Reinforcement Learning
Reinforcement Learning: Advanced Techniques in the Tabular Setting

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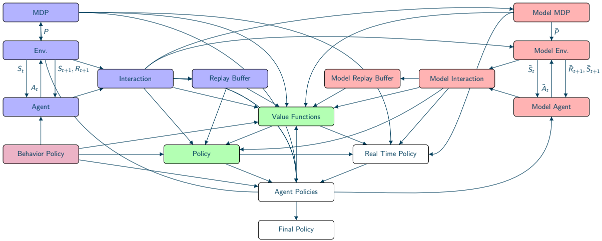
M2 DS - Fall 2022

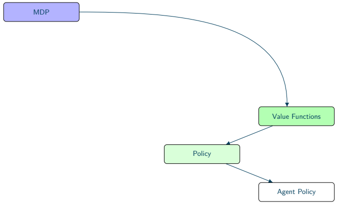
RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View

Operations Research and MDP

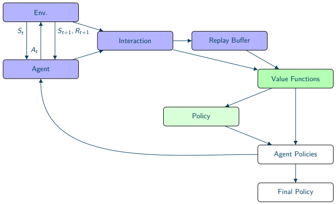


How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Most results for the discounted setting.

Reinforcement Learning and Interactions

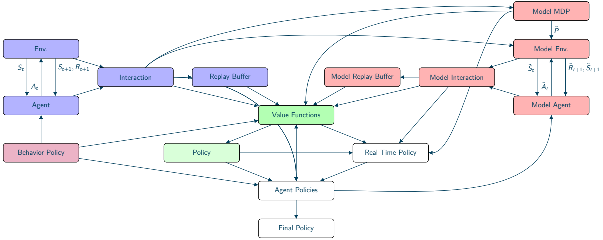


How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA. . .)
- Policy deduced by a statewise optimization of the value function over the actions.

More Tabular Reinforcement Learning



Can We Do Better?

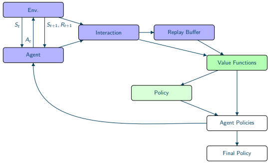
- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).

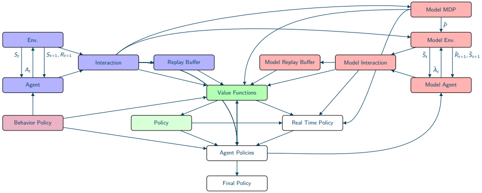
Outline

Advanced Tabular Reinforcemcent Learning

From



to



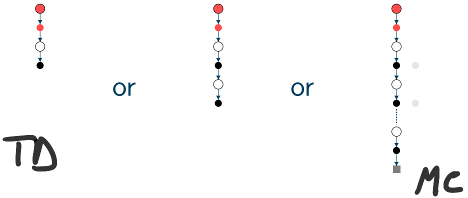
- Core idea: Approximate Bellman Operators with Stochastic Approximation...

Advanced Ideas?

- Between MC and TD?
- Off-policy vs on-policy?
- Exploration vs Exploitation?
- Model? Replay?
- Real Time Planning?

Outline

n-steps



How many steps before backup?

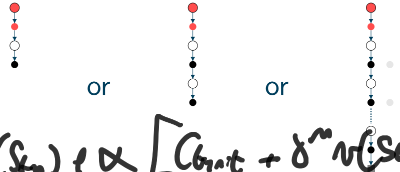
- One step: TD.
- As many steps as required to end the episod: MC.
- n-steps: n-steps TD.

$$\left(\mathcal{T}^{\Pi}\right)^{n} v(s) = \mathbb{E}_{\Pi}\left[\underbrace{R_{t+1} + \gamma R_{t+2} + \gamma^{n-1} R_{t+n} + \gamma^{n} v(S_{t+n})}_{G_{t:t+n}} \middle| S_t = s\right]$$

- Family of stochastic approximation algorithms:

$$v(S_t) \leftarrow v(S_t) + \alpha(N(S_t))\left(G_{t:t+n} - v(S_t)\right)$$

n-steps TD



$$v(S_{t-n}) \leftarrow v(S_{t-n}) + \alpha \left[G_{t-n:t} + \gamma^n v(S_t) - v(S_{t-n}) \right]$$

$$v(S_t) \leftarrow v(S_t) + \alpha(N(S_t))\left(G_{t:t+n} + \gamma^n \cancel{v}(S_{t+n}) - v(S_t)\right)$$

n-steps TD

- Convergence for prediction.
- Need to be combined with Policy Improvement for planning: n-steps SARSA.
- n-steps Q-learning could be an extension of API… but this means following the optimized policy Π… i.e. SARSA!

