Operations Research and MDP.
Reinforcement learning and interactions.
More tabular reinforcement learning.
Reinforcement and approximation of value functions.
Actor/Critic: a Policy Point of View
Operations Research and MDP

How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interactive methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Most results for the discounted setting.
How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA...)
- Policy deduced by a statewise optimization of the value function over the actions.
More Tabular Reinforcement Learning

Can We Do Better?

- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).
Reinforcement and Approximation of Value Functions

How to Deal with a Large/Infinite states/action space?

- How to approximate value functions?
- How to estimate good approximation of value functions?

- Finite action space setting.
- Stochastic algorithm *(Deep Q Learning...)*.
- Policy deduced by a statewise optimization of the value function over the actions.
Actor/Critic: a Policy Point of View

Could We Directly Parameterized the Policy?

- How to parameterize a policy?
- How to optimize this policy?
- Can we combine parametric policy and approximated value function?

- State Of The Art Algorithms (DPG, PPO, SAC...)

Note: State Of The Art Algorithms include DPG, PPO, SAC...
Outline

1. Policy Gradient Theorems
2. Monte Carlo Based Policy Gradient
3. Actor / Critic Principle
4. 3 SOTA Algorithms
5. Average Return
6. References
Policy Point of View

- Optimize policy directly instead of deriving it from a value function.
- Avoid the argmax operator.
- Most natural POV?

- Pontryagin vs Hamilton-Jacobi(-Bellman) in control!
Outline

1 Policy Gradient Theorems

2 Monte Carlo Based Policy Gradient

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4 3 SOTA Algorithms

5 Average Return

6 References
Goal: average expected return over the states

- Target used to define the linear programming formulation of an optimal policy in the tabular setting.
- $\mu$ can be the initial distribution of the states (independent of $\pi$)...
- but may also depends on $\pi$ (for instance the associated stationary measure)
- Other choices will appear.

- Goal: optimize $J_\mu(\pi)$ in $\pi$!
Policy Gradient Theorems

Parametric Policy

\[ \pi_\theta(a|s) = \begin{cases} 
\frac{e^{h_\theta(a,s)}}{\sum_{a'} e^{h_\theta(a',s')}} & \text{(softmax)} \\
P_{h_\theta(s)}(a) & \text{(parametric conditional model)} \\
1_{a=h_\theta(s)} & \text{(deterministic)} 
\end{cases} \]

- Restriction of the set of policy to a parametrized one.
- Most classical parametrizations:
  - Soft-max with a preference function \( h_\theta(a, s) \),
  - Parametric conditional model with parameter \( h_\theta(s) \)
- To be useful need to be able to sample the distribution.
- \( h_\theta \): from linear model to deep learning...
- Most of our result will assume that \( \pi_\theta(a|s) \) is differentiable with respect to \( \theta \).
- Deterministic policies will be considered with a different analysis.
Episodic Setting: Gradient of Expected Returns

\[ v_{\pi_{\theta}}(s) = \mathbb{E}_{\pi_{\theta}}[G_0|S_0 = s] \]

\[ \nabla_{\theta} v_{\pi_{\theta}}(s) = \mathbb{E}_{\pi_{\theta}} \left( \left( \sum_{t=0}^{T_{\tau}-1} \nabla \log \pi_{\theta}(A_t|S_t) \right) G_0 \bigg| S_0 = s \right) \]

Expected Returns

- Rely on \( v_{\pi_{\theta}}(s) = \sum_{\tau} \mathbb{P}_{\pi_{\theta}}(\tau|S_0 = s) G_0(\tau) \) and

\[ \nabla \mathbb{P}_{\pi_{\theta}}(\tau|S_0 = s) = \mathbb{P}_{\pi_{\theta}}(\tau|S_0 = s) \nabla \log \mathbb{P}_{\pi_{\theta}}(\tau|S_0 = s) \]

\[ = \mathbb{P}_{\pi_{\theta}}(\tau|S_0 = s) \sum_t (\nabla \log \pi(A_t|S_t) + \nabla p(R_{t+1}, S_{t+1}|S_t, A_t)) \]

\[ = \mathbb{P}_{\pi_{\theta}}(\tau|S_0 = s) \sum_t \nabla \log \pi(A_t|S_t) \]

