Multifrequency Channel Decompositions of Images and Wavelet Models

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Abstract—In this paper we review recent multichannel models developed in psychophysics, computer vision, and image processing. In psychophysics, multichannel models have been particularly successful in explaining some low-level processing in the visual cortex. The expansion of a function into several frequency channels provides a representation which is intermediate between a spatial and a Fourier representation. We describe the mathematical properties of such decompositions and introduce the wavelet transform. We review the classical multiresolution pyramidal transforms developed in computer vision and show how they relate to the decomposition of an image into a wavelet orthonormal basis. In the last section we discuss the properties of the zero crossings of multifrequency channels. Zero-crossings representations are particularly well adapted for pattern recognition in computer vision.

I. INTRODUCTION

WITHIN the last 10 years, multifrequency channel decompositions have found many applications in image processing. In the psychophysics of human vision, multichannel models have also been particularly successful in explaining some low-level biological processes. The expansion of a function into several frequency channels provides a representation which is intermediate between a spatial and a Fourier representation. In harmonic analysis, this kind of transform appeared in the work of Littlewood and Payley in the 1930's. More research has recently been focused on this domain with the modeling of a new decomposition called the wavelet transform. In this paper we review the recent multichannel models developed in psychophysics, computer vision, and image processing. We describe the motivations of the models within each of these disciplines and show how they relate to the wavelet transform.

In psychophysics and the physiology of human vision, evidence has been gathered showing that the retinal image is decomposed into several spatially oriented frequency channels. In the first section of this paper, we describe the experimental motivations for this model. Biological studies of human vision have always been a source of ideas for computer vision and image processing research. Indeed, the human visual system is generally considered to be an optimal image processor. The goal is not to imitate the processings implemented in the human brain, but rather to understand the motivations of such processings and analyze their application to computer vision problems. From this point of view, the recent experimental findings in psychophysics and physiology open challenging questions. In order to get a better understanding of multichannel decompositions, we review the main mathematical results in this domain. The best-known decomposition which is intermediate between a spatial and a frequency representation is the window Fourier transform. The window Fourier transform is used in signal processing for coding and pattern detection [47]. We describe its properties but also show why it is not a convenient decomposition for image analysis. The wavelet transform was introduced by Morlet to overcome the shortcomings of the window Fourier transform. It is computed by expanding the signal into a family of functions which are the dilations and translations of a unique function \( \psi(x) \). Grossmann and Morlet [20] have shown that any function in \( L^2(R) \) can be characterized from its decomposition on the wavelet family \( \{ \sqrt{s} \psi(s(x-u)) \}_{s,u} \in R^2 \). A wavelet transform can be interpreted as a decomposition into a set of frequency channels having the same bandwidth on a logarithmic scale. We review the most important properties of a wavelet transform and describe its discretization as studied by Daubechies [11]. A very important particular case of discrete wavelet transform was found by Meyer [45] and Stromberg [55]. They proved that there exist some wavelets \( \psi(x) \) such that \( \{ \sqrt{2} \psi(2^j(x-2^{-jn})) \}_{j,n} \in Z^2 \) is an orthonormal basis of \( L^2(R) \). Wavelet orthonormal bases provide an important new tool in functional analysis. Indeed, it was believed that we could not build simple orthonormal bases of \( L^2(R) \) whose elements have a good localization both in the spatial and Fourier domains. These bases have already found many applications in pure and applied mathematics [27, 33, 57], in quantum mechanics [15, 48], and in signal processing [30].

In computer vision, multifrequency channel decompositions are interpreted through the concept of multiresolution. Generally, the structures that we want to recognize have very different sizes. Hence, it is not possible to define a priori an optimal resolution for analyzing images. Several researchers [22, 42, 52] have developed pattern matching algorithms which process the image at different resolutions. Some pyramidal implementations have been developed for computing these decompositions [4, 10, 50]. A multiresolution transform also decomposes the signal into a set of frequency channels of constant...
bandwidth on a logarithmic scale. It can be interpreted as a discrete wavelet transform. We review the wavelet multiresolution model [38] which provides a mathematical interpretation of the concept of resolution. We see in particular that a large class of wavelet orthonormal bases can be computed from quadrature mirror filters [39].

Multifrequency channel decompositions are well adapted for data compression in image coding. We show that this efficiency is due to the intrinsic statistical properties of images and to the ability of such representations to match the sensitivity of human vision. For pattern recognition applications, it is also necessary to build a signal representation which translates when the signal translates. Indeed, the representation of a pattern should not depend upon its position. When a pattern is translated, its representation should be translated without being modified. The pyramidal multiresolution representations as well as discrete wavelet transforms do not have this translation property. In the last section, we study the properties of rep-

Fig. 1. Illustration of the anatomical visual pathway. The higher level processes are the least understood and are difficult to evaluate in psychophysical experiments.
Contrast Sensitivity

-1 2 3 5 10 20
Frequency [c/deg]

Retina Image

\( H_n \) : combined with the band-pass filter \( H_p \)

The diagram illustrates the multifrequency channel decompositions in vision.
A. Definition of a Window Fourier Transform

From the Fourier transform of a function $f(x)$, we get a measure of the irregularities (high frequencies) but this information is not spatially localized. Indeed, the Fourier
\[ Gf(\omega_0, u_0) = \int_{-\infty}^{+\infty} f(x) \overline{g_{\omega,ue}(x)} \, dx \]
\[ = \int_{-\infty}^{+\infty} \tilde{f}(\omega) g_{\omega,ue}(\omega) \, d\omega. \]  

(12)

The first integral shows that in the spatial domain, \( Gf(\omega_0, u_0) \) essentially depends upon the values of \( f(x) \) for \( x \in [u_0 - \sigma_u, u_0 + \sigma_u] \). The second integral proves that in the frequency domain, \( Gf(\omega_0, u_0) \) depends upon the values of \( \tilde{f}(\omega) \) for \( \omega \in [\omega_0 - \sigma_\omega, \omega_0 + \sigma_\omega] \). The spatio-frequency domain which is covered by \( Gf(\omega_0, u_0) \) can thus be represented in the phase-space by the resolution cell \([u_0 - \sigma_u, u_0 + \sigma_u] \times [\omega_0 - \sigma_\omega, \omega_0 + \sigma_\omega]\) as shown in Fig. 7.

