Zero-Crossings of a Wavelet Transform
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Abstract—Sharp variation points are among the most meaningful features for characterizing transient signals. For a particular class of wavelets, the zero-crossings of a wavelet transform provide the locations of the signal sharp variation points at different scales. The completeness and stability of a signal representation based on zero-crossings of a wavelet transform at the scales $2^j$, for integer $j$ are studied. An alternative projection algorithm is described. It reconstructs a signal from a zero-crossing representation which is stabilized. The reconstruction algorithm has a fast convergence and each iteration requires $O(N \log^2 N)$ computations for a signal.

Index Terms—Multiscale, pattern matching, signal representation, wavelet transform, zero-crossings.

I. INTRODUCTION

A N IMPORTANT problem in signal processing is to define a representation that is well adapted for extracting the information content of signals. The sharp variations of a signal amplitude are generally among the most meaningful features. For example, the discontinuities of an image intensity provide the contours of the different objects. When the signal includes important structures that belong to different scales, it is often helpful to reorganize the signal information into a set of “detail components” of varying size [17]. Marr and Hildreth [14] have shown that one can obtain the position of multiscale sharp variations points from the zero-crossings of the signal convolved with the Laplacian of a Gaussian. This edge detection procedure has been used in many pattern recognition applications [4]. An important practical and theoretical issue is to understand whether the multiscale edges carry all the information of the original signal. Indeed, for pattern recognition applications, we do not want to remove some important components of the signal, when representing it with multiscale zero-crossings. Completeness by itself is not sufficient as for most applications the representation must also be stable. This means that a small perturbation of the representation should correspond to a small modification of the original signal. While reviewing some previous work, we shall see that the positions of multiscale zero-crossings may provide a complete representation under certain restrictive assumptions but such a representation is not stable. We show that one can stabilize a zero-crossing representation by adding a complement of information that measures the “size” of the structure between two consecutive zero-crossings. This new signal representation is based on the wavelet transform reformalization of multiscale decompositions. We introduce the most important results of the wavelet theory in order to study the properties of multiscale zero-crossings. The central result of this article is an algorithm that reconstructs one-dimensional signals from a stabilized zero-crossing representation. This algorithm iterates on a nonexpansive projector on a convex set and an orthogonal projector on a Hilbert space, hence the convergence is guaranteed. The numerical results show that the reconstruction is independent from the choice of the initial point at the beginning of the iteration but this has not been proven mathematically. The convergence is fast and each iteration requires $O(N \log^2 N)$ computations, for a signal of $N$ samples.

In order to illustrate the application of this new zero-crossing representation to pattern recognition, we describe the results of a stereo-matching algorithm. The stereo matching problem consists of finding a point by point correspondence between two one-dimensional signals that are shifted from one another and have some local distortions. In image processing, we must solve such a correspondence problem when trying to recover a depth information from a pair of stereo images. We introduce a simple distance based on our multiscale zero-crossing representation and derive a coarse to fine matching algorithm to compute the stereo correspondence. Matching results on two epipolar lines of real images are given.

A. Notation

$\mathbb{Z}$ denotes the set of integers, $L^2$ denotes the Hilbert space of measurable, square-integrable one-dimensional functions. For $f(x) \in L^2$ and $g(x) \in L^2$, the inner product of $f(x)$ with $g(x)$ is

$$\langle g(x), f(x) \rangle = \int_{-\infty}^{\infty} g(x)f(x) \, dx.$$
The norm of \( f(x) \in L^2 \) is given by
\[
\| f \| = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.
\]
We also denote by \( L^2(L^2) \) the Hilbert space of all sequences of functions \((g_j(x))_{j \in \mathbb{Z}}\), such that for all \( j \in \mathbb{Z}, g_j(x) \in L^2 \) and
\[
\sum_{j=-\infty}^{\infty} |g_j(x)|^2 < +\infty.
\]
This infinite sum is the norm of the sequence \((g_j(x))_{j \in \mathbb{Z}} \) in \( L^2(L^2) \).

We denote the convolution of two functions \( f(x) \in L^2 \) and \( g(x) \in L^2 \) by
\[
f \ast g(x) = \int_{-\infty}^{\infty} f(u)g(x-u) \, du.
\]
The Fourier transform of \( f(x) \in L^2 \) is written \( \hat{f}(\omega) \) and is defined by
\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx.
\]

II. PROPERTIES OF THE WAVELET TRANSFORM

The wavelet transform is a linear operation that decomposes a signal into components that appear at different scales. This transform is based on the convolution of the signal with a dilated filter. Such a decomposition has been studied in signal processing [19] and computer vision [20] but has recently been reformalized in mathematics. For a thorough presentation, the reader is referred to general reviews [2], [12] and an advanced functional analysis book of Meyer [16]. A wavelet is a function \( \psi(x) \in L^2 \) such that
\[
\int_{-\infty}^{\infty} \psi(x) \, dx = 0.
\]
Let us denote by \( \psi_s(x) \) the dilation of \( \psi(x) \) by a factor \( s \):
\[
\psi_s(x) = \frac{1}{s} \psi \left( \frac{x}{s} \right).
\]
The wavelet transform of a function \( f(x) \) at the scale \( s \) and position \( x \) is given by the convolution product
\[
W_s f(x) = f \ast \psi_s(x).
\]
Morlet and Grossmann [5] have shown that the wavelet transform satisfies an energy conservation equation and that \( f(x) \) can be reconstructed from its wavelet transform. When the scale \( s \) decreases, the support of \( \psi_s(x) \) decreases so the wavelet transform \( W_s f(x) \) is sensitive to finer details. The scale \( s \) characterizes the size and regularity of the signal features extracted by the wavelet transform.