- Best convergence often for intermediate n.
- No proof beside TD for $n > 1$!

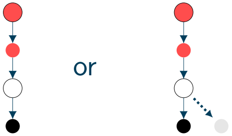
n-steps TD

Discounted: Prediction by *n*-steps TD

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, \tilde{Q}(s, a), N(s, a) = 0$, n=0, $t' = 0$
repeat
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $\tilde{Q}(S_{t-n}, A_{t-n}) \leftarrow \tilde{Q}(S_{t-n}, A_{t-n}) + \alpha(N(S_t, A_t)) \left(G_{t-n:t} + \gamma^n \cancel{\tilde{Q}(S_t}, A_t) - \tilde{Q}(S_t, A_t)\right)$
  - $t \leftarrow t + 1$
  - $t' \leftarrow t' + 1$
- **until** *episod ends at time* T' *or* $t' == T$

until $t' == T$
output: State-Action value function $\tilde{Q}$

Expected SARSA



Expected SARSA

- The policy Π is known so that we can use it in a formula:

$S_{t+1}, A_{t+1} \Rightarrow$ $$R_t + \gamma Q(S_t, A_t) \longrightarrow R_t + \gamma \sum_a \pi(a|S_t) Q(S_t, a)$$

- Make the update independent of the action chosen (and thus of the policy used to play).
- Reduce the variance for a computational cost.
- Amount to use the current estimate for $V(S_t)$...

Expected SARSA

Discounted: Prediction by Expected SARSA

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, \tilde{Q}(s, a), N(s, a) = 0$, n=0, $t' = 0$
repeat
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $\tilde{Q}(S_t, A_t) \leftarrow \tilde{Q}(S_t, A_t) + \alpha(N(S_t, A_t)) \left(R_{t+1} + \gamma \sum_a \pi(a|S_t)\tilde{Q}(S_{t+1}, a) - \tilde{Q}(S_t, A_t)\right)$
  - $t \leftarrow t + 1$
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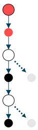
- **until** *episod ends at time* T' *or* $t' == T$

until $t' == T$
output: State-Action value function $\tilde{Q}$

n-steps Tree Backup



n-steps Tree Backup

- At each time step, use the expected SARSA average over the action while replacing the Q value for the picked action by a deeper estimate.
  - 1-step return (Expected Sarsa)

$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$

  - 2-step return:

$$\begin{aligned} G_{t:t+2} = R_{t+1} &+ \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) \\ &+ \gamma \pi(A_{t+1}|S_{t+1}) \left(R_{t+2} + \gamma \sum_a \pi(a|S_{t+2}) Q(S_{t+2}, a) \right) \\ = R_{t+1} &+ \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2} \end{aligned}$$

n-steps Tree Backup

- 1-step return (Expected Sarsa)

$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$

- 2-step return:

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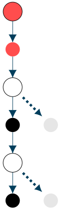
- Recursive definition of n-step return:

$$\begin{aligned} G_{t:t+n} = R_{t+1} &+ \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \\ &+ \gamma \pi(A_{t+1}|S_{t+1}) \left(G_{t+1:t+n} - Q(S_{t+1}, A_{t+1})\right) \end{aligned}$$

- TD update

$$Q(S_{t-n}, A_{t-n}) = Q(S_{t-n}, A_{t-n}) + \alpha(N(S_{t-n}, Q_{t-n})) \left(G_{t-n:t} - Q(S_{t-n}, A_{t-n})\right)$$

Q(σ)

Between  and 

Sampling or Averaging

- Unifying algorithm!
- Recursive definition of *n*-step return:

$$
\begin{aligned}
G_{t:t+n} = R_{t+1} &+ \sigma \left(G_{t+1:t+n} - Q(\cancel{S_{t+1}, A_{t+1}}) \right) \\
&+ (1-\sigma)\Big(\gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \\
&\qquad + \gamma\pi(A_{t+1}|S_{t+1}) \left(G_{t+1:t+n} - Q(S_{t+1}, A_{t+1}) \right) \Big)
\end{aligned}
$$

λ-Return

Averaged n-steps return?

- n-step return:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- Averaged n-step return: (compound update)

$$G_t^\omega = \sum_{n=1}^{\infty} \omega_n G_{t:t+n} \quad \text{with} \sum_{i=1}^{\infty} \omega_n = 1$$

- TD(λ): specific averaging

$$G_{t:t+n} = R_{t+1} + \gamma G_{t+1:t+n}$$

$$G_t^\lambda = (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n} = \text{"}\cancel{R_{t+1} + \gamma G_t}\text{"}$$

$$= (1-\lambda) \sum_{n=1}^{T-t} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t} G_t \quad \text{(Episodic)}$$

interpolating between TD (a.k.a TD(0)) and MC for $\lambda = 1$.

