# Reinforcement Learning
# Reinforcement Learning: Approximation of the Value Functions

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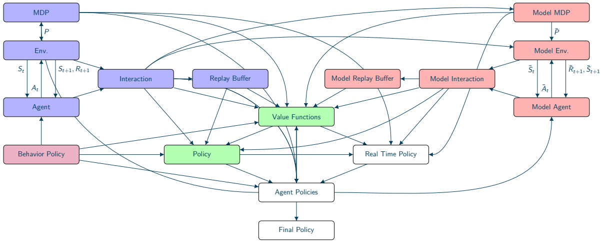
M2DS - Reinforcement Learning – Fall 2023

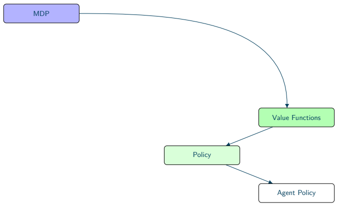
# RL: What Are We Going To See?




## Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View

# Operations Research and MDP

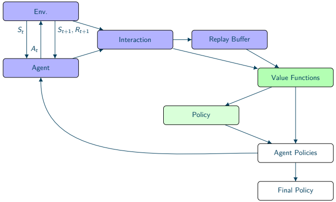



**How to find the best policy knowing the MDP?**

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.

# Reinforcement Learning and Interactions

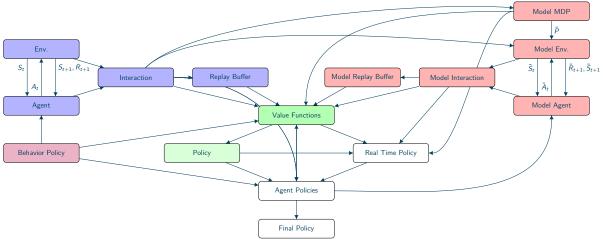



## How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions ($Q$ learning, SARSA…)
- Policy deduced by a statewise optimization of the value function over the actions.

# More Tabular Reinforcement Learning

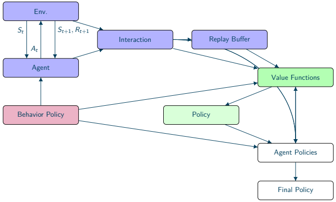



## Can We Do Better?

- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).

# Reinforcement and Approximation of Value Functions



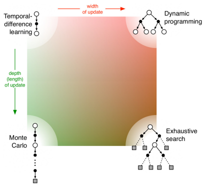

## How to Deal with a Large/Infinite states/action space?

- How to approximate value functions?
- How to estimate good approximation of value functions?

- Finite action space setting.
- Stochastic algorithm (Deep $Q$ Learning...).
- Policy deduced by a statewise optimization of the value function over the actions.

# Outline

# Approximation?




## Tabular Setting

- Require to store the state(-action) values (a table).
- Requirement in both OR and RL.

## Approximation!

- Use instead approximated value functions.
- What is a good approximation?
- How to use them?

- Focus on value-functions...

# Outline

# Approximated Value Functions



$$V(s) \Longrightarrow V_{\boldsymbol{w}}(s)$$
$$Q(s,a) \Longrightarrow Q_{\boldsymbol{w}}(s,a)$$

## Parametric Model

- Reduce dimensionality by storing $\boldsymbol{w}$ instead of all the values.
- Linear: $V_{\boldsymbol{w}}(s) = \langle \Phi(s), \boldsymbol{w} \rangle$ and $Q_{\boldsymbol{w}}(s,a) = \langle \Phi(s,a), \boldsymbol{w} \rangle$
  - $\Phi(s)$ and $\Phi(s,a)$ are features associated to the states(-actions).
  - Tabular setting corresponds to $(\Phi)_{s'(,a')}(s(,a)) = \mathbf{1}_{s'=s(,a'=a)}$.
  - Often used in theoretical analysis.
- Deep Learning: $V_{\boldsymbol{w}}(s) = \mathrm{NN}_{\boldsymbol{w}}(\Phi(s))$ and $Q_{\boldsymbol{w}}(s,a) = \mathrm{NN}_{\boldsymbol{w}}(\Phi(s,a))$
  - NN is any (deep) learning network.
  - Often used in practice.

- Other parametrization (or even non parametric coding) could be used (at least in theory. . . ).

# Approximated Value Functions Usage

$$v_\pi(s) \simeq V_{\mathbf{w}_\pi}(s) \qquad v_\star(s) \simeq V_{\mathbf{w}_\star}(s)$$

$$q_\pi(s,a) \simeq Q_{\mathbf{w}_\pi}(s,a) \qquad q_\star(s,a) \simeq Q_{\mathbf{w}_\star}(s,a)$$

$$\operatorname*{argmax}_a q_\pi(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\pi}(s,a) \qquad \operatorname*{argmax}_a q_\star(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\star}(s,a)$$

## Approximated Value Functions Usage

- *Drop-in* replacements for all the value functions?
- Prediction and Planning?
- Quality and Stability?