- In an episodic setting, any trajectory \( \tau \) ends at a finite time \( T_{\tau} \).
Episodic Setting: Policy Gradient Theorem

\[ J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s) \nu_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \right] \]

Policy Gradient Theorem

- Natural \( \mu \): initial state distribution.
- Gradient is an expectation: MC type algorithm...
- Can be interpreted as the gradient of a the maximum likelihood of the actions weighted by the return.
- Favors good actions over bad ones.
Baseline and Variance Reduction

\[ J_{\mu_0}(\pi_\theta) = \sum_s P(S_0 = s) v_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \right] \]

Variance Reduction and Baseline

- The previous formulae are valid if one replace \( G_0 \) by any function of \( \tau \).
- For any constant \( b \), this leads to

\[ \nabla \mathbb{E}_{\pi_\theta} [b] = 0 = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right) b \right] \]

- Optimal value for

\[ b = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right)^2 G_0 \right] / \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right)^2 \right] \]

- Most used value \( b = \mathbb{E}_{\pi_\theta} [G_0] \).
Episodic/Discounted: Gradient(s) of Expected Return

\[ v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta} \left[ \sum \gamma^t R_t \mid S_0 = s \right] \]

\[ \nabla v_{\pi_\theta}(s) = \sum \gamma^t \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t'=0}^{t-1} \nabla \log \pi_\theta(A_{t'} \mid S_{t'}) \right) R_t \mid S_0 = s \right] \]

\[ = \sum \mathbb{E}_{\pi_\theta} \left[ \nabla \log \pi_\theta(A_{t'} \mid S_{t'}) \left( \sum_{t \geq t'} \gamma^t R_t \right) \mid S_0 = s \right] \]

\[ = \sum \gamma^{t'} \mathbb{E}_{\pi_\theta} \left[ \nabla \log \pi_\theta(A_{t'} \mid S_{t'}) Q_{\pi_\theta}(S_{t'}, A_{t'}) \mid S_0 = s \right] \]

\[ = \sum \gamma^{t'} \mathbb{E}_{\pi_\theta} \left[ \nabla \log \pi_\theta(A_{t'} \mid S_{t'}) \left( Q_{\pi_\theta}(S_{t'}, A_{t'}) - V_{\pi_\theta}(S_{t'}) \right) \mid S_0 = s \right] \]

Expected Returns

- Several gradients!
Episodic/Discounted: More Gradient(s)

\[ \nabla v_{\pi_\theta}(s) = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) Q_{\pi_\theta}(S_{t'}, A_{t'}) | S_0 = s] \]

\[ = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) (Q_{\pi_\theta}(S_{t'}, A_{t'}) - V_{\pi_\theta}(S_{t'})) | S_0 = s] \]

\[ = \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s | S_0 = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) Q_{\pi_\theta}(s, a) \right) \]

\[ = \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s | S_0 = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (Q_{\pi_\theta}(s, a) - V_{\pi_\theta}(s, a)) \right) \]

Focus on states

- More gradients!
Episodic/Discounted: Policy Gradient(s)

\[ J(\mu_0)(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \]

\[ \nabla J(\mu_0)(\pi_\theta) = \sum_s \left( \sum_t \gamma^t p_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) Q_{\pi_\theta}(s, a) \right) \]

\[ = \sum_s \left( \sum_t \gamma^t p_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (Q_{\pi_\theta}(s, a) - V_{\pi_\theta}(s, a)) \right) \]

Discounted Setting

- Average (discounted) return from the beginning.
- Focus on early steps in discounted setting...
Policy Improvement Lemma

\[ J_{\mu_0}(\pi') - J_{\mu_0}(\pi) = \sum_t \gamma^t P_{\pi'}(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) Q_{\pi}(s, a) \right) \]

\[ = \sum_t \gamma^t P_{\pi'}(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) (Q_{\pi}(s, a) - V_{\pi}(s)) \right) \]