The surface and shape of the resolution cell is independent from \( u_0 \) and \( \omega_0 \). The uncertainty principle applied to the function \( g(x) \) implies that

\[ \sigma_u^2 \sigma_\omega^2 \geq \frac{\pi}{2}. \]  

(13)

The resolution cell can therefore not be smaller than \( \sqrt{\pi} \). The uncertainty inequality reaches its upper limit \( \sqrt{\pi} \).

The discrete window Fourier transform is equivalent to a division of the frequency axis into intervals separated by \( \omega_0 \) (see Fig. 6). In each of these intervals, the signal is sampled at a rate \( 1/\omega_0 \). Daubechies [11] made a thorough study of the completeness and stability of a discrete window Fourier transform. Intuitively, the sampling intervals \( u_0 \) and \( \omega_0 \) must be chosen in order to cover the whole phase-space with the resolution cells shown in Fig. 7. Formally, to reconstruct any function \( f(x) \in L^2(\mathbb{R}) \) from the set of samples \( \{G_{\omega,ue}(m)\} \), the operator

\[ \begin{bmatrix} \omega_0 \\ u_0 \end{bmatrix} \]
We saw that the spatial and frequency resolution of a window Fourier transform is constant. In the spatial domain, the information provided by this decomposition is therefore unlocalized within intervals of size $\sigma_x$. The standard deviation $\sigma_x$ of $g(x)$ defines a resolution of reference. If the signal has a discontinuity such as an edge, with a window Fourier transform, it is difficult to locate this edge with a precision better than $\sigma_x$ (see Fig. 9). This localization limit is generally not acceptable. If the signal has important features of very different sizes, we cannot define an optimal resolution for analyzing the signal. This is typically the case with images. For example, in the image of a house, the pattern we want to analyze might range from the overall structure of the house to the details on one of the window curtains. With a given window size, it is difficult to analyze both the fine and the large structures. This fixed resolution also introduces misleading high frequencies when decomposing local features. Let $e(x)$ be an edge as shown in Fig. 9, and suppose that

$$
e(x) = \begin{cases} 
0 & \text{if } x \leq x_0 - \frac{\Delta x}{2} \\
\frac{1}{2} + \frac{1}{2} \sin \left( \frac{\pi}{\Delta x} (x - x_0) \right) & \frac{\Delta x}{2} < x < x_0 + \frac{\Delta x}{2} \\
1 & \text{if } x \geq x_0 + \frac{\Delta x}{2}.
\end{cases}$$

Let us denote $\omega_0 = \pi / \Delta x$. One would expect that at the point $x_0$, the decomposition coefficients $Ge(\omega, x_0) = \langle e(x), e^{i\omega x}g(x-x_0) \rangle$ decrease very quickly when $\omega$ gets larger than $\omega_0$. Indeed, in the neighborhood of $x_0$, the edge $e(x)$ is a sinusoidal wave of frequency $\omega_0$. In reality, this property does not hold because the edge is very localized and has only half of the sinusoidal wave period. As a consequence, when the frequency $\omega$ is large with respect to $\omega_0$, the modulus of the coefficients $Ge(x_0, \omega)$ decreases slowly. Although the signal $e(x)$ is locally a pure sinusoidal wave of frequency $\omega_0$, at a frequency $2\omega_0$, the window Fourier coefficient $|Ge(x_0, 2\omega_0)|$ is still about half the value of $|Ge(x_0, \omega_0)|$. This numerical property makes it hard to interpret the window Fourier coefficients when the features are very localized with respect to the size of the support of $g(x)$. More details about this property can be found in the article of Daubechies [11]. A window Fourier transform is better suited for analyzing signals where all the patterns appear approximately at the same scale.

In order to avoid the inconvenience of a transform having a fixed resolution in the spatial and frequency domains, Morlet defined a decomposition based on dilations. In the next section, we describe the properties of this decomposition which is called the wavelet transform.

C. Definition of a Wavelet Transform

Morlet [20] defined the wavelet transform by decomposing the signal into a family of functions which are the translation and dilation of a unique function $\psi(x)$. The function $\psi(x)$ is called a wavelet and the corresponding wavelet family is given by $(\sqrt{s} \psi(s(x - u)))_{(s, u) \in R^2}$. The wavelet transform of a function $f(x) \in L^2(R)$ is defined by

$$Wf(s, u) = \int_{-\infty}^{+\infty} f(x) \sqrt{s} \psi(s(x - u)) \, dx.$$  \hspace{1cm} (19)

The idea behind the wavelet decomposition is not new. It is very much related to some other types of spatial-frequency decompositions such as the Wigner-Ville transform. Some versions of the wavelet transform have been studied independently under other names such as the scale-space decomposition of Witkin [62], and in mathematics its origin can be traced back to being of the century. However, the formalization effort of Morlet and Grossmann [20] opened a broader field of applications and has led to important new mathematical results. Let us denote the dilation of $\psi(x)$ with a factor $s$ by

$$\psi_s(x) = \sqrt{s} \psi(sx).$$  \hspace{1cm} (20)

A wavelet transform can be rewritten as inner products in $L^2(R)$

$$Wf(s, u) = \langle f(x), \psi_s(x - u) \rangle.$$  \hspace{1cm} (21)

It thus corresponds to a decomposition of $f(x)$ on the family of functions $(\psi_s(x - u))_{(s, u) \in R^2}$. As shown in Fig. 10, the functions $\psi_s(x)$ have the same type as $\psi(x)$, but have a support $s$ times smaller. In the following, we suppose that the wavelet $\psi(x)$ and the signal $f(x)$ have real values. As explained later, in order to reconstruct $f(x)$ from its wavelet transform, the Fourier transform $\hat{\psi}(\omega)$ of $\psi(x)$ must satisfy

$$C_\psi = \int_{-\infty}^{+\infty} \frac{\mid \hat{\psi}(\omega) \mid^2}{\omega} \, d\omega < \infty.$$ \hspace{1cm} (22)