The wavelet transform depends on two parameters \( s \) and \( x \) that vary continuously over the set of real numbers. For practical applications these parameters must be discretized. For a particular class of wavelets, the scale parameter can be sampled along the dyadic sequence \((2^j)_{j \in \mathbb{Z}}\), without modifying the overall properties of the transform. The principles of such a dyadic scale decomposition was studied in mathematics by Littlewood and Paley in the 1930's. The wavelet transform at the scale \( 2^j \) is given by
\[
W_{2^j} f(x) = f \ast \psi_{2^j}(x).
\]
(3)
At each scale \( 2^j \), the function \( W_{2^j} f(x) \) is continuous since it is equal to the convolution of two functions in \( L^2 \). The Fourier transform of \( W_{2^j} f(x) \) is
\[
\hat{W}_{2^j} (\omega) = \hat{f}(\omega) \hat{\psi}_{2^j}(2^j \omega).
\]
(4)
By imposing that
\[
\sum_{j=-\infty}^{\infty} |\hat{\psi}(2^j \omega)|^2 = 1,
\]
we insure that the whole frequency axis is covered by a dilation of \( \psi(\omega) \) by the scales factors \((2^j)_{j \in \mathbb{Z}}\). Any wavelet satisfying equation (5) is called a dyadic wavelet. We also call dyadic wavelet transform the sequence of functions \((W_{2^j} f(x))_{j \in \mathbb{Z}}\).

We denote by \( W \) the dyadic wavelet operator defined by \( W = (W_{2^j} f(x))_{j \in \mathbb{Z}} \).

From (4) and (5) and by applying the Parseval theorem, we obtain an energy conservation equation
\[
\| f \|^2 = \sum_{j=-\infty}^{\infty} \| W_{2^j} f(x) \|^2.
\]
(7)
Let \( \hat{\psi}_{2^j}(x) = \hat{\psi}_{2^j}(-x) \). The function \( f(x) \) can be reconstructed from its dyadic wavelet transform:
\[
f(x) = \sum_{j=-\infty}^{\infty} W_{2^j} f \ast \hat{\psi}_{2^j}(x).
\]
(8)
This equation is proved by computing its Fourier transform and inserting (4) and (5).

Let \( V \) be the space of the dyadic wavelet transforms \((W_{2^j} f(x))_{j \in \mathbb{Z}}\), for all functions \( f(x) \in L^2 \). Let us denote by \( L^2(L^2) \) the Hilbert space of all sequences of functions \((g_j(x))_{j \in \mathbb{Z}}\), such that
\[
g_j(x) \in L^2 \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \| g_j(x) \|^2 < +\infty.
\]
Equation (7) proves that \( V \) is a subspace of \( L^2(L^2) \). We denote by \( W^{-1} \) the operator from \( L^2(L^2) \) to \( L^2 \) defined by
\[
W^{-1}(g_j(x))_{j \in \mathbb{Z}} = \sum_{j=-\infty}^{\infty} g_j \ast \hat{\psi}_{2^j}(x).
\]
(9)
The reconstruction formula (8) shows that the restriction of \( W^{-1} \) to the wavelet space \( V \) is the inverse of the dyadic wavelet transform operator \( W \).

Any sequence of functions \((g_j(x))_{j \in \mathbb{Z}} \in L^2(L^2)\) is not a priori the dyadic wavelet transform of some function \( f(x) \in L^2 \). Indeed, if there exists a function \( f(x) \in L^2 \) such that \((g_j(x))_{j \in \mathbb{Z}} = W f(x)\), then clearly we should have
\[
W(W^{-1}(g_j(x))_{j \in \mathbb{Z}}) = (g_j(x))_{j \in \mathbb{Z}}.
\]
(10)
If we replace the operators \( W \) and \( W^{-1} \) by their expres-
Let us now study in more detail the properties of the wavelet transform zero-crossings. We call smoothing function the impulse response of a low-pass filter. The convolution of a function $f(x)$ with a smoothing function attenuates part of its high frequencies without modifying the lowest frequencies and hence smooths $f(x)$. Let us show that if the wavelet is the second derivative of a smoothing function, the zero-crossings of a wavelet transform indicate the location of the signal sharper variation points.
lytic functions. The zero-crossing characterization as explained by Logan is not stable: “the problem of actually recovering (the signal) from its sign changes appears to be very difficult and impractical.”

