BANDELET IMAGE APPROXIMATION AND COMPRESSION

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Abstract. Finding efficient geometric representations of images is a central issue to improve image compression and noise removal algorithms. We introduce bandelet orthogonal bases and frames that are adapted to the geometric regularity of an image. Images are approximated by finding a best bandelet basis or frame that produces a sparse representation. For functions that are uniformly regular outside a set of edge curves that are geometrically regular, the main theorem proves that bandelet approximations satisfy an optimal asymptotic error decay rate. A bandelet image compression scheme is derived. For computational applications, a fast discrete bandelet transform algorithm is introduced, with a fast best basis search which preserves asymptotic approximation and coding error decay rates.

Key words. Wavelets, Bandelets, Geometric Representation, Non-Linear Approximation

AMS subject classifications. 41A25, 42C40, 65T60

1. Introduction. 2. Mern ep r e e ppro m on fM lny f fM 2 C M a elsewhere k k 2 CM where k k stands for the L2 norm. The decay exponent is optimal in the sense that no other approximation scheme can improve it for all such functions. If f is C 1 over [0, 1/2] f can be decomposed into a sum of functions that are C 1 on each of the curves C i, i = 1, 2, ..., l, where the C i are regular curves. Several approaches [1, 6, 8, 18] have already been proposed to improve the decay of this wavelet approximation error, for C 1 unknown, the issue addressed by this paper is to find an approximation scheme that is asymptotically as efficient as if f was Holderian of order over its whole support, and to derive an image compression scheme. This is particularly important to approximate and compress images, where the contours of objects create edge transitions along piecewise regular curves.

Section 2 reviews non-linear approximation results for piecewise regular images including edges. In the neighborhood of an edge, the image gray levels vary regularly in directions parallel to the edge, but they have sharp transitions across the edge. This anisotropic regularity is specified by a geometric flow that is a vector field that indicates the local direction of regularity. Section 3 constructs bandelets, which are anisotropic wavelets that are warped along this geometric flow, and bandelet orthonormal bases in bands around edges. We study the approximation in bandelet bases of functions including edges over such bands. Bandelet frames of L2 [0, 1/2] are defined in Section 4.1 as a union of bandelet bases in different bands. A dictionary of bandelet frames is constructed, and the best bandelet frame is found by a fast search algorithm that preserves asymptotic approximation and coding error decay rates.
delet frames is constructed in Section 4.2 with dyadic square segmentations of $[0, 1]^2$ and parameterized geometry. The main theorem of Section 4.2 proves that a best bandelet frame obtained by minimizing an appropriate Lagrangian cost function in the bandelet dictionary yields asymptotically optimal approximations of piecewise regular functions. If $f$ is H"olderian of order $\alpha$ over $[0, 1]^2$, then the main theorem proves that an approximation from $M$ parameters in a best bandelet frame satisfies

$$\|f - f_M\| \leq CM^{-\alpha}.$$

To compress images in bits, an image transform code is defined in Section 5. It is proved that a best bandelet frame yields a distortion rate that nearly reaches the Kolmogorov asymptotic lower bound, up to a logarithmic factor. For numerical implementations over digital images, Section 6.1 discretizes bandelet bases and frames. Section 6.2 describes a fast algorithm that finds a best discrete bandelet frame with an approximation error that decays like $M^{-\alpha}$ up to a logarithmic factor.

2. Geometric Image Model.

We begin by establishing a mathematical model for geometrically regular images using the notion of edge. This model incorporates the fact that the image intensity is not necessarily singular at edge locations, which is why edge detection is an ill-posed problem. We then review existing constructive procedures to approximate such geometrically regular functions. Functions that are regular everywhere outside a set of regular edge curves define a first simple model of geometrically regular functions. Let $C^{\alpha}(\mathbb{R}^n)$ be the space of H"olderian functions of order $\alpha$ over $\mathbb{R}^n$ defined for $\alpha > 0$:

$$C^{\alpha}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \middle| \forall \beta, \| \alpha \|_{\infty} \right\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\alpha_i \in \mathbb{R}$. For integer, the space $C^{\alpha}(\mathbb{R}^n)$ is slightly larger than the space of functions having bounded derivatives up to order $\alpha$. The norm $\|f\|_{C^{\alpha}(\mathbb{R}^n)}$ used throughout this paper is defined by:

$$\|f\|_{C^{\alpha}(\mathbb{R}^n)} = \max_{(x,y) \in \mathbb{R}^n} \left| \sum_{\alpha} \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f(x) \right| \times \|x - y\|^{-\alpha}.$$

We say that an edge curve is H"olderian of order $\alpha$ if the coordinates in $\mathbb{R}^n$ of the points along this curve has a parameterization by arc length which is H"olderian of order $\alpha$. An image model with geometrically regular edges is obtained by imposing that $f \in C^{\alpha}(\mathbb{R}^2)$ for $\alpha = (\alpha_1, \alpha_2)$ and $\alpha_1, \alpha_2 > 0$. For most images, this model is too simplistic because most often the image intensity has a sharp variation but is not singular across an edge. In particular, discontinuities of the image intensity created by occlusions in the visual scene are blurred by optical effects along edges. This blurring effect along edges can be modeled by

$$\left\{ f, \mathbb{R}^n \rightarrow \mathbb{R} \middle| \forall \beta, \| \alpha \|_{\infty} \right\}$$

For order $\alpha \in \mathbb{R}^n$, the norm $\|f\|_{C^{\alpha}(\mathbb{R}^2)}$ is defined by:

$$\|f\|_{C^{\alpha}(\mathbb{R}^2)} = \max_{(x,y) \in \mathbb{R}^2} \left| \sum_{\alpha} \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \cdots \partial x_2^{\alpha_2}} f(x) \right| \times \|x - y\|^{-\alpha}.$$
through a convolution with an unknown kernel of compact support $h(x)$. This means that we can write $f(x) = \tilde{f} \ast h(x)$ where $\tilde{f} \in C^\infty$ for all $x \in [0,1]^2$.

When $f \neq \tilde{f}$, finding from $f$ the exact locations of the edges $C$ is an ill-posed problem, especially since $h$ is unknown. The difficulty to locate blurred edges is well known in image processing [3]. The goal of this paper is to find an approximation $f_M$ from $M$ parameters which satisfies

$$\|f - f_M\|^2 \leq C M^{-\alpha},$$

with a constant $C$ that does not depend on the blurring kernel $h$.

A wavelet approximation decomposes $f$ in an orthonormal wavelet basis and reconstructs $f_M$ from a partial sum of $M$ wavelets corresponding to the largest amplitude coefficients. Over a class of functions whose total variation are uniformly bounded then one can prove [5] that

$$\|f - f_M\|^2 \leq C M^{-1},$$

$$\|f - f_M\|^2 \leq C M^{-2} \log^2 M.$$
The goal of this paper is to find an approximation \( f_M \) from \( M \) parameters that satisfies
\[
\| f - f_M \| \leq C M^{-\alpha} \] for any \( \alpha \) and any blurring kernel \( h \).

3. Approximation in Orthonormal Bandelet Bases.

3.1. Geometric Flow and Bandelet Bases.

An image having geometrically regular edges, as in the model of Section 2, has sharp transitions when moving across edges but has regular variations when moving parallel to these edges. This displacement parallel to edges can be characterized by a geometric flow which is a field of parallel vectors that give the local direction in which \( f \) has regular variations. Bandelet orthonormal bases are constructed by warping anisotropic wavelets bases with this geometric flow.

A geometric flow is a vector field \( \tilde{v}(x_1, x_2) \) which gives directions in which \( f \) has regular variations in the neighborhood of each \((x_1, x_2)\). In the neighborhood of an edge, the flow is typically parallel to the tangents of the edge curve. To construct an orthogonal basis with a geometric flow, we shall impose that the flow is locally either parallel in the vertical direction and hence constant in this direction or parallel in the horizontal direction. To simplify the explanations, we shall first consider horizontal or vertical models [9] (or boundary fragments [13]), which are functions \( f \) that include a single edge \( C \) whose tangents have an angle with the horizontal or vertical that remains smaller than \( \pi/3 \), so that \( C \) can be parameterized horizontally or vertically by a function \( g \).

Suppose that \( f \) is a horizontal horizon model. We define a vertically parallel flow whose angle with the horizontal direction is smaller than \( \pi/3 \). Such a flow can be written:
\[
\tilde{v}(x_1, x_2) = \tilde{v}(x_1) = (1, g(x_1)) \quad \text{with} \quad |g(x_1)| < 2.
\]

4
Horizontal horizon model with a flow induced by the edge and the corresponding band

\[ WB = \{ x_1, x_2 \mid x_1, x_2 + g(x_1) \in B \} \]

A band \( B \) and its warped band \( WB \)

\[ W f(x_1, x_2) = W f(x_1, x_2 - g(x_1)), \]

where \( W \) is an orthogonal operator, an orthonormal family warped with \( W \) remains orthonormal. By inverse warping, an orthogonal wavelet basis of the rectangle \( WB \) yields thus an orthogonal basis over the band \( B \) with basis functions having vanishing moments along the \( \psi \) lines.

A separable wavelet basis is defined from one-dimensional wavelet \( \psi_j \) and a scaling function \( \phi_j \), that are here chosen compactly supported, which are dilated and translated:

\[ \psi_{j,m}(t) = \sqrt{2} \psi\left(\frac{t - jm}{2^j}\right) \quad \text{and} \quad \phi_{j,m}(t) = \sqrt{2} \phi\left(\frac{t - jm}{2^j}\right). \]
whose tangents have an angle smaller than

deleting in the warped wavelet basis (3.9) yields a bandelet orthonormal basis of

\[ g \]

is an orthonormal basis of a multiresolution space which also admits an orthonormal basis of wavelets

\[ f \]

Following [15], the index

\[ j \]

could thus already include some anisotropic functions. This ensures that a polynomial

\[ l \]

is typically a negative integer. The family of separable wavelets

\[ m \]

vanishing moments but

\[ n \]

has an angle smaller than

\[ o \]

Following [15], the index

\[ p \]

has an angle smaller than

\[ q \]

is typically a negative integer. The family of separable wavelets

\[ r \]

vanishing moments but

\[ s \]

has an angle smaller than

\[ t \]

is typically a negative integer. The family of separable wavelets

\[ u \]

vanishing moments but

\[ v \]

has an angle smaller than

\[ w \]
We study the bandelet approximation of \( f \) where \( f \) has a support included in \([-s, s]^2\) and is \( C^\alpha \) with \( \|h\|_{C^\alpha} \leq s^{-2(2+\alpha)} \). The edge curve \( C \) is Hölderian of order \( \alpha \) and its tangents have an angle with the horizontal or vertical direction that remains smaller than \( \pi/\alpha \), with a distance larger than \( s \) from the horizontal (respectively vertical) boundary.

The bandeletization replaces each family of scaling functions \( \{\phi_k\}_{k \in \mathbb{Z}^2} \) by:

\[
\{ \phi_{j,m_1} x_1 - g x_2, \psi_{j,m_2} x_2 \} = \{ \psi_{j,m_1} x_1 - g x_2, \phi_{j,m_2} x_2 \} \quad (j,m_1,m_2) \in \mathbb{I}_{WB}.
\]

Definition 3.1. A function \( f \) is a \( C^\alpha \) horizon model over \( \mathcal{O} \) if

- \( f \) or \( f/h \) with \( f \in C^\alpha \) for 2 \(-\{\mathcal{O}\}\),
- the blurring kernel \( h \) has a support included in \([-s, s]^2\) and is \( C^\alpha \) with \( \|h\|_{C^\alpha} \leq s^{-2(2+\alpha)} \),
- the edge curve \( C \) is Hölderian of order \( \alpha \) and its tangents have an angle with the horizontal or vertical direction that remains smaller than \( \pi/\alpha \), with a distance larger than \( s \) from the horizontal (respectively vertical) boundary.