This condition implies that $\hat{\psi}(0) = 0$, and that $\hat{\psi}(\omega)$ is small enough in the neighborhood of $\omega = 0$. The function $\psi(x)$ can be interpreted as the impulse response of a band-pass filter. For normalization purposes, we suppose that the energy of $\psi(x)$ is equal to 1. Let us denote $\hat{\psi}_s(x) = \psi_s(-x)$. We can rewrite the wavelet transform at a point $u$ and a scale $s$ as a convolution product with $\hat{\psi}_s(x)$

$$Wf \circ u \lambda = f \ast \hat{\psi}_s(u)$$ \hspace{1cm} (23)
A wavelet transform can therefore be viewed as a filtering of $f(x)$ with a band-pass filter whose impulse response is $\hat{\psi}_{s}(x)$. From (20), we derive that the Fourier transform of $\hat{\psi}_{s}(x)$ is given by

\begin{equation}
\hat{\psi}_{s}(\omega) = \frac{1}{\sqrt{S_{0}}} e^{i\omega t_{0}} \frac{\sin(\pi \sigma_{0} \omega)}{\pi \omega} \delta(\omega - \sigma_{0}) \theta(\omega) \theta(\omega + \sigma_{0}).
\end{equation}

Fig. 10. (a) Graph of a wavelet $\hat{\psi}(x)$. (b) Graph of $\hat{\psi}_{s}(x)$ for $s_{1} > 1$. (c) Graph of $\hat{\psi}_{s}(x)$ for $s_{1} < 1$. The curves (a'), (b'), and (c') are, respectively, the Fourier transform of the function shown in (a), (b), and (c). They have the same rms bandwidth on a logarithmic scale.

Fig. 11. In the phase-space, the shape of a wavelet resolution cell depends upon the scale. When the scale increases, the resolution increases in the spatial domain and decreases in the frequency domain. The surface of all the resolution cells is the same.

ables the wavelet transform to zoom into the irregularities of the signal and characterize them locally.

For some applications, it can be useful to use a complex wavelet $\psi(s)$ in order to separate a phase and modulus component from the wavelet transform. For this purpose, Morlet and Grossmann are using wavelets whose Fourier transform is equal to zero for $\omega < 0$. Such
Equation (23) is proved by applying the Parseval theorem and using the definition of \( Wf(s, u) \) given in (19). Similarly, we can derive that the reconstruction of \( f(x) \) from \( Wf(s, u) \) is given by
\[
f(x) = \frac{1}{C_0} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} Wf(s, u) \psi_s(x - u) \, ds \, du.
\] (24)

Like a window Fourier transform, a wavelet transform is redundant. In other words, the value of \( Wf(s', u') \) depends upon the values of \( Wf(s, u) \) for \( s \neq s' \) and \( u \neq u' \). By inserting (24) in the definition (19) of a wavelet transform, one can show that the function \( Wf(s, u) \) satisfies the following reproducing kernel equation [21]:
\[
\forall (s', u') \in \mathbb{R}^+ \times \mathbb{R}, \quad Wf(s', u') = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} Wf(s, u) K(s, s', u', u') \, ds \, du,
\] (25)

where
\[
K(s, s', u, u') = \frac{1}{C_0} \int_{-\infty}^{+\infty} \psi_s(x - u) \psi_{s'}(x - u') \, dx.
\]

\( K(s, s', u, u') \) is called a reproducing kernel. It expresses the redundancy between \( Wf(s, u) \) and \( Wf(s', u') \) for any two pairs of points \( (s, u) \) and \( (s', u') \). Equation (25) shows that, a priori, any function \( F(s, u) \in L^2(\mathbb{R}^+ \times \mathbb{R}) \) is not the wavelet transform of some function \( f(x) \in L^2(\mathbb{R}) \). One can easily prove that there exists a function \( f(x) \in L^2(\mathbb{R}) \) such that \( F(s, u) = Wf(s, u) \), if and only if
\[
\forall (s', u') \in \mathbb{R}^+ \times \mathbb{R}, \quad F(s', u') = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} F(s, u) K(s, s', u', u') \, ds \, du.
\] (26)

The function \( f(x) \) is then given by
\[
f(x) = \frac{1}{C_0} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} F(s, u) \psi_s(x - u) \, ds \, du.
\] (27)

The reproducing kernel equation is an important characterization of a wavelet transform that we use later.

The wavelet transform can be discretized by sampling both the scale parameter \( s \) and the translation parameter \( u \). In order to build a complete representation, we must cover the phase-space with the resolution cells shown in Fig. 11. This can be done with an exponential sampling of the scale parameter. We first select a sequence of scales \( (\alpha^t)_{t \in \mathbb{Z}} \), where \( \alpha \) is the elementary dilation step. We saw in (22) that the wavelet transform \( Wf(\alpha^t, u) \) can be re-written
\[
Wf(\alpha^t, u) = f * \tilde{\psi}_{\alpha^t}(u).
\] (28)

For each scale \( \alpha^t \), \( \tilde{\psi}_{\alpha^t}(x) \) has a Fourier transform centered in \( \alpha^t \omega_0 \) with an rms bandwidth of \( \alpha^t \sigma_0 \). Equation (28) can be posed signal in each channel, we must sample it uniformly at a rate proportional to \( \alpha^t \). Let \( \alpha^t / \beta \) be the sampling rate at the scale \( \alpha^t \). The discrete wavelet transformed is defined by
\[
W_d f(j, n) = Wf\left(\alpha^j, \frac{n \beta}{\alpha^j}\right) = \int_{-\infty}^{+\infty} f(x) \psi_{\alpha^j}(x - \frac{n \beta}{\alpha^j}) \, dx
\]
\[
= f * \tilde{\psi}_{\alpha^j}(\frac{n \beta}{\alpha^j}).
\] (29)

Fig. 12 illustrates this sampling pattern in the phase-space. When the scale increases, the density of samples increases.

