Reinforcement Learning
Reinforcement Learning: Policy Approach

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M2DS - Reinforcement Learning – Fall 2023
RL: What Are We Going To See?

Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View
How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interactive methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.
Reinforcement Learning and Interactions

How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA...)
- Policy deduced by a statewise optimization of the value function over the actions.
More Tabular Reinforcement Learning

Can We Do Better?

- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).
Reinforcement and Approximation of Value Functions

How to Deal with a Large/Infinite states/action space?

- How to approximate value functions?
- How to estimate good approximation of value functions?

- Finite action space setting.
- Stochastic algorithm (Deep Q Learning...).
- Policy deduced by a statewise optimization of the value function over the actions.
Could We Directly Parameterized the Policy?

- How to parameterize a policy?
- How to optimize this policy?
- Can we combine parametric policy and approximated value function?

- State Of The Art Algorithms (DPG, PPO, SAC...)

Actor/Critic: a Policy Point of View
Outline

1. Policy Gradient Theorems
2. Monte Carlo Based Policy Gradient
3. Actor / Critic Principle
4. 3 SOTA Algorithms
5. References
Policy Point of View

Optimize policy directly instead of deriving it from a value function.
Avoid the argmax operator.
Most natural POV?

Pontryagin vs Hamilton-Jacobi(-Bellman) in control!
Outline

1. Policy Gradient Theorems
2. Monte Carlo Based Policy Gradient
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**Policy and Goal**

\[ J_\mu(\pi) = \sum_s \mu(s) v_\pi(s) \]

**Goal:** average expected return over the states

- Target used to define the linear programming formulation of an optimal policy in the tabular setting.
- \( \mu \) can be the initial distribution of the states (independent of \( \pi \))...
- but may also depends on \( \pi \) (for instance the associated stationary measure)
- Other choices will appear.

- Goal: optimize \( J_\mu(\pi) \) in \( \pi \)!
Parametric Policy

\[ \pi_\theta(a|s) = \begin{cases} \frac{e^{h_\theta(a,s)}}{\sum_{a'} e^{h_\theta(a',s')}} & \text{(softmax)} \\ P_{h_\theta(s)}(a) & \text{(parametric conditional model)} \\ 1_{a=h_\theta(s)} & \text{(deterministic)} \end{cases} \]

Parametric Policy

- Restriction of the set of policy to a parametrized one.
- Most classical parametrizations:
  - Soft-max with a preference function \( h_\theta(a, s) \),
  - Parametric conditional model with parameter \( h_\theta(s) \)
- To be useful need to be able to sample the distribution.
- \( h_\theta \): from linear model to deep learning...
- Most of our result will assume that \( \pi_\theta(a|s) \) is differentiable with respect to \( \theta \).
- Deterministic policies will be considered with a different analysis.
Episodic Setting: Gradient of Expected Returns

\[ v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta} [G_0|S_0 = s] \]

\[ \nabla_\theta v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \bigg| S_0 = s \right] \]

Expected Returns

- Rely on \( v_{\pi_\theta}(s) = \sum_\tau \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) G_0(\tau) \) and

\[ \nabla \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) = \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \nabla \log \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \]

\[ = \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \sum_t \left( \nabla \log \pi_\theta(A_t|S_t) + \nabla p(R_{t+1}, S_{t+1}|S_t, A_t) \right) \]

\[ = \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \sum_t \nabla \log \pi_\theta(A_t|S_t) \]

- In an episodic setting, any trajectory \( \tau \) ends at a finite time \( T_\tau \).
Episodic Setting: Policy Gradient Theorem

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s) v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \right]$$

**Policy Gradient Theorem**

- **Natural \( \mu \):** initial state distribution.
- **Gradient is an expectation:** MC type algorithm.
- Can be interpreted as the gradient of the maximum likelihood of the actions weighted by the return.
- Favors good actions over bad ones.
Baseline and Variance Reduction

\[
J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s) \nu_{\pi_\theta}(s)
\]

\[
\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \right]
\]

Variance Reduction and Baseline

- The previous formulae are valid if one replace \( G_0 \) by any function of \( \tau \).
- For any constant \( b \), this leads to

\[
\nabla \mathbb{E}_{\pi_\theta} [b] = 0 = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) b \right]
\]

- Optimal value for

\[
b = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right)^2 G_0 \right] / \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right)^2 \right]
\]