The bandelet approximation of \( f \) includes polynomials of degree \( p \) and \( g \) and \( \alpha \) times differentiable. The decomposition (3.15) defines a parameterized curve over a family of orthogonal scaling functions \( \{\phi_k\}_{k \in \mathbb{Z}^2} \) by parameterizing the curve integral:

\[
\int f(x) \phi_k(x) \, dx = \int f(x) \phi_k(x) \, dx = 0 \quad (k \in \mathbb{Z}^2) \quad \text{or} \quad f(x) = \int f(x) \phi_k(x) \, dx.
\]

A bandelet basis is constructed from a geometric curve. To optimize the approximation of \( f \) by setting \( f \), the edge curve \( C \) is specified with as few parameters as possible. A vertically parallel curve is specified according to the following definition introduces such geometrical regular functions according to the parameters in a bandelet basis.

3.2. Bandelet Approximation in a Band. We require \( \mathcal{O} \) on of \( \mathcal{O} \) and \( \mathcal{O} \) on. The space \( \mathcal{O} \) is defined as the set of points within a distance \( s \) from the horizontal (respectively vertical) boundary.

The bandelet approximation replaces each family of scaling functions \( \{\phi_k\}_{k \in \mathbb{Z}^2} \) by:

\[
\{ \phi_{j,m_1} x_1 - g x_2, \psi_{j,m_2} x_2 \} = \{ \psi_{j,m_1} x_1 - g x_2, \phi_{j,m_2} x_2 \} \quad (j,m_1,m_2) \in \mathbb{I}_{WB}.
\]
one can use a bandelet orthonormal basis over a band $P$.

Let us now construct a bandelet orthonormal basis defined in (3.10). An approximation of the orthonormal bandelet basis defined in (3.10) is obtained by keeping only the bandelet coefficients above a threshold $\tau$. The approximation error in a bandelet basis defined by the integral $g$ on $\mathcal{D}$ depends upon the order of the algorithm that is used.

Let $P_{V_k}$ be an orthonormal bandelet basis defined in (3.10). An approximation of the bandelet coefficients above a threshold $\tau$ on an edge is defined by

$$\tilde{g}(\theta, k, m) = \alpha_m \langle \hat{g}, \theta, k, m \rangle \| \theta, k, m \|^2.$$ 

Theorem 3.2. Let $f$ be a $C^p$ horizon model with an edge parameterized by $c$ and $\alpha \leq \alpha < p$. Let $\hat{g}$ be an edge estimation such that $\| \hat{g} - c \|_{\infty} \leq C_d \, s$. There exists $C$ such that for any threshold $T$, the approximation error in a bandelet basis defined by the flow integral $g$ $P_{V_k}$ $\hat{g}$ with $k$ on $\mathcal{D}$ satisfies

$$\| f - \tilde{f} \|_{\infty}^2 \leq C C_f^2 \alpha + 1 \, M^{-\alpha}$$

with $C_f, \alpha, C_d, \alpha, C_s,$ and $\alpha$.

Proof [Theorem 3.2]
e. The approximation error satisfies

\[ \| f - f_M \|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)} \]

and

\[ M \leq m \quad C C_f^{2/(\alpha+1)} \ell_1 T^{-2/(\alpha+1)}, C | o_2 T| \]

**Proposition 3.3.** Under the hypotheses of Theorem 3.2, there exists \( C \) that only depends upon the edge geometry such that for any threshold \( T \leq s^{(\alpha+1)/(2\alpha)} \)

\[ \sum_{b_m \in B_1, \| (f,b_m) \| < T} |(f,b_m)|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)} \]

and

\[ M_{B,1} \leq m \quad C C_f^{2/(\alpha+1)} \ell_1 T^{-2/(\alpha+1)}, C \]

**Lemma 4.** Under the hypotheses of Theorem 3.2, there exists a constant \( C \) such that

\[ \sum_{b_m \in B_2, \| (f,b_m) \| < T} |(f,b_m)|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)} \]

and

\[ M_{B,2} \leq m \quad C C_f^{2/(\alpha+1)} \ell_1 T^{-2/(\alpha+1)}, C | o_2 T| \]

**Lemma 5.** Under the hypotheses of Theorem 3.2, there exists a constant \( C \) such that

\[ \sum_{b_m \in B_2, \| (f,b_m) \| < T} |(f,b_m)|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)} \]

and

\[ M_{B,2} \leq m \quad C C_f^{2/(\alpha+1)} \ell_1 T^{-2/(\alpha+1)}, C | o_2 T| \]
\[\langle f, b_{i,j,m} \rangle = \langle f, x_1 + x_2 - x_1, \psi_{i,m_1} x_1, \psi_{i,m_2} x_2 \rangle = \langle W f, x_1, 2x_1, \psi_{i,m_1} x_1, \psi_{i,m_2} x_2 \rangle.\]

A common way to prove \[\|\|f\|\|_{L^p} \leq C \|\|g\|\|_{L^p}\]
for \(f \in C^k\) and \(g \in C^\alpha\) is to use a decay of \(\|\|f\|\|_{L^1} \leq C \|\|g\|\|_{L^1}\)
for \(f \in C^k\) and \(g \in C^\alpha\) as stated by the following lemma proved in Appendix A.3:

\[\exists C\|\|f\|\|_{L^1} \leq C \|\|g\|\|_{L^1}\]

Moreover, The proof of Lemma 3.4 in Appendix A.1 combines this lemma with the regularity along the axis, a control on the regularity of \(f\) in a neighborhood of a point \((x_1, x_2, \psi_{i,m_1} x_1, \psi_{i,m_2} x_2)\) as stated by the following lemma proved in Appendix A.3:

\[\exists C\|\|f\|\|_{L^1} \leq C \|\|g\|\|_{L^1}\]

An immediate consequence of this lemma is that \(f\) is regular in a neighborhood of the smoothed singularity is required. This is given by the following lemma proved in Appendix A.4:

\[\exists C\|\|f\|\|_{L^1} \leq C \|\|g\|\|_{L^1}\]

Combining (3.22), (3.25), (3.26) (3.27) and (3.28) allows to conclude.
Lemma 3.7. Suppose $f$ is a $C^\alpha$ horizon model with $s > 1$. If $g$ satisfies

$$\forall \beta < \alpha, \quad \|g - c^{(\beta)}\|_\infty \leq C \|c\|_{C^\alpha, C_d}, \quad s^{1-\beta/\alpha}$$

and

$$\forall x \quad \|g - c^{(\beta)}(x) - g - c^{(\beta)}(x_0)\|_\infty \leq C \|c\|_{C^\alpha, C_d}, \quad |x - x_0|^{\alpha - \beta}.$$

then

$$\left| \frac{\partial |c|}{\partial x_1} \right| Wf(x_1, x_2) - \frac{\partial |c|}{\partial x_1} \right| Wf(x_1, x_2) \leq C \|f\|_{C^\alpha(L)} \|c\|_{C^\alpha, C_d}, \quad \|c\|_{C^\alpha, C_d}, \quad s^{-1} |x'_1 - x_1|^{\alpha - \beta}.$$

Proof [Proposition 3.8]

Under the hypotheses of Theorem 3.2, there exists a constant $C$ such that for all threshold $T$ the resulting bandelet approximation error satisfies

$$\|f - f_M\|_2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}$$

and

$$M \leq C C_f^{2/(\alpha+1)} \ell_1 T^{-2/(\alpha+1)}, C \|2 T\|.$$

LEMMA 3.9. If $f$ is as $C^\alpha$ horizon model, there exists a constant $C$ such that

$$\|\|f\|_{C^\alpha(L)}\|_1 \leq C \|f\|_{C^\alpha(L)} \ell_1 s.$$

Proof [Lemma 3.9]

$$\|f - f_{mod}\|_2^2 \leq C \|f\|_{C^1(L)}^2 \ell_1 T^{2\alpha/(\alpha+1)}$$

and

$$M_{mod} \leq C C_f^{2/(\alpha+1)} \ell_1 T^{-2/(\alpha+1)}, C \|2 T\|.$$
Let $J'$ be a set of bandelets such that

\[ |\langle f, u_m \rangle| \geq T \]

\[ \| f - f_{\text{mod}} \|_2^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)} \]

one can verify

\[ \sum_{m \in J} |\langle f, u_m \rangle|^2 \leq \sum_{m \in J} |\langle f_{\text{mod}}, u_m \rangle|^2 + \sum_{m \notin J'} |\langle f - f_{\text{mod}}, u_m \rangle|^2 \]

\[ \leq \| f_{\text{mod}} - f_{\text{mod, Mmod}} \|_2^2 + \| f - f_{\text{mod}} \|_2^2 \]

\[ \sum_{m \notin J'} |\langle f, u_m \rangle|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)} \]

**Lemma 3.10.** Let $\{u_m\}_{m \in J}$ be a family of functions and $J_T \{m \mid |\langle f, u_m \rangle| > T\}$. If $J' \subset J$ then

\[ \sum_{m \notin J'} |\langle f, u_m \rangle|^2 + T^2 \text{Card } J_T \leq \sum_{m \notin J'} |\langle f, u_m \rangle|^2 + T^2 \text{Card } J' \]

**Proof** [Lemma 3.10] or exactly $J' \subset J$

\[ \sum_{m \notin J'} |\langle f, u_m \rangle|^2 + T^2 \text{Card } J' \sum_{m \in J} - 1_{m \in J'} |\langle f, u_m \rangle|^2 + 1_{m \in J'} T^2 \]

**4. Bandelet Frames and Approximations.**

**4.1. Bandelet Frames.**
or a single edge whose tangents have a smaller angle than $=3\pi$ with the horizontal or with the vertical, or $f$ has an edge junction. This is illustrated by Figure 4.1 with square regions. The size of squares become smaller in the neighborhood of junctions. Bandelet bases are defined over bands that include the regions $i$ and it is shown that their union defines a frame of $L^2$. [Fig. 4.1. Partition of an image]

If $f$ is uniformly regular in a region $i$ then there is no need to define a geometric ow. Similarly, if $f$ has an edge junction in $i$ then there is no geometric regularity and no appropriate geometric ow can be defined. In both cases, $f$ is decomposed in a separable wavelet basis $B_i = \{b_{i;m}\}$, that is constructed over the smallest rectangle $B_i$ that includes $i$. If $f$ is a horizon model over $i$ then a vertically or horizontally parallel geometric ow is defined over $i$. Let $B_i$ be the most narrow band parallel to the ow in $i$ and that includes $i$. Figure 4.2 gives an example. Section 3.1 explains how to construct a bandelet orthonormal basis $B_i = \{b_{i;m}\}$ of $L^2(B_i)$. The following proposition proves that the union of such orthonormal bases over a segmentation of $[0;1]^2$ defines a frame of $L^2([0;1]^2)$, that is called a bandelet frame. We write $P_i f x \begin{cases} f x \in \Omega_i & f x \notin \Omega_i \end{cases}$. [Fig. 4.2. A square $\Omega_i$ with a flow and its associated minimum band $B_i$.]
Proposition 4.1. For any segmentation, \( f \subseteq \bigcup_i \mathcal{B}_i \) and \( f \in L^2 \),

\[
f \sum_{i,m} \langle f, b_{i,m} \rangle b_{i,m} \quad \text{with} \quad b_{i,m} \quad P_{i,m} b_{i,m}.
\]

and

\[
\|f\|^2 \leq \sum_{i,m} |\langle f, b_{i,m} \rangle|^2.
\]

If the sup over all \( x \in \mathbb{R}^2 \) of the number of bands \( B_i \) that includes \( x \) is a finite number \( A \) then

\[
\|f\|^2 \geq \frac{A}{A} \sum_{i,m} |\langle f, b_{i,m} \rangle|^2.
\]

and the union of bandelet bases \( \mathcal{F} = \bigcup_i \mathcal{B}_i \) is a frame of \( L^2 \).

Proof. [Proposition 4.1] since \( B_i = \{b_{i,m}\}_m \) and \( \sum_{m} \langle f, b_{i,m} \rangle b_{i,m} \) More so \( \mathcal{F} \subseteq \mathcal{B}_i \) and \( \sum_{m} \langle f, b_{i,m} \rangle b_{i,m} \).

\[
f \sum_{i} P_{i,m} f \sum_{i,m} \langle f, b_{i,m} \rangle P_{i,m} b_{i,m}.
\]

The scheme provides a direct reconstruction of an image from its bandelet coefficients. The discontinuous nature of the border \( \sim b_{i,m} \) leads to blocking effects which have to be avoided in image processing, but which do not degrade the error decay.