It is not possible to understand the properties of this transform by using the Nyquist theorem since the Fourier transform of \( \psi(x) \) does not have a compact support (it is not strictly band-limited). With an approach similar to her study of the discrete window Fourier transform, Daubechies [11] analyzed the main properties of a discrete wavelet transform. She made a clear comparison of these two types of multichannel decompositions from a mathematical point of view. In order to reconstruct a function \( f(x) \) from the discrete wavelet transform \( W_d f(j, n) \), the operator
\[
L^2(\mathbb{R}) \xrightarrow{W_d} L^2(\mathbb{Z}^2)
\] (30)

must be invertible on its range and have a bounded inverse. Since
\[
W_d(j, n) = \left( f(x), \psi_{\alpha^j}(x - \frac{n \beta}{\alpha^j}) \right),
\] (31)

the properties of the operator \( W_d \) depend upon the family of functions \( (\psi_{\alpha^j}(x - (n \beta / \alpha^j)))_{(0, \beta) \in \mathbb{Z}^2} \). Daubechies [11] studied the properties of this family of functions and gave some necessary and sufficient conditions on \( \alpha, \beta, \) and \( \psi(x) \) so that the operator \( W_d \) admits a bounded inverse.

A very important class of discrete wavelet transform was found independently by Meyer [45] and Stromberg [55]. They showed that there exist some wavelets \( \psi(x) \in L^2(\mathbb{R}) \) such that for \( (\alpha^t, \omega^t) = (\omega_0, \omega_0) \), ...
\( \beta = 1 \). Wavelet orthonormal bases can be built for sequences of scales other than \( (2^j)_{j \in \mathbb{Z}} \), but we will concentrate on dyadic scales which lead to simpler decomposition algorithms. These new orthonormal bases had a striking impact in functional analysis. It was indeed believed that one could not find simple orthonormal bases whose elements have a good localization both in the spatial and frequency domains. Any function can be reconstructed from its decomposition into a wavelet orthonormal basis with the classical expansion formula of a vector into an orthonormal basis

\[
f(x) = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f(u), \psi_{2j}(u - n2^{-j}) \rangle \psi_{2j}(x - n2^{-j}).
\]  

(32)

The Haar basis is a well-known particular case of wavelet orthonormal basis. The orthogonal wavelet corresponding to the Haar basis is given by

\[
\psi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} \leq x < 1 \\
0 & \text{otherwise.}
\end{cases}
\]  

(33)

The Haar wavelet is not continuous, which is a major inconvenience for many applications. Meyer [45] showed that we can find some orthogonal wavelets \( \psi(x) \) which are infinitely continuously differentiable and whose decay at infinity are faster then any power \( x^{-n}, n > 0 \). In Section IV-A, we show that the Fourier transform of a large class of orthogonal wavelets can be expressed from the transfer function of a quadrature mirror filter [38]. The decomposition of a function in such a wavelet orthonormal basis can be computed with a quadrature mirror filter bank. Fig. 13 gives the graph of a particular orthogonal wavelet and its Fourier transform. This wavelet is a cubic spline studied independently by Lemarie [32] and Battle [3].

An important property of a wavelet transform is to easily characterize the local regularity of a function. This can have a particularly interesting application for discriminating image textures. In mathematics, it leads to a simple characterization of the classical functional spaces such as the \( L^p(\mathbb{R}) \) spaces, the Sobolev spaces, the Holder spaces, etc. Let us give an example. One way to measure the local regularity of a function is to measure the Lipschitz exponent. A function \( f(x) \) is lipschitz \( \alpha \) in the neighborhood of a point \( x_0 \), if and only if, for any point \( x \) in a neighborhood of \( x_0 \),

\[
|f(x) - f(x_0)| = O(|x - x_0|^{\alpha}).
\]  

(34)

A function which is differentiable in \( x_0 \) is lipschitz 1. The larger the Lipschitz coefficient \( \alpha \), the smoother the function is in the neighborhood of \( x_0 \). Let us now suppose that the wavelet \( \psi(x) \) is continuously differentiable. We also assume that our signal \( f(x) \) is continuous and that there exist \( \epsilon > 0 \) such that \( f(x) \) is lipschitz \( \epsilon \) everywhere. Jaffard [26] proved that for any \( \alpha > 0 \), one can find whether

\[
f(x) \text{ is lipschitz } \alpha \text{ at } x_0 \text{ by measuring the decay of wavelet coefficients in the neighborhood of } x_0. \text{ More precisely, } f(x) \text{ is lipschitz } \alpha \text{ at } x_0 \text{ if and only if}
\]

\[
\exists C > 0, \quad \forall n \in \mathbb{Z}, \quad \left| \langle f(x), \psi_{2j}(x - n2^{-j}) \rangle \right| 
\leq C 2^{-j(1/2 + \alpha)} (1 + |2^j x_0 - n|).
\]  

(35)

The regularity of a function at a point \( x_0 \) thus depends upon the decay rate of the wavelet coefficients in the neighborhood of \( x_0 \), when the scale increases. Other kinds of regularity, such as the derivability at any order (in the sense of Sobolev), can be derived similarly [33]. These results show that it is necessary to combine the information at different scales in order to analyze the local properties of a function. In the next section, we describe the extension of the wavelet model to two-dimensional signals. We come back to orthonormal wavelets in Section IV-A to explain their relation to the concept of multi-resolution in computer vision.