- Can be mixed with tree backup strategies (TB(λ))

λ-return and Temporality

True λ-return

- Require to wait until the end of an episode before we can update.
- Unusable in a non episodic setting!

Truncated λ-return

- Truncated λ-return:

$R_{t+1} + \dots + \gamma^{H-t-1} R_H + \gamma^{H-t} Q(S_H, A_H)$

$$G_t^\lambda = (1-\lambda) \sum_{n=1}^{H-t} \lambda^{n-1} G_{t:t+n} + \lambda^{H-t} G_{t:H}$$

- The virtual horizon H may vary during the algorithm.

λ-return and Temporality

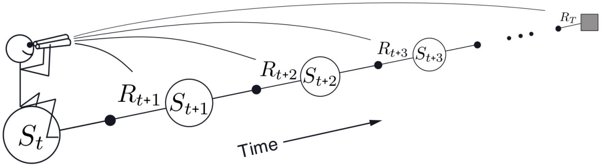
Temporality

- *n*-step return
$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$
depends on a current estimate V (or Q)!
- In G, should we use
  - an estimate available at time t?
  - an estimate available at time $t+n$?
  - an estimate available at time H?
- Off-Line vs On-Line!
  - Off-line: keep V constant during the episodes.
  - On-line: Used updated V when available.
  - True on-line (Sutton and Barto): restart algorithm with a growing horizon.

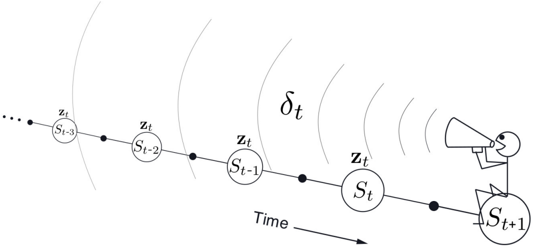
Outline

Forward and Backward Point of View

From a forward view



To a backward one:



Source: Sutton and Barto

Returns and Temporal Differencies

- n-step returns:

$$
\begin{aligned}
G_{t:t+n} - Q(S_t, A_t) &= R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} \\
&\quad + \gamma^n Q(S_{t+n}, A_{t+n\cancel{-1}}) - Q(S_t, A_t) \\
&= \sum_{l=1}^{n} \gamma^{l-1} (R_{t+l} + \gamma Q(S_{t+l}, A_{t+l\cancel{-1}}) - Q(S_{t+l-1}, A_{t+l-1})) \\
&= \sum_{l=0}^{n-1} \gamma^{l-1} \delta_{t+l}
\end{aligned}
$$

$\delta_{t:t+n}$

$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$

- λ return:

$$
\begin{aligned}
G_t^\lambda - Q(S_t, A_t) &= (1-\lambda) \sum_n \lambda^n (G_{t:t+n} - Q(S_t, A_t)) \\
&= \sum_{n=0} \lambda^n \gamma^n \delta_{t+n}
\end{aligned}
$$

Forward View and Backward View

Forward View

- Updates:

$$Q_t(s,a) = Q_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_{t'},A_{t'})}\alpha_{t'}(s,a)\left(\sum_{t''\geq t'} \lambda^{t''-t'}\gamma^{t''-t'}\delta_{t''}\right)$$

- Cumulative updates:

$$Q_t(s,a) = Q_0(s,a) + \sum_{t'\leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})}\alpha_{t'}(s,a)\left(\sum_{t''\geq t'} \lambda^{t''-t'}\gamma^{t''-t'}\delta_{t''}\right)$$

- Limit:

$$Q_\infty(s,a) = Q_0(s,a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})}\alpha_{t'}(s,a)\left(\sum_{t''\geq t'} \lambda^{t''-t'}\gamma^{t''-t'}\delta_{t''}\right)$$

- Focus on the update place.

Limit(s)

- Limit:

$$
\begin{aligned}
Q_\infty(s,a) &= Q_0(s,a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right) \\
&= Q_0(s,a) + \sum_{t''} \delta_{t''} \sum_{t' \leq t''} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t''-t'} \gamma^{t''-t'}
\end{aligned}
$$

- Focus on the update place or and the temporal differencies...