To suppress these discontinuities, the reconstruction can be computed with the classical iterative frame algorithm on the full bandelet frame. This would yield a smaller approximation error and may avoid the blocking effect but requires an iterative reconstruction algorithm. Another solution is proposed in [14], where the bandelets themselves are modified in order to cross the boundaries, which removes the blocking artefacts. The bandelet lifting scheme removes the blocking effects but we then have no proof that the resulting best basis algorithm described in the next section yields an approximation whose error decay is optimal for geometrically regular functions.

4.2. Approximation in a Dictionary of Bandelet Frames. A \( \mathcal{D} \) of \( \mathcal{F} \) is defined by geometric parameters that specify the segmentation of the image support into subregions \( i \) and by the geometric \( \mathcal{D} \) in each \( i \). A dictionary of bandelet frames is constructed with segmentations in dyadic square regions and a parameterization of the geometric \( \mathcal{D} \) in each region. Within this dictionary, a best bandelet frame is defined by minimizing a Lagrangian cost function. The main
Theorem computes the error when approximating a geometrically regular function in a best bandelet frame.

Like in the wedgelet bases of Donoho [9], the image is segmented in dyadic square regions obtained by successive subdivisions of square regions into four squares of twice smaller width. For a square image support of width $L$, a square region of width $L/2$ is represented by a node at the depth $j$ of a quad-tree. A square subdivided into four smaller squares corresponds to a node having four children in the quad-tree. Figure 4.3 gives an example of a dyadic square image segmentation with the corresponding quad-tree.

In each dyadic square $i$ of size $2^i$, a variable indicates if the basis is a wavelet or bandelet basis and in this last case whether the geometric $\nu$ is constant vertically or horizontally. If it exists, the $\nu$ is characterized by the decomposition (3.15) of an integral curve over a family of scaling functions at a scale $2^k_i$.

Let $F = \{B_i\}$ be the bandelet frame resulting from such a dyadic segmentation of $[0;1]$ and such a parameterized $\nu$. An approximation of $f$ in $F$ is obtained by keeping only the bandelet coefficients above a threshold $T$:

$$f_M = \sum_{i,m} \langle f, b_{i,m} \rangle \tilde{b}_{i,m},$$

with $\tilde{b}_{i,m} = \sum_{i} \langle f, b_{i,m} \rangle \tilde{b}_{i,m}$.

The total number of parameters is $M = M_S + M_G + M_B$ where $M_S$ is the number of geometric parameters that define the dyadic image segmentation, $M_G$ is the number of parameters that define the geometric $\nu$s in all the dyadic squares and $M_B$ is the number of bandelet coefficients above $T$. Optimizing the frame means finding $F$, that depends upon $f$ and $M$, in a dictionary $D$ of bandelet frames so that

$$|f - f_M| \leq CM^{-\beta},$$

for some $M$. For $|f - f_M|$, one gets

$$|f - f_M| \leq \sum_{i,m} |\langle f, b_{i,m} \rangle|^2.$$

Figure 4.3: A dyadic square segmentation and its corresponding quad-tree.
The Lagrangian multiplier is of wedgelet approximations:

\[ L \approx \frac{1}{T^2} \sum_{(f,b_i,m)} |(f,b_i,m)|^2 + T^2 M \quad M = M_S + M_G + M_B \]

where \( M \) is the blurring kernel \( M \) and \( M \) is the approximation error of the blurring kernel \( M \).

There exists a best bandelet frame \( \mathcal{F} \) such that only depends upon the edges geometry such that for any \( T \) > the thresholding approximation of \( f \) in a best bandelet frame of \( D_{T^2} \) yields an approximation \( f_{T^2} \) that satisfies

\[ \| f - f_{T^2} \|^2 \leq C T^2 \| \mathcal{F} \| \| \mathcal{F} \|^2 \| f \|^2 \]
The remarks on the non-optimality of the constant \( C_f \) of Theorem 3.2 apply here. In addition, the dependency of \( C \) on the edges geometry is not explicitly controlled but involves the number of curves \( G \) as well as the geometric configuration (distance, angle of the crossing, ...).

Proof [Theorem 4.3] The theorem proof is based on the following lemma which exhibits a bandelet frame in \( \mathbb{D}_T^2 \) having suitable approximation properties when \( \alpha > 1 \) and thus the edges have tangents. The remaining case, \( \alpha = 1 \), is obtained with the classical wavelet basis that is included in the dictionary.

Lemma 4.4. Under the hypotheses of Theorem 4.3 and if \( \alpha > 1 \), for any \( T > 0 \) there exists a bandelet frame \( F \in \mathbb{D}_T^2 \) such that

\[
\mathcal{L}(f, T, F) \leq C C_f^{2/\alpha} \ell_C T^{2\alpha/(\alpha+1)},
\]

for a constant \( C \) that only depends upon the edges geometry.

\[
\sum_{\substack{b_{i,m}' \in F' \setminus (f, b_{i,m}) < T}} |(f, b_{i,m}')|^2 + T^2 M \leq C C_f^{2/\alpha} \ell_C T^{2\alpha/(\alpha+1)}.
\]

A more general result is

\[
\sum_{\substack{b_{i,m}' \in F' \setminus (f, b_{i,m}) < T}} |(f, b_{i,m}')|^2 \leq C C_f^{2/\alpha} \ell_C T^{2\alpha/(\alpha+1)} + M^{-\alpha}.
\]

Proof [Lemma 4.4] The core of Lemma 4.4 is designed in each square so that the resulting bandelet frame \( F \) satisfies (4.13). This segmentation is constructed by separating squares that are close to an edge from the others. Regular squares \( i \in I_R \) are squares which are distant by more than \( s \) from all edges \( C \). In such squares, \( f \) is uniformly regular. No geometric knot is defined in regular squares, which means that the bandelet basis is a separable wavelet basis.
Let \( q \) be a \( \in \mathbb{N} \) if \( e \) is a \( \in \mathbb{N} \) then we define \( q \) and \( e \) in \( \mathbb{N} \) or \( \mathbb{Z} \) and \( \mathbb{C} \) respectively.

By definition, if \( e \) and \( q \) are both \( \in \mathbb{N} \), then \( q \) is called \( \mathbb{N} \) and \( e \) is called \( \mathbb{Z} \) or \( \mathbb{C} \).

We shall verify that \( e \) and \( q \) are both \( \in \mathbb{N} \) and \( e \) is called \( \mathbb{Z} \) or \( \mathbb{C} \).

The following algorithm constructs a dyadic image segmentation by labeling regions.

1. **Initialization:** Label the square \([0, 1] \times [0, 1]\) and \( g \) as \( I \) if its size is larger than \( \eta \).

2. **Step 1:** Split in four every square otherwise.

3. **Step 2:** If \( g \) is a square and \( e \) is a \( \in \mathbb{N} \), then we can define \( g \) as a horizontal edge square by default.

4. **Temporary:** If \( e \) is an even \( \in \mathbb{N} \), then we define \( g \) as a horizontal edge square by default.

5. **Vertical edge:** If \( e \) is an odd \( \in \mathbb{N} \), then we define \( g \) as a vertical edge square by default.

6. **Junction:** If \( e \) is neither even nor odd \( \in \mathbb{N} \), then we define \( g \) as a junction square.

7. **Regular:** If \( e \) is a \( \in \mathbb{N} \) and \( q \) is a \( \in \mathbb{N} \) then we define \( g \) as a regular square.

8. **Temporary:** If \( q \) is a \( \in \mathbb{N} \) and \( e \) is a \( \in \mathbb{N} \) then we define \( g \) as a temporary square.

\[ Q_{T_x} x = qT^2 \quad q - / \quad T^2 \leq x < q + / \quad T^2 \quad x \in \mathbb{Z}. \]
Fig. 4.4. Partition with dyadic square labeled $\mathcal{J}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{V}$.

Fig. 4.5. Close up on a junction zone with an intermediate partition with dyadic square labeled $\mathcal{R}$, $\mathcal{H}$, $\mathcal{V}$ and temporary square.

- **Step 3**

$\mathcal{T} = \{\text{temporary squares}\}$

From such a dyadic image partition, we associate a bandelet frame $\mathcal{F}$ as the union of the bandelet bases defined with the geometric construction (or not) in each square depending upon their label. We now prove that the resulting Lagrangian satisfies the property (4.13) of Lemma 4.4. To evaluate the Lagrangian,

$$L(f;T;\mathcal{F}) = \sum_{b, i, m} |\langle f, b_i, m \rangle|^2 + T^2 M_{S} + M_{G} + M_{G}$$

we separate the geometrical cost $M_{S} + M_{G}$ and decompose it into:

$$L(f;T;\mathcal{F}) = T^2 M_{S} + M_{G} + \sum_{i \in I_{\mathcal{R}}} \hat{L} f, T, B_{i} + \sum_{i \in I_{\mathcal{J}}} \hat{L} f, T, B_{i}$$

$$+ \sum_{i \in I_{\mathcal{H}}} \hat{L} f, T, B_{i} + \sum_{i \in I_{\mathcal{V}}} \hat{L} f, T, B_{i}$$

\[19\]

$$\hat{L} f, T, B_{i} = \sum_{b, i, m \in B_{i}} |\langle f, b_i, m \rangle|^2 + T^2 M_{B,i}$$
where $M_{B,i}$ is the number of inner products $j_h f; b_{i,m}$ for a fixed $i$.

To prove that $L(f; T, B_i) = C C_f^2(\alpha + 1) T^{2\alpha/(\alpha + 1)}$, each of the four partial sums corresponding to different classes of squares in (4.21) shall be proved.

The first lemma characterizes the dyadic segmentation obtained by our splitting algorithm and computes the resulting number of geometric parameters. Its proof can be found in Appendix B.1.

Lemma 4.5. There exists a constant $C$ that depends upon the edges geometry such that the resulting dyadic image segmentation defined recursively includes at most $C j \log_2 \max\{s, T^2\} = (\alpha + 1) j s$ squares with at most $C$ junction squares. Furthermore

\begin{equation}
T^2 (M_S + M_G) \leq C \ell_\infty \max\{1, C\} T^2 = (\alpha + 1). \tag{4.23}
\end{equation}

For junction squares the Lagrangian value is calculated using the fact that there are few such squares and they have a small size bounded by $\max\{s, T^2\}$. The lemma's proof is in Appendix B.3.

Lemma 4.7. There exists a constant $C$ that depends upon the edges geometry such that the sum of the partial Lagrangian $\hat{L}_i$ over all junction square $i$ satisfies

\begin{equation}
\sum_{i \in \mathcal{J}} \hat{L}_i f, T, B_i \leq C C_f^2(\alpha + 1) T^{2\alpha/(\alpha + 1)}. \tag{4.25}
\end{equation}

Transposing this result to vertical edge squares proves that

\begin{equation}
\sum_{i \in \mathcal{V}} \hat{L}_i f, T, B_i \leq C C_f^2(\alpha + 1) T^{2\alpha/(\alpha + 1)}. \tag{4.27}
\end{equation}

Inserting (4.23), (4.24), (4.25), (4.26) and (4.27) in equation (4.21) proves the result of Lemma 4.4.
This theorem provides a constructive approximation scheme with an optimal approximation bound. The decay rate $M$ of the error is optimal as it is the same as the optimal one for uniformly $C$ functions. It is much better than the decay rate $M_1$ for the wavelets and improves the decay rate $(\log_2 M)^3$ of the curvelets even if $\lambda = 2$. Furthermore, the Lagrangian minimization does not require any information on the regularity parameter $\alpha$ or on the smoothing kernel $\eta$. However, an exhaustive search to minimize the Lagrangian in the dictionary $D_T^2$ requires an exponential number of operations which prohibits its practical use. Section 6.2 introduces a modified dictionary and a fast algorithm that finds a best bandelet basis with a polynomial complexity, at the cost of adding a logarithmic factor in the resulting approximation error.