E. Wavelet Transform in Two Dimensions

The wavelet transform can be generalized in \( \mathbb{R}^n \), but we only consider the two-dimensional case for image processing applications. The model can first be extended without distinguishing any spatial orientation. Let \( \tilde{\Psi}(x, y) \in L^2(\mathbb{R}^2) \) be a function whose Fourier transform \( \tilde{\Psi}(\omega_x, \omega_y) \) satisfies

\[
\Psi(\omega_x, \omega_y) \in \mathbb{R}^2 \int_0^{+\infty} \frac{\left| \tilde{\Psi}(s\omega_x, s\omega_y) \right|^2}{s} \, ds = C_{\Psi} < +\infty.
\]  

(36)
The value of the integral (36) must be finite and constant for all \((\omega, \omega') \in \mathbb{R}^2\). For example, this property is satisfied for a wavelet \(\Psi(x, y)\) which is isotropic \((\Psi(x, y) = \rho(\sqrt{x^2 + y^2})\) and whose Fourier transform is null at the origin \((\hat{\Psi}(0, 0) = 0)\). For normalization purposes, we suppose that \(\|\Psi\| = 1\). The function \(\Psi(x, y)\) can be interpreted as the impulse response of a band-pass filter having no preferential spatial orientation. The wavelet transform of a function \(f(x, y) \in L^2(\mathbb{R}^2)\) at the scale \(s\) and a point \((u, v)\) is defined by

\[
W_f(s, (u, v)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) s\Psi(s(x - u), s(y - v)) \, dx \, dy.
\]

(37)

Let \(\Psi_s(x, y) = s\Psi(sx, sy)\) and \(\hat{\Psi}_s(x, y) = \Psi_s(-x, -y)\). The wavelet transform of \(f(x, y)\) at the scale \(s\) and a point \((u, v)\) can be rewritten as a convolution product

\[
W_f(s, (u, v)) = f \ast \hat{\Psi}_s(u, v).
\]

(38)

It can be interpreted as a two-dimensional band-pass filtering with no orientation selectivity. The wavelet transform in two dimensions has the same properties as a one-dimensional wavelet transform. There is an energy conservation equation

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| W_f(s, (u, v)) \right|^2 \, ds \, du = C_\phi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x, y)|^2 \, dx \, dy.
\]

(39)

As in the one-dimensional case, this equation is proved with the Parseval theorem. We can also reconstruct a function \(f(x, y)\) from its wavelet transform with a simple two-dimensional extension of (24):

\[
f(x, y) = \frac{1}{C_\phi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_f(s, (u, v)) \cdot \Psi'_s(x - u, y - v) \, ds \, du \, dv.
\]

(40)

In two dimensions, a wavelet transform also satisfies a reproducing kernel equation similar to (25).

For image recognition applications, it is often necessary to have a decomposition which depends on \(s \cdot \theta\).

Fig. 14. Decomposition in the Fourier domain of the support of \(\hat{\Psi}(\omega, \omega')\) into 6 wavelets \(\hat{\Psi}^i(\omega, \omega') (1 \leq i \leq 6)\) having different orientation selectivities. In this example, the supports of the functions \(\hat{\Psi}^i(\omega, \omega')\) are symmetrical about 0 and are rotated from one another.

The discretization of a wavelet transform in two dimensions is similar to the discretization in one dimension. We choose a sequence of scales \((\alpha')_{\mathbb{Z}}\) where \(\alpha\) is the elementary dilation step. For each scale \(\alpha'\), the translation
resolution transforms and show how it relates to a discrete wavelet decomposition.

Multiresolution transforms have been thoroughly studied in computer vision since the work of Rosenfeld and Thurston [51] on multiscale edge detection, and the Marr theory of low-level vision [40]. At different resolutions, the details of an image generally characterize different types of physical structures. For example, a coarse resolution satellite image of a coast gives a description of only the overall shape of the coast. When the resolution of the image is increased, we are able to successively distinguish the local relief of the region, and if the resolution gets even finer, we can recognize the different types of local vegetation. In order to process these different structures separately, researchers in computer vision have tried to extract the difference of information between the approximation of an image at two different resolutions. Given a sequence of increasing resolutions \((r_j)_{j \in \mathbb{Z}}\), the details of \(f(x)\) at the resolution \(r_j\) are defined as the difference of information between the approximation of \(f(x)\) at the resolution \(r_{j+1}\) and the approximation at the resolution \(r_j\).

A multiresolution representation also provides a simple hierarchical framework for interpreting the image information [29]. In some sense, the details of the image at a coarse resolution provide the “context” of the image, whereas the finer details correspond to the particular “modalities.” For example, it is difficult to recognize that a small rectangle inside an image is the window of a house if we did not previously recognize the house “context.” It is therefore natural to first analyze the image details at a coarse resolution and then increase the resolution. This is called a coarse-to-fine processing strategy. At a coarse resolution, the image details are characterized by very few samples. Hence, the coarse information processing can be performed quickly. The finer details are characterized by more samples, but the prior information, derived from the context, constrains and thus speeds up the computations. With a coarse-to-fine strategy, we process the minimum amount of details which are necessary to perform a recognition task. Indeed, if we can recognize an object from a coarse description, we do not need to analyze the finer details. For example, in order to distinguish a car from a house, the coarse details of the image should be enough. Such a strategy is efficient for pattern recognition algorithms. It has already been widely studied for low-level image processing tasks such as stereo matching and template matching [19], [22].

### A. Pyramidal Multiresolution Decompositions

The approximation of a signal \(f(x)\) at a resolution \(r\) is defined as an estimate of \(f(x)\) derived from \(r\) measurements per unit length. These measurements are computed by uniformly sampling a rate \(r \) the function \(f(x)\) smoothed by a low-pass filter whose bandwidth is proportional to \(r\). In order to be consistent when the resolution varies, these low-pass filters are derived from a unique function \(\theta(x)\) which is dilated by the resolution factor \(r : \theta_{r} = \sqrt{r} \theta(r x)\). The set of measurements \(A_r f = (f \ast \theta_{r}(n/r))_{n \in \mathbb{Z}}\) is called a discrete approximation of \(f(x)\) at the resolution \(r\). In the following, we study the approximation of a function on a dyadic sequence of resolutions \((2^j)_{j \in \mathbb{Z}}\). The discrete approximation of a function \(f(x)\) at the resolution \(2^j\) is thus given by

\[
A_{2^j} f = \left(f \ast \theta_{2^j}(n/2^j)\right)_{n \in \mathbb{Z}}. \quad (45)
\]