# Approximation Quality



$$v_\pi(s) \simeq V_{\mathbf{w}_\pi}(s) \qquad v_\star(s) \simeq V_{\mathbf{w}_\star}(s)$$

$$q_\pi(s,a) \simeq Q_{\mathbf{w}_\pi}(s,a) \qquad q_\star(s,a) \simeq Q_{\mathbf{w}_\star}(s,a)$$

$$\operatorname*{argmax}_a q_\pi(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\pi}(s,a) \qquad \operatorname*{argmax}_a q_\star(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\star}(s,a)$$

## Approximation Quality Norm

- Ideal loss:
  $$\|v - V_{\mathbf{w}}\|_\infty \quad \text{or} \quad \|q - Q_{\mathbf{w}}\|_\infty$$
  as this is the error used in all the previous analysis.
- Practical loss:
  $$\|v - V_{\mathbf{w}}\|_{\mu,p}^p = \sum_s \mu(s)|v(s) - V_{\mathbf{w}}(s)|^p$$
  $$\text{or} \quad \|q - Q_{\mathbf{w}}\|_{\mu,p}^p = \sum_{s,a} \mu(s,a)|q(s,a) - Q_{\mathbf{w}}(s,a)|^p$$
  often with $p = 2$ and $\mu$ related to the behavior policy.

# Approximation Target(s)



$$q(s,a) = \mathcal{T} q(s,a) \sim Q_{\mathbf{w}}(s,a) \longrightarrow \begin{cases} \|q - Q_{\mathbf{w}}\|_{\mu,p} \text{ small} \\ \|\mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_{\mu,p} \text{ small} \end{cases}$$

**Approximation Targets(s)**

- Direct measurement.
- Bellman residual error.

**Extended Measurement**

- Projection (with linear parametrization): $\|P_\Phi \left(\mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}\right)\|_{\mu,p}$ small
- Probes $Z$:

$$\mathbb{E}_Z[|\langle \mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}, Z\rangle|^p]$$

- Lots of freedom but hard to link with optimality of derived policy!

# Outline

# Prediction, Approximation and Gradient Descent



$$\min_{\boldsymbol{w}} \sum_{s,a} \mu_\pi(s,a) |q_\pi(s,a) - Q_{\boldsymbol{w}}(s,a)|^2$$

## Prediction, Approximation and Gradient Descent

- Prediction objective:
$$\overline{\mathrm{VE}}(\boldsymbol{w}) = \sum_q \mu_\pi(s,a) |q_\pi(s,a) - Q_{\boldsymbol{w}}(s,a)|^2$$
- Gradient:
$$\nabla \overline{\mathrm{VE}}(\boldsymbol{w}) = -2 \sum_{s,a} \mu_\pi(s,a) \left(q_\pi(s,a) - Q_{\boldsymbol{w}}(s,a)\right) \nabla Q(s,a)$$
- Stochastic gradient:
$$\widehat{\nabla} \overline{\mathrm{VE}}(\boldsymbol{w}) = -2 \left(q_\pi(S_t,A_t) - Q_{\boldsymbol{w}}(S_t,A_t)\right) \nabla Q_{\boldsymbol{w}}(S_t,A_t)$$

- Not a practical algorithm as $q_\pi$ is unknown.

# Prediction, Approximation and MC



$$\mathbf{w}_{t+1} = \mathbf{w}_t + 2\alpha_t \left(G_t - Q_{\mathbf{w}_t}(S_t, A_t)\right) \nabla Q_{\mathbf{w}_t}(S_t, A_t)$$

## Monte Carlo Approach

- Replace $q_\pi(S_t, A_t)$ by its Monte Carlo estimate $G_t$.
- Still a Stochastic Gradient of the original problem with limit (if it exists) satisfying

$$\begin{aligned}\mathbb{E}_\pi[(G_t - Q_{\mathbf{w}_\infty}(S_t, A_t))\nabla Q_{\mathbf{w}_\infty}(S_t, A_t)] \\ = \mathbb{E}[(q_\pi(S_t, A_t) - Q_{\mathbf{w}_\infty}(S_t, A_t))\nabla Q_{\mathbf{w}_\infty}(S_t, A_t)] = 0\end{aligned}$$

- Convergence ensured for the linear parametrization as it is a convex problem.

- Correspond exactly to the tabular MC prediction algorithm for the tabular parametrization.
- For the linear parametrization:

Limiting equation: $\mathbb{E}_\pi[q_\pi(S_t)\Phi(S_t, A_t)] = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right] \mathbf{w}_\infty$

# Prediction, Approximation and TD



$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + 2\alpha_t \left(R_{t+1} + \gamma Q_{\boldsymbol{w}_t}(S_{t+1}, A_{t+1}) - Q_{\boldsymbol{w}_t}(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

## Temporal Differencies Approach

- Replace $q_\pi(S_t, A_t)$ by $R_{t+1} + \gamma Q_{\boldsymbol{w}_t}(S_{t+1}, A_{t+1})$.
- Not a Stochastic Gradient of the original problem but a Stochastic Approximation algorithm with limit (if it exists) satisfying

$$\begin{aligned}\mathbb{E}_\pi[(R_t + \gamma Q_{\boldsymbol{w}_\infty}(S_{t+1}, A_{t+1}) - Q_{\boldsymbol{w}_\infty}(S_t, A_t)) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)]\\ = \mathbb{E}_\pi[((\mathcal{T}^\pi Q_{\boldsymbol{w}_\infty} - Q_{\boldsymbol{w}_\infty})(S_t, A_t)) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)] = 0\end{aligned}$$

- No simple argument to justify the convergence...

- In general, no straightforward relation with Bellman operator.
- Correspond exactly to the tabular TD prediction algorithm for the tabular parametrization.