5. Image Compression.

For image compression, we must minimize the total number of bits $R$ required to encode the approximation as opposed to the number of parameters $M$. An image is compressed in a bandelet frame by first coding the segmentation of the image support and a geometric "now" in each region of the segmentation. The decomposition coefficients of the image in the resulting bandelet frame are then quantized and stored with a binary code. This very simple algorithm does not provide a scalable scheme but does give an almost optimal distortion-rate.

We denote by $R$ the resulting total number of bits to encode a bandelet frame and the bandelet coefficients $f$ in this frame. It can be decomposed into

$$R = R_S + R_G + R_B$$

where $R_S$ is the number of bits to encode the dyadic square segmentation, $R_G$ is the number of bits to encode the "now" in each square region and $R_B$ is the number of bits to encode the quantized bandelet coefficients. We saw in Section 4.2 that a dyadic square segmentation of $[0; 1]$ is represented by a quadtree whose leaves are the square regions of the partition. Each interior node of the tree corresponds to a square that is split in four subsquares. This splitting decision is encoded with a bit equal to 1. The leaves of the tree correspond to squares which are not split, which is encoded with a bit equal to 0. With this code, the number of bits $R_S$ that specifies the segmentation quadtree is thus equal to the number of nodes of this quadtree.

Over a square of size $2^k$, the geometric "now" is parameterized at a scale $2^k$ in (3.15) by $2^k$ quantized coefficients $m = qT$ with $|q| \leq 2$. We thus need $2^k \log_2 (C T)$ bits to encode these $m$. The number of bits $R_G$ to encode the "nows" is the sum of these values over all squares where a "now" is defined plus the number of bits required to specify the scale.

In a bandelet frame $F = f_{b; i;m}$, all bandelet coefficients $h f_{b; i;m}$ are uniformly quantized with a uniform quantizer $Q_T$ of step $T$:

$$Q_T(x) = q_T if \frac{q_T - 1}{2} \leq x < \frac{q_T}{2}$$

The indices $i,m$ of the $M_B$ non-zero quantized coefficients are encoded together with the value $Q_T(h f_{b; i;m})$. The proof of Theorem 4.3 shows that the $M_B$ non-zero quantized bandelet coefficients whose amplitude is larger than $T$ appear at a scale larger than $C T$. Since there are $C T^2$ such coefficients, to encode an index $i,m$ is equivalent to encode an integer in $[1; C T^2]$ which requires $\log_2 (C T^2)$ bits. Since $\|h f_{b; i;m}\| > T$, each non-zero quantized coefficient is
encoded \( R_B \leq M_B \| f \|_2 \) and from (5.1) we encode \( f \) in the frame that minimizes

\[
J(f) = \sum_{i,m} |\langle f, b_{i,m} \rangle|^2 + M_B T^2 / \ell_c.
\]

Inserting this bound in (5.6) implies

\[
D \leq C J^{2/(\alpha + 1)} + \ell_c T^{-2/(\alpha + 1)} | f R^\alpha |
\]

with

\[
R \leq C J^{2/(\alpha + 1)} + \ell_c T^{-2/(\alpha + 1)} | f R^\alpha |
\]

where \( C, J \) are the constants from Lemma 4.4. The following theorem computes the resulting decay of \( D(R) \) as a function of \( R \) for a geometrically regular image in a best bandelet frame.

**Theorem 5.1.** Let \( f \) be a \( C^\alpha \) geometrically regular function and \( \alpha < \beta \). There exists \( C \) that only depends upon the edges geometry such that for any \( T > 0 \), coding \( f \) in a best bandelet frame in \( D T^2 \) that minimizes \( J(f, T^2, \beta) \) over \( D T^2 \) yields a distortion-rate that satisfies

\[
D \leq C J^{2/(\alpha + 1)} + \ell_c T^{-2/(\alpha + 1)} | f R^\alpha |
\]
A CCD camera is an array of pixels obtained by averaging the input analog intensity. We know that for such a class of images the Kolmogorov lower bound of the distortion reaches the Kolmogorov lower bound up to the factor \(C_{\max} = \frac{1}{2} \log_2 f + 1\) for \(f > 1\) and \(C_{\max} = 0\) for \(f = 1\). The resulting discretized image values are \(R \leq 2 \cdot 2 \cdot C_{\psi} \|f\|_\infty T^{-2} + T^{-2} T^{2n/(\alpha+1)} \) for \(n \geq 1\).

**Lemma 6.1**: Discrete Orthogonal Bandelet Bases. This theorem proves that the asymptotic decay of a bandelet transform code \(D \leq C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)}\) for \(n \geq 1\).

As the number of splits that specify the best segmentation is of order \(\log_2 n\), the resulting discretized image values are obtained by quantizing the coefficients. As shown in Lemma 4.5, there are \(R \leq C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)}\) bits needed to specify the scale \(2^{n+1}\) of the quantized coefficients. As \(n \to \infty\), the number of bits to encode the dyadic square segmentation, the indicator function of the square \([0,1]^2\), is decomposed in (5.1) in 3 terms: (i) the number of bits to encode the quantized coefficients \(\ell_C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)}\), (ii) the number of bits to encode the quantized coefficients \(\ell_C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)}\), and (iii) the number of bits to encode the quantized coefficients \(\ell_C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)}\).

6. **Discretized Image and Discretized Bandelets.**

6.1. **Discrete Orthogonal Bandelet Bases.** A discrete bandelet basis \(\psi_{n,m}\) can be obtained by extending the dyadic square \([0,1]^2\) and by choosing \(\psi_{n,m}\) to be \(C\) or \(\infty\), depending on the desired number of coefficients. The resulting discrete bandelet basis \(\psi_{n,m}\) is a discrete version of the bandelet transform, with \(\psi_{n,m}\) denoting the bandelet coefficients.

\[
\ell_C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)} \leq C \|f\|_\infty T^{-2} T^{2n/(\alpha+1)}
\]
The operator $T$ defined from this warping over the sampling grid by:

$$
T \phi_{j,m} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \phi_{j,m} \ast \psi_{j,m,n}
$$

with the fast discrete wavelet transform of $f$.

Parameterized in a basis of scaling functions according to (3.15). Suppose that the analog image $x$ is an orthonormal wavelet basis of signals defined in the band $B_{1/2}$. In the following we shall suppose that $x$ has a support in $B_{1/2}$.

If $x$ has a support in $B_1$, then

$$
\phi_{j,m} = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \phi_{j,m} \ast \psi_{j,m,n}
$$

where $\psi_{j,m,0} = \phi_{j,m}$.

For $0 \leq n_1, n_2 \leq 2^{-j}$, define a parallelogram $R_{n_1,n_2}$.

Discretized orthogonal bandelets are defined with the same approach as in Section 6.1. The discrete separable orthonormal wavelet basis corresponding warped wavelet coefficients of $f$ are computed by applying a one-dimensional discrete wavelet transform over the rectangle $B_{1/2} \times B_1$.

In the following, a discrete bandelet basis with a horizontally parallel or vertically parallel wavelet is defined. By transposing the procedure previously described and exchanging $x$ correspoding equivalent family of discrete bandelets.

We define a vertically parallel wavelet that approximates the tangents of this edge. $C \subset B$ parallel to this wavelet is defined from the warped over $R_{n_1,n_2}$.

$$
W_{f,n_1,n_2} f \left[ n_1, n_2 + \lfloor \epsilon^{-1} g n_1 \rfloor \right],
$$

where $n_1, n_2 \in \mathbb{Z}$, $g \geq 1$, and $B$ is the surface of the band $B_{1/2}$. Using the fact $B_{1/2} \times B_1$ operations are used for computing $x$, some of these operations are redundant. We can compute equivalent bands in this way.

The edge $C \subset B$ parallel to this wavelet is defined from $B_{1/2}$.

$$
\psi_{j,m,n_1} \psi_{j,m,n_2}, \quad \psi_{j,m,n_1} \psi_{j,m,n_2}
$$

$$
\left( \psi_{j,m,n_1} \psi_{j,m,n_2} \right)_{(j,m_1,m_2) \in \mathbb{Z}^2}
$$

and hence

$$
\left( \psi_{j,m,n_1} \psi_{j,m,n_2} \right)_{(j,m_1,m_2) \in \mathbb{Z}^2}
$$

and $o_{\psi_{j,m,n_1} \psi_{j,m,n_2}}$. In the following we shall suppose that $x$ has a support in $B_{1/2}$.

$$
\left( \psi_{j,m,n_1} \psi_{j,m,n_2} \right)_{(j,m_1,m_2) \in \mathbb{Z}^2}
$$

and $o_{\psi_{j,m,n_1} \psi_{j,m,n_2}}$. In the following we shall suppose that $x$ has a support in $B_{1/2}$.

$$
\left( \psi_{j,m,n_1} \psi_{j,m,n_2} \right)_{(j,m_1,m_2) \in \mathbb{Z}^2}
$$

and $o_{\psi_{j,m,n_1} \psi_{j,m,n_2}}$. In the following we shall suppose that $x$ has a support in $B_{1/2}$.
6.2. Approximation in a Best Discrete Bandelet Frame. A discrete ban-
edele frame is defined as a union of discrete orthogonal bandelet bases constructed 
on the regions of a dyadic square image segmentation: \([0, 1)^2 = [i, i]. This section 
proves that the approximation result of Theorem 4.3 remains valid for discrete images 
decomposed in a discrete bandelet frame.

In each square \(i\), we either construct a separable discrete wavelet basis or a 
bandelet basis over the smallest band \(B_i\) including \(i\) and which is parallel to the 
now \(i\). One can verify that \(\#B_i \leq \#i\). The wavelet or bandelet basis associated 
to \(i\) is written \(B_i = \{b_{i,m}\}\) and the union of these bases \(F = \{B_i\}\) is a discrete 
frame. The corresponding discrete wavelet or bandelet coefficients of \(f\) are computed 
with a fast algorithm that requires \(O(\#i^2)\) operations. Since \(\sum_i \#B_i = 1\) the fast 
transforms over all dyadic squares of \([0, 1)^2\) requires \(O(2^2)\) operations.

Let \(P_i\) be the orthogonal projector over signals having a support in 
\(i\). We verify, as in Proposition 4.1, that \(f = \sum_i \sum_m h_{f, b_{i,m} i} P_{i, b_{i,m}}\) \(\tag{6.6}\)

and \(\|f\|^2 \leq \sum_i \sum_m |(f, b_{i,m})|^2\). \(\tag{6.7}\)

An approximation \(f_M\) is obtained by keeping all coefficients above a threshold 
\(T\): \(f_M = \sum_i \sum_m h_{f, b_{i,m} i} P_{i, b_{i,m}}(\tag{6.8}\)

The total number of parameters is \(M = M_S + M_G + M_B\) where 
\(M_S\) is the number of 
parameters describing the segmentation, \(M_G\) is the number of geometric parameters 
describing the now and \(M_B\) is the number of bandelet coefficients above \(T\).

To minimize the error \(\|f - f_M\|\), as in Section 4.2, we search for a best bandelet 
frame which minimizes the Lagrangian:

\[L(f; T; F) = \sum_{b_{i,m}} (f, b_{i,m})^2 + T^2 M \text{ with } M = M_S + M_G + M_B. \tag{6.9}\]

The complexity of the best bandelet frame search in Section 4.2 is driven by the 
complexity to find the best geometric now in a square, which is exponential. This 
exponential complexity makes it impossible to use such a best bandelet search algo-
rithm in numerical computations. A polynomial complexity algorithm is introduced 
by choosing geometric nows that are piecewise polynomial.

In the bandelet dictionary of Section 4.2, over a square \(n\) of size \(2^k\), the now is 
parameterized in a family of scaling functions (3.15) at a scale \(2^k\). We replace such a 
ow by a piecewise polynomial now over the \(2^k\) intervals of sizes \(2^k\):

\[g(t) = \sum_{n=1}^{r} \sum_{r;n} (2^k t - r)^{n-1} \sum_{r;f} \sum_{r;m} (2^k t - r)^{m-1}. \tag{6.10}\]
where $f_{n}^{1}$ is an orthogonal basis of the space of polynomials of degree $p$ over $[0, 1]$ and which vanish at $0$. However, instead of considering this as a single square of width $2$, we shall view it as a subdivision of into rectangles of length $2$ and width $2$, inside each of which the now is a polynomial parameterized by $f_{n;r}^{1} p$.

