**Compressed Sensing**

- Acquire few measurements and reconstruct a high resolution signal, if the signal has a sparse representation in a dictionary.

- A super-resolution problem, where the measurement operator can be chosen.

- **Key idea:** use random measurement operators to construct incoherent transformed dictionaries.
**Representation from Linear Sampling**

- **Linear sampling**: an analog signal $\bar{f}(x)$ is projected on a basis $\{\bar{\phi}_n\}_{n<N}$ of an approximation space $V_N$, which specifies a linear approximation:

  $$P_{\mathcal{U}} \bar{f}(x) = \sum_{n<N} \langle \bar{f}, \bar{\phi}_n \rangle \tilde{\bar{\phi}}_n$$

- **Uniform sampling**: $\bar{\phi}_n(x) = \bar{\phi}(x - Tn)$

- **Sparse representation of** the discrete signal

  $$f[n] = \langle \bar{f}, \bar{\phi}_n \rangle \in \mathbb{R}^N$$

  in a basis $\{g_p\}_{p \in \Gamma}$ with the $M$ largest coefficients

  $$\{\langle f, g_p \rangle\}_{p \in \Lambda} \text{ with } |\Lambda| = M.$$
Compressive Sensing

• Sparse random analog measurements

\[ Y[q] = \bar{U}\bar{f}[q] + W[q] = \langle \bar{f}, \bar{u}_q \rangle + W[q] \]

where \( \bar{u}_q(x) \) are realizations of a random process.

• Discretization: if \( \bar{u}_q(x) \in V_N \) then

\[ Y[q] = Uf[q] + W[q] = \langle f, u_q \rangle + W[q] \]

where \( f[n] = \langle \bar{f}, \bar{\phi}_n \rangle \) and \( u_q[n] \in \mathbb{R}^N \) is a random vector.

• If \( f \) is sparse in \( \{g_p\}_{p \in \Gamma} \) can we recover \( f \) from \( Y \)?
From measurements with a random operator

\[ Y[q] = U f[q] + W[q] \]

sparse super-resolution estimation:

\[ \tilde{F} = \sum_{p \in \tilde{\Lambda}} \tilde{a}[p] g_p \]

where \( \tilde{a}[p] \) has a support \( \tilde{\Lambda} \) computed with a sparse decomposition of \( Y \) in \( D_U = \{U g_p\}_{p \in \Gamma} \) by minimizing:

\[
\frac{1}{2} \left\| \sum_{p \in \Gamma} a[p] U g_p - Y \right\|^2 + T \sum_{p \in \Gamma} |a[p]|.
\]

or with an orthogonal matching pursuit.
• Riesz basis condition for recovery stability

\[(1 - \delta_\Lambda) \sum_{p \in \Gamma} |a[p]|^2 \leq \left\| \sum_{p \in \Lambda} a[p] U g_p \right\|^2 \leq (1 + \delta_\Lambda) \sum_{p \in \Gamma} |a[p]|^2 .\]

• Restricted isometry condition: \(\delta_\Lambda \leq \delta_M(D_U) < 1\) if \(|\Lambda| \leq M\)

\[(1 - \delta_M) \sum_{p \in \Gamma} |a[p]|^2 \leq \left\| \sum_{p \in \Lambda} a[p] U g_p \right\|^2 \leq (1 + \delta_M) \sum_{p \in \Gamma} |a[p]|^2 .\]

any such family \(\{U g_p\}_{p \in \Lambda}\) is “nearly” orthogonal.

• Relation to incoherence:

\[\delta_M(D_U) \leq (M - 1) \mu(D_U) \text{ with } \mu(D_U) = \max_{p \neq q} \langle U g_p, U g_q \rangle.\]
Exact Recovery

- **Theorem:**

  If \( f = \sum_{p \in \Lambda} a[p] g_p \) with \( |\Lambda| = M \) and \( \delta_{3M} < 1/3 \)

  then \( a = \arg \min_b \frac{1}{2} \| \sum_{p \in \Gamma} b[p] U g_p - Y \|^2 + T \| b \|_1 \)

- Sparse signals are exactly recovered.
Gaussian Random Matrices

- We want to have \( \{U g_p\}_{p \in \Gamma} \) nearly uniformly distributed over the unit sphere of \( \mathbb{R}^Q \) so that \( \{U g_p\}_{p \in \Lambda} \) is as orthogonal as possible even for \( \Lambda \) not small.

- The distribution of a Gaussian white noise of variance 1 is a uniform measure in the neighborhood of the unit sphere of \( \mathbb{R}^Q \).

- If \( \{g_p\}_{p \in \Gamma} \) is an orthonormal basis of \( \mathbb{R}^N \) and if \( U \) is a matrix of \( Q \) by \( N \) values taken by independant Gaussian random variables (white noise) then \( \{U g_p\}_{p \in \Gamma} \) are values taken by \( Q \) independant Gaussian random variables.
**Theorem:** If $U$ is a Gaussian random matrix then for any $\delta < 1$ there exists $\beta > 0$ such that

$$\delta_M(D_U) \leq \delta \quad \text{if} \quad M \leq \frac{\beta Q}{\log(N/Q)}.$$ 

- Valid for random Bernoulli matrices (random 1 and -1).