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + 2\alpha_t \left(\tilde{G}_t - Q_{\boldsymbol{w}_t}(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

### Temporal Differencies Approach

- Replace $q_\pi(S_t, A_t)$ by any advanced return $\tilde{G}_t$.
- Not a Stochastic Gradient of the original problem but a Stochastic Approximation algorithm with limit (if it exists) satisfying

$$\begin{aligned}\mathbb{E}_\pi&\left[\left(\tilde{G}_t - Q_{\boldsymbol{w}_t}(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)\right] \\ &= \mathbb{E}_\pi\left[\left((\widetilde{\mathcal{T}}^\pi Q_{\boldsymbol{w}_\infty} - Q_{\boldsymbol{w}_\infty})(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)\right] = 0\end{aligned}$$

- No simple argument to justify the convergence...

- In general, no straightforward relation with Bellman operator.
- Correspond exactly to the tabular TD prediction algorithm for the tabular parametrization.

# Prediction, Approximation and Eligibility Trace

$$\begin{aligned} z_t &= \gamma\lambda z_{t-1} + \nabla Q_{\mathbf{w}_t}(S_t, A_t) \\ \delta_t &= R_{t+1} + \gamma Q_{\mathbf{w}_t}(S_{t+1}, A_{t+1}) - Q_{\mathbf{w}_t}(S_t, A_t) \\ \mathbf{w}_{t+1} &= \mathbf{w}_t + \alpha_t \delta_t z_t \end{aligned}$$

## Eligibility Trace

- Rewrite the TD($\lambda$) updates using the backward point of view.
- No strict equivalence due to time evolution of the parameterization.
- Stochastic Approximation with limit (if it exists) satisfying

$$\begin{aligned} &\mathbb{E}_\pi[(R_{t+1} + \gamma Q_{\mathbf{w}_\infty}(S_{t+1}, A_{t+1}) - Q_{\mathbf{w}_\infty}(S_t, A_t))\, z_t] \\ &= \mathbb{E}_\pi[(\mathcal{T}^\pi Q_{\mathbf{w}_\infty} - Q_{\mathbf{w}_\infty})(S_t, A_t) z_t] = 0 \end{aligned}$$

- No simple argument to justify the convergence.

# Outline

# Linear Parametrization



$$Q_{\boldsymbol{w}}(S_t, A_t) = \Phi(S_t, A_t)^\top \boldsymbol{w} \quad \text{and} \quad \nabla Q_{\boldsymbol{w}}(S_t, A_t) = \Phi(S_t, A_t)$$

## Linear Parametrization

- Extension of the tabular setting.
- Derivative is independent of $\boldsymbol{w}$.
- Analysis of Stochastic Approximation often possible!
- More than a toy model as an algorithm not converging in the linear case will almost certainly not converge in a more general setting.

# Linear Parametrization and MC



$$\text{Iteration:} \boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t (G_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t)\Phi(S_t, A_t)$$

$$\text{Limiting equation: } \mathbb{E}_\pi[q_\pi(S_t, A_t)\Phi(S_t, A_t)] = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right] \boldsymbol{w}_\infty$$

$$\text{ODE: } \frac{d\boldsymbol{w}}{dt} = -\mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right](\boldsymbol{w} - \boldsymbol{w}_\infty)$$

## Linear Parametrization and MC

- Limiting equation is a linear equation.
- Under asymptotic stationarity assumption, convergence of ODE as $\mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right]$ is a Gram Matrix with positive eigenvalues (provided $\Phi$ is not redundant and under an ergodicity assumption).
- Need to explore all state-action pairs!

# Linear Parametrization and TD

Iteration: $\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t (R_{t+1} + \gamma \Phi(S_{t+1}, A_{t+1})^\top \boldsymbol{w}_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t) \Phi(S_t, A_t)$

Lim. eq.: $\mathbb{E}_\pi[r(S_T, A_t)\Phi(S_t, A_t)] = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \Phi(S_{t+1}, A_{t+1})^\top\right)\right] \boldsymbol{w}_\infty$

ODE: $\dfrac{d\boldsymbol{w}}{dt} = -\mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \Phi(S_{t+1}, A_{t+1})^\top\right)\right] (\boldsymbol{w} - \boldsymbol{w}_\infty)$

## Linear Parametrization and TD

- Convergence of ODE if $\mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \Phi(S_{t+1}, A_{t+1})^\top\right)\right]$ has complex eigenvalues with positive real parts...
- which can be proved to be true under an ergodicity assumption!
- Need to explore all state-action pairs!
- Different solution than MC! Minimization of the Projected Bellman Residual...
- **Prop:**

$$\overline{VE}(\boldsymbol{w}_{\mathrm{TD}}) \leq \frac{1}{1-\gamma} \overline{VE}(\boldsymbol{w}_{\mathrm{MC}}) = \frac{1}{1-\gamma} \min_{\boldsymbol{w}} \overline{VE}(\boldsymbol{w})$$

# Least-Squares TD



$$b = \mathbb{E}_\pi[r(S_T, A_t)\Phi(S_t, A_t)] \sim \frac{1}{t}\sum_{t'=0}^{t-1} R_{t'+1}\phi(S_{t'}, A_{t'})$$

$$A = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma\Phi(S_{t+1}, A_{t+1})^\top\right)\right]$$

$$\sim \frac{1}{t}\sum_{t'=0}^{t-1} \Phi(S_{t'}, A_{t'})\left(\Phi(S_{t'}, A_{t'})^\top - \gamma\Phi(S_{t'+1}, A_{t'+1})^\top\right)$$

## Least-Squares TD

- Bypass the Stochastic Approximation scheme by estimating directly its limit:
$$\boldsymbol{w}_\infty = A^{-1}b$$
- Much more sample efficient.
- Recursive implementation possible.
- Recursive implementation maintaining an estimate of $A^{-1}$ is also possible.