To construct an image partition, we first begin with a dyadic square segmentation $[0, 1] = \bigcup_{i} B_{i}$. If $i$ has a geometric now, then it is subdivided into $2^{k_{i}}$ sub-rectangles $i;r$ having polynomial geometric nows. An orthogonal bandelet basis $B_{i;r} = \{b_{i;r;m}\}$ is defined over each band $B_{i;r}$ associated to a sub-rectangle $i;r$. The union of these bandelet orthogonal bases defines a bandelet frame $F_{i}$ over $i$:

$$F_{i} = \bigcup_{r=1}^{2^{k_{i}}} B_{i;r}.$$

If $i$ has no geometric now then $F_{i}$ is the discrete separable wavelet basis defined over $i$. The union of these families of bandelets for all $i$ defines a bandelet frame $F = \bigcup_{i} F_{i}$ over the image support such that

$$f = \sum_{b_{i;r;m} \in F} h_{f; b_{i;r;m}}(6.11)$$

and

$$\|F\|^{2} = \sum_{b_{i;r,m} \in F} |(F, b_{i;r,m})|^{2}.\]
with the corresponding best segmentation configuration. Appendix C proves that the complexity of this algorithm is polynomial in $O(T_2^2p)$. The exponent $p$ of this polynomial complexity algorithm depends upon maximum the degree of the polynomial $f$ now.

In the following, we shall suppose that $p$ to be equal to the number of vanishing moments of the wavelet that is used to construct the bandelet bases. Despite the image discretization, the following theorem proves that the best bandelet frame computed with piecewise polynomial $f$ nows yields an approximation error that has the same optimal asymptotic decay as in Theorem 4.3, up to a logarithmic factor.

**Theorem 6.1.** Let $f$ be a $C^\alpha$ geometrically regular function and $\overline{f}$ be its discretization at a scale $\epsilon$. For any $\epsilon > 0$, there exists a constant $C_\epsilon$ that only depends upon the edges geometry and such that for any $T$, one can construct a bandelet frame $F_M$ that satisfies

$$
\|\overline{f} - \overline{f}_M\|^2 \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

with

$$
M \leq C_\epsilon C_f^2(\alpha + 1)M_1, \ell_c T^{-2/(\alpha + 1)} | o_2 T |,
$$

were $C_f$ is $\|f\|_{C^\alpha(\Lambda)}$, $\|c\|_{C^\alpha(\Lambda)} - 2$, $\|c\|_{C^\alpha(\Lambda)}$, $\|f\|_{C^\alpha(\Lambda)}$, and $\ell_c$ is the total length of the curves.

One could note that if $T_2^2 = (1 + 1)\log_2 T$ a stronger result holds as the logarithmic factor disappears.

**Proof.** (Theorem 6.1)

To prove (6.15), it is sufficient to prove the existence of a discrete frame $F_0$ such that

$$
\|f - f_{\overline{F}_0}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

Indeed as in the proof of Theorem 4.3, the best frame $F_{\overline{F}_0}$ thus also satisfies

$$
\|f - f_{\overline{F}_0}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

This implies

$$
\|f - f_{\overline{F}_0}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

and

$$
M \leq C_\epsilon C_f^2(\alpha + 1)M_1, \ell_c T^{-2/(\alpha + 1)} | o_2 T |.
$$

The proof of the existence of a bandelet frame $F_0$ of discrete images that satisfies (6.17) relies on the existence of a bandelet frame $F_0$ of $L^2[0,1]$ that satisfies a condition similar to (6.17), which is given by the following lemma.

**Lemma 6.2.** Under the hypotheses of Theorem 6.1, there exists a constant $C$ that depends upon the edges such that, for any $T$, one can construct a bandelet frame $F'$ of $L^2$, satisfying

$$
\|f - f'_{\overline{F'}}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

**Proof.** (Theorem 6.1)

To prove (6.15), it is sufficient to prove the existence of a discrete frame $F_0$ such that

$$
\|f - f_{\overline{F}_0}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

Indeed as in the proof of Theorem 4.3, the best frame $F_{\overline{F}_0}$ thus also satisfies

$$
\|f - f_{\overline{F}_0}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

This implies

$$
\|f - f_{\overline{F}_0}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$

and

$$
M \leq C_\epsilon C_f^2(\alpha + 1)M_1, \ell_c T^{-2/(\alpha + 1)} | o_2 T |.
$$

The proof of the existence of a bandelet frame $F_0$ of discrete images that satisfies (6.17) relies on the existence of a bandelet frame $F_0$ of $L^2[0,1]$ that satisfies a condition similar to (6.17), which is given by the following lemma.

**Lemma 6.2.** Under the hypotheses of Theorem 6.1, there exists a constant $C$ that depends upon the edges such that, for any $T$, one can construct a bandelet frame $F'$ of $L^2$, satisfying

$$
\|f - f'_{\overline{F'}}\| \leq C_\epsilon C_f^2(\alpha + 1)M^{-\alpha} o_2 M^{\alpha + 1}.
$$
with a number of edge rectangles bounded by $C \ell_c T^{-2/(\alpha + 1)}$.

Lemma 6.3. There exists a constant $C$ that depends upon the edges geometry such that

$$\sum_{i,m} |(\mathcal{F}, \mathcal{B}_{i,m}) - (f, b_{i,m})|^2 \leq C \|\hat{f}\|_{\infty} \ell_c + \|\hat{f}\|_{C^2(A)} \epsilon \epsilon.$$

To prove (6.17) let us consider $J^T = f(i,m) ; jh f; b_i,m; T g$ and

$$L(f, T, F) \leq \sum_{(i,m) \in J'} |(f, b_{i,m})|^2 + T^2 C \text{ rd } J + T^2 M_S + M_G$$

$$\leq \left( \sum_{(i,m) \in J'} |(f, b_{i,m})|^2 + T^2 C \text{ rd } J + T^2 M_S + M_G + \sum_{(i,m) \in J'} |(f, b_{i,m}) - (f, b_{i,m})|^2 \right).$$

By Lemma 3.10 and $J^T = f(i,m) ; jh f; b_i,m; T g$, we get

$$\sum_{i,m} |(f, b_{i,m})|^2 + T^2 C \text{ rd } J + T^2 M_S + M_G + \sum_{i,m} |(f, b_{i,m}) - (f, b_{i,m})|^2.$$

The condition $T = 2$ of Theorem 6.1 is a consequence of Lemma 6.3 which controls the differences between the discrete bandelet coefficients of the discrete image and the bandelet coefficients of the original analog image. From the discretization process (6.1) any linear reconstruction of the samples yields a square error with respect to $f$ that can only be bounded by an order of $2$. Indeed, $f$ can be discontinuous along curves whose locations are unknown. Since we are using a linear warping operator, we cannot reduce this error.
Numerically, in the regular regions of \( f \), the discrete bandelet scheme is improved if we replace the 0 order interpolation of (6.2) with an higher order interpolation \([14]\).

Unfortunately, this destroys the energy conservation properties of the bandelet basis proposed here, and it does not improve the asymptotic decay of the error because of the presence of discontinuities.

The choice of a \( L^2 \) normalized averaging for the discretization can be relaxed: the result holds indeed for any \( f_{[n_1; n_2]} = f ?_{(n_1; n_2)} \) with a local averaging function defined from a compactly supported function such that \( k_{k^2} = 1 \) and \( k_{k^1} = 0 \) by \((x_1; x_2) = 1_{(x_1; x_2)}\). A possible choice for \( \phi \) is thus a compactly supported scaling function associated to a wavelet \([11]\).

This theorem provides a constructive approximation scheme with a polynomial complexity and a decay rate optimal up to a logarithmic factor. As in Section 5, this implies a compression result. The same coding strategy can be used in this context and the logarithmic factor of Theorem 6.1, which does not appear in Theorem 4.3, just modifies the exponent of the \( j \log R \) factor. Under the hypotheses of Theorem 6.1, we get:

\[
D(R) \leq CC^2 f \ell C^{a+1} R - \circ R^{2a+1}.
\]

Hence, the distortion rate of the discrete bandelet coder reaches the Kolmogorov lower bound \( R \) up to a logarithmic factor \( j \log R \).

As in Theorem 4.3, the choice of a different basis in each square region yields a fast algorithm to optimize the geometry, but this can create discontinuities at the boundaries of the region. An implementation of the bandelet transform \([14]\) overcomes this difficulty with an adapted lifting scheme, for which there is no proof of optimality.

Finally, although the algorithm is polynomial, it is still computationally intensive and most of the bandelets algorithm implementation \([14]\) replace the full geometry exploration with a faster geometry exploration obtained from a geometry estimation similar to the one described in Section 3.2. As long as the jumps of the discontinuities do not vanish to zero, the estimation remains precise enough to obtain a good geometry. The corresponding optimization algorithm yields an error decay of order \( M - a \) with a low order polynomial complexity.

Appendix A. Proofs of Lemmas for Theorem 3.2.


\[
(f, h_{l,j,m}) = (f_{x_1, x_2 + g \ x_1, \psi_{l,m}, x_1, \psi_{j,m}, x_2).
\]

We need to prove that \( f \) is \( n \) times \( C^a \) on \( \Omega \) and \( f \) is \( n \) times \( C^a \) on \( \Omega \). We can do this by induction on \( n \).

\[
\|f\|_{C^a((x_1, x_2))} \leq \|h\|_{C^a((\Lambda))}.
\]

We use \( \|f\|_{C^a((x_1, x_2))} \) to denote the norm of \( f \) on \( \Omega \) and \( \|h\|_{C^a} \leq s^{-(2+a)} \) to denote the norm of \( h \) on \( \Omega \).
Now \( (A.3) \) yields along
\[
\left| \frac{\partial^{\alpha} W f}{\partial x_1^{\alpha}} \right| (x_1, x_2) \leq \| \hat{f} \|_{C^\alpha(A)} \| c \|_{C^\alpha} \quad | x_1 - x_1' |^{\alpha - \| \alpha \|}
\]

\( A \),

and along
\[
\left| \frac{\partial^{\alpha} W f}{\partial x_2^{\alpha}} \right| (x_1, x_2') \leq \| \hat{f} \|_{C^\alpha(A)} \| c \|_{C^\alpha} \quad | x_2 - x_2' |^{\alpha - \| \alpha \|}
\]

\( A \)

Indeed, assume
\[
\| f \| W_{l_1}^{\alpha(j,j,m)} (x_1, x_2) = l_1^{\alpha(j,j,m)} (x_1)
\]

and
\[
\| f \| W_{l_1}^{\alpha(j,j,m)} (x_1, x_2) = l_1^{\alpha(j,j,m)} (x_2)
\]

for any \( d \).