Let us now explain how Logan’s theorem can be integrated in the wavelet model. Let \( \phi(x) \) be the function equal to the impulse response of a perfect bandpass filter of one octave. Its Fourier transform is given by

\[
\hat{\phi}(\omega) = \begin{cases} 
1, & \text{if } |\omega| \leq 2\pi, \\
0, & \text{otherwise.}
\end{cases}
\]  

(17)

The function \( \phi(x) \) clearly satisfies (5) and is therefore a dyadic wavelet. Let \( f(x) \in L^2 \); the Fourier transform of \( W_2 f(x) \) is given by \( \hat{W}_2 f(\omega) = f(\omega)\hat{\phi}(2\omega) \). The support of \( \hat{W}_2 f(\omega) \) is thus included in the one octave intervals \([-2^{-j+1}\pi, -2^{-j}\pi] \cup [2^{-j}\pi, 2^{-j+1}\pi]\). From Logan’s theorem we derive that each function \( W_2 f(x) \) is characterized by its zero-crossings. Since we can reconstruct \( f(x) \) from \( (W_2 f(x))_{j \in \mathbb{Z}} \), the original function \( f(x) \) is also characterized by the zero-crossings of all the functions \( (W_2 f(x))_{j \in \mathbb{Z}} \). This characterization is however not stable as previously explained.

Although Logan’s theorem is an important result, we want now to emphasize the reason why it cannot be used for the type of wavelets we are interested in. We need a wavelet equal to the second derivative of some smoothing function so that zero-crossings indicate the position of the signal shaper variation points. If \( \phi(x) = (d^2 \theta(x))/dx^2 \), then its Fourier transform can be written \( \hat{\phi}(\omega) = -\omega^2 \hat{\theta}(\omega) \). Since \( \theta(x) \) is the impulse response of a low-pass filter, it satisfies \( \theta(0) \neq 0 \) so \( \hat{\phi}(\omega) \) has a zero of order two at \( \omega = 0 \). Similarly, one can show that a wavelet is the \( n \)-th-order derivative of some smoothing function only if its Fourier transform \( \hat{\phi}(\omega) \) has a zero of order \( n \) at \( \omega = 0 \).

The Logan wavelet \( \phi(x) \) given in (17) cannot be written as a finite-order derivative of some smoothing function since its Fourier transform has an infinite-order zero in \( \omega = 0 \). Hence, the zero-crossings of the wavelet transform \( W_2 f(x) \) can not be interpreted as any particularly interesting features of \( f(x) \). In fact, there are too much zero-crossings since \( W_2 f(x) \) changes sign in almost all intervals of length \( 2^j \), for any function \( f(x) \). Logan as well as other researchers who extended this result, use the band-limited properties of the signal for computing its analytic extension. All these proofs do not provide any stability result since they are based on nonstable characterization of analytical functions \([1, 18, 23]\). The reader is referred to a review by Hummel and Moniot for more details \([7]\).

Many studies have also described the properties of zero-crossings of functions convolved with the Laplacian of a Gaussian. This convolution is equivalent to the wavelet transform built with a wavelet \( \phi(x) \) equal to the Laplacian of a Gaussian. Such a wavelet transform can be interpreted as the result of a heat diffusion process \([8]\). Indeed, the Gaussian is the Green function of the heat diffusion equation. Let \( t = s^2 \) be the diffusion time, one can show that the wavelet transform \( W_2 f(x) \) built with the Laplacian of a Gaussian satisfies the heat differential equation

\[
\frac{\partial W_2 f(x)}{\partial t} = \frac{\partial^2 W_2 f(x)}{\partial x^2}.
\]  

(18)

The wavelet transform \( W_2 f(x) \) is therefore equal to a heat distribution after a diffusion time \( t = s^2 \) with an initial heat distribution at \( t = 0 \) equal to \( \Delta f(x) \) (the Laplacian is taken in the sense of distributions). By using the maximum principle, several authors have proved interesting properties of the propagation of zero-crossings across scales \([6, 8, 22]\). Hummel and Moniot as well as Yuille and Poggio have also proven that the position of the zero-crossings of \( W_2 f(x) \) give a complete characterization of any function \( f(x) \) equal to a polynomial of arbitrary high order \([6]\). If \( f(x) \) is a polynomial then the function \( F(s, x) = W_2 f(x) \) is a polynomial in \((s, x) \in \mathbb{R}^+ \times \mathbb{R} \), so the problem is reduced to the characterization of a polynomial from the locus of its real roots. The proof is based on an analytic continuation result so the stability of the reconstruction is unlikely \([7]\). The polynomial assumption can not be extended by a density argument because of this instability. Numerical results \([7]\) show that one can build signals which are quite different although the zero-crossings of their wavelet transform are very close. It is difficult to make a formal proof of the instability of a zero-crossing representation because the notion of instability is not well defined. A representation is said to be unstable if a small perturbation of the representation may correspond to an arbitrary large perturbation of the original function. In order to measure the modification of the representation, we must define a metric on zero-crossings. The problem is that there is no satisfactory metric based only on the position of multiscale zero-crossings.