- Most used value \( b = \mathbb{E}_{\pi_\theta} [G_0] \).
Gradient(s) of Expected Return

\[ \nu_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}[\sum \gamma^t R_t | S_0 = s] \]

\[ \nabla \nu_{\pi_\theta}(s) = \sum_t \gamma^t \mathbb{E}_{\pi_\theta}\left[ \left( \sum_{t'=0}^{t-1} \nabla \log \pi_\theta(A_{t'}|S_{t'}) \right) R_t | S_0 = s \right] \]

\[ = \sum_{t'} \mathbb{E}_{\pi_\theta}\left[ \nabla \log \pi_\theta(A_{t'}|S_{t'}) \left( \sum_{t \geq t'} \gamma^t R_t \right) | S_0 = s \right] \]

\[ = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}\left[ \nabla \log \pi_\theta(A_{t'}|S_{t'}) q_{\pi_\theta}(S_{t'}, A_{t'}) | S_0 = s \right] \]

\[ = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}\left[ \nabla \log \pi_\theta(A_{t'}|S_{t'}) \left( q_{\pi_\theta}(S_{t'}, A_{t'}) - \nu_{\pi_\theta}(S_{t'}) \right) | S_0 = s \right] \]

**Expected Returns**

- Several formulas of stochastic gradients!
More Gradient(s)

\[ \nabla v_{\pi_\theta}(s) = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta} [\nabla \log \pi_\theta(A_{t'}|S_{t'}) q_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s] \]

\[ = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta} [\nabla \log \pi_\theta(A_{t'}|S_{t'}) a_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s] \]

\[ = \sum_s \left( \sum_t \gamma^t P_{\pi_\theta}(S_t = s|S_0 = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a) \right) \]

\[ = \sum_s \left( \sum_t \gamma^t P_{\pi_\theta}(S_t = s|S_0 = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) a_{\pi_\theta}(s, a) \right) \]

Focus on states
- Even more stochastic gradients!
Policy Gradient(s)

\[ J(\mu_0)(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a) \right) \]

\[ = \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s)(q_{\pi_\theta}(s, a) - v_{\pi_\theta}(s, a)) \right) \]

Discounted Setting

- Average (discounted) return from the beginning.
- Focus on early steps in discounted setting...
Policy Improvement Lemma

\[ J_{\mu_0}(\pi') - J_{\mu_0}(\pi) = \sum_t \gamma^t P_{\pi'}(S_t = s) \left( \sum_a (\pi'(a|s) - \pi(a|s)) q_\pi(s, a) \right) \]

\[ = \sum_t \gamma^t P_{\pi'}(S_t = s) \left( \sum_a (\pi'(a|s) - \pi(a|s)) a_\pi(s, a) \right) \]

**Proof**

- By construction, if \( S_t \) is a trajectory using policy \( \pi' \):
  \[ v_{\pi'}(S_t) - v_\pi(S_t) = \sum_a (\pi'(a|S_t) - \pi(a|S_t)) q_\pi(S_t, a) + \sum_a \pi'(a|S_t) (q_{\pi'}(S_t, a) - q_\pi(S_t, a)) \]
  \[ = \sum_a (\pi'(a|S_t) - \pi(a|S_t)) v_\pi(S_t, a) + E_{\pi'}[V_{\pi'}(S_{t+1}) - v_{\pi'}(S_{t+1})|S_t] \]

- Discounted setting shortcut
  \[ v_{\pi'} - v_\pi = r_{\pi'} + \gamma P^\pi v_{\pi'} - r_\pi - \gamma P^\pi v_\pi = r_{\pi'} - r_\pi + \gamma \left( P^\pi - P^\pi \right) v_\pi + \gamma P^\pi (v_{\pi'} - v_\pi) \]
  \[ v_{\pi'} - v_\pi = (I - \gamma P^\pi) v_\pi \]

Episodic / Discounted
Approximate Policy Improvement Lemma

\[ \left| J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_t \gamma^t P_{\pi}(S_t = s) \left( \sum_a (\pi'(a|s) - \pi(a|s)) a_{\pi}(s, a) \right) \right| \]

\[ = \left| \sum_t \gamma^t (P_{\pi'}(S_t = s) - P_{\pi}(S_t = s)) \left( \sum_a (\pi'(a|s) - \pi(a|s)) a_{\pi}(s, a) \right) \right| \]

\[ \leq \frac{2\gamma}{(1 - \gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_{\pi}(s, a)| \]