*Proof in the discounted setting rely on*

\[ v_{\pi'} - v_{\pi} = r_{\pi'} + \gamma P_{\pi'} v_{\pi'} - r_{\pi} - \gamma P_{\pi} v_{\pi} \]

\[ = r_{\pi'} - r_{\pi} + \gamma \left( P_{\pi'} - P_{\pi} \right) v_{\pi} + \gamma P_{\pi'} (v_{\pi'} - v_{\pi}) \]

\[ v_{\pi'} - v_{\pi} = (I - \gamma P_{\pi'})^{-1} \left( r_{\pi'} - r_{\pi} + \gamma \left( P_{\pi'} - P_{\pi} \right) v_{\pi} \right) \]
Approximate Policy Improvement Lemma

\[
J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_t \gamma^t P_{\pi}(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) (Q_{\pi}(s, a) - V_{\pi}(s)) \right)
\]

\[
= \sum_t \gamma^t (P_{\pi'}(S_t = s) - P_{\pi}(S_t = s)) \left( \sum_a (\pi'(s|a) - \pi(s|a)) (Q_{\pi}(s, a) - V_{\pi}(s)) \right)
\]

\[
\leq \sum_t \gamma^t 2 \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1 \max_{s, a} |Q_{\pi}(s, a) - V_{\pi}(s)|
\]

Approximate Policy Improvement Lemma

- If \( \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1 \leq \epsilon \)

\[
P_{\pi'}(S_t = s) = (1 - \epsilon)^t P_{\pi}(S_t = s) + (1 - (1 - \epsilon)^t) P_{\text{mistake}}(S_t = s)
\]

\[
\rightarrow |P_{\pi'}(S_t = s) - P_{\pi}(S_t = s)| \leq 2(1 - (1 - \epsilon)^t) \leq 2\epsilon t
\]
Approximate Policy Improvement Lemma

\[ J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) (Q_\pi(s, a) - V_\pi(s)) \right) \]

\[ \leq \sum_t \gamma^t 2t \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |Q_\pi(s, a) - V_\pi(s)| \]

Approximate Policy Improvement Lemma and Policy Gradient Theorem

- Let \( \pi' = \pi_{\theta+h} \) and \( \pi_\theta \)
  - \( \pi_{\theta+h}(s, a) - \pi_\theta(s, a) = \pi(s, a) \langle \nabla \log \pi(s, a), h \rangle + O(\|h\|^2) \)
  - \( \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1 \leq \|h\| \max_a \|\nabla \log \pi(s, a)\| + O(\|h\|^2) \)

- Implies Policy Gradient Theorem:
  \[ J_{\mu_0}(\pi_{\theta+h}) = J_{\mu_0}(\pi_\theta) + \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left( \sum_a \pi(s|a) \langle \nabla \log \pi(s, a), h \rangle (Q_\pi(s, a) - V_\pi(s)) \right) + O(\|h\|^2) \]
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Monte Carlo Approach

\[ G_t = \sum_{t' \geq t} R_{t+1} \]

\[ Q_{t, \pi_0}(s, a) = \mathbb{E}[G_t | S_t = s, A_t = a] \]

**Monte Carlo**

- Replace every return by an empirical estimate along episodes.
- Need to wait until the end of the episodes.
REINFORCE: Monte Carlo Based Policy Gradient

\[ J_{\mu_0}(\pi_\theta) = \sum_s P(S_0 = s) v_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \right] \]

\[ = \sum_s \left( \sum_t P_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) Q_{\pi_\theta}(s, a) \right) \]

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) G_0 \quad \text{or} \quad \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t) G_t \]

**REINFORCE**

- Plain MC (SGD) algorithm.
- Need to wait until the end of the episodes.
- Convergence guarantees (even in off-line setting with importance sampling).
REINFORCE with Baseline

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \right] \]

\[ = \sum_s \left( \sum_t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (Q_{\pi_\theta}(s, a) - b(s)) \right) \]