# Advanced Returns



$$\text{Return: } \tilde{G}_t = \tilde{R}_{t+1} + \tilde{\Phi}_t^\top \boldsymbol{w} \quad \text{(affine formula)}$$

$$\text{Iteration: } \boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t (\tilde{R}_t + \tilde{\Phi}_t^\top \boldsymbol{w}_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t) \Phi(S_t, A_t)$$

$$\text{Lim. eq.: } \mathbb{E}_\pi \left[ \tilde{R}_t \Phi(S_t, A_t) \right] = \mathbb{E}_\pi \left[ \Phi(S_t, A_t) \left( \Phi(S_t, A_t)^\top - \Phi_t{}^\top \right) \right] \boldsymbol{w}_\infty$$

$$\text{ODE: } \frac{d\boldsymbol{w}}{dt} = -\mathbb{E}_\pi \left[ \Phi(S_t, A_t) \left( \Phi(S_t, A_t)^\top - \Phi_t{}^\top \right) \right] (\boldsymbol{w} - \boldsymbol{w}_\infty)$$

## Linear Parametrization and TD

- Convergence of ODE if $\mathbb{E}_\pi \left[ \Phi(S_t, A_t) \left( \Phi(S_t, A_t)^\top - \Phi_t{}^\top \right) \right]$ has complex eigenvalues with positive real parts...
- which can be proved to be true for the advanced returns under an ergodicity assumption!

# Outline

# On-Policy Prediction

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + 2\alpha_t \left( \tilde{G}_t - Q_{\boldsymbol{w}_t}(S_t, A_t) \right) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

## On-line TD Algorithm

- Use the policy $\Pi$ to obtain the interactions $S_t A_t R_{t+1} S_{t+1} A_{t+1} \ldots$
- Convergence. . . for linear parametrization under stationarity and coverage assumptions!
- Appear to *converge* even with more complex parametrization.

- Monte Carlo can be used if the episods are short.
- Similar observations with elegibility trace.

# On-Policy Control



$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + 2\alpha_t \left(\tilde{G}_t - Q_{\boldsymbol{w}_t}(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

$$\pi_{t+}(s) = \operatorname{argmax} Q_{\boldsymbol{w}_t}(s, \cdot) \quad \text{(plus exploration)}$$

## On-Policy Control

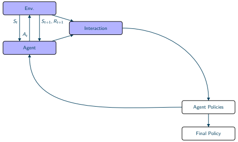
- SARSA type algorithm: update $Q$ values and policy $\pi$ while using policy $\pi$.
- Not a Stochastic Approximation algorithm anymore. . .
- Not approximate policy improvement as no sup-norm control. . .
- No proof of convergence... but appear to work well in practice.

- Non trivial scheduling issue in the definition of $\tilde{G}_t$.
- More constraints with eligibility trace.

# Outline

# On-Policy vs Off-Policy

From

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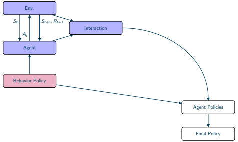



## On-Policy vs Off-Policy

- On-Policy: the policy $b$ used to interact is the same than the policy $\Pi$ evaluated or optimized.
- Off-Policy: the policy $b$ used to interact may be different from the policy $\Pi$ evaluated or optimized.

- Off-Policy correction available for the return.

# Off-Policy Prediction



$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha_t \left(\tilde{G}_t - Q_{\mathbf{w}_t}(S_t, A_t)\right) \nabla Q_{\mathbf{w}_t}(S_t, A_t)$$

**Off-policy TD Algorithm**

- Use a policy $b$ to obtain the interactions $S_t A_t R_{t+1} S_{t+1} A_{t+1} \ldots$
- Compute an (importance-sampling based) corrected return.
- Use it in the algorithm.

- **Can fail spectacularly!**
- Monte Carlo will work.

Discounted

# Off-Policy Divergence


## Simplest Example?

- Simple transition with a reward 0.
- TD error:

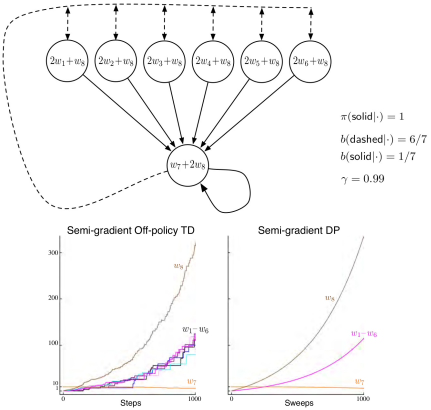
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V_{\boldsymbol{w}_t}(S_{t+1}) - V_{\boldsymbol{w}_t}(S_t)\\ &= 0 + \gamma 2\boldsymbol{w}_t - \boldsymbol{w}_t = (2\gamma - 1)\boldsymbol{w}_t\end{aligned}$$

- Off-policy semi-gradient TD(0) update:

$$\begin{aligned}\boldsymbol{w}_{t+1} &= \boldsymbol{w}_t + \alpha_t \rho_t \delta_t \nabla V(S_{t+1}, \boldsymbol{w}_t)\\ &= \boldsymbol{w}_t + \alpha_t \times 1 \times (2\gamma - 1)\boldsymbol{w}_t = (1 + \alpha_t(2\gamma - 1))\boldsymbol{w}_t\end{aligned}$$

- Explosion if this transition is explored without $\boldsymbol{w}$ being update on other transitions as soon as $\gamma > 1/2$.