Approximate Policy Improvement Lemma

- If \( \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1 \leq \epsilon \)

\[ P_{\pi'}(S_t = s) = (1 - \epsilon)^t P_{\pi}(S_t = s) + (1 - (1 - \epsilon)^t) P_{\text{mistake}}(S_t = s) \]

\[ \rightarrow |P_{\pi'}(S_t = s) - P_{\pi}(S_t = s)| \leq 2(1 - (1 - \epsilon)^t) \leq 2\epsilon t \]

- \[ \sum_t 2\gamma^t t = \frac{2\gamma}{(1-\gamma)^2} \]
Approximate Policy Improvement Lemma

\[ |J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left( \sum_a (\pi'(a|s) - \pi(a|s)) a_\pi(s, a) \right)| \]

\[ \leq \frac{2\gamma}{(1 - \gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s, a} |a_\pi(s, a)| \]

Approximate Policy Improvement Lemma and Policy Gradient Theorem

- Let \( \pi' = \pi_{\theta+h} \) and \( \pi_\theta \)
  - \( \pi_{\theta+h}(a|s) - \pi_\theta(a|s) = \pi_\theta(a|s)\langle \nabla \log \pi_\theta(a|s), h \rangle + O(\|h\|^2) \)
  - \( \|\pi_{\theta+h}(\cdot|s) - \pi_\theta(\cdot|s)\|_1 \leq \|h\| \max_a \|\nabla \log \pi_\theta(a|s)\| + O(\|h\|^2) \)

- Implies Policy Gradient Theorem:
  \[ J_{\mu_0}(\pi_{\theta+h}) \]

  \[ = J_{\mu_0}(\pi_\theta) + \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \left( \sum_a \pi_\theta(a|s)\langle \nabla \log \pi_\theta(s, a), h \rangle a_\pi(s, a) \right) + O(\|h\|^2) \]
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Monte Carlo Approach

$G_t = \sum_{t' \geq t} R_{t+1}$

$Q_{t, \pi_\theta}(s, a) = \mathbb{E}[G_t|S_t = s, A_t = a]$
REINFORCE: Monte Carlo Based Policy Gradient

\[ J_{\mu_0}(\pi_\theta) = \sum_s P(S_0 = s) v_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \right] \]

\[ = \sum_s \left( \sum_t P_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a) \right) \]

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \quad \text{or} \quad \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t) G_t \]

REINFORCE

- Plain MC (SGD) algorithm.
- Need to wait until the end of the episodes.
- Convergence guarantees (even in off-line setting with importance sampling).
REINFORCE with Baseline

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \right] \]

\[ = \sum_s \left( \sum_t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_\pi(s, a) - b(s)) \right) \]

\[ \widehat{\nabla J}_{\mu_0}(\pi_\theta) = \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \]

or \[ \widehat{\nabla J}_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t) (G_t - b(S_t)) \]

- Several choices for \( b \)...
- and for \( b(s) \) which can be any function (a crude estimate of \( V_{t,\pi}(s) \) for instance)!
- Convergence guarantees (even in off-line setting with importance sampling).
Discounted REINFORCE?

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_{\tau}-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \right] \\
= \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s, a) - b(s)) \right) \\
\hat{\nabla} J_{\mu_0}(\pi_\theta) = \left( \sum_{t=0}^{T_{\tau}-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \\
or \quad \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t|S_t) (G_t - b(S_t)) \\
\]

Discounted REINFORCE

- Can be defined...
- but still requires an episodic setting for the discounted return \( G_t \) to be computed.
Monte Carlo Based Policy Gradient

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t | S_t) \left( G_t - b(S_t) \right) \]

\[ \rightarrow \hat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) = \frac{1}{1 - \gamma} \nabla \log \pi_\theta(A_t | S_t) \left( G_t - b(S_t) \right) \]

Discounted Measure?

- Much less weights for later states!
- Probability independent of \( t \) if the initial distribution is the stationary distribution \( \mu_{\pi_\theta} \) corresponding to \( \pi_\theta \) (it it exists).
- Approximately true after a burning stage if we reach stationarity...
- Better handled by the average return!

- More on this later...
**Actor/Critic**

- **Actor**: Parametric policy $\pi_\theta$ used.
- **Critic**: $Q$-value function $Q_w(\cdot, \cdot)$ approximating $Q_{\pi_\theta}$.
- **Critic follows the Actor**, which is optimized using the Critic.