In order to stabilize the reconstruction of a function from its zero-crossings, Hummel records the gradient of the wavelet transform along each zero-crossing. Hummel and Moniot \([6]\) have implemented an algorithm for reconstructing the signal from the zero-crossings and gradient values. The algorithm is essentially based on the differential equation (18) that gives the evolutionary properties of \( W_2 f(x) \) when the scale \( s \) and the abscissa \( x \) vary. The zero-crossing information of \( W_2 f(x) \) is computed for \( s \) varying along a uniform discrete sequence with a scale interval \( \Delta s : (s \mathcal{J}, s \mathcal{J}) \in \mathbb{Z} \). The convergence of the reconstruction algorithm is not proven but the numerical experiments show that it converges slowly. This reconstruction procedure is computationally intensive. The differential equation approach is only valid for a wavelet equal to the Laplacian of a Gaussian and it is required to record the zero-crossing information on a dense sequence of scales.

In the following sections, we show that the reproducing kernel equation of a wavelet transform provides a general procedure to reconstruct a function from a stabilized zero-crossing representation, for any type of wavelet. This approach enables us to record the zero-crossing information only along the sparse sequence of scales \((2^j)_{j \in \mathbb{Z}}\), and
the corresponding reconstruction algorithm has a fast convergence.

IV. STABILIZED ZERO-CROSSING REPRESENTATION

Instead of considering the zero-crossings of a wavelet transform on a continuum of scales \(s\), we restrict ourselves to dyadic scales \((2^j)_{j \in \mathbb{Z}}\). In order to stabilize the zero-crossing representation, we also record the value of the wavelet transform integral between two zero-crossings. We compute an integral measure instead of a gradient value because it will then enable us to define a simple \(L^1\) norm on the zero-crossing representation. This is particularly important for pattern recognition applications, as explained in Sections VIII and IX.

Let \(f(x) \in L^2\) and \((W_j f(x))_{j \in \mathbb{Z}}\) be its dyadic wavelet transform. For any pair of consecutive zero-crossings of \(W_j f(x)\), whose abscissas are respectively \((z_{n-1}, z_n)\), we record the value of the integral

\[
e_n = \int_{z_{n-1}}^{z_n} W_j f(x) \, dx.
\]  

(19)

Equation (16) proves that

\[
W_j f(x) = 2^{j/2} \frac{d^2}{dx^2} (f \ast \theta_{\varepsilon}) (x).
\]  

(20)

Since \(z_{n-1}\) and \(z_n\) are two zero-crossings of \(W_j f(x)\), these abscissas correspond to two consecutive extrema of \((d/dx)^2 f \ast \theta_{\varepsilon}(x))\). Equations (19) and (20) yield

\[
e_n = 2^{j/2} \left\{ \frac{d}{dx} (f \ast \theta_{\varepsilon}) (z_n) - \frac{d}{dx} (f \ast \theta_{\varepsilon}) (z_{n-1}) \right\}.
\]

The integral \(e_n\) is proportional to the difference between two consecutive extrema of the derivative of \(f(x)\) smoothed at the scale \(2^j\). This value gives an estimate of the size of the structure which is between the two “edges” located at \(z_{n-1}\) and \(z_n\). If \(W_j f(x)\) has a zero-crossing \(z_0\) of minimum abscissa, then we consider that \(-\infty\) is also a zero-crossing and we record the integral of \(W_j f(x)\) between \(-\infty\) and \(z_0\). The equivalent is done if there exists a zero-crossing of maximum abscissa. In order to make sure that these integrals are finite, we suppose that \(f(x)\) is absolutely integrable.

For any function \(W_j f(x)\), the position of the zero-crossings \((z_j)_{j \in \mathbb{Z}}\) and the integral values \((e_n)_{n \in \mathbb{Z}}\), can be represented by a piece-wise constant function \(Z_j f(x)\) defined by

\[
Z_j f(x) = \frac{e_n}{z_n - z_{n-1}}, \quad \text{for } x \in [z_{n-1}, z_n].
\]  

(21)

(22)

Satisfy the constraints

\[
\int_{-\infty}^{z_0} Z_j f(x) \, dx = \int_{-\infty}^{z_0} W_j f(x) \, dx,
\]  

\[
\int_{-\infty}^{z_0} |Z_j f(x)|^2 \, dx \leq \int_{-\infty}^{z_0} |W_j f(x)|^2 \, dx.
\]  

(23)

If there exists a zero-crossing of maximum abscissa, \(Z_j f(x)\) is defined similarly between this zero-crossing and \(+\infty\). Equation (23) enables us to prove in Appendix V that \(||Z_j f|| \leq ||W_j f||\) and that \((Z_j f(x))_{n \in \mathbb{Z}} \in L^2\). The sequence of piece-wise constant functions \(Zf = (Z_j f(x))_{j \in \mathbb{Z}}\) is called a zero-crossing representation of \(f(x)\). Fig. 5(c) shows the zero-crossing representation of the signal in Fig. 5(a). As expected, the zero-crossings indicate the position of the sharper variation points of \(f(x)\) smoothed at different scales.