Tanimoto and Pavlidis [56], Burt [4], and Crowley [10] have developed efficient algorithms to compute the approximation of a function at different resolutions. We first describe these decompositions and then explain the Burt and Crowley algorithms for computing the details at different resolutions. The details are regrouped in a pyramid data structure called a Laplacian pyramid. This simple and elegant algorithm does not define the details from the difference of information between \(A_{2^j+1} f\) and \(A_{2^j} f\). At different resolutions, the details computed with this algorithm are correlated. It is thus difficult to know whether a similarity between the image details at different resolutions is due to a property of the image itself or to the intrinsic redundancy of the representation. We review the multiresolution wavelet model which shows that the difference of information between two successive resolutions can be computed by decomposing the signal in a wavelet orthonormal basis.

In pyramidal multiresolution algorithms, the low-pass filter function \(\theta(x)\) is chosen such that its Fourier transform can be written

\[
\hat{\theta}(\omega) = \prod_{p=1}^{\omega} U(e^{-i2^p \omega}), \quad (46)
\]

where \(U(e^{-i\omega})\) is the transfer function of a low-pass discrete filter \(U = (u_n)_{n \in \mathbb{Z}}\). Daubechies [11] studied the regularity and decay at infinity of the function \(\theta(x)\) depending upon the properties of the filter \(U(e^{-i\omega})\). In general, we want to have a function \(\theta(x)\) which is as smooth as possible and which is well concentrated around 0 in the spatial domain.

Let us suppose that we have already computed the discrete approximation of a function \(f(x) \in L^2(\mathbb{R})\) at the resolution \(2^j+1\): \(A_{2^{j+1}} f = (f \ast \theta_{2^{j+1}}(n/2^{j+1}))_{n \in \mathbb{Z}}\). One can show [4], [11], [38] that the discrete approximation of \(f(x)\) at a resolution \(2^j\) is calculated by filtering \(A_{2^{j+1}} f\) with the discrete low-pass filter \(U = (u_n)_{n \in \mathbb{Z}}\) and keeping every other sample of the convolution product. Let \(\Lambda = (\lambda_n)_{n \in \mathbb{Z}}\) be such that

\[
\Lambda = A_{2^{j+1}} f \ast U, \quad (47)
\]

then

\[
A_{2^j} f = (\lambda_{2n})_{n \in \mathbb{Z}}. \quad (48)
\]

A measuring device provides the approximation of an input signal at a finite resolution. Let us suppose for normalization purposes that this resolution is equal to one. The approximation of this signal at any resolution \(2^{-j}, J\)
where $\Theta(x, y)$ is a two-dimensional low-pass filter, and $\Theta_2(x, y) = 2^j \Theta(2^j x, 2^j y)$. For image processing, the pyramidal algorithm is extended with separable convolutions along the rows and columns of the image \cite{4}. The low-pass filter $\Theta(x, y)$ is chosen such that its Fourier transform can be written

$$
\hat{\Theta}(\omega_x, \omega_y) = \prod_{p=1}^{+\infty} U(\epsilon^{-2^{p} \omega_p}) U(\epsilon^{-2^{j} \omega_p})
$$

Let us suppose that the video camera provides an image approximated at the resolution 1: $A_1 f = (f * \Theta(n, m))_{(n, m) \in Z^2}$. With a separable extension of the algorithm described in (47) and (48), we can compute the approximation of an image at any resolutions $2^j$, $(j < 0)$. Fig. 16 shows an image approximated at the resolution $2^j$ for $0 \geq j \geq -3$ (Gaussian pyramid). The detail signals $(D_2 f)_{0 < j \leq -3}$ can also be computed with a straightforward extension of the one-dimensional algorithm. Fig. 17 shows the Laplacian pyramid of the image given in Fig. 16. If the original image has $N^2$ pixels, each detail image $D_2^j f$ has $2^{2j}N^2$ pixels and $A_2^j f$ has $2^{2j}N^2$ pixels. Hence, the total number of pixels of this representation is approximately $N^2$.

In a Laplacian pyramid, the signals $D_2 f$ do not correspond to the difference of information between $A_2 f$ and $A_2 f$. If they did, the total number of pixels representing the signal would be the same as in the original signal. We saw that the number of samples representing the signal is increased by a factor of 2 in one dimension and by a factor of $\frac{3}{2}$ in two dimensions. This is due to the correlation between the detail signals $D_2 f$ at different resolutions. The correlation can be understood and suppressed with the multisresolution wavelet model described in [39] and [38]. It is indeed possible to extract exactly the difference of information between $A_2 f$ and $A_2 f$ by decomposing the signal into a wavelet orthonormal basis.

Let us first explain the multisresolution wavelet model in one dimension. We saw in (45) that the discrete approximation of a function $f(x)$ at the resolution $2^j$ is defined by $A_2 f = (f * \Theta_2(2^j n))_{n \in \mathbb{Z}}$. Let us denote $\tilde{\theta}_2(x) = \theta_2(x - x)$. Each convolution product in a point can be rewritten as an inner product in $L^2(R)$

$$
A_2 f = \left( f(x), \tilde{\theta}_2(x - 2^{-j} n) \right)_{n \in \mathbb{Z}}
$$