- In Value Approximation, the Actor follows the Critic (through the argmax operator).
- In on-line methods, the Actor is used to interact with the environment.
Actor / Critic Principle

\[
J(\mu_0)(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s)
\]

\[
\nabla J_{\mu_0}(\pi_\theta) = \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) \left( q_{\pi_\theta}(s, a) - v_{\pi_\theta}(s, a) \right) \right)
\]

\[
\hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left( q_{\pi_\theta}(S_t, A_t) - \sum_a \pi(a|S_t) q_{\pi_\theta}(S_t, A_t) \right) \approx \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left( Q_w(S_t, A_t) - \sum_a \pi(a|S_t) Q_w(S_t, A_t) \right)
\]

**Actor / Critic**

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating \( q_{\pi_\theta} \).
- Requires a two-scales algorithm so that \( Q_w \) is always a good estimate of \( q_{\pi_\theta} \).
- Is this a real algorithm in a non-episodic setting?
Actor / Critic Principle

\[ J_{\mu_{\pi_\theta}}(\pi_\theta) = \sum_s \mu_{\pi_\theta}(s) \nu_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_{\pi_\theta}}(\pi_\theta) = \sum_s \frac{1}{1-\gamma} \mathbb{P}_{\pi_\theta}(S_t = s) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s)(q_{\pi_\theta}(s, a) - \nu_{\pi_\theta}(s, a)) \right) \]

\[ \hat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) \approx \frac{1}{1-\gamma} \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left( Q_w(S_t, A_t) - \sum_a \pi(a|S_t) Q_w(S_t, A_t) \right) \]

### Actor / Critic

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any \(Q\)-value methods estimating \(q_{\pi_\theta}\).
- Requires a two-scales algorithm so that \(Q_w\) is always a good estimate of \(q_{\pi_\theta}\).
- Require the existence of a stationary measure... and that this stationary measure is reached quickly.
- Much harder to do off-policy algorithm as the stationary measure is not known!
Critic in Actor/Critic

\[ Q_w \simeq q_{\pi_\theta} \]

Critic

- On-line TD learning with interaction following \( \pi_\theta \).
- Off-Policy TD learning is possible if the policy used for any action is stored.
- Approximate off-policy TD learning is possible using a replay buffer providing \( \pi_\theta \) is changing slowly.

- May lead to 3 scales algorithm (Actor/Critic Target/Critic)
- As mentioned in the previous slide, much harder to do off-line update for the actor.
Off-Line Actor

\[ J'_{\mu}(\pi) = \sum_s \mu(s) v_{\pi}(s) \]

Idea proposed in 2012.

Key lemma in the paper

\[ \nabla J'_{\mu}(\pi_\theta) \approx \sum_s \mu(s) \sum_a \pi_\theta(a|s) \nabla \pi_\theta(a|s) q_{\pi_\theta}(s, a) \]

turns out to be wrong!

Still used as a heuristic justification of many algorithms!

Explicit formula for \( \nabla J'_{\mu}(\pi_\theta) \) can be obtained but much harder to use...
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PPO: Minorize-Majorization Algorithm

\[ J_{\mu_0}(\pi') \geq J_{\mu_0}(\pi) + \sum_t \gamma^t P_{\pi}(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) a_\pi(s, a) \right) \]

\[ - \frac{2\gamma}{(1 - \gamma)^2} \max_s ||\pi'(\cdot|s) - \pi(\cdot|s)||^2 \max_{s,a} |a_\pi(s, a)| \]

Ideal Minorize-Majorization Algorithm

- At step \( k \), find \( \theta_{k+1} \) maximizing

\[ J_{\mu_0}(\pi_{\theta}|\pi_{\theta_k}) = \sum_s \sum_t \gamma^t P_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi_{\theta}(s|a) - \pi_{\theta_k}(s|a)) a_{\pi_{\theta_k}}(s, a) \right) \]

\[ - \frac{2\gamma}{(1 - \gamma)^2} \max_s ||\pi_{\theta}(\cdot|s) - \pi_{\theta_k}(\cdot|s)||^2 \max_{s,a} |a_{\pi_{\theta_k}}(s, a)| \]

- By construction, \( J_{\mu_0}(\pi_{\theta_{k+1}}) \geq J_{\mu_0}(\pi_{\theta_k}) \)

- Sample efficient algorithm as the same trajectory can be (re)used in the optimization.
\[
J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) a_{\pi_{\theta_k}}(s, a) \right) \\
- \frac{2\gamma}{(1 - \gamma)^2} \max_s ||\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)||^2_1 \max_{s,a} |a_{\pi_{\theta_k}}(s, a)|
\]

**Optimization**

- Gradient descent is possible.
- Gradient of the first term can be approximated using a critic by
  \[
  \sum_s \sum_t \gamma^t \mathbb{P}_{\pi}(S_t = s) \left( \sum_a \pi_\theta \nabla \pi_\theta(s|a) A_{\pi_{\theta_k}}(s, a) \right)
  \]
- Gradient of the second term more involved.
  