Forward View and Backward View

Backward View

- Same limit with cumulative udpates over temporal differencies

$$Q_t(s,a) = Q_0(s,a) + \sum_{t'' \leq t} \delta_{t''} \sum_{t' \leq t''} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t''-t'} \gamma^{t''-t'}$$

- Updates

$$Q_t(s,a) = Q_{t-1}(s,a) + \delta_t \underbrace{\sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t-t'} \gamma^{t-t'}}_{z_t(s,a)}$$

- Pseudo Eligibility trace:

$$\begin{aligned} z_t(s,a) &= \sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t-t'} \gamma^{t-t'} \\ &= \lambda \gamma z_{t-1}(s,a) + \alpha_t(s,a) \mathbf{1}_{(s,a)=(S_t,A_t)} \end{aligned}$$

- Proof of convergence toward the same target.

Eligibility Trace

$$Q_t(s,a) = Q_{t-1}(s,a) + \alpha_t \delta_t z_t(s,a)$$

Eligibility Trace

- Focus on temporal differencies with simultaneous update on all states.
- TD(λ) eligibility trace: $z_t(s,a) = \lambda\gamma z_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$
- Strictly equivalent to the previous scheme for constant stepsize
- Other eligibility trace:
  - Replacing trace:
  $$z_t(s,a) = \begin{cases} 1 & \text{if } (s,a) = (S_t,A_t) \\ \lambda\gamma z_{t-1}(s,a) & \text{otherwise} \end{cases}$$
  - Time dependent trace:
  $$z_t(s,a) = c_t\gamma z_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$$
  where c_t is defined *in a appropriate way* to ensure the convergence of the algorithm.

- Need to store (and update) this information. . .

Temporal Differencies

$$\delta_t?$$

Temporal Differencies

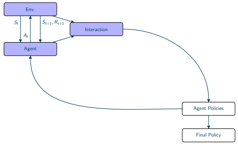
- Basic temporal differencies:
$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$
- Expected temporal differencies:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$
- Average of both:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma\sigma Q(S_{t+1}, A_{t+1}) + \gamma(1-\sigma) V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma\sigma \left(Q(S_{t+1}, A_{t+1}) - V(S_{t+1})\right) - Q(S_t, A_t)\end{aligned}$$

- Only expected temporal average is independent of the next action.
- No generic proof of convergence...

Outline

On-Policy vs Off-Policy

From



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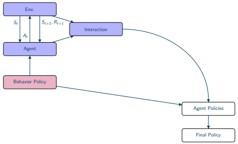


On-Policy vs Off-Policy

- On-Policy: the policy b used to interact is the same than the policy Π evaluated or optimized.
- Off-Policy: the policy b used to interact may be different from the policy Π evaluated or optimized.

- Off-Policy allows in particular to (re)use interactions from previous experiments.
- Q-learning was possible in off-policy setting.

Importance Sampling

$$\rho_{t:t'} = \frac{\mathbb{P}_\Pi(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} | S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} | S_t)} = \frac{\pi(A_t|S_t)\dots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\dots\pi(Q_{t'}|S_{t'})}$$

Importance Sampling

- For any law p and q, and any function g
$$\mathbb{E}_p[g(x)] = \mathbb{E}_q\left[\frac{p(x)}{q(x)} g(x)\right]$$
provided $q(x) = 0$ implies $p(x) = 0$.
- $\mathbb{V}\mathrm{ar}_q\left[\frac{p(x)}{q(x)} g(x)\right]$ may be large with respect to $\mathbb{V}\mathrm{ar}_p\left[g(x)\right]$ if the ratio $p(x)/q(x)$ is large...

$$\mathbb{E}_p[g(x)] = \sum p(x)\, g(x)$$
$$= \sum q(x) \frac{p(x)}{q(x)} g(x)$$
$$= \mathbb{E}_q\left[\frac{p(x)}{q(x)} g(x)\right]$$

Importance Sampling for Trajectories

- For any trajectory $\tau_{t:t'} = S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1}),$,
$$\frac{\mathbb{P}_\Pi(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1}) | S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1}) | S_t)} = \frac{\pi(A_t|S_t)\dots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\dots b(A_{t'}|S_{t'})}$$

Importance Sampling and Returns

$$\mathbb{E}_{\Pi}[g(\tau_{t:t'})|S_t = s] = \mathbb{E}_b[\rho_{t:t'} g(\tau_{t:t'})|S_t = s] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t)\ldots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\ldots\pi(Q_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}
\mathbb{E}_\pi[G_{t:t'}|S_t = s] &= \mathbb{E}_\pi\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} V(S_{t'}) \middle| S_t = s\right] \\
&= \mathbb{E}_b\left[\rho_{t:(t'-1)} \left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} V(S_{t'})\right) \middle| S_t = s\right] \\
&= \mathbb{E}_b\left[\sum_{t''=t+1}^{t'} \rho_{t:(t''-1)} \gamma^{t''-t-1} R_{t''} + \rho_{t:(t'-1)} \gamma^{t'-t} V(S_{t'}) \middle| S_t = s\right]
\end{aligned}$$