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \left( \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \]

or \[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t) (G_t - b(S_t)) \]

REINFORCE with baseline

- Several choices for \( b \ldots \)
- and for \( b(s) \) which can be any function (a crude estimate of \( V_{t, \pi}(s) \) for instance)!
- Convergence guarantees (even in off-line setting with importance sampling).
Discounted REINFORCE?

\[ \nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \right] \]

\[ = \sum_s \left( \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (Q_{\pi_\theta}(s, a) - b(s)) \right) \]

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \left( \sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t) \right) (G_0 - b) \]

or \[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t|S_t) (G_t - b(S_t)) \]

Discounted REINFORCE

- Can be defined...
- but still requires an episodic setting for the discounted return \( G_t \) to be computed.
Monte Carlo Based Policy Gradient

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t | S_t) \left( G_t - b(S_t) \right) \]

\[ \text{\rightarrow} \hat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) = \frac{1}{1 - \gamma} \nabla \log \pi_\theta(A_t | S_t) \left( G_t - b(S_t) \right) \]

Discounted Measure?

- Much less weights for later states!
- Probability independent of \( t \) if the initial distribution is the stationary distribution \( \mu_{\pi_\theta} \) corresponding to \( \pi_\theta \) (if it exists).
- Approximately true after a burning stage if we reach stationarity...
- Better handled by the average return!

- More on this later...
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Actor/ Critic

- **Actor**: Parametric policy $\pi_\theta$ used.
- **Critic**: $Q$-value function $Q_w(\cdot, \cdot)$ approximating $Q_{\pi_\theta}$.
- **Critic follows the Actor**, which is optimized using the Critic.

- In **Value Approximation**, the Actor follows the Critic (through the argmax operator).
- In **on-line methods**, the Actor is used to interact with the environment.
Actor/Critic Principle

\[ J(\mu_0)(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \sum_s \left( \sum_t \gamma^t P_{\pi_\theta}(S_t = s) \right) \left( \sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (Q_{\pi_\theta}(s, a) - V_{\pi_\theta}(s, a)) \right) \]

\[ \hat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left( Q_{\pi_\theta}(S_t, A_t) - \sum_a \pi(a|S_t) Q_{\pi_\theta}(S_t, A_t) \right) \]

\[ \simeq \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left( Q_w(S_t, A_t) - \sum_a \pi(a|S_t) Q_w(S_t, A_t) \right) \]

**Actor/Critic**

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating \( Q_{\pi_\theta} \).
- Requires a two scale algorithm so that \( Q_w \) is always a good estimate of \( Q_{\pi_\theta} \).

- Is this a real algorithm in a non episodic setting?
Actor/Critic Principle

\[ J_{\mu_{\pi\theta}}(\pi_{\theta}) = \sum_{s} \mu_{\pi\theta}(s) v_{\pi\theta}(s) \]

\[ \nabla J_{\mu_{\pi\theta}}(\pi_{\theta}) = \sum_{s} \frac{1}{1 - \gamma} P_{\pi\theta}(S_t = s) \left( \sum_{a} \pi_{\theta}(a|s) \nabla \log \pi_{\theta}(a|s) (Q_{\pi\theta}(s, a) - V_{\pi\theta}(s, a)) \right) \]

\[ \hat{\nabla} J_{\mu_{\pi\theta}}(\pi_{\theta}) \simeq \frac{1}{1 - \gamma} \pi_{\theta}(A_t|S_t) \nabla \log \pi_{\theta}(A_t|S_t) \left( Q_w(S_t, A_t) - \sum_{a} \pi(a|S_t) Q_w(S_t, A_t) \right) \]

**Actor/Critic**

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating \( Q_{\pi\theta} \).
- Requires a two scale algorithm so that \( Q_w \) is always a good estimate of \( Q_{\pi\theta} \).
- Require the existence of a stationary measure...and that this stationary measure is reached *quickly*.
- Much harder to do off-line algorithm as the stationary measure is not known!
Critic in Actor/Critic

\[ Q_w \sim Q_{\pi_\theta} \]

**Critic**

- On-line TD learning with interaction following \( \pi_\theta \).
- Off-Policy TD learning is possible if the policy used for any action is stored.
- Approximate off-policy TD learning is possible using a replay buffer providing \( \pi_\theta \) is changing slowly.