- Simpler (TRPO like) strategy: optimize
  \[
  \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) a_{\pi_{\theta_k}}(s, a) \right)
  \]
  under \(\max_s ||\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)||^2_1 \leq \epsilon\) and reduce \(\epsilon\) there is no gain.
PPO: KL Relaxation

\[ J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) a_{\pi_{\theta_k}}(s,a) \right) \]

\[- \frac{2\gamma R_{\max}}{(1 - \gamma)^2} \max_s KL(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s))\]

TRPO/PPO Optimization

- Replace the $\ell_1$ norm by a KL divergence.
- In practice, replace the max by an average and replace $\frac{2\gamma R_{\max}}{(1 - \gamma)^3}$ by parameter $\beta$ and replace the $a_{\pi_k}$ by an estimate $A_{\pi_k}$.
- PPO: Gradient descent of the relaxed goal.
- TRPO: Constrained optimization.

- Adaptive scheme to set $\beta$.
- Can be used with continuous action.
PPO: Clipped Objective

\[
\sum_{s} \sum_{t} \gamma^{t} P_{\pi_k}(S_{t} = s) \left( \sum_{a} \pi_{\theta_k}(s|a) \min \left( \frac{\pi_{\theta}(s|a)}{\pi_{\theta_k}(s,a)} a_{\pi_{\theta_k}}(s,a), \operatorname{clip}(1 - \epsilon, \frac{\pi_{\theta}(s|a)}{\pi_{\theta_k}(s,a)}, 1 + \epsilon) a_{\pi_{\theta_k}}(s,a) \right) \right)
\]

Clipped Objective

- Insight by (re)substracting \( \sum_{a} \pi_{\theta_k}(s|a) a_{\theta_k}(s,a) = 0 \):
  \[
  \sum_{a} \min \left( (\pi_{\theta}(s|a) - \pi_{\theta_k}(s,a)) a_{\pi_{\theta_k}}(s,a), \operatorname{clip}(\epsilon \pi_{\theta}(s|a) - \pi_{\theta_k}(s,a), \epsilon) a_{\pi_{\theta_k}}(s,a) \right)
  = \sum_{a} \operatorname{clip}(\epsilon \pi_{\theta_k}(s,a) - \pi_{\theta_k}(s,a), \epsilon a_{\pi_{\theta_k}}(s,a)) a_{\pi_{\theta_k}}(s,a)
  \]
  \[
  - \max \left( 0, -(\pi_{\theta}(s|a) - \pi_{\theta_k}(s,a)) a_{\pi_{\theta_k}}(s,a) - \epsilon \pi_{\theta_k}(s,a) a_{\pi_{\theta_k}}(s,a) \right)
  \]
- First term amount to replace \( \pi_{\theta} \) by a policy
  \[
  \tilde{\pi}_{\theta}(a|s) = \operatorname{clip}(\pi_{\theta_k}(a|s)(1 - \epsilon), \pi_{\theta}(a|s), \pi_{\theta_k}(a|s)(1 + \epsilon)) + \eta_{s} \pi_{\theta_k}(a|s)
  \]
  where \( \eta \) is so that \( \tilde{\pi} \) is a probability for all \( s \) and \( ||\tilde{\pi}_{\theta}(\cdot, s) - \pi_{\theta_k}(\cdot, s)||_{1} \leq \epsilon \)
- Second term: hinge loss type penalization of policy \( \pi_{\theta} \) penalizing bad actions.