V. RECONSTRUCTION FROM A ZERO-CROSSING REPRESENTATION

Let us now study the reconstruction of a function from its zero-crossing representation. We reformulate the completeness problem within the wavelet framework and then derive an algorithm to perform the reconstruction. Let \(f(x) \in L^2\) and \((W_j f(x))_{j \in \mathbb{Z}}\) be its dyadic wavelet transform. Since \(f(x)\) can be recovered from its dyadic wavelet transform, we first try to reconstruct \((W_j f(x))_{j \in \mathbb{Z}}\) given the zero-crossings and integral values of each function \(W_j f(x), j \in \mathbb{Z}\). Clearly, for any scale \(2^j\), there exists an infinite number of functions \(g(x)\) that have the same zero-crossings and integral values as \(W_j f(x)\). The piece-wise constant function \(Z_j f(x)\) is an example. However, any such sequence of functions \((g_j(x))_{j \in \mathbb{Z}}\) is not necessarily the wavelet transform of some function in \(L^2\). Indeed, we saw in Section II that a dyadic wavelet transform must satisfy the reproducing kernel conditions (11). We thus have two types of information for reconstructing the functions \((W_j f(x))_{j \in \mathbb{Z}}\). We know the zero-crossings and integral values of each function \(W_j f(x)\) and we want to reconstruct a sequence of functions that satisfies the inner redundancy given by the reproducing kernel (11). Let us recall that \(F(L^2)\) is the space of all sequence of functions \((g_j(x))_{j \in \mathbb{Z}}\) such that \(\sum_{j \in \mathbb{Z}} \|g_j(x)\|^2 < \infty\). The space of all dyadic wavelet transforms \((W_j f(x))_{j \in \mathbb{Z}}\) is denoted \(F\) and is a subspace of \(F(L^2)\). In order to express the conditions given by the zero-crossings of the wavelet transform of \(f(x)\), we define the set \(\Gamma\) of all
sequences \( (g_s(x))_{s \in \mathbb{Z}} \) in \( L^2(\mathbb{R}) \) such that for all scales \( 2^j \), \( g_s(x) \) and \( W_{2^j}(f(x)) \) have the same position of zero-crossings and the same integral value between all consecutive zero-crossings \((z_{s_{k-1}}, z_s)\) 

\[
\int_{z_{s_{k-1}}}^{z_s} W_{2^j}(f(x)) \, dx = \int_{z_{s_{k-1}}}^{z_s} g_s(x) \, dx.
\]

We explain in Appendix IV how to define the zero-crossing of a function in \( L^2 \) so that \( \Gamma \) is a closed convex set. The zero-crossing representation is complete if and only if there exists no dyadic wavelet transform different from \( (W_{2^j}(f(x))_{s \in \mathbb{Z}} \) that has the same zero-crossings and integral values. In other words, the intersection of \( \Gamma \) with \( V \) must be reduced to one element

\[
\Gamma \cap V = \{(W_{2^j}(f(x))_{s \in \mathbb{Z}}\}.
\]  

In order to verify numerically this assertion, we describe an algorithm that reconstructs the intersections of \( \Gamma \) with \( V \).

A classical technique for recovering the intersection of a convex set with a linear space is to iterate on alternative projections on the convex and the linear space. Voula and Webb [21] wrote a review of the mathematical properties of these algorithms. For any \((g_s(x))_{s \in \mathbb{Z}} \) in this Hilbert space, we can define a projection \( P_\Gamma \) on \( \Gamma \) that transforms \((g_s(x))_{s \in \mathbb{Z}} \) into the sequence of functions \((h_s(x))_{s \in \mathbb{Z}} \in \Gamma \) that is the closest to \((g_s(x))_{s \in \mathbb{Z}} \). Since \( \Gamma \) is convex, the projection \( P_\Gamma \) is unique. The characterization of \( P_\Gamma \) is given in Appendix IV. Let \( P_\Gamma \) be the orthogonal projection on the space \( V \), we saw in Section II that this operator can be written \( P_\Gamma = W^*W^{-1} \). Let \( P = P_V \circ P_\Gamma \) be the composition of \( P_\Gamma \) and \( P_\Gamma \). Clearly any element at the intersection of \( \Gamma \) and \( V \) is a fixed point of \( P \). To compute such a fixed point, we iterate on the operator \( P \) as illustrated in Fig. 3. Let \( P^n \) be the composition \( n \) times of the operator \( P \). Since \( P_\Gamma \) is a nonexpansive projection on a closed convex and \( P_\Gamma \) is an orthogonal projection, one can prove [21] that for any initial sequence of functions \((g_s(x))_{s \in \mathbb{Z}} \), when \( n \) tends to \( +\infty \), \( P^n(g_s(x))_{s \in \mathbb{Z}} \) converges weakly to an element in \( \Gamma \cap V \). This ensures that the iterative algorithm converges, but in order to prove that it reconstructs the dyadic wavelet transform of \( f(x) \), for all initial sequences \((g_s(x))_{s \in \mathbb{Z}} \), we must prove that the intersection of \( \Gamma \) and \( V \) is reduced to one element. We as yet have no mathematical proof of this uniqueness; however, the numerical experiments described in Section VII show that the algorithm does reconstruct the wavelet transform of \( f(x) \) for any initial sequence.