- May lead to 3 scales algorithm (Actor/Critic Target/Critic)
- As mentionned in the previous slide, much harder to do off-line update for the actor.
Off-Line Actor

\[ J'_\mu(\pi) = \sum_s \mu(s) v_\pi(s) \]

Idea proposed in 2012.

Key lemma in the paper

\[ \nabla J'_\mu(\pi_\theta) \sim \sum_s \mu(s) \sum_a \pi_\theta(a|s) \nabla \pi_\theta(a|s) Q_{\pi_\theta}(s, a) \]

turns out to be wrong!

Still used as a heuristic justification of many algorithms!

Explicit formula for \( \nabla J'_\mu(\pi_\theta) \) can be obtained but much harder to use...
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PPO: Minorize-Majorization Algorithm

\[
J_{\mu_0}(\pi') \geq J_{\mu_0}(\pi) + \sum_{t} \gamma^t P_{\pi}(S_t = s) \left( \sum_{a} (\pi'(s|a) - \pi(s|a)) A_{\pi}(s, a) \right) \\
- \sum_{t} \gamma^t 2t \max_s \| \pi'(\cdot|s) - \pi(\cdot|s) \|_2 \max_{s,a} |A_{\pi}(s, a)|
\]

Ideal Minorize-Majorization Algorithm

- At step \(k\), find \(\theta_{k+1}\) maximizing

\[
J_{\mu_0}(\pi_{\theta}|\pi_{\theta_k}) = \sum_s \sum_t \gamma^t P_{\pi_{\theta_k}}(S_t = s) \left( \sum_{a} (\pi_{\theta}(s|a) - \pi_{\theta_k}(s|a)) A_{\pi_{\theta_k}}(s, a) \right) \\
- \sum_{t} \gamma^t 2t \max_s \| \pi_{\theta}(\cdot|s) - \pi_{\theta_k}(\cdot|s) \|_2 \max_{s,a} |A_{\pi_{\theta_k}}(s, a)|
\]

- By construction, \(J_{\mu_0}(\pi_{\theta_{k+1}}) \geq J_{\mu_0}(\pi_{\theta_k})\)

- Sample efficient algorithm as the same trajectory can be (re)used in the optimization.
PPO: Optimization

\[ J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t P_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) A_{\pi_{\theta_k}}(s,a) \right) \]

\[ - \sum_t \gamma^t 2t \max_s ||\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)||_2^2 \max_{s,a} |A_{\pi_{\theta_k}}(s,a)| \]

Optimization

- Gradient descent is possible.
- Gradient of the first term is

\[ \sum_s \sum_t \gamma^t P_{\pi}(S_t = s) \left( \sum_a \pi_\theta \nabla \pi_\theta(s|a) A_{\pi_{\theta_k}}(s,a) \right) \]

- Gradient of the second term more involved.

- Simpler (TRPO like) strategy: optimize

\[ \sum_s \sum_t \gamma^t P_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi'(s|a) - \pi(s|a)) A_{\pi_{\theta_k}}(s,a) \right) \]

under \( \max_s ||\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)||_2^2 \leq \epsilon \) and reduce \( \epsilon \) there is no gain.
PPO: KL Relaxation

\[ J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t P_{\pi_{\theta_k}}(S_t = s) \left( \sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) A_{\pi_{\theta_k}}(s, a) \right) \]

\[ - \frac{\gamma R_{\text{max}}}{(1 - \gamma)^3} \max_s \text{KL}(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s)) \]

TRPO/PPO Optimization

- Replace the \( \ell_1 \) norm by a KL divergence.
- In practice, replace the max by an average and replace \( \frac{2\gamma R_{\text{max}}}{(1 - \gamma)^3} \) by parameter \( \beta \).
- PPO: Gradient descent of the relaxed goal.
- TRPO: Constrained optimization.