- Very efficient for discrete actions.
PPO: Stationary Objective

\[
\sum_{s,t} \mathbb{P}_{\pi_k} (S_t = s) \left( \sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) a_{\pi_{\theta_k}}(s, a) \right) - \beta \max_s \text{KL}(\pi_{\theta_k}(.|s), \pi_\theta(.|s))
\]

\[
\sum_{s,t} \mathbb{P}_{\pi_k} (S_t = s) \left( \sum_a \pi_{\theta_k}(s|a) \min \left( \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)} a_{\pi_{\theta_k}}(s, a), \text{clip}(1 - \epsilon, \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)}, 1 + \epsilon) a_{\pi_{\theta_k}}(s, a) \right) \right)
\]

Stationary Objective

- Amount to replace \( J_{\mu_0}(\pi) \) by \( J_{\mu_\pi}(\pi) \)
- Most common implementation of PPO...
- But no way to justify it mathematically!
- May explain the (possible) instabilities of PPO.
DPG: Deterministic Policy Gradient

\[ J_{\mu_0}(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \quad \text{with deterministic policy} \quad \pi_\theta(a|s) = 1_{a=h_\theta(s)} \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \sum_s \sum_t \gamma^t P_{\pi_\theta}(S_t = s) \nabla_a q(S_t, h_\theta(S_t)) \nabla h_\theta(S_t) \]

**Deterministic Policy Gradient**

- Deterministic policy replaced by a randomized one centered on \( h_\theta(s) \) in the interactions!
- Critic trained with a TD variant of DQN.
- Same formula by using a policy \( \pi_\theta = N(h_\theta(s), \sigma^2 \text{Id}) \) and letting \( \sigma \) goes to 0.
- Off-Policy as claimed?
- Yes for the actor but no theoretical justification for the critic!
- In practice, the buffer contains only samples using a policy close to the current one...
SAC: A New Goal

\[ R_t \rightarrow R_t + \lambda \mathcal{H}(\pi(S_t)) \]

**A Modified Reward**

- Modification of the reward to favor high entropy policy:
  \[ R_t \rightarrow R_t + \lambda \mathcal{H}(\pi(S_t)) \]

- Goal:
  \[ J(\pi) = \mathbb{E}_\pi \left[ \sum_t \gamma^t (R_t + \lambda \mathcal{H}(\pi(S_t))) \right] \]

- Soft value function implicitly defined as the fixed point of
  \[ T^\pi q_\pi(s, a) = r_\pi(s, a) + \gamma \sum_{s'} p(s'|s, a) v_\pi(s') \]

  where \[ v_\pi(s, a) = \sum_a \pi(a|s) (q_\pi(s, a) - \log \pi(a|s)) \]
SAC: Policy Improvement and Optimal Policy

$$R_t \rightarrow R_t + \lambda \mathcal{H}(\pi(S_t))$$

A Modified Policy Improvement Lemma

- Policy improvement rule:

$$\pi^+ (\cdot | s) = \arg \max_{\pi \cdot | s} \sum_a \pi(a | s) (q(s, a) - \lambda \log(\pi(a | s)))$$

$$\pi^+(a | s) \propto \exp(-\frac{1}{\lambda} q(s, a))$$

implies $$G_{\pi^+}(s, a) \geq G_\pi(s, a)$$.

- At convergence, $$J(\pi^*)$$ is optimal!

- Convergence in the finite setting.

Discounted
SAC: Parametrization

\[ \pi \sim \pi_\theta \quad \text{and} \quad q(s, a) \sim Q_w \]

**SAC Choices**

- **Fitted TD learning for** \( Q \):
  \[ w \simeq \arg\min \sum_{(S,A,R,S') \in B} \left( R + \mathbb{E}_{\pi_\theta} \left[ \gamma Q_{\overline{w}}(S', a) - \lambda \log \pi_\theta(a|S') \right] - Q_w(S, A) \right)^2 \]
  where the trajectory pieces are samples from a replay buffer and \( \overline{w} \) is a slowdown version of \( w \) (two-scales algorithm).

- **Online version rather than batch...**

- **Fitted KL for** \( \pi \):
  \[ \theta \simeq \arg\min \sum_{(S,A,R,S') \in B} \text{KL}(\pi_\theta(\cdot|S)| \exp -\lambda Q_{\overline{w}}(S, \cdot)/Z_{\overline{w}}(S)) \]
  \[ \simeq \sum_{(S,A,R,S') \in B} \mathbb{E}_{\pi_\theta} \left[ \frac{1}{\lambda} \log \pi_\theta(a|S) - Q_\theta(a|s) \right] \]
Outline

1. Policy Gradient Theorems
2. Monte Carlo Based Policy Gradient
3. Actor / Critic Principle
4. 3 SOTA Algorithms
5. References

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