Let \( J \) be the union of \( J' \) and \( J_T \) such that
\[
J' \{ l, j, m \} \in B_1 \quad \cap \quad C \cap \left( (A.1) \right) \quad \ni \quad (f, l_1^{\alpha(j,j,m)}) \leq T
\]

\( A \)

and
\[
J_T \{ l, j, m \} \in J' \quad \ni \quad (f, l_1^{\alpha(j,j,m)}) \geq T
\]

\( A \)
Since there are at most $\max(2l, K)$ and $\max(2j, K)$, one can verify with a summation over $l$ and $j$ that

$$\text{Card}\left(J_0\right) \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.13)$$

so

$$\text{Card}\left(J_T\right) \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.14)$$

Combining the bound on the number of coefficients with (A.7) yields for any scale of index $(l;j)$

$$\sum_{m} |(f, b_{l,j,m})|^2 \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.15)$$

Summing over $l$ and $j$, it results that

$$\sum_{(l,j,m) \notin J_T} |(f, b_{l,j,m})|^2 + \sum_{(l,j,m) \in J_T} |(f, b_{l,j,m})|^2 \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.16)$$

and as $(l;j;m)$ implies $J_T$,

$$\sum_{(l,j,m) \notin J_T} |(f, b_{l,j,m})|^2 \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.17)$$

No

$$\sum_{(l,j,m) \notin J_T} |(f, b_{l,j,m})|^2 \leq \sum_{(l,j,m) \notin J_T} |(f, b_{l,j,m})|^2 \cdot (A.18)$$

and as for any scale of index $(l;j)$

$$\sum_{(l,j,m) \notin J_T} |(f, b_{l,j,m})|^2 \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.19)$$

A.2. Lemma 3.5: the inner bandelets. Proof [Lemma 3.5]

Let $J = \{l, j, m : b_{l,j,m} \in B_2\}$ and $J_T = \{l, j, m \in J : |(f, b_{l,j,m})| \geq T\}$

and $A$ conc de

$$\sum_{(l,j,m) \notin J_T} |(f, b_{l,j,m})|^2 \leq C\ell_1\ell_2 C_f^2 2^{2\alpha l} \cdot (A.20)$$

$\Box$
We first prove that the total energy of the bandelet coefficients $h_{f;b;l;j;m}$ with $j \leq j_*$ is small:

$$\sum_{(l,j,m) \in J} |\langle f, h_{l;j,m} \rangle|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}. \quad (A.23)$$

On one hand, as $J_T \subseteq J$, (A.23) implies

$$\sum_{(l,j,m) \in J_T} |\langle f, h_{l;j,m} \rangle|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}. \quad (A.24)$$

On the other hand, as $(l,j,m) \in J_T$ implies $\langle h_{f;b;l;j;m} \rangle \in J_T$, (A.23) also implies

$$\text{Card} \{ (l,j,m) \in J_T : j \leq j_* \} \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}. \quad (A.25)$$

To prove (A.23), we use that the bandelets $b_{l;j;m}$ with $j \leq j_*$ are obtained with an orthogonal change of bases from warped wavelets of scale $\leq j_*$ as described in section 4.1 so the energy of the corresponding bandelet coefficients is bounded by the one of the wavelet coefficients. Let $f_d^{j;m}$ be the wavelet basis of $B$, where $d_{j;m}$ stands for $d_{j;m,1}$, $d_{j;m,2}$ and $d_{j;m,3}$ depending on $d \in \{1, 2, 3\}$, and define

$$J_\delta = \{ j,m : j \leq j_* \}. \quad (A.26)$$

As the space generated by $(l,j,m) \in J_T$ is included in the space generated by $(l,j,m) \in J$, one verifies that

$$\sum_{(l,j,m) \in J} |\langle f, h_{l;j,m} \rangle|^2 \leq \sum_{(j,m) \in J_\delta} |\langle f, W^{-1} d_{j,m} \rangle|^2. \quad (A.27)$$

To prove (A.23), it is thus sufficient to prove that

$$\sum_{(j,m) \in J_\delta} |\langle f, W^{-1} d_{j,m} \rangle|^2 \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}. \quad (A.28)$$

or equivalently, as $h_{f;W^{-1} d_{j,m}} = h_{Wf; d_{j,m}}$, $\langle f, W^{-1} d_{j,m} \rangle$ and $\langle Wf, d_{j,m} \rangle$ depend on $d \in \{1, 2, 3\}$.

In $B$, one can verify that $Wf$ has the same regularity as $f$. Three different kinds of wavelets are distinguished in (A.29):

1. The wavelets that do not intersect the smoothed singularities: there are at most $\ell_1 \ell_2$ such wavelets at the scale $2^j$ and the bound (A.4) and (A.5) on $Wf$ implies

$$|\langle f, W^{-1} d_{j,m} \rangle| \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}. \quad (A.30)$$

2. The wavelets that intersect the smoothed singularities: there are at most $\ell_1 \ell_2 - 2^j$ such wavelets and the bound on $Wf$ implies

$$|\langle f, W^{-1} d_{j,m} \rangle| \leq C C_f^{2/(\alpha+1)} \ell_1 T^{2\alpha/(\alpha+1)}. \quad (A.31)$$
The wavelets that do intersect the singularities with a scale $2^j$: there are at most $C'_{\alpha} 2^{2j}$ such wavelets at the scale $2^j$ and, using $\tilde{W}_f k f k f 1$, one verifies $\|Wf\| \leq C'_{\alpha} 2^{2j} (A.32)$

$\|Wf\| \leq C'_{\alpha} 2^{2j} (A.33)$

The wavelets that do intersect the singularities with a scale $2^j$: there are at most $C'_{\alpha} 2^{2j}$ such wavelets at the scale $2^j$ and, as $f = \tilde{f} \ast h$ and $h$ is $C^\infty$, we get $\|f\| \leq C'_{\alpha} 2^{2j} (A.34)$

so yields $\|Wf\| \leq C'_{\alpha} 2^{2j} (A.35)$

Combining (A.31), (A.33) and (A.35) with the respective bounds on the number of coefficients gives bounds on the energy of the coefficients that eventually yields (A.29).

The remaining bandelets are the one that intersect the smoothed singularities at a scale $2^j$ and the corresponding coefficients are controlled with the regularity of the geometry.

As stated in the proof of Theorem 3.2, Lemma 3.7 implies when $J_0 \leq m \leq C^2 f_s T^{-2/(\alpha+1)} |j_1| (A.36)$

$J_0 \leq m \leq C^2 f_s T^{-2/(\alpha+1)} |j_1| (A.37)$

Using the vanishing moment of the bandelets and the size of their support, (A.37) implies $\|Wf\| \leq C'_{\alpha} 2^{2j} (A.38)$

which is sufficient for $s \geq j$. Let $J_0 = \{ l, j, m \in J : s \geq j, C^2 f_s T^{-2/(\alpha+1)} |j_1| (A.39)$

and $J_0 \leq m \leq C^2 f_s T^{-2/(\alpha+1)} |j_1| (A.40)$

and thus by a summation over $l, j$ yields $\sum_{m} |\langle f, b_l j, m \rangle|^2 \leq C \ell_1 C^2 f_s T^{-2/(\alpha+1)} (A.41)$
A \ s \geq i \geq j \ s  J_T \ \mathfrak{m} \ e \ l, j, m \notin J_T \ \mathfrak{m} \ e \ l, j, m \notin J \ or \ l, j, m \in J \ nd \ |\langle f, b_{l,j,m} \rangle| \leq T \ \com\ n\ A^n \ nd \ A^n \ \mathfrak{m} \ e \\
\sum_{(l,j,m) \notin J_T, s \geq 2^s \geq 2^s} |\langle f, b_{l,j,m} \rangle|^2 \leq \sum_{(l,j,m) \notin J_T, s \geq 2^s \geq 2^s} |\langle f, b_{l,j,m} \rangle|^2 \\
+ \sum_{(l,j,m) \notin J_T, s \geq 2^s \geq 2^s} |\langle f, b_{l,j,m} \rangle|^2 \ \mathfrak{m} \ e \\
\sum_{(l,j,m) \notin J_T, s \geq 2^s \geq 2^s} |\langle f, b_{l,j,m} \rangle|^2 \leq CCC_f^{2/(\alpha+1)} \epsilon_1 T^{2\alpha/(\alpha+1)} \ \mathfrak{m} \ e \\
nd \ l, j, m \in J_T \ \mathfrak{m} \ e \ |\langle f, b_{l,j,m} \rangle| \geq T \ nd \ i, j, m \in J \\
C \{ all \ \text{cases} \} \ \mathfrak{m} \ e \ C \{ all \ \text{cases} \} \ \mathfrak{m} \ e \\
\sum_{x_1 \geq x_1} \langle f, x_1, x_2, b_{l,j,m} \rangle \ & \sum_{x_1 \geq x_1} \langle f, x_1, x_2, b_{l,j,m} \rangle \\
\sum_{x_1 \geq x_1} \langle f, x_1, x_2, b_{l,j,m} \rangle \ & \sum_{x_1 \geq x_1} \langle f, x_1, x_2, b_{l,j,m} \rangle
Let \( J_+ \) be the set of \( l, j, m \in J \) such that \( s^{1/\alpha} \geq j \geq s, C C T \) \( (\alpha+1/2)(l - 1/2) \geq T \). There exists a constant \( C \) such that for all \( l \) and \( j \),

\[
C r d \left\{ l, j, m \in J_+ \text{ s.t. } s^{1/\alpha} \geq j \geq s, C C T \right\} \leq m \leq K \}
\]

Here, \( K \) is a constant that depends on the parameters \( \alpha \) and \( \beta \). The proof follows from the continuity and boundedness of the coefficients.

In particular, for \( l, j, m \in J_+ \) and \( l \neq m \), we have

\[
\sum_{(l, j, m) \in J_+, j \geq s} \left| (f, P_{1,2} b_{l,j,m}) \right|^2 \leq C C T \leq 2^{(\alpha+1)/2} T^{1/2} \left( l - 1/2 \right)
\]

Moreover, for \( l, j, m \in J_+ \) and \( l \neq m \), we have

\[
\sum_{(l, j, m) \in J_+, j \geq s} \left| (f, P_{1,2} b_{l,j,m}) \right|^2 \leq C C T \leq 2^{(\alpha+1)/2} T^{1/2} \left( l - 1/2 \right)
\]

If \( j \geq l, j, m \in J_+ \) and \( l \neq m \) or \( l, j, m \in J \) and \( \left| (f, b_{l,j,m}) \right| \leq T \), then

\[
\sum_{(l, j, m) \in J_+, j \geq s} \left| (f, b_{l,j,m}) \right|^2 \leq C C T \leq 2^{(\alpha+1)/2} T^{1/2} \left( l - 1/2 \right)
\]

and

\[
\sum_{(l, j, m) \in J_+, j \geq s} \left| (f, b_{l,j,m}) \right|^2 \leq C C T \leq 2^{(\alpha+1)/2} T^{1/2} \left( l - 1/2 \right)
\]

In the case where \( b_{l,j,m} \) is not zero, we can apply Lemma 3.6: Regularity of the approximated curve, which states that

\[
C r d \left\{ l, j, m \in J_+, j \geq s \right\} \leq C C T \leq 2^{(\alpha+1)/2} T^{1/2} \left( l - 1/2 \right)
\]

Proof: [Lemma 3.6]

In summary, we have shown that for \( \alpha \) and \( \beta \) such that \( 0 < \alpha \leq 1 \) and \( 0 < \beta \leq 1 \), the coefficients are bounded.

\[ P_{V_k} u^{(\beta)} x_0 \sum_k (u, \theta_{k,n}) \theta_{k,n}^{(\beta)} \]

and \( \theta_{k,n} \) are properly zeroed \( K^k \)

\[ P_{V_k} u^{(\beta)} x_0 \sum_{|n^{2^k} - x_0| \leq K^{2^k}} (u, \theta_{k,n}) \theta_{k,n}^{(\beta)} \]

\[ | P_{V_k} u^{(\beta)} x_0 | \leq K \max_{|n^{2^k} - x_0| \leq K^{2^k}} |(u, \theta_{k,n})| \|\theta_{k,n}^{(\beta)}\|_\infty \]

By hypothesis \( \|\alpha\|\) \( C^\alpha \) regular on \( |x - x_0| \leq K^k \)

\[ |P_{V_k} u^{(\alpha)} x - P_{V_k} u^{(\alpha)} x_0| \leq C^{-k\alpha} \max_{|n^{2^k} - x_0| \leq K^{2^k}} |(u, \theta_{k,n})| |x - x_0|^\alpha - \|\alpha\| \]

By definition of the Taylor polynomial, \( u \) is polynomial on \( |x - x_0| \leq K^k \)

\[ P_{V_k} \pi_{x_0} - P_{V_k} c^{(\beta)} x_0 | \leq C \|c\| \|\alpha\|^{-k\beta} \]