# Off-Policy Divergence




## Baird's Counterexample

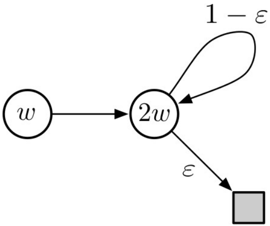
- Divergence of off-policy algorithm even without sampling, i.e. in Dynamic Programming.

Discounted

Source: Sutton and Barto

# Off-Policy Divergence






## Tsistiklis and Van Roy's Counterexample

- Exact minimization of bootstrapped $\overline{VE}$ at each step:

$$\begin{aligned}
\boldsymbol{w}_{t+1} &= \operatorname*{argmin}_{\boldsymbol{w}} \sum_{s} \left(V_{\boldsymbol{w}_t}(s) - \mathbb{E}_\pi[R_{t+1} + \gamma V_{\boldsymbol{w}_t}(S_{t+1}) | S_t = s]\right)^2 \\
&= \operatorname*{argmin}_{\boldsymbol{w}} (\boldsymbol{w} - \gamma 2\boldsymbol{w}_t)^2 + (2\boldsymbol{w} - (1-\epsilon)\gamma 2\boldsymbol{w}_t)^2 \\
&= \frac{6-4\epsilon}{5}\gamma \boldsymbol{w}_t
\end{aligned}$$

- Divergence if $\gamma > 5/(6-4\epsilon)$.

# Linear Parametrization and TD

Iteration: $\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t(R_{t+1} + \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, a)^\top \boldsymbol{w}_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t)\Phi(S_t, A_t)$

Lim. eq$\mathbb{E}_b[r(S_T, A_t)\Phi(S_t, A_t)] = \mathbb{E}_b\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, a)^\top\right)\right]\boldsymbol{w}_\infty$

ODE: $\frac{d\boldsymbol{w}}{dt} = -\mathbb{E}_b\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, q^\top\right)\right](\boldsymbol{w} - \boldsymbol{w}_\infty)$

## Linear Parametrization and TD

- Convergence of ODE if
  $$\mathbb{E}_b\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, q^\top\right)\right] = \Phi\Xi(I - \gamma P^\pi)\Phi^\top$$
  (with $\Phi = (\Phi(s, a))$, $\Xi = \mathrm{diag}(\mu(s, a)))$ and $P\pi$ the transition matrix associated to $\pi$) has complex eigenvalues with positive real parts...
- Proof for on-policy relies on $\mu = \mu_\pi$ which satisfies $\mu_\pi{}^\top P_\pi = \mu_\pi{}^\top$.
- Not true anymore with an arbitrary behavior policy!

Discounted

# Deadly Triad

## Deadly Triad

- **Function approximation**
- **Bootstrapping**
- **Off-policy training**

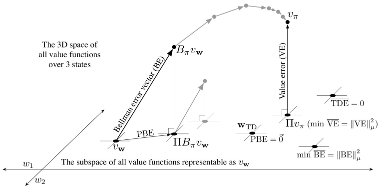
- **Instabilities as soon as the three are present!**

## Issue

- Function approximation is unavoidable.
- Bootstrap is much more computational and data efficient.
- Off-policy may be avoided... but essential when dealing with extended setting (learn from others or learn several tasks)

- Dead End?

# Objective?




## Linear Parametrization Target?

- Prediction objective $\overline{VE}$:

$$\|q_\pi - Q_{\mathbf{w}}\|_\mu^2$$

- Bellman Error $\overline{BE}$:

$$\|\mathcal{T}^\pi Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_\mu^2$$

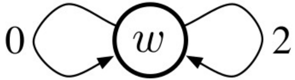
- Projected Bellman Error $\overline{PBE}$:

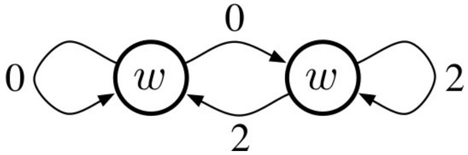
$$\|\operatorname{Proj} \mathcal{T}^\pi Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_\mu^2$$

with $\operatorname{Proj} = \Phi(\Phi^\top \Xi \Phi)\Phi\,(\Phi)\,\Xi$.

# Prediction Objective





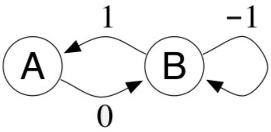

## Prediction Objective

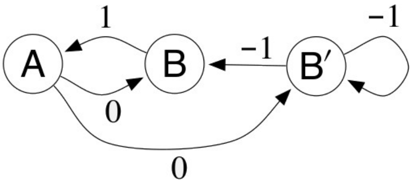
- Two MRP with the same outputs (because of approximation).
- but different $\overline{VE}$.
- Impossibility to learn $\overline{VE}$.
- Minimizer however is learnable:

$$\begin{aligned}\overline{RE}(\boldsymbol{w}) &= \mathbb{E}\left[(G_t - V_{\boldsymbol{w}_t}(S_t))^2\right] \\ &= \overline{VE}(\boldsymbol{w}) + \mathbb{E}\left[(G_t - v_\pi(S_t))^2\right]\end{aligned}$$

- MC method target.