Let us call the continuous approximation of $f(x)$ at the resolution of $2^j$ the best estimate of $f(x)$ given the sequence of inner products $A_2 f$. By “best” we mean as close as possible to $f(x)$ with respect to the $L^2(R)$ distance (mean square distance). One can easily derive from the projection theorem that this best estimate is equal to the orthogonal projection of $f(x)$ on the vector space $V_2$, generated by the family of functions $(\tilde{\theta}_2(x - 2^{-j} n))_{n \in \mathbb{Z}}$. The vector space $V_2$ can be viewed as the set of all possible approximations of functions at resolution $2^j$. The sequence of vector spaces $(V_2)_{j \in \mathbb{Z}}$ is called a multisresolution approximation of $L^2(R)$. The proper-
ties of the vector space $V_2$, are further studied in [38] and [39]. For any function $f(x) \in L^2(\mathbb{R})$, the continuous approximation of $f(x)$ at the resolution $2^j$ is thus given by the orthogonal projection of $f(x)$ on $V_2$. In order to compute this approximation, we need an orthonormal basis of $V_2$. One can show [39] that we can build such an orthonormal basis by dilating and translating a particular function $\phi(x)$ called a scaling function. For any resolution $2^j$, let us denote $\phi_{2j}(x) = \sqrt{2} \phi(2^j x)$. The family of functions $(\phi_{2j}(x-2^{-j} n))_{n \in \mathbb{Z}}$ is then an orthonormal basis of $V_{2j}$. The Fourier transform of $\phi(x)$ is characterized by

$$\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} H(e^{-i2^{-p}\omega}),$$

where $H(e^{-i\omega})$ is the transfer function of a discrete filter [39]. One can show that $H(e^{-i\omega})$ satisfies the condition

$$|H(e^{-i\omega})|^2 + |H(-e^{-i\omega})|^2 = 1. \quad (53)$$

The discrete filters $H = (h_n)_{n \in \mathbb{Z}}$ whose transfer function satisfy (53) are called quadrature mirror filters [14].

The orthogonal projection of a function $f(x) \in L^2(\mathbb{R})$ on $V_j$ can now be computed by decomposing $f(x)$ into the orthonormal basis $(\phi_{2j}(x-2^{-j} n))_{n \in \mathbb{Z}}$. Let $P_{V_j}$ be the orthogonal projection operator on $V_{2j}$:

$$P_{V_j}(f)(x) = \sum_{n \in \mathbb{Z}} \langle f(u), \phi_{2j}(u-2^{-j}n) \rangle \phi_{2j}(x-2^{-j}n). \quad (54)$$

Let us denote $\hat{\phi}(x) = \phi(-x)$. Since $\phi(x)$ is a low-pass filter, we can redefine the discrete approximation $A_{2j}f$ with the function $\hat{\phi}(x)$ instead of $\theta(x)$

$$A_{2j}f = (f \ast \hat{\phi}_{2j}(2^{-j}n))_{n \in \mathbb{Z}} = ((f(x), \phi_{2j}(x-2^{-j}n))_{n \in \mathbb{Z}}. \quad (55)$$

The best estimate of $f(x)$ can easily be derived from this discrete approximation by using (54). Let $\hat{H}$ be the discrete filter whose impulse response is $(h_{-n})_{n \in \mathbb{Z}}$. From (52) and (55), one can show [38] that the discrete approximations, $A_{2j}f$, are computed with the same pyramidal algorithm described in (47) and (48), by using the discrete filter $\hat{H}$ instead of $U$. Fig. 18 gives the graph of a scaling function $\phi(x)$.

Let us now explain how to extract exactly the difference of information between the approximations of a function at the resolutions $2^j$ and $2^{j+1}$. The approximations of a function $f(x) \in L^2(\mathbb{R})$ at the resolutions $2^j$ and $2^{j+1}$ are given by the orthogonal projection of $f(x)$ on the vector spaces $V_{2j}$ and $V_{2^{j+1}}$, respectively. Intuitively, the approximation at the resolution $2^{j+1}$ must give a better estimate of $f(x)$ than the approximation at the resolution $2^j$. Hence, the vector spaces $V_{2j}$ and $V_{2^{j+1}}$ should satisfy

$$V_{2j} \subset V_{2^{j+1}}. \quad (56)$$

The difference of information between the approximations at the resolutions $2^j$ and $2^{j+1}$ is therefore equal to the
orthogonal projection of \( f(x) \) on the orthogonal complement of \( V_1 \) in \( V_0 \). Let \( O \) be this orthogonal projection.
characterized by the sequences of inner products

\[ D^H_1 f = \left\{ \left\langle f(x, y), \psi^H_1(x - 2^{-j}n, y - 2^{-j}m) \right\rangle \right\} \] 
\[ (n, m) \in \mathbb{Z}^2 \]

\[ D^H_2 f = \left\{ \left\langle f(x, y), \psi^H_1(x - 2^{-j}n, y - 2^{-j}m) \right\rangle \right\} \] 
\[ (n, m) \in \mathbb{Z}^2 \]

\[ D^H_3 f = \left\{ \left\langle f(x, y), \psi^H_1(x - 2^{-j}n, y - 2^{-j}m) \right\rangle \right\} \] 
\[ (n, m) \in \mathbb{Z}^2 \]

Each of these sequences of inner products can be considered as an image. \( D^H_1 f \) gives the vertical higher frequencies (horizontal edges), \( D^H_2 f \) gives the horizontal higher frequencies (vertical edges), and \( D^H_3 f \) gives the higher frequencies in both directions (corners) (see Fig. 21). Let us suppose that initially we have an image \( A_1 f \) measured at the resolution 1. For any \( J > 0 \), this discrete image can be decomposed between the resolutions 1 and \( 2^{-J} \), and completely represented by the \( 3J + 1 \) discrete images

\[ (A_{2^j} f, (D^H_{2j} f)_{-J \leq j \leq 1}, (D^H_{3j} f)_{-J \leq j \leq 1}) \]

This set of images is called an orthogonal wavelet representation in two dimensions [38]. The image \( A_{2^j} f \) is a coarse approximation, and the images \( D^H_{2j} f \) give the image details for different orientations and resolutions. If the original image has \( N^2 \) pixels, each image \( A_{2^j} f, D^H_{2j} f, D^H_{3j} f, D^H_{3j} f \) has \( 2^j \cdot N^2 \) pixels (\( j < 0 \)). The total number of pixels of an orthogonal wavelet representation is therefore equal to \( N^2 \). It does not increase the volume of data. This is due to the orthogonality of the representation.