- We need $Q \sim CM \left\lfloor \alpha N/M \right\rfloor$ measurements to recover $M$ values and $M$ unknown indices among $N$.
- Coding would require of the order of $M \log(N/M)$ bits.
• Monte-Carlo experiments for recovering signals with $M$ non-zero coefficients out of $N$ with $Q$ random Gaussian measurements: $Q \sim C M \| \alpha N / M \|$

Ratio of recovered signals with $Q = 100$

- $N=512$, “Perfect” for $Q/M > 6.2$
- $N=1024$, $Q/M > 7.7$
- $N=2048$, $Q/M = 11$
- $N=4096$, $Q/M = 16$
Other Random Operators

- Storing a random Gaussian matrix $U$ and computing $Uh$ requires $O(N^2)$ memory and calculations, too much.

- RIP theorem valid for Bernouilli matrices (random 1 and -1). Still too much memory and computations.

- Similar RIP theorem valid for a random projector in an orthonormal basis $\{g'_m\}_m$ which is highly incoherent with the sparsity basis $\{g_p\}_p$. May require only $O(N)$ memory and $O(N \log N)$ computations.
Random Sparse Spike Inversion

- Measurements

\[ Y = u \ast f + W \quad \text{with} \quad f[n] = \sum_{p \in \Lambda} a[p] \delta[n - p]. \]

- Random wavelet makes a random sampling of the Fourier coefficients of \( f \) : \( \hat{u}[k] \) is the indicator of random set of frequencies.

- Fourier and Dirac bases have a low coherence.
Random Sparse Spike Inversion

Seismic wavelet $u[n]$

Incoherence $u * \hat{u}[p]$

Original signal

Recovery

Recovery
Non-Linear Approximation Error

- \{ \langle f, g_{m_k} \rangle \}_k \text{ sorted with decreasing amplitude}
  \quad |\langle f, g_{m_{k+1}} \rangle| \leq |\langle f, g_{m_k} \rangle|.

- Non-linear approximation in an orthonormal basis:

  \begin{align*}
    f_M &= \sum_{k=1}^{M} \langle f, g_{m_k} \rangle g_{m_k} \\
    \text{and} \\
    \|f - f_M\|^2 &= \sum_{k=M+1}^{N} |\langle f, g_{m_k} \rangle|^2.
  \end{align*}

  If \quad |\langle f, g_{m_k} \rangle| = O(k^{-\alpha}) \quad \text{then} \quad \|f - f_M\|^2 = O(M^{1-2\alpha}).
• **Theorem:** There exists $C$ such that if $\delta_{3M} < 1/3$ and

$$\tilde{a} = \arg \min_b \frac{1}{2} \| Y - \sum_{p \in \Gamma} b[p] U g_p \|^2 + T \| b \|_1$$

then

$$\| f - \sum_{p \in \Gamma} \tilde{a}[p] g_p \|^2 \leq \frac{C}{\sqrt{M}} \sum_{k=M}^{N-1} | \langle f, g_{m_k} \rangle | + C \| W \|$$

and if $| \langle f, g_{m_k} \rangle | = O(k^{-s})$ with $s > 1$ and $\| W \| = 0$

$$\| f - \sum_{p \in \Gamma} \tilde{a}[p] g_p \|^2 = O(M^{-2s+1}).$$

• Requires $Q \sim C' M \log(N/M)$ random measurements.
For $Q$ random measurements, $M$ is the number basis coefficients defining an approximation having the same error. The ratio $Q/M$ is computed with a Monte-Carlo experiment for different decay exponents $s$ for $N=1024$. 

![Graph showing $Q/M$ for different $s$ values]

- $s = 0.8$
- $s = 0.9$
- $s = 1$
- $s = 1.1$
- $s = 1.2$
Recovery Efficiency

Compressive sensing efficiency in random Gaussian and Fourier dictionaries.

Comparison of Basis Pursuit, Matching Pursuit and Orthogonal Matching Pursuits.
• Wavelet coefficients of images often have the decay exponent $s = 1$ of bounded variation images.

• $Q$ measurements with a linear uniform sampling satisfy $Q/M < 5$, relatively to a non-linear approximation with $M$ coefficients.

• Direct compressive is worst with typically $Q/M > 7$.

• Improvements by using some prior information on coefficients. Grouping wavelet coefficients scale by scale improves image approximations.
Compressive Sensing Applications

- **Robustness**: compressive sensing is a “democratic” acquisition process where all samples are equally important. A missing sample introduce an error that is diluted across the signal.

- **Analog to Digital Converters**: for signals that have a sparse Fourier transform, with a random time sampling. For ultra-wide band signals having a Nyquist rate which is too large.

- **Single pixel camera**: random Bernouilli (1 or -1) measurements of images with a single pixel at very high sampling rate.

- **Medical Resonance Imaging**: randomize as much as possible the Fourier sampling of images obtained with MRI, and use their wavelet sparsity to improve their resolution.
• Sparse super-resolution becomes stable with randomized measurements. Large potential applications.

• Asymptotic performance equivalent to a non-linear approximation with a known signal.

• The devil is in the constants, compared to a linear uniform sampling.

• Technological difficulties for the signal recovery: large amount of memory and computations are required.