- Adaptive scheme to set \( \beta \).
- Can be used with continuous action.
PPO: Clipped Objective

\[
\sum_s \sum_t \gamma^t \mathbb{P}_{\pi_k} (S_t = s) \left( \sum_a \pi_k(s|a) \min \left( \frac{\pi(s|a)}{\pi_k(s,a)} A_{\pi_k}(s,a), \text{clip}(1 - \epsilon, \frac{\pi(s|a)}{\pi_k(s,a)}, 1 + \epsilon) A_{\pi_k}(s,a) \right) \right)
\]

Clipped Objective

- Insight by (re)substracting \( \sum_a \pi_k(s|a) A_{\pi_k}(s,a) = 0 \):
  \[
  \sum_a \min \left( (\pi(s|a) - \pi_k(s,a)) A_{\pi_k}(s,a), \text{clip}( -\epsilon, \pi(s|a) - \pi_k(s,a), \epsilon) A_{\pi_k}(s,a) \right)
  \]
  \[
  = \sum_a \text{clip}( -\epsilon \pi_k(s,a), \pi(s|a) - \pi_k(s,a), \epsilon \pi_k(s,a)) A_{\pi_k}(s,a)
  \]
  \[
  - \max \left(0, -(\pi(s|a) - \pi_k(s,a)) A_{\pi_k}(s,a) - \epsilon \pi_k(s,a) | A_{\pi_k}(s,a)| \right)
  \]

- First term amount to replace \( \pi \) by a policy
  \[
  \tilde{\pi}(a|s) = \text{clip}(\pi_k(a|s)(1 - \epsilon), \pi(a|s), \pi_k(a|s)(1 + \epsilon)) + \eta s \pi_k(a|s)
  \]
  where \( \eta \) is so that \( \tilde{\pi} \) is a probability for all \( s \) and \( \| \tilde{\pi}(\cdot, s) - \pi_k(\cdot, s) \|_1 \leq \epsilon \)

- Second term is a hinge loss time penalizing policy \( \pi \) picking bad actions.

- Very efficient for discrete actions.
PPO: Stationary Objective

\[
\sum_s P_{\pi_k}(S_t = s) \left( \sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) A_{\pi_\theta_k}(s, a) \right) - \beta \max_s KL(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s))
\]

\[
\sum_s P_{\pi_k}(S_t = s) \left( \sum_a \pi_{\theta_k}(s|a) \min \left( \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s, a)} A_{\pi_\theta_k}(s, a), \text{clip}(1 - \epsilon, \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s, a)}, 1 + \epsilon) A_{\pi_\theta_k}(s, a) \right) \right)
\]

**Stationary Objective**

- Amount to replace \( J_{\mu_0}(\pi) \) by \( J_{\mu_\pi}(\pi) \)
- Most common implementation of PPO...
- But no way to justify it mathematically!
- May explain the (possible) instabilities of PPO.
DPG: Deterministic Policy Gradient

\[ \pi_\theta(a|s) = 1_{a=m_\theta(s)} \] (deterministic policy)

\[ J_{\mu_0}(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \]

\[ \nabla J_{\mu_0}(\pi_\theta) = \sum_s \sum_t \gamma^t P_{\pi_\theta}(S_t = s) \nabla_a Q(S_t, m_\theta) \]

**Deterministic Policy Gradient**

- Deterministic policy replaced by a randomized one centered on \( m_\theta(s) \).
- Critic trained with a TD variant of DQN.
- Gradient can be obtained by use a policy \( \pi_\theta = \mathcal{N}(m_\theta(s), \sigma^2 \text{Id}) \) and letting \( \sigma \) goes to 0.
- Off-Policy as claimed?
- Yes for the actor but no theoretical justification for the critic!
- In practice, the buffer contains only samples using a policy close to the current one...
SAC: A New Goal

\[ R_t \rightarrow R_t + \lambda H(\pi(S_t)) \]

**A Modified Reward**

- Modification of the reward to favor high entropy policy:
  \[ R_t \rightarrow R_t + \lambda H(\pi(S_t)) \]