By hypothesis \( c - g \|\alpha\| \leq C_d s \) \( |(c - g, \theta_{k,n})| \leq C C_d s \) \( \|c - g\| \leq C C_d s \)

\[ |P_{V_k} c - P_{V_k} g^{(\beta)} x_0| \leq C C_d s^{-k\beta} \]

\[ A^{-k} \|c\|^{-1/\alpha} s^{1/\alpha} c \|\alpha\| n n A A \]
\textbf{A.4. Lemma 3.7: Regularity of the warped function.} \textit{Proof} \ [Lemma 3.7]

\begin{align*}
f(x_1, x_2 + g x_1) &= \int_u f(x_1 - u_1, x_2 + g x_1 - u_2) \, h(u_1, u_2) \, du \\
&= A \quad \text{for each } u_1 \text{ on } x_1^\alpha \text{ measurable and } \partial \text{ on } x_1^\alpha \text{ regular}
\end{align*}

\begin{align*}
\frac{\partial^n}{\partial x_1^n} f(x_1 - u_1, x_2 + c x_1 - u_1 - u_2) \, h(u_1, u_2 + g x_1 - c x_1 - u_1) &= \sum_{d=0}^{n} \binom{n}{d} \frac{\partial^{n-d}}{\partial x_1^{n-d}} f(x_1 - u_1, x_2 + c x_1 - u_1 - u_2) \, h(u_1, u_2 + g x_1 - c x_1 - u_1) \\
&\leq C \left\| f \right\|_{C^0(\Lambda)} m \left\| c \right\|_{C^0}, \\
&\text{nd for any } x_1'
\end{align*}

\begin{align*}
\frac{\partial^{[\alpha]}}{\partial x_1^{[\alpha]}} f(x_1 - u_1, x_2 + c x_1 - u_1 - u_2) - \frac{\partial^{[\alpha]}}{\partial x_1^{[\alpha]}} f(x_1 - u_1, x_2 + c x_1 - u_1 - u_2) &= \frac{\partial^{[\alpha]}}{\partial x_1^{[\alpha]}} f(x_1 - u_1, x_2 + c x_1 - u_1 - u_2) \\
&\leq C \left\| f \right\|_{C^0(\Lambda)} m \left\| c \right\|_{C^0}, \left\| x_1' - x_1 \right\|^{\alpha - [\alpha]} \\
&\text{nd for any } x_1'
\end{align*}

\begin{align*}
\frac{\partial^d}{\partial x_1^d} h(u_1, u_2 + g x_1 - c x_1 - u_1) &= \sum_{k_1 + 2k_2 + \cdots + dk_d = d} \frac{d}{k_1 \cdots k_d} \frac{\partial^{k}}{\partial x_1^{k}} h(u_1, u_2 + g x_1 - c x_1 - u_1) \\
&\times \left( \left( \frac{g(1) x_1 - c(1) x_1 - u_1}{d} \right)^{k_1} \cdots \left( \frac{g(d) x_1 - c(d) x_1 - u_1}{d} \right)^{k_d} \right) \\
&\text{for } k = k_1 + k_2 + \cdots + k_d
\end{align*}
For any $g - c$ and $w$, the derivative of $h$ with respect to $u_1$ is bounded by $|u_1| \leq s$

$$g^{(d)} x_1 - c^{(d)} x_1 - u_1$$

$$\leq |g^{(d)} x_1 - c^{(d)} x_1| + |c^{(d)} x_1 - c^{(d)} x_1 - u_1|$$

$$\leq \begin{cases} C \cdot m \|c\|_{C^0, C_d}, & s^{1-d/\alpha} + \|c\|_{C^0} s^{d} f d < \|a\| \quad A \\ C \cdot m \|c\|_{C^0, C_d}, & s^{1-\|a\|/\alpha} + \|c\|_{C^0, s^{\alpha-\|a\|}} f d < \|a\| \quad A \end{cases}$$

\[ \text{and} \]

$$|g^{(d)} x_1 - c^{(d)} x_1 - u_1| \leq C \cdot m \|c\|_{C^0, C_d}, \quad s^{1-d/\alpha}. \quad A$$

\[ \text{Moreover, for any } x'_1, |x'_1 - x_1| \leq K s^{1/\alpha} \]

$$\left| g^{(\|a\|)} x'_1 - c^{(\|a\|)} x'_1 - u_1 - g^{(\|a\|)} x_1 - c^{(\|a\|)} x_1 - u_1 \right|$$

$$\leq C \cdot m \|c\|_{C^0, C_d}, \quad |x'_1 - x_1|^{\alpha-\|a\|}. \quad A$$


No $h$ is defined $C^\alpha$, $\|h\|_{C^\alpha} \leq s^{-(2+\alpha)}$ on

$$\left| \frac{\partial^k}{\partial x_1^k} h u_1, u_2 + x_1 \right| \leq s^{-2} s^{-k}$$

\[ \text{and} \]

$$\left| \frac{\partial^{\|a\|}}{\partial x_1^{\|a\|}} h u_1, u_2 + x_1 + \frac{\partial^{\|a\|}}{\partial x_1^{\|a\|}} h u_1, u_2 + x_1 \right| \leq s^{-2} s^{-\alpha} |x'_1 - x_1|^{\alpha-\|a\|}. \quad A$$

\[ \text{Comment: for any } A \text{ and } A \text{ and } A \text{ are defined on} \]

$$\left| \frac{\partial^d}{\partial x_1^d} h u_1, u_2 + g x_1 + c x_2 - u_2 \right| \leq C s^{-2} m \|c\|_{C^0, C_d}, \quad s^{-d/\alpha} \quad A$$

\[ \text{and for any } x'_1, |x'_1 - x_1| \leq K s^{1/\alpha} \]

$$\left| \frac{\partial^d}{\partial x_1^d} h u_1, u_2 + g x_1 + c x_2 - u_2 - \frac{\partial^d}{\partial x_1^d} h u_1, u_2 + g x_1 + c x_2 - u_2 \right|$$

$$\leq C s^{-2} m \|c\|_{C^0, C_d}, \quad s^{-1} |x'_1 - x_1|^{\alpha-\|a\|}. \quad A$$

\[ \text{and for any } A \text{ and } A \text{ and } A \text{ are defined on } \]

$$\left| \frac{\partial^a}{\partial x_1^a} \left( f x_1 - u_1, x_2 + c x_1 - u_1 - u_2 h u_1, u_2 + g x_1 - c x_1 - u_1 \right) \right|$$

$$\leq C m \|c\|_{C^0, C_d}, \quad s^{-2} s^{-\alpha/\alpha} \quad A$$
and A nd A for ny \(x'_1, |x'_1 - x_1| \leq K s^{1/\alpha}\)

\[
\frac{\partial \hat{f}}{\partial x^0}(\hat{f} x'_1 - u_1, x_2 + c x'_1 - u_1 - u_2 h u_1 + g x'_1 - c x_1 - u_1)
- \frac{\partial \hat{f}}{\partial x^1}(\hat{f} x_1 - u_1, x_2 + c x_1 - u_1 - u_2 h u_1 + g x_1 - c x_1 - u_1) \\
\leq C \mathfrak{m}_1 \|c\|_{C_\alpha}, C_{\alpha}^p, s^{-2} s^{-1} |x'_1 - x_1|^{\alpha - \alpha}.
\]

\[A.5, \text{Lemma 3.9 : Smoothing effect.} \]

**Proof [Lemma 3.9]** By con on

\[
\|h\|_1 \hat{f} x - f x \quad \|h\|_1 \hat{f} x - \int \hat{f} x - u h u \ du \quad A
\]

\[
\int \hat{f} x - \hat{f} x - u h u \ du.
\]

\[A
\]

\[
\|h\|_1 \hat{f} x - f x \leq \mathfrak{m}_1 \|\hat{f} - \hat{f} x - u h u \|_1 \leq \mathfrak{m}_1 \|\hat{f} x - \hat{f} x - u h u \|_1.
\]

\[A
\]

No \(f x \notin C_s\) \text{ re r y of } \hat{f} ye d

\[
\forall x \notin C_s \|h\|_1 \hat{f} x - f x \leq \|\hat{f}\|_{C_s} s
\]

\[A
\]

\[
\forall x \in C_s \|h\|_1 \hat{f} x - f x \leq \|\hat{f}\|_s \leq \|\hat{f}\|_{C_1}.
\]

\[A
\]

No \(\text{re of } C_s \) \text{ nd } B c n \text{ con ro ed } \#C_s \leq C \ell_1 s \text{ nd } \#B \leq C \ell_1 \ell_2

\[
\|h\|_1 \hat{f} x - f x \|_1^2 \int_{x \notin C_s} \|h\|_1 \hat{f} x - f x \|_1^2 \, dx + \int_{x \in C_s} \|h\|_1 \hat{f} x - f x \|_1^2 \, dx
\]

\[A
\]

\[
\leq C \ell_1 \ell_2 \|\hat{f}\|_{C_1}^2 s^2 + C \ell_1 s \|\hat{f}\|_{C_1}^2.
\]

\[A
\]

\[
\|h\|_1 \hat{f} x - f x \|_1^2 \leq C \|\hat{f}\|_{C_1}^2 \ell_1 s.
\]

\[\square
\]

**Appendix B. Proofs of Lemmas for Theorem 4.3.**
B.1. Lemma 4.5: Geometry construction. Proof [Lemma 4.5] e e $a_n r_{\text{men}} \text{ of } a\text{e proof } a_0 a_{\text{de ere e pro e } a_{\text{rec r e p n occ r f er } h n a_n \text{er of ep on y r e } a_{\text{es nc on } a_{\text{ o o con ro } a_n n_{\text{er of } e c n d of } q r e e a_n n_{\text{er of } p r_{\text{er req red o de cr e } a_{\text{em}}}}}$

We give the main arguments of the proof without the details. We prove that the recursive splitting occurs, after a finite number of steps, only near the junctions. This allows to control the number of each kind of squares as well as the number of parameters required to describe them.

The non tangency condition implies that there is a minimum angle $0 > \theta_0 > 0$ between the tangents of the edges at the junctions. The regularity of the curves allows to define a neighborhood of each junction in which the tangents do not vary by more than $0 = 3$. The angle between the tangents of two different curves remains thus larger than $0 = 3$ and the geometry around the junction is close to the geometry of the junction of half-lines as illustrated in the close-up of Figure 4.5. Outside this neighborhood, there is a minimal distance $d > 0$ between the curves and so in a finite number, independent of the geometric precision $\tau^2 = \max(\tau, T^2) = (\tau + 1)$, of steps each dyadic square is either a regular square or an edge square. The corresponding number of square is thus uniformly bounded.

In the neighborhood of the junction, the recursive splitting continues until the size of the squares is of order $\tau$, but one can verify that, after a few steps, the number of squares around each junction that can be labeled as temporary square is bounded by a constant. As there is only a finite number of such junctions, this implies that the number of the edge squares as well as the number of regular squares is bounded by $C_j \log_2 \tau$ and that the number of junction square is bounded by $C_j$.

The segmentation is specified by the $M_S$ inner nodes of the corresponding dyadic tree. There is a finite number, independent of $\tau$, of splits outside the neighborhood of the junction and at most a constant number of splits at each scale near the junctions. So, as $T^2 = (\tau + 1)$,

\[
M_{\text{G}} \leq C_{\tau} \leq C_{\text{min}(\tau, 1)} + C_{\text{log}_2 T} \leq C_{\text{f}} T^{2/\alpha(1)}
\]

where $\tau$ is the total length of the edge curves.

Combining (B.1) and (B.3) gives (4.23).