Importance Sampling and Returns

$$\mathbb{E}_{\Pi}[g(\tau_{t:t'})|S_t, A_t] = \mathbb{E}_b\left[\rho_{(t+1):t'} g(\tau_{t:t'})\middle|S_t, A_t\right] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t)\ldots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\ldots\pi(Q_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}
\mathbb{E}_\pi[G_{t:t'}|S_t, A_t] &= \mathbb{E}_\pi\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\middle|S_t, A_t\right] \\
&= \mathbb{E}_b\left[\rho_{(t+1):(t'-1)}\left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\right)\middle|S_t, A_t\right] \\
&= \mathbb{E}_b\left[\rho_{(t+1):(t''-1)} \sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \rho_{(t+1):t'}\gamma^{t'-t} Q(S_{t'}, A_{t'})\middle|S_t, A_t\right]
\end{aligned}$$

- No correction if $t' = t + 1$

λ-return

λ-return

- Recursive definition of the λ-return:

$$G_t^\lambda | S_t = R_{t+1} + \gamma \left((1-\lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \Big((1-\lambda) (\sigma Q(S_{t+1}, A_{t+1}) + (1-\sigma) (\sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \\ + \pi(A_{t+1}|S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right))) + \lambda G_{t+1}^\lambda \Big)$$

- Off-line correction

$$G_t^\lambda | S_t = \rho_{t:t} \left(R_{t+1} + \gamma \left((1-\lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right) \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \Big((1-\lambda) (\sigma Q(S_{t+1}, A'_{t+1}) + (1-\sigma) (\sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \\ + \pi(A_{t+1}|S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right))) \\ + \lambda \rho_{t+1:t+1} G_{t+1}^\lambda \Big)$$

where A'_{t+1} is drawn following π (or multiply by $\rho_{t+1:t+1}$ to use A_{t+1}).

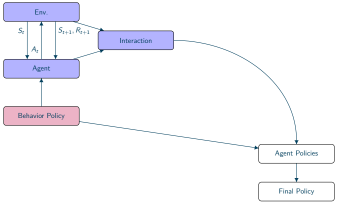
Temporal Differencies

$$\delta_t?$$

Temporal Differencies

- Basic temporal differencies:
$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A'_{t+1}) - Q(S_t, A_t)$$
with A'_{t+1} drawn using π.
- Expected temporal differencies:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t) \\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$
without any correction.
- Average of both:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma\sigma Q(S_{t+1}, A_{t+1}) + \gamma(1-\sigma) V(S_{t+1}) - Q(S_t, A_t) \\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma\sigma \left(Q(S_{t+1}, A'_{t+1}) - V(S_{t+1})\right) - Q(S_t, A_t)\end{aligned}$$
with A'_{t+1} drawn using π.

Off-Policy Algorithm



Off-Policy Correction

- Replace any estimate of the gain by an importance-sampling corrected one.
- Works well for prediction.
- Can be combined with policy improvement (a la SARSA) but less (no?) theoretical guarantees.

Retrace(λ)

$$\widetilde{\mathcal{T}}Q(s,a) = Q(s,a) + \mathbb{E}_b\left[\sum_{t\geq 0}\gamma^t \left(\prod_{t'=1}^{t} c_{t'}\right)\delta_t \middle| S_0 = s, A_0 = a\right]$$

$$c_t = c(A_t, S_t, A_{t-1}, S_{t-1}, \cdots, A_0, S_0)$$

$$\mathbb{E}_b[\delta_t|S_t, A_t] = \mathbb{E}[R_{t+1} + \gamma\mathbb{E}_\pi[Q(S_{t+1},\cdot)] - Q(S_t, A_t)|S_t, A_t]$$

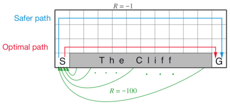
Generic Off-Policy Algorithm

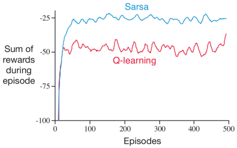
- Generic off-line algorithm including
  - Importance sampling: $c_t = \rho_{t:t} = \pi(A_t|S_t)/b(A_t|S_t)$
  - TB(λ): $c_t = \lambda\pi(A_t|S_t)$
  - Retrace(λ): $c_t = \lambda\min(1, \pi(A_t|S_t)/b(A_t/S_t))$
- **Prop:** Q_π is a fixed point as $\mathbb{E}_b[\delta_t|S_t, A_t] = \mathbb{E}[\mathcal{T}^\pi Q(S_t, A_t) - Q(S_t, A_t)|S_t, A_t]$.
- **Prop:** $\widetilde{\mathcal{T}}$ is a contraction provided $c_t \leq \rho_{t:t} = \pi(A_t|S_t)/b(A_t|S_t)$.
- Convergence for Importance sampling, TB(λ) and Retrace(λ) for any b.
- Partial results for policy improvement under more assumption.