# Bellman Error



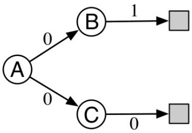




## Bellman Error

- Two MRP with the same outputs (because of approximation).
- Different $\overline{BE}$.
- Different minimizer!
- $\overline{BE}$ is not learnable!

# TD Error

$$\overline{TDE}(\mathbf{w}) = \|\mathbb{E}_\pi\left[\delta_t^2 | S_t, A_t\right]\|_\mu$$




## Mean-Squares TD Error

- $\overline{TDE}(\mathbf{w}) = \mathbb{E}_b\left[\rho_t \delta^2\right]$
- Gradient: $\nabla \overline{TDE}(\mathbf{w}) =$
  $\mathbb{E}_b[\rho_t\left(R_t + \gamma Q_{\mathbf{w}}(S_{t+1}, A_{t+1})\right) - Q_{\mathbf{w}_t}(S_t, A_t))\left(\gamma \nabla Q_{\mathbf{w}_t}(S_{t+1}, A_{t+1}) - \nabla Q_{\mathbf{w}_t}(S_t, A_t)\right)]$
- SGD algorithm...
- but solutions often converge to not to a *desirable place* even without approximation!

# Projected Bellman Error

$$\| \operatorname{Proj} \mathcal{T}^{\pi} Q_{\boldsymbol{w}} - Q_{\boldsymbol{w}} \|_{\mu}^{2} \quad \text{with } \operatorname{Proj} = \Phi(\Phi^{\top}\Xi\Phi)^{-1}\Phi^{\top}\Xi.$$

## Projected Bellman Error

- Rewriting

$$\begin{aligned}\overline{PBE}(\boldsymbol{w}) &= \| \operatorname{Proj} \mathcal{T}^{\pi} q_{\boldsymbol{w}} - q_{\boldsymbol{w}} \|_{\mu}^{2} = \| \operatorname{Proj} \delta_{\boldsymbol{w}} \|_{\mu}^{2} \\ &= (\operatorname{Proj} \delta_{\boldsymbol{w}})^{\top} \Xi (\operatorname{Proj} \delta_{\boldsymbol{w}}) = (\Phi^{\top}\Xi\delta_{\boldsymbol{w}})^{\top} \left(\Phi^{\top}\Xi\Phi\right)^{-1} (\Phi^{\top}\Xi\delta_{\boldsymbol{w}})\end{aligned}$$

- Gradient:

$$\nabla \overline{PBE}(\boldsymbol{w}) = 2\nabla(\Phi^{\top}\Xi\delta_{\boldsymbol{w}})^{\top} \left(\Phi^{\top}\Xi\Phi\right)^{-1} (\Phi^{\top}\Xi\delta_{\boldsymbol{w}})$$

- Expectations:

$$\begin{aligned}\Phi^{\top}\Xi\delta_{\boldsymbol{w}} &= \mathbb{E}_b[\rho_t \delta_t \Phi(S_t, A_t)] \\ \nabla(\Phi^{\top}\Xi\delta_{\boldsymbol{w}})^{\top} &= \mathbb{E}_b\left[\rho_t(\gamma\Phi(S_{t+1}, A_{t+1}) - \Phi(S_t, A_t))\Phi(S_t, A_t)^{\top}\right] \\ \Phi^{\top}\Xi\Phi &= \mathbb{E}_b\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^{\top}\right]\end{aligned}$$

- Not yet a SGD/SA as the gradient is a product of several terms...

Discounted

# Projected Bellman Error

## Gradient and Stochastic Approximation

- Gradient:

$$\nabla \overline{PBE}(\boldsymbol{w}) = 2\mathbb{E}_b\left[\rho_t(\gamma\Phi(S_{t+1},A_{t+1}) - \Phi(S_t,A_t))\Phi(S_t,A_t)^\top\right]$$
$$\left(\mathbb{E}_b\left[\Phi(S_t,A_t)\Phi(S_t,A_t)^\top\right]\right)^{-1}\mathbb{E}_b[\rho_t\delta_t\Phi(S_t,A_t)]$$

- Least-squares inside:

$$v = \left(\mathbb{E}_b\left[\Phi(S_t,A_t)\Phi(S_t,A_t)^\top\right]\right)^{-1}\mathbb{E}_b\left[\rho_t\delta_t\Phi(S_t,A_t)^\top\right]$$
$$\Leftrightarrow v = \operatorname*{argmin}_v \mathbb{E}_b\left[\left(\Phi(S_t,A_t)^\top v_t - \rho_t\delta_t\right)^2\right]$$

  which can be estimated by

$$v_{t+1} = v_t + \beta_t\Phi(S_t,A_t)(\delta_t - \rho_t\Phi(S_t,A_t)^\top v_t)$$

- Plugin pseudo gradient (SA):

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - 2\alpha_t\rho_t(\gamma\Phi(S_{t+1},A_{t+1}) - \Phi(S_t,A_t))\Phi(S_t,A_t)^\top v_t$$

- Same target than Pseudo Gradient but converging algorithm provided $\alpha_t \ll \beta_t$.

# Gradient TD Algorithm

## GTD

- Simultaneous update:

$$v_{t+1} = v_t + \beta_t \Phi(S_t, A_t)(\delta_t - \rho_t \Phi(S_t, A_t)^\top v_t)$$

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - 2\alpha_t \rho_t (\gamma \Phi(S_{t+1}, A_{t+1}) - \Phi(S_t, A_t))\Phi(S_t, A_t)^\top v_t$$

- As $\alpha_t \ll \beta_t$, $\boldsymbol{w}$ is seen as constant by $v$...