VI. DISCRETE DYADIC WAVELET TRANSFORM

A proper implementation of the zero-crossing representation and of the reconstruction algorithm raises several important questions. The input signal is generally measured with a finite resolution that imposes a finer scale when computing the wavelet transform. In practice, the scale parameter must also vary on a finite range. This section explains how to interpret mathematically a dyadic wavelet transform on a finite range of scales. In all previous sections, our model was based on functions of a continuous parameter \( x \). We discretize the abscissa \( x \) and describe efficient algorithms for computing a discrete wavelet transform and its inverse. The results of the reconstruction algorithm from the zero-crossing representation is described in the next section.

In practice, we cannot compute the wavelet transform at all scales \( 2^j \) for \( j \) varying from \( -\infty \) to \( +\infty \). We are limited by a finite larger scale and a nonzero finer scale. Let us suppose for normalization purposes that the finer scale is equal to \( 1 \) and that \( 2^2 \) is the largest scale. Let \( f(x) \in L^2 \). We first show that between the scales \( 1 \) and \( 2^2 \), the wavelet transform \((W_{2^j}(f(x))_{s \in \mathbb{Z}} \) can be interpreted as the details available when smoothing \( f(x) \) at the scale \( 1 \) but which have disappeared when smoothing \( f(x) \) at the larger scale \( 2^2 \). Let us introduce a function \( \phi(x) \) whose Fourier transform is given by

\[
|\hat{\phi}(\omega)|^2 = \sum_{j=1}^{+\infty} |\hat{\phi}(2^j\omega)|^2.
\]

Since the wavelet \( \psi(x) \) satisfies \( \sum_{n=1}^{+\infty} |\phi(2^n\omega)|^2 = 1 \), one can derive that \( \lim_{\omega \to 0} |\phi(\omega)| = 1 \). The energy of the Fourier transform \( \phi(\omega) \) is concentrated in the low frequencies so \( \phi(x) \) is a smoothing function. Let us define the smoothing operator \( S_{2^j} \) by

\[
S_{2^j}(f(x)) = f * \phi(2^j\omega), \quad \phi(2^j\omega) = \frac{1}{2^j} \phi \left( \frac{x}{2^j} \right).
\]

The larger the scale \( 2^j \), the more details of \( f(x) \) are removed by the smoothing operator \( S_{2^j} \). Let us prove that the dyadic wavelet transform \((W_{2^j}(f(x))_{s \in \mathbb{Z}} \) between the scales \( 1 \) and \( 2^2 \) provide the details available in \( S_{2^j}(f(x)) \) but not in \( S_{2^j}(f(x)) \). The Fourier transform of \( S_{2^j}(f(x)) \) and \( S_{2^j}(f(x)) \) are respectively given by

\[
\hat{\Phi}(\omega) = \hat{\phi}(\omega)^2, \quad \hat{\phi}(2^j\omega) = \hat{\phi}(2^j\omega)^2.
\]
and
\[ \hat{W}_2 f(\omega) = \hat{\phi}(2\omega) \hat{f}(\omega). \] (28)

Equation (25) yields
\[ |\hat{\phi}(\omega)|^2 = \sum_{j=1}^{J} |\hat{\phi}(2^j \omega)|^2 + |\hat{\phi}(2^J \omega)|^2. \] (29)

Using Parseval's theorem, we derive from (27)–(29) the following energy conservation equation
\[ \|S_j f(x)\|^2 = \sum_{j=1}^{J} \|W_2 f(x)\|^2 + \|S_2 f(x)\|^2. \] (30)

This equation proves that the higher frequencies of \( S_j f(x) \) that have disappeared in \( S_2 f(x) \) can be recovered from the dyadic wavelet transform \( (W_2 f(x))_{k \in \mathbb{Z}} \) between the scales 1 and \( 2^J \). The functions \( S_j f(x) \), \( (W_2 f(x))_{k \in \mathbb{Z}} \) are called the finite-scale wavelet transform of \( S_2 f(x) \). In practice, the signal we process is given by a discrete sequence of values. The following lemma proves that any discrete signal of finite energy can be interpreted as the uniform sampling of some function smoothed at the scale 1.