- For Q(λ), $c_t = \lambda$, convergence if $\|\pi(|s) - b(|s)\|_1 \leq \epsilon$ and $\lambda \leq (1-\gamma)/(\gamma\epsilon)$.

Outline

Q-Learning vs SARSA





How different are they?

- In Q-learning, the exploratory policy used is decoupled from the optimized policy.
- This exploratory policy may yield low rewards on average.
- In SARSA, the two policies are linked with the hope on having higher rewards during the learning step.
- Subtle different behavior even if we modify the exploratory policy in Q-Learning.

Exploration vs Exploitation

Exploration vs Exploitation

- Exploration: explore new policies to be able to discover the best ones.
- Exploitation: use good policies to obtain a good return.
- Exploration is a requirement.

- No tradeoff if we optimize only the final result!
- Tradeoff between the two if we consider that the returns during training matters!
- Q-learning use the first approach and SARSA try to tackle the second.
- Tradeoff if we study a regret:
$$\sum_t \mathbb{E}_{\Pi_\star}[R_t] - \mathbb{E}_{\Pi_t}[R_t]$$
which forces us to be good as fast as possible.
- No natural definition in the discounted setting.

Bandits

$$\mathcal{S} = \{0\} \quad \text{and} \quad A = \{1, \dots k\} \quad \text{and} \quad r(s,a) = r_a$$

Bandits

- Very simple toy model where there is only one state!
- Optimal policy: pick $a_\star \in \operatorname{argmax} r_a$.
- Q estimation: estimate r_a by playing action a.
- Strategy:
  - Every arm has to be played until we are sure they are bad.
  - Best arm should be played as often as possible to maximime the rewards during the learnig phase.
- Simple enough setting to obtain result on the regret

$$r_T = \sum_{t \leq T} (r_{a_\star} - R_t)$$

- We will use $\Delta_a = r_{a_\star} - r_a$ and assume that $R|a$ is 1-subgaussian.

Explore Then Commit

Explore Then Commit (Random Exploration)

- Play the arm successively during Km steps and then play the optimal one during $T - Km$ steps.
- **Prop:**

$$r_T \leq \min(m, T/K) \sum_{a=1}^{k} \Delta(a) + \max(T - mK, 0) \sum_{a=1}^{k} \Delta(a) \exp(-m\Delta(a)^2/4)$$

Furthermore,

$$\mathbb{P}(a_T = a_*) \geq 1 - \sum_{a \neq a_*} \exp(-m\Delta(a)^2/4)$$

ϵ-greedy Strategy

ϵ-greedy Strategy

- Estimate $Q(a) = r_a$ by MC:

$$Q_t(a) = \frac{\sum_{t'=1}^{t-1} \mathbf{1}_{A_{t'}=a} R_{t'}}{\sum_{i=1}^{t-1} \mathbf{1}_{A_{t'}=a}}$$

- Pick arm a at time t using

$$\pi(a) = \begin{cases} \epsilon_t/k + (1-\epsilon) & \text{if } a = \operatorname{argmax}_{a'} Q_t(a') \text{ (only the smallest if necessary)} \\ \epsilon_t/k & \text{otherwise} \end{cases}$$

- **Prop:**

$$r_T \geq \sum_{t=1}^{T} \frac{\epsilon_t}{k} \sum_{a=1}^{k} \Delta(a)$$

ϵ-greedy Strategy

ϵ-greedy Strategy

- **Prop:**

$$\mathbb{P}(A_T = a_*) \geq 1 - \epsilon_T - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

with $\Sigma_T = \sum_{s=1}^{T} \epsilon_s$.
Furthermore,

$$\mathbb{P}(a_* = \operatorname{argmax} Q_{T,a}) \geq 1 - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

If $\epsilon_t = c/t$,

$$r_T \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T) + 1}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

as soon as $c/(6k) > 1$ and $c \min_{a \neq a_*} \Delta(a)/4k < 1$.
If $\epsilon_t = c \log(t)/t$ then

$$r_T \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T)(\log(T) + 1)}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

UCB Strategy

Upper Confidence Bound

- Use an optimistic strategy to pick the best arm

$$A_t = \text{argmax}\, Q_t(a) + \sqrt{\frac{c \log t}{N_t(a)}}$$

- **Prop:**

$$r_n(t) \leq C_c \sum_a \Delta(a) + \sum_a \frac{4c \ln t}{\Delta(a)}.$$

with $C_c < +\infty$ as soon as $c > 3/2$
Furthermore

$$\mathbb{P}(A_t = a_*) \geq 1 - 2kt^{-2c+2}$$

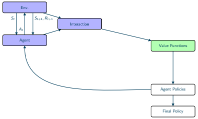
as soon as $t \geq \max_a \frac{4c \ln t}{\Delta(a)^2}$.