## TDC

- Simultaneous update:

$$v_{t+1} = v_t + \beta_t \Phi(S_t, A_t)(\delta_t - \rho_t \Phi(S_t, A_t)^\top v_t)$$

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - 2\alpha_t \rho_t (\delta_t \Phi(S_t, A_t) - \gamma \Phi(S_{t+1}, A_{t+1}))\Phi(S_t, A_t)^\top v_t$$

- Obtained by a similar derivation but faster in practice...
- As $\alpha_t \ll \beta_t$, $\boldsymbol{w}$ is seen as constant by $v$...

- Restricted to the linear setting but interesting insight.

Discounted

# Outline

# Stochastic Approximation

$$\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$$
$$\implies \theta_k \to \{\theta, H(\theta) = 0\}$$

## Stochastic Approximation

- Family of sequential stochastic algorithm converging to a zero of a function.
- Classical assumptions:
  - $\mathbb{E}[\epsilon_k] = 0$, $\mathbb{V}\text{ar}\,[\epsilon_k] < \sigma^2$, and $\mathbb{E}[\|\eta_k\|] \to 0$,
  - $\sum_k \alpha_k \to \infty$ and $\sum_k \alpha_k^2 < \infty$,
  - the algorithm converges if we replace $h_k$ by $H$.

- Convergence toward a neighborhood if $\alpha$ is kept constant (as often in practice).
- Most famous example are probably Robbins-Monro and SGD.
- Proof quite technical in general.
- The convergence with $H$ is easy to obtain for a contraction.

# Stochastic Approximation and ODE



From $\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k)$ with $h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$

to $\dfrac{d\tilde{\theta}}{dt} = H(\tilde{\theta})$

## ODE Approach

- General proof showing that the algorithm converges provided the ODE converges.
- Rely on the rewriting the equation
$$\frac{\theta_{k+1} - \theta_k}{\alpha_k} = h_k(\theta_k) = H(\theta_k) + \epsilon_k + \eta_k$$
- $\alpha_k$ can be interpreted as a time difference allowing to define a time $t_k = \sum_{t' \leq t} \alpha_k$.
- $\theta(t)$ is piecewise affine and defined through its derivative at time $t \in (t_k, t_{k+1})$.
- This piecewise function remains close to any solution of the ODE starting from $\theta_k$ for an arbitrary amount of time provided $k$ is large enough.

- More general proofs based on martingale.

# Stochastic Approximation

$$\begin{cases}\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k, \nu_k) \\ \nu_{k+1} = \nu_k + \beta_k g_k(\theta_k, \nu_k)\end{cases} \quad \text{with} \quad \begin{cases} h_k(\theta, \nu) = H(\theta, \nu) + \epsilon_k + \eta_k \\ g_k(\theta, \nu) = G(\theta, \nu) + \epsilon'_k + \eta'_k\end{cases}$$

$$\Longrightarrow \theta_k \to \{\theta, H(\theta, \nu(\theta)) = 0, \nu(\theta) \in \{\nu, G(\theta, \nu) = 0\}\}$$

## Stochastic Approximation

- Family of sequential stochastic algorithm converging to a zero of a function.
- Classical assumptions:
  - $\mathbb{E}[\epsilon_k] = 0$, $\mathbb{V}\mathrm{ar}\,[\epsilon_k] < \sigma^2$, and $\mathbb{E}[\|\eta_k\|] \to 0$,
  - $\sum_k \alpha_k \to \infty$ and $\sum_k \alpha_k^2 < \infty$,
  - $\sum_k \beta_k \to \infty$ and $\sum_k \beta_k^2 < \infty$,
  - $\alpha_k / \beta_k \to 0$ (two-scales assumption),
  - the algorithm converges if we replace $h_k$ and $g_k$ by $H$ and $G$.

- Convergence toward a neighborhood if $\alpha \ll \beta$ are kept constant (as often in practice).

# Stochastic Approximation and ODE



From $\begin{cases} \theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k, \nu_k) \\ \nu_{k+1} = \nu_k + \beta_k + g_k(\theta_k, \nu_k) \end{cases}$ with $\begin{cases} h_k(\theta, \nu) = H(\theta, \nu) + \epsilon_k + \eta_k \\ g_k(\theta, \nu) = G(\theta, \nu) + \epsilon'_k + \eta'_k \end{cases}$

to $\dfrac{d\tilde{\theta}}{dt} = H(\tilde{\theta}, \tilde{\nu}(\tilde{\theta}))$ with $\tilde{\nu}(\theta)$ the limit of $\dfrac{d\tilde{\nu}}{dt} = G(\theta, \tilde{\nu})$

## ODE Approach

- General proof showing that the algorithm converges provided the two ODE converge.
- Quite generic setting and source of new algorithm or insight on existing ones.
- Importance of having two scales...

- Can be used to prove the convergence of GTD and TDC!

# Outline

# Simplified Deep $Q$-Learning

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \beta_t(R_{t+1} + \gamma \max_a Q_{\nu_t}(S_{t+1}, a) - Q_{\mathbf{w}_t}(S_t, A_t))\nabla Q_{\mathbf{w}_t}(S_t, A_t)$$

$$\nu_t = \mathbf{w}_{\lceil t/T \rceil T}$$

## Simplified Deep $Q$-Learning

- Stochastic Approximation for a fixed $\nu$:
  - Limiting equation:
    $$\mathbb{E}_b[(\mathcal{T}^\star Q_\nu(S_t, A_t) - Q_{\mathbf{w}_\infty}(S_t, A_t))\nabla Q_{\mathbf{w}_\infty}(S_t, A_t)] = 0$$
  - Stochastic Gradient Descent of
    $$\mathbb{E}_b\left[(\mathcal{T}^\star Q_\nu(S_t, A_t) - Q_{\mathbf{w}}(S_t, A_t))^2\right]$$
  - $Q_{\mathbf{w}} \to \mathcal{T}^\star Q_\nu$
- Approximate Value Iteration Scheme!