B.2. Lemma 4.6: Wavelets over Regular squares. Proof [Lemma 4.6] e e $a_n r_{\text{men}} \text{ of } a\text{e proof } a_0 a_{\text{de ere e pro e } a_{\text{rec r e p n occ r f er } h n a_n \text{er of ep on y r e } a_{\text{es nc on } a_{\text{ o o con ro } a_n n_{\text{er of } e c n d of } q r e e a_n n_{\text{er of } p r_{\text{er req red o de cr e } a_{\text{em}}}}}$

Let $\lambda$ be a regular square of size $2^{i_0}$. By definition, $f$ is $C_0$ in $\lambda$, and this regularity implies

\[
\|f - \tilde{f}\|_{C^0} \leq C_k\|f\|_{C^0} (\alpha + 1) j
\]

Inserting (A.3) in (B.5) yields

\[
\|f - \tilde{f}\|_{C^0} \leq C_k\|f\|_{C^0} (\alpha + 1) j
\]

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We define now a cutting scale $j_0$ as the largest scale such that

$$C_{k} \sim f_k C_{k}(2 + 1)^{j_0} < T$$

so

$$1 < \frac{T}{C_{k}} \sim f_k C_{k}(2 + 1)^{j_0} = \frac{1}{2}.$$  \hspace{1cm} (B.7)

Let $J \{ j, m \mid j \geq j_0 \}$ be

$$\hat{L}_{i} f, T, B_i \leq \sum_{(j, m) \notin J} |(f, d_{j, m})|^2 + T^2 C \text{ rd } J' \hspace{1cm} \text{B}$$

for any $j, m \in J$ for $j \geq j_0$

$$\sum_{(j, m) \notin J} |(f, d_{j, m})|^2 \leq C \|\hat{f}\|_{C^0(\Lambda)}^2 2^{\lambda_2 j_0} \hspace{1cm} \text{B}$$

and hence $j, m \in J$ for $j \geq j_0$

$$\sum_{(j, m) \in J} |(f, d_{j, m})|^2 \leq C \|\hat{f}\|_{C^0(\Lambda)}^2 2^{\lambda_2 j} \hspace{1cm} \text{B}$$

By definition, the size of $i$ is smaller than $s = \max(s, T^2(1 + 1))$. If $s = T^2(1 + 1)$, the result holds immediately as $k_f f_k 1 \leq \frac{1}{2}$ and $k_f f_k 1$. Otherwise $s = s T^2(1 + 1)$ and we use the regularity of $f$ to obtain the bound.

Indeed, combining (A.34) and (B.5) yields to

$$|\langle f, d_{j, m} \rangle|^2 \leq C \|\hat{f}\|_{C^0(\Lambda)}^2 2^{\lambda_2 j} \hspace{1cm} \text{(B.14)}$$

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\[ -\left( \frac{Ts^\alpha}{C\|f\|_\infty} \right)^{1/(\alpha+1)} \leq j_0 \leq \left( \frac{Ts^\alpha}{C\|f\|_\infty} \right)^{1/(\alpha+1)}. \]

or \( J' \{ j, m \ j \geq j_0 \) Lemma \( \lambda \) e \[ \hat{\lambda}_i f, T, B_i \leq \sum_{(j,m) \notin J'} |\langle f, d_{j,m} \rangle|^2 + T^2 C \text{ rd } J'. \]

\[ \leq C_m \ s^2 - 2j, K^2 \]

\[ \text{Lemma 3.10 implies } 1 \leq 2C \]

\[ \max(2^s, 2^\alpha) \]

\[ \text{Combining (3.35) and (3.36) implies thus that } \]

\[ \lambda_i f, T, B_i \leq C s^{2/(\alpha+1)} \|\hat{f}\|_{\infty}^{2/(\alpha+1)} T^{2\alpha/(\alpha+1)}. \]

\[ \lambda_i f, T, B_i \leq \sum_{i \in I_T} \lambda_i \ C^{2/(\alpha+1)} \left( T^{\alpha/(\alpha+1)} \right)^{1/2} \]
which proves (4.26) and concludes Lemma 4.8.

Appendix C. Complexity of the CART Algorithm.

We denote by $C(2^2)$ the numerical complexity to find this best geometric ow over a square $i$ of size $2^2$. Since there are $2^2$ square regions of size $2^2$ in $[0; 1]^2$, the total numerical complexity to find the best bandelet bases in all dyadic squares is

$$
\sum_{\lambda=\log_2 T^2}^0 -2\lambda C + O(T^{-4}).
$$

Since the minimum size of squares is $T/2$, the segmentation quad-tree has a depth at most equal to $j \log_2 T/2j$. The number of comparisons of the bottom-up optimization algorithm is proportional to the number of nodes of a full quad-tree of depth $j \log_2 T/2j$, which is $O(T^{4p})$. The total number of operations to find the best bandelet frame is therefore

$$
C = \sum_{\lambda=\log_2 T^2}^0 -2\lambda C + O(T^{-4}) + O(T^{4p}).
$$

Appendix D. Proof of the lemmas for Theorem 6.1.


Proof. To show the existence of a frame satisfying the conditions of the lemma, we use the same strategy as in the proof of Lemma 4.4 and focus on the case $T/2 = (s+1)/s$ with $s > 1$. The dyadic squares of this frame are exactly the ones of Lemma 4.5. A wavelet basis is still used in the regular squares and the junction squares. The only difference is that a bandelet basis is used in the dyadic squares, which yields a minimum Lagrangian value

$$
L(f; T, F_{i})
$$

over a square $i$ of size $2^2$. For this purpose all possible geometric flows are explored. If there is no flow then $F_i$ is a discrete wavelet basis of $i$. The wavelet coefficients and the corresponding Lagrangian value are computed with $O(\#i) = O(2^{2^2})$ operations. If there is a horizontally parallel flow then $i$ is subdivided into $2^k$ rectangles $i;m$ of length $2^k$ and height $2^p$, where $2^k$ is an adjustable scale variable. There are $O((2^k T/2)^p)$ different polynomial flows over each $i;m$. Computing the corresponding coefficients of $f$ for each polynomial flow and its Lagrangian cost with the algorithm of Section 4.1 requires $O(2^{k+2})$ operations. Among the $O((2^k T/2)^p)$ polynomial flows finding the one that minimizes the Lagrangian can thus be done with $O(2^{k+2} \cdot (p+1) \cdot 2^p)$ operations. Hence, the total number of operations to find the best horizontally parallel flow is

$$
\sum_{k=\log_2 T^2}^0 -2\lambda C + O(T^{-4}) + O(T^{4p}).
$$

The same argument applies to vertically parallel flows. Combining all possibilities (no flow, horizontally or vertically parallel flows), it results that the total number of operations to find the best bandelet frame over a square of size $2^2$ is $C = O(2^{(p+2)2^p})$. Inserting this in (C.1) shows that the numerical complexity to find the best bandelet frame over $[0; 1]^2$ is

$$
C = \sum_{\lambda=\log_2 T^2}^0 -2\lambda C + O(T^{-4}) + O(T^{4p}) = O(2^{(p+2)2^p}).
$$
\[\pi_{2^k m} x \sum_{\beta \leq \alpha} \frac{c^{(\beta)} k^m}{\beta} x - k^m \beta \]

\[g x \sum_{n=1}^{p} Q_{T^2} \left( \frac{\pi_{2^k m} c, \theta_n}{\theta_n^k m} \right) \theta_n^k m \quad \text{D} \]

\[|c - g^{(\beta)} x| \leq C m \quad |c|_{C^0}, C_d \quad s^{-1/\alpha} \quad \forall \beta \leq \|\alpha\| \quad \text{D} \]

\[C_d \quad \|\theta\|_{C^p} \quad \text{D} \]

\[\hat{f}, T, B_{i,r} \leq C m \quad C^{2/(\alpha+1)} \quad T^{2/\alpha/(\alpha+1)}, T^2 K \quad o_T \quad \text{D} \]

\[\sum_{i,r} \hat{f}, T, B_{i,r} \leq C C^{2/(\alpha+1)} \quad T^{2/\alpha/(\alpha+1)} + C C m \quad |c|_{C^0}, s^{-1/\alpha} T^2 K \quad o_T \quad \text{D} \]
\[ \sum_{i,r} \mathcal{L} f, T, B_{i,r} \leq C \mathcal{C}^2_{f_{i,r}} \ell_c \{ T^{2/(\alpha+1)}, s^{-1/\alpha} T^2 \} | o_{2T} | \leq C \mathcal{C}^2_{f_{i,r}} \ell_c T^{2/(\alpha+1)} | o_{2T} | \] 

A \ s \geq T^{2/(\alpha+1)} \implies 

\[ \sum_{i,r} \mathcal{L} f, T, B_{i,r} \leq C \mathcal{C}^2_{f_{i,r}} \ell_c T^{2/(\alpha+1)} | o_{2T} | \]

\[ \text{nd conc de } \psi_e \text{ proof } \psi_{en} T^{2/(\alpha+1)} \leq s \text{ On } \psi_{eo} \text{ d no e } \psi \text{ on } T^{2/(\alpha+1)} | o_{2T} | \leq s \text{ } \psi_{en} \text{ or } \psi_{en} \text{ for } T^{2/(\alpha+1)} \geq s \text{ } \psi_{en} \text{ ned } n \text{ proof of } \psi_{en} \text{ proof of } \psi_{en} \]

**D.2, Lemma 6.3: Discretization.** Proof [Lemma 6.3] A \( \psi_e \) \( \text{ n e r o } \) \( g \) \( \text{ de } \psi_{en} \text{ y } \text{ con } \text{ no } \text{ c e e } \) \( \{ b_{l,j,m_1,m_2} \} \) \( \text{ c n } \text{ e } \) \( \text{ c r e d } \) \( \text{ o e r } \) \( \text{ n } \) \( \text{ d } \) \( \text{ B } \) \( \text{ O } \) \( \text{ er } \) \( \psi_{e r} \) \( \text{ r p e d } \) \( \text{ nd } \) \( \text{ WB } \) \( \psi_{e r} \) \( \text{ no } \) \( \text{ r p } \) \( \text{ n } \) \( \text{ e } \) \( \text{ d } \) \( \psi_{e r} \)

\[ \psi_{WB} x_1, x_2 \sum \psi_f n_1, n_2 \phi_{j_0}, n_1 x_1 \phi_{j_0}, n_2 x_2 \]

or

\[ \psi_{WB} x_1, x_2 \sum \psi_f n_1, n_2 + [g n_1 \epsilon \epsilon^{-1}] \phi_{j_0}, n_1 x_1 \phi_{j_0}, n_2 x_2 \]

\[ \psi_{j_0} \epsilon \psi_{j_0} \epsilon \]

\[ \langle \psi_f, b_{l,j,m_1,m_2} \rangle \langle \psi_f, b_{l,j,m_1,m_2} \rangle \]

\[ \langle \psi_f, b_{l,j,m_1,m_2} \rangle \langle \psi_f, b_{l,j,m_1,m_2} \rangle \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]

\[ \sum_{l \geq j > j_0} |\langle \psi_f, b_{l,j,m_1,m_2} \rangle - \langle \psi_f, b_{l,j,m_1,m_2} \rangle|^2 \sum_{l \geq j > j_0} |\langle \psi_f, \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \psi_{j_0,n_1} x_1 \phi_{j_0}, n_2 x_2 \rangle|^2 \]
The provided text contains mathematical expressions and formulas that are not properly formatted. The text appears to be discussing concepts related to scaling functions, integral inequalities, and possibly some results from a specific domain of mathematics. Here is a rough attempt at rendering the content:

\[ \sum_{m} |\langle f, j_{0,m} \rangle - j_{0} f_{j_{0},m} \rangle | \leq K_2^2 \left| f_{x_{j_{0},m}} \right| \leq K_2^0 \left| f_{x_{j_{0},m}} \right| \leq \| f \|_{C^1(A)} K_{j_{0}}. \]

\[ \sum_{m} |\langle f, j_{0,m} \rangle - j_{0} f_{j_{0},m} \rangle | \leq \| f \|_{C^1(A)} K_{j_{0}}. \]

\[ \sum_{m} |\langle f, j_{0,m} \rangle - j_{0} f_{j_{0},m} \rangle | \leq \| f \|_{\infty} \leq C \| \hat{f} \|_{C^1(A)}. \]

\[ \sum_{m} |\langle f, j_{0,m} \rangle - j_{0} f_{j_{0},m} \rangle | \leq \| f \|_{\infty} \leq C \| \hat{f} \|_{C^1(A)}. \]

The exact meaning and implications of these expressions depend on the specific context in which they are used, which is not clear from the text provided.
\[ \sum_m \left| \langle f, j_0, m \rangle - \hat{f} \right| \leq C \frac{2^{j_0}}{s^{2j_0}} \| f \|_{C^1(\Lambda)}^2 s^{-2j_0} 2^{j_0} + C \ell s^{-2j_0} \| \hat{f} \|_{C^1(\Lambda)}^2 s^{-1} 2^{j_0} \]

\[ \sum_m \left| \langle f, j_0, m \rangle - \hat{f} \right| \leq C \left( \sum_k \left( \frac{1}{s_k} \right)^{j_0} + \frac{1}{s_k} \right)^{j_0} \]

REFERENCES