**Lemma 1:** Let \( D = \{d_n\}_{n \in \mathbb{Z}} \) be a discrete signal of finite energy, \( \sum_{n=-\infty}^{\infty} |d_n|^2 < \infty \). Let us suppose that for strictly positive constants \( C_1 \) and \( C_2 \) and all real \( \omega \), the Fourier transform \( \hat{\phi}(\omega) \) satisfies
\[ C_1 \leq \sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2 \leq C_2. \]

There exists a (nonunique) function \( f(x) \in L^2 \) such that for any integer \( n \)
\[ S_1 f(n) = d_n. \] (31)

The proof of this lemma is in Appendix I. The discrete signal \( D \) can thus be rewritten \( D = (S_1 f(n))_{n \in \mathbb{Z}} \). For a particular class of wavelets \( \phi(x) \) described in Appendix II, the samples \( (S_1 f(n))_{n \in \mathbb{Z}} \) enable us to compute a uniform sampling of the finite scale wavelet transform of \( S_1 f(x) \)
\[ \{(S_2 f(n))_{n \in \mathbb{Z}}, (W_2 f(n))_{n \in \mathbb{Z}}\}_{k \leq j \leq J}. \] (32)

Let us denote
\[ W_2^J f = (W_2 f(n))_{n \in \mathbb{Z}} \quad \text{and} \quad S_2^J f = (S_2 f(n))_{n \in \mathbb{Z}}. \] (33)

The sequence of discrete signals \( (S_2^J f, W_2^J f)_{k \leq j \leq J} \) is called a discrete dyadic wavelet transform of the signal \( D = (S_1 f(n))_{n \in \mathbb{Z}} \). If the signal \( D \) has \( N \) nonzero-samples, each discrete signal \( W_2^J f \) has \( N \) nonzero samples so discrete dyadic wavelet transform has at most \( N \log(N) \) nonzero samples. We denote by \( W^d \) the discrete wavelet transform operator that associates to a signal \( D \) the discrete wavelet transform previously defined. Appendix III describes a fast algorithm for implementing this operator. The complexity of this algorithm is \( O(N \log(N)) \). It is based on a cascade of convolutions with two discrete wavelet transforms.

Fig. 4. Graph of the dyadic wavelet \( \phi(x) \) used in the numerical experiments shown in this article. Wavelet is characterized numerically in Appendix II.

The zero-crossings of the functions \( W_2 f(x) \) are estimated from the sign changes of the samples of \( W_2 f \). The position of each zero-crossing is estimated with a linear interpolation between the two samples of different sign. The value of the integral \( e_n \) between two consecutive zero-crossings is estimated with the integral on the piecewise linear function that interpolates the samples of \( W_2 f \). If \( D \) has \( N \) nonzero samples, since there are at most \( N \log(N) \) samples in the discrete wavelet representation, the number of operations to obtain the position of the zero-crossings as well as the integral values, is \( O(N \log(N)) \). From a discrete dyadic wavelet transform, we can only compute the zero-crossing positions and the integral values along the scales \( 2^J \) such that \( 1 < 2^J \leq 2^{J-1} \). In order to keep the signal information at the scales larger than \( 2^J \), we need to keep the coarse signal \( S_2^J f \) in the zero-crossing representation. When \( J \) is large enough, this coarse signal is almost constant and equal to the average value of \( f(x) \). We call discrete zero-crossing representation the set of signals
\[ \{(Z_j f(x))_{k \leq j \leq J}, S_2^J f\}. \] (34)

The signal in Fig. 5(a) is an image scan-line of 256 samples and Fig. 5(b) is its discrete wavelet transform computed with the wavelet shown in Fig. 4. The curves in both figures are linear interpolations between the samples of each discrete signal. The curve at the top of Fig. 5(b) is the coarse signal \( S_2^J f \). Since the wavelet used is the second derivative of a smoothing function, the zero-crossings of the wavelet transform indicate the points of sharper
Fig. 6. (a) Reconstruction of the dyadic wavelet transform of the zero-crossing representation given in Fig. 5(c). This reconstruction was obtained with 15 iterations on the operator $P^T$. (b) Reconstruction of the signal by applying the inverse wavelet operator $W^{-1}$ on the reconstructed wavelet transform of Fig. 6(a). Quality of the reconstruction can be appreciated by comparing this graph with Fig. 5(a).

verse wavelet transform. Within a discrete framework,
satisfies
\[ d(W_{Zf}, Zg) \leq \|f\|^2 + \|g\|^2. \] (38)

The distance \( d \) makes a global comparison of two zero-crossing representations over the entire spatial domain. A pattern is often a local feature embedded in the signal. For pattern matching purposes, we need to define a local distance which compares locally two zero-crossing representations. In order to derive such a distance from \( d \), we study the decomposition at all scales of a local feature such as a Dirac delta function \( \delta_u(x) \) centered at \( u \).

\[ W_{Zf}\delta_u(x) = \delta_u(x) * \psi_{Zf}(x) = \psi_{Zf}(x-u). \] (39)

Let \( 2\sigma \) be the size of an interval where the energy of \( \psi(x) \) is mostly concentrated,

\[ \int_{-\infty}^{\alpha} |\psi(x)|^2 \, dx = \int_{-\infty}^{\alpha} |\psi(x)|^2 \, dx. \] (40)