- Two-scales algorithm flavour as $\nu$ is kept constant.
- Explicit two scales with $\nu_{t+1} = \nu_t + \alpha_t(\mathbf{w}_t - \nu_t)$ variation.
- Could be used for prediction with $R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q_{\nu_t}(S_{t+1}, a)$

# Deep $Q$-Learning



$$\mathbf{w}_{t+1} = \mathbf{w}_t + \beta_t(R_t + \gamma \max_a Q_{\nu_t}(S_{t+1}, a) - Q_{\mathbf{w}}(S_t, A_t))\nabla Q_{\mathbf{w}}(S_t, A_t)$$

$$\nu_t = \mathbf{w}_{\lceil t/T \rceil T}$$

- **Who are $S_t, A_t, R_{t+1}, S_{t+1}$?** and thus to what corresponds $\mathbb{E}_b$?

## Simplified Deep $Q$-Learning

- Use a behaviour policy $b$.
- The current greedy plus exploration $Q$-policy can be used.

## Neural Fitted-$Q$

- Instead of a policy $b$, use a fix dataset $\mathcal{D}$ of $S_t, A_t, R_{t+1}, S_{t+1}$.
- Several pass on the data can be made.

## Deep $Q$-Learning

- Use the current greedy plus exploration $Q$-policy to populate a FIFO buffer $\mathcal{D}$.
- Use random samples of the buffer $\mathcal{D}_t$ (more than one per interaction is OK).

# Deep $Q$-Learning



$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \beta_t (R_t + \gamma \max_a Q_{\nu_t}(S_{t+1}, a) - Q_{\boldsymbol{w}}(S_t, A_t)) \nabla Q_{\boldsymbol{w}}(S_t, A_t)$$

$$\nu_t = \boldsymbol{w}_{\lceil t/T \rceil T}$$

Plus tricks

## Deep $Q$-Learning Tricks

- Replay buffer
- Double $Q$-Learning
- Better Exploration
- Advanced Return and Distributional
- Network Architecture

- Rainbow paper...

# Replay Buffer



## Replay Buffer

- Replace an expectation over real trajectories by an empirical average over past (short) sub-trajectories stored in a replay buffer.
- The empirical average corresponds to uniform sampling.
- If the policy is changing across time, we should use a importance sampling correction to be faithful with the theory. . .
- Not necessary for one-step $Q$ learning but required for most of the other methods where replay buffer is used.
- Often no correction in practice if the policies used in the buffer are closed to the current one.
- Prioritized sweeping variant possible. . .

- Buffer can be constructed in parallel of the learning part.
- Only requires to transmit the *current* greedy plus exploration $Q$-policy.

# Double $Q$-Learning



## $Q$-Learning and overestimation

- Target: $R_{s,a} + \gamma \max_{a'} Q_{\mathbf{w}}(s', a')$
- Approximation issue: $Q_{\mathbf{w}}(s', a') \sim Q(s, a) + \epsilon(s, a)$
- Consequence: $\mathbb{E}[\max_a Q_{\mathbf{w}}(S_t, a)] \geq \max\left(Q(s, a) + \mathbb{E}[\epsilon(s, a)]\right)$

## Double $Q$-Learning with two $Q$ functions: $Q_{\mathbf{w}_1}$ and $Q_{\mathbf{w}_2}$

- Used in a crossed way for the target of $Q_{\mathbf{w}_i}$:
$$R_{s,a} + \gamma Q_{\mathbf{w}_{i'}}(s', \operatorname*{argmax}_{a'} Q_{\mathbf{w}_i}(s', a'))$$
- Mitigates the bias.

## Clipped $Q$-Learning with several $Q$ functions: $Q_{\mathbf{w}_i}$

- Used in a pessimistic way for the target of $Q_{\mathbf{w}_i}$:
$$R_{s,a} + \gamma \min_{i'} Q_{\mathbf{w}_{i'}}(s', \operatorname*{argmax}_{a'} Q_{\mathbf{w}_i}(s', a'))$$
- Seems even more efficient.

# Outline

# Continuous Action



- Case (almost) not yet covered in the lectures.
- Most complex theoretical extension.

## Prediction

- No algorithmic issue if one can sample $\pi$.
- Off-policy can be considered under a domination assumption.

## Planning

- Main issue is the argmax of the greedy policy (or the sampling of Gibbs policy).
- May be impossible to compute.
- Possible if the parametrization of $Q$ with respect to $a$ is simple (e.g. explicit quadratic dependency in $a$).
- An alternative could be to approximate the argmax operator, or to learn how to approximate the argmax directly, which is very close to approximating directly the policy itself...

# Outline

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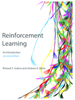


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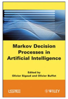


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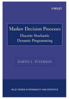


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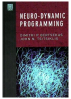


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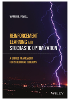


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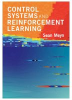


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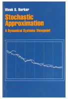


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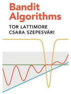


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