- Optimal regret!
- Hard to extend to RL setting but shows that ϵ-greedy may not be optimal.

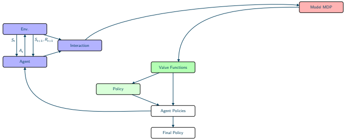
Outline

Model Based Approach

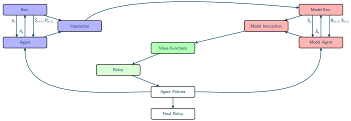
From



to



or

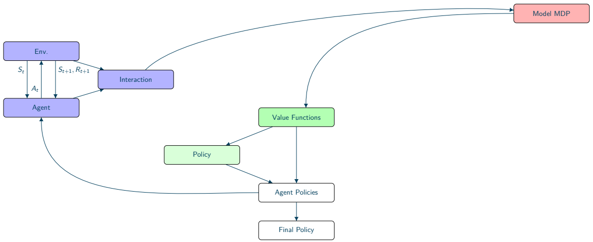


Model Based Approach

- Use the interactions to learn a model. . .
- that can be used to learn a good policy.
- This model can be:
  - a MDP,
  - a simulator.

- Often easier to obtain a simulator.

Model based and MDP

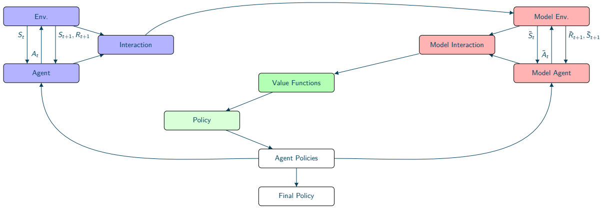


Estimated MDP: back to OR

- MDP can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated MDP, prediction and planning can be done using OR.

- Implicitely done by TD(0) when doing several passes.
- Model should be checked/improved as much as possible when new trajectories arrive.

Model based and RL

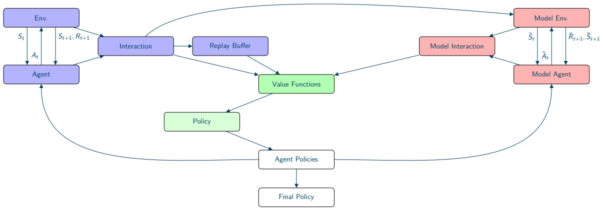


Estimated Simulator: back to RL

- Simulator can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated simulator, prediction and planning can be done using RL.

- Model should be checked/improved as much as possible when new trajectories arrive.

Model Free and Model Based Approach

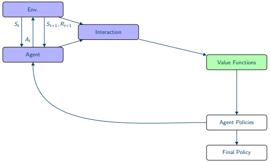


Dyna

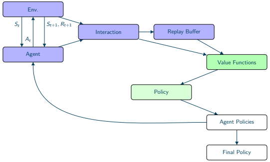
- Combine true interactions with simulated ones.
- Simultaneous acting, model learning, OR learning and RL learning.
- Search for a tradeoff between the (slow) learning RL algorithm and the (wrong) model OR algorithm.
- Need to deal with schedule!

Outline

Replay Buffer and Prioritized Sweeping

From 

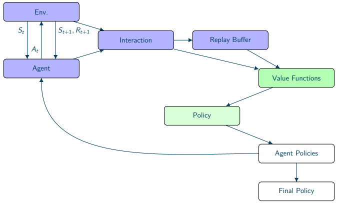
to



Replay Buffer and Prioritized Sweeping

- Can we reuse previous interactions?
- In which order?

Replay Buffer

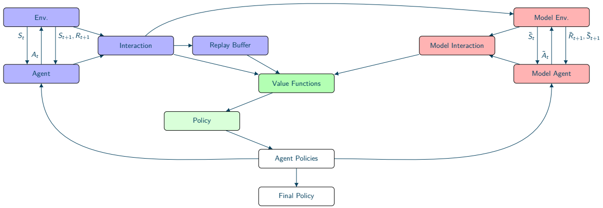


Replay Buffer

- Store previous interactions (trajectories) in a first-in first-out buffer.
- Draw a subsequence from those interactions (trajectories) and use it in a RL algorithm:
  - On-line: if the trajectory comes from the same policy.
  - Off-line: if the trajectory comes from a different policy.

