
Cambridge University Press, 2020

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