A wavelet representation can be computed with a separable extension of the algorithm illustrated in Fig. 19(a) [38]. This extension corresponds to a separable quadrature mirror filter decomposition as described by Woods [63]. Fig. 21(b) gives the wavelet representation of the image in Fig. 16. From this representation, we can reconstruct the original image with a two-dimensional separable extension of the algorithm illustrated in Fig. 19(b) [38]. Fig. 21(c) is the reconstructed image from the wavelet representation shown in Fig. 21(b). The reconstruction is numerically stable. It enables us to use this type of representation for image coding. A more general nonseparable extension of the wavelet model was studied by Meyer [43]. Such extensions are, however, more difficult to implement and are computationally more expensive.

B. Applications of Multiresolution Transforms

The wavelet model gives a precise understanding of the concept of multiresolution by introducing the sequence of vector spaces \( (V_j)_{j \in \mathbb{Z}} \). A noncorrelated multiresolution representation can be built by decomposing the signal into a wavelet orthonormal basis. A difficult problem when using a multiresolution representation for analyzing a scene is to relate the details appearing at different resolutions. Many ad hoc techniques have been developed for this purpose. We saw in Section III-D that the local regularity of a function is provided by the decay rate of the
wavelet coefficients when the resolution increases. These theorems give a first approach for comparing the value of the decomposition at different resolutions.

Multiband image decompositions are also well adapted for coding images because it is possible to match the human visual system sensitivity and take advantage of the intrinsic statistical properties of images. The contrast sensitivity function (Fig. 3) shows that the sensitivity of human vision depends upon the frequency of the stimulus. We want to quantize each frequency band with the minimum number of bits, and at the same time try to reconstruct the best possible image for the human visual perception. For this purpose, we adapt the quantization noise to the human sensitivity along each frequency band. The more sensitive the human system, the less quantization noise is introduced. This enables us to introduce a minimum amount of perceivable distortion in the reconstructed image. Watson has done some particularly detailed psychophysical experiments to test this type of approach for image coding [60].

The statistical properties of images give another reason for using multiband decompositions in image coding. It is well known that the intensity of images is locally correlated. Predictive codings have been particularly successful to compress the number of bits used in coding an image. The wavelet coefficients give a measure of the local contrast at different scales. Since the image intensity is locally correlated, these local contrasts generally have a small amplitude [38]. We can take advantage of this property for coding the wavelet coefficients on fewer bits without introducing any noticeable distortion. As explained in the previous section, a wavelet orthogonal representation can be interpreted as a decomposition into a quadrature mirror filter bank. Several studies in image processing have already shown the efficiency of these filter banks for data compression [1], [63].

In order to use a multiresolution representation for pattern recognition applications, we must be able to build models of patterns within the multiresolution representation. The patterns might be located anywhere in the image. Hence, the models must be independent from the pattern location. When a pattern is translated, its model should only be translated but not modified. Let us show that a multiresolution representation does not verify this translation property. To simplify the explanation, we consider the particular case of a one-dimensional orthogonal wavelet transform. At the resolution level 

![Wavelet Coefficients](image)

Fig. 22. This drawing shows that the sampling of a wavelet transform (given by the crosses) can be very different after translating the signal. The wavelet transform is translated but the sampling does not translate if the translation is not proportional to the sampling interval (adapted from [37]).

However, the sampling of $Wg(2^j, u)$ does not correspond to a translation of the sampling of $Wf(2^j, u)$ unless $\tau = k2^{-j}, k \in \mathbb{Z}$ (see Fig. 22).

This distortion through translation implies that the wavelet coefficients of a pattern at the resolution $2^j$ depend upon the position of the pattern modulo $2^{-j}$. This property is inherent to the notion of resolution. Indeed, at the resolution $2^j$, we cannot measure anything smaller than $2^{-j}$ so we cannot represent a displacement smaller than $2^{-j}$. One can find the same problem in all the pyramidal multiresolution representations and any uniform sampling of a wavelet transform.

A first solution to this translation problem is to sample the wavelet transform $Wf(2^j, u)$ at a rate much larger than $2^j$. The samples then translate approximately when the signal translates. However, this solution considerably increases the redundancy of the representation and the translation is still not perfect. This technique is often adopted for pattern recognition algorithms based on pyramid decompositions. A second solution consists of defining a representation based on adaptive sampling of the functions $Wf(2^j, u)$ which translates when the signal translates.

V. Zero Crossings of Multifrequency Channels

In the previous sections we studied the properties of the decomposition of a function into multifrequency channels of constant size on a logarithmic scale. We saw that such a decomposition can be interpreted as a wavelet transform. We then described the properties and applications of a discrete wavelet transform built from a uniform sampling of the continuous wavelet transform. However, we showed that such a discretization is difficult to use for pattern recognition applications because it is not invariant through translation. Here, we review the characterization
zero crossing of \( W_f(s, u) \) corresponds to a point of abrupt change in the function \( f(x) \) smoothed by \( \xi_s(x) = \sqrt{s} \xi(sx) \). Indeed, if \( \psi(x) = \xi''(x) \)

\[
W_f(s, u) = f * \psi_s(u) = s^2 (f * \xi_s)'(u). 
\]

Fig. 23. The zero crossings of a wavelet transform provide the location of the reflection points (edges) of \( f * \xi_s(x) \) (adapted from [22]).
wavelets and quadrature mirror filters. We can compute the decomposition of a function into a wavelet orthonormal basis with a quadrature mirror filter bank. A third motivation for using multiband decomposition is due to the intrinsic statistical properties of images. Images have a relatively simple decomposition into frequency subbands. These bands can be coded with fewer bits with no visible distortions.

A uniform sampling of each multifrequency channel defines a representation which is not translation invariant. It is therefore difficult to build pattern recognition algorithms from such decompositions. We reviewed the properties of zero crossing in multiband decompositions. This adaptive sampling is translation invariant but is much more difficult to analyze. We described some previous results and gave the wavelet formalization of this problem through the reproducing kernel equation.

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