- Goal:
  \[ J(\pi) = \sum_t (R_t + \lambda H(\pi(S_t))) \]

- Soft value function implicitly defined as the fixed point of
  \[ T^\pi Q_\pi(s, a) = r_\pi(s, a) + \sum_{s'} p(s'|s, a) V_\pi(s') \]

where

\[ V_\pi(s, a) = \sum_a \pi(a|s) (Q_\pi(s, a) - \log \pi(a|s)) \]
SAC: Policy Improvement and Optimal Policy

\[ R_t \rightarrow R_t + \lambda \mathcal{H}(\pi(S_t)) \]

**A Modified Policy Improvement Lemma**

- **Policy improvement rule:**
  \[ \pi^+(\cdot|s) = \arg\max_{\pi(\cdot|s)} \sum_a \pi(a|s) \left( q(s, a) - \lambda \log(\pi(a|s)) \right) \]

  \[ \pi^+(a|s) \propto \exp\left(-\frac{1}{\lambda} q(s, a)\right) \]

  implies \( G_{\pi^+}(s, a) \geq G_{\pi}(s, a) \).

- At convergence, \( J(\pi^*) \) is optimal!

- Convergence in the finite setting.
3 SOTA Algorithms

SAC: Parametrization

$$\pi \sim \pi_\theta$$
$$Q(s, a) \sim Q_w$$

SAC Choices

- Fitted TD learning for $Q$:
  $$w \simeq \arg\min_w \sum_{(S,A,R,S') \in B} (R + \mathbb{E}_{\pi_\theta} [\gamma Q_w(S', a) - \lambda \log \pi_\theta(a|S') - Q_w(S,A)])^2$$
  where the trajectory pieces are samples from a replay buffer and $\overline{w}$ is a slowdown version of $w$ (two scale algorithm).

- Online version rather than batch...

- Fitted KL for $\pi$:
  $$\theta \simeq \arg\min_\theta \sum_{(S,A,R,S') \in B} \text{KL}(\pi_\theta(\cdot|S) | \exp -\lambda Q[\overline{w}](S, \cdot)/Z_w(S))$$
  $$\simeq \sum_{(S, A, R, S') \in B} \mathbb{E}_{\pi_\theta} \left[ \frac{1}{\lambda} \log \pi_\theta(a|S) - Q_w(a|S) \right]$$
Outline

1. Policy Gradient Theorems
2. Monte Carlo Based Policy Gradient
3. Actor / Critic Principle
4. 3 SOTA Algorithms
5. Average Return
6. References
Continuing Tasks and Average Return

- Most natural performance measure:

  \[ J(\pi) = r(\pi) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_\pi[R_t|S_0] \]

  \[ = \sum_s \left( \lim_{T \to \infty} \sum_{t=1}^{T} \mathbb{P}_\pi(S_t = s|S_0) \right) \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)r \]

- \( \mu(s) \) if it exists is such that

  \[ \sum_s \mu(s) \sum_a \pi(a|s)p(s'|s, a) = \mu(s') \]

- Gradient of \( J(\pi|\theta) \):

  \[ \nabla J(\pi|\theta) = \sum_s \mu(s) \sum_a \pi\theta(a|s) \nabla \log \pi_{\pi|\theta}(a|s)q_{\pi|\theta}(s, a) \]

- Beware \( q_{\pi|\theta} \) are the relative Q-value functions and not the classical one.
Average Return and Relative Value Functions

\[ r(\pi) = \sum_s \mu(s) \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)r \]

\[ G_t = \sum_{t' \geq t} R_t - r(\pi) \]

\[ V_\pi(s) = \mathbb{E}_\pi[G_t|S_t = s] \quad \text{and} \quad Q_\pi(s) = \mathbb{E}_\pi[G_t|S_t = s] \]

- Numerical algorithm to estimate those relative value functions.
- Leads to another family of Policy Gradient algorithm.
Outline

1. Policy Gradient Theorems
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5. Average Return
6. References
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- **Main contributor**: E. Le Pennec