- Similar to a simulator but no arbitrary choice of state or action.
- Often use with on-line algorithm if the policy has only mildy evolved...

Prioritized Sweeping



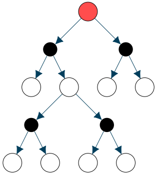
Prioritized Sweeping

- Plain Replay Buffer: subsequence drawn uniformly.
- Prioritized Sweeping: subsequence drawn favoring states with large temporal differencies.

- Can be combined with a model approach.

Outline

Real Time Planning

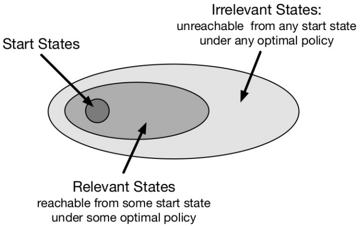


Real Time Planning

- Can we optimize the policy at the current state?
- Do we need to optimize it everywhere?
- What is required?

- Planning at decision time. . .

Real-Time Dynamic Programming



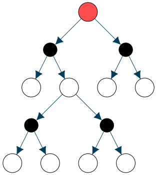
- Warmup in Dynamic Programming...

RT DP

- Use trajectories to sample the states to update.
- Convergence holds with exploratory policy.
- Optimal policy does not require to specify the action in irrelevant states.
- Convergence holds even without full exploration in some specific cases!

- In practice, seems to be computationaly efficient.

Planning At Decision Time

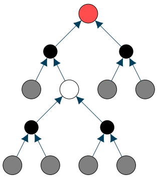


Planning At Decision Time

- Can we find a good action A_t at $S_t \ldots$ without having it precomputed?
- Policy Improvement
$$A_t = \operatorname{argmax} Q_t(S_t, \cdot)$$
can be seen as a first step.
- How to go deeper?

- **A model or a simulator will be required!**

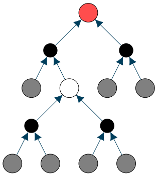
Heuristic Search



Heuristic Search

- Requires the knowledge of the MDP and of a heuristic based value function V.
- Strategy:
  - Build a limited depth tree by stopping after a few steps and at some specific states.
  - Backup the heuristic based value function using Dynamic Programming (Optimal Bellman operator).
  - Pick the action having the hight value.
- The deeper the better... but the more expensive due to branching!
- Requires a *suitable* heuristic...

Rollout Algorithm

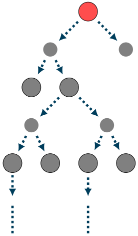


Rollout Policy

- Use a MC estimate with a default policy instead of a heuristic.
- Backup those estimates using Dynamic Programming.
- Simulation can even start after the first action (as in Policy Improvement).

- The values are (most of the time) discarded for the next state.

Monte Carlo Tree Search



Monte Carlo Tree Search

- Simultaneour tree growing, rollout and backup by DP.
- Repeat 4 steps:
  - Selection of a sequence of actions according to the current values with a tree policy.
  - Expansion of the tree at the last node without values.
  - Simulation with a rollout policy to estimate the values at this node.
  - Backup of the value by relaxed Dynamic Programming.
- MCTS focuses on promising paths using a UCB approach.

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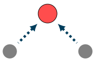

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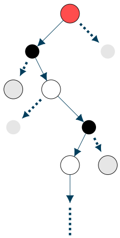

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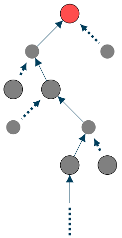

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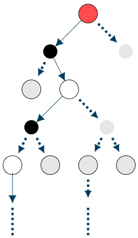

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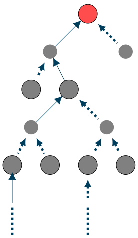

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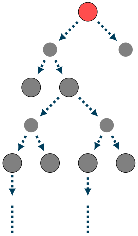

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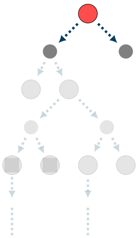

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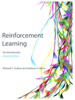

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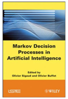

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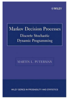

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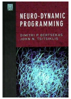

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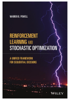

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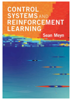

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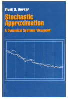

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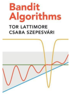

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