Equations (39) and (40) show that the energy of \( W_{Zf}\delta_u(x) \) is mainly concentrated on the interval \([u-2\sigma, u+2\sigma]\). This interval defines the domain of influences of the point \( u \) at the scale \( 2^1 \). In order to compare two zero-crossing representations \( Zf \) and \( Zg \) in the neighborhood of a point \( u \), we define the local distance \( d_u \)

\[ d_u(Zf, Zg) = \sum_{j=-\infty}^{+\infty} d_j(Z_{2^j}f, Z_{2^j}g)^2. \] (41)

with

\[ d_j(Z_{2^j}f, Z_{2^j}g)^2 = \int_{u-2^j\alpha}^{u+2^j\alpha} |Z_{2^j}f(x) - Z_{2^j}g(x)|^2 \, dx. \] (42)

\( d_j(Z_{2^j}f, Z_{2^j}g) \) is a measure of the local distortion between \( f(x) \) and \( g(x) \) around the point \( u \), at the scale \( 2^j \). The integral of (42) is computed with few operations since the functions \( Z_{2^j}f(x) \) and \( Z_{2^j}g(x) \) are piece-wise constant. For a discrete zero-crossing representation, the local distance \( d_u \) is redefined with a finite sum as

\[ d_u(Zf, Zg) = \sum_{j=-1}^{j} d_j(Z_{2^j}f, Z_{2^j}g)^2. \] (43)

IX. APPLICATION TO STEREO-MATCHING

In order to illustrate the application of the zero-crossing representation to pattern matching, we study the implementation of a stereo-matching algorithm. Through...
of stereo cameras from the difference of positioning $\tau$ between $P_L$ and $P_R$ (see Fig. 7). This difference of positioning is called disparity. The goal of a stereo-matching algorithm is to find for each point $P_L$ of the left image, the matching point $P_R$ of the right image such that $P_L$ and $P_R$ are the projections of the same point $P$ on the scene. The principle of such an algorithm is to look for a point $P_R$ in the right image such that locally around $P$, the image is the most similar to the neighborhood around $P_L$ in the left image. Although this matching problem is a priori a two-dimensional search, it can be reduced to a one-dimensional search by using the epipolar geometry of the cameras. An epipolar plane is a plane that contains the point $P$ and the optical centers of the left and right cameras. The intersections of such a plane with the left and the right images define a pair of epipolar lines. The stereo match of any point that is on a left epipolar line can be found on the corresponding right epipolar line. The problem is thus reduced to a one-dimensional matching problem along each pair of epipolar lines. Much research has been devoted to finding efficient algorithms for matching these epipolar lines [4], [15]. In particular, Grimson has developed a coarse to fine matching algorithm based on multiscale zero-crossings. The principle of a coarse to fine strategy is to use first the information at large scales to perform the matching. Then the result of the matching are refined by progressively using the information at finer scales. A main difficulty of the Grimson algorithm is that we can not define a stable distance based on the zero-crossings only. In this section, we show that one can easily adapt the Grimson algorithm within a stabilized zero-crossing representation and that the distance described in Section VIII enables us to implement a simple and efficient matching procedure.

Let us now explain in more detail how to match two epipolar lines from their zero-crossing representation. The epipolar line is a discrete one-dimensional signal. Let

\[
\left\{ (Z_{2L}(x))_{x \in [1, 2^L]}, S_{2L}^L \right\} \quad \text{and} \quad \left\{ (Z_{2R}(x))_{x \in [1, 2^R]}, S_{2R}^R \right\}
\]

be respectively the discrete zero-crossing representation of the left and right epipolar lines. Fig. 8(a) gives an example of pair of epipolar lines and Fig. 8(b) shows the corresponding zero-crossing representations. These epipolar lines were obtained from real images and as it can be observed, they are not only translated from one another but also distorted due to the perspective effect and the noise. We need to make a correspondence between the zero-crossings of both representations, at all the scales $2^i$.

A coarse to fine strategy consists of matching first the coarse details of the two epipolar lines and then using the finer details to get more precise matches. Within the zero-crossing representation, we are first going to make the correspondence between the zero-crossings of $Z_{2L}(x)$ and $Z_{2R}(x)$ at the largest scale $2^1$ and then progressively decrease $2^i$ while using the information provided by the matches at the coarser scales in order to compute the matches at the finer scales. Given a zero-crossing $z_n$ of $Z_{2L}(x)$ we want to find a zero-crossing $\hat{z}_n$ of $Z_{2R}(x)$ such that if $\tau = z_n - \hat{z}_n$ then $Z_{2L}(x)$ and $Z_{2R}(x - \tau)$ are as similar as possible in the neighborhood of $x = z_n$. Hence, the disparity $\tau$ is the value that minimizes the
and a zero-crossing of $Z_j \tau(r(x))$ gives a local estimate of the disparity $\tau$. At the next finer scale $2^{j-1}$, we use this local estimate of the disparity in order to constrain the search when trying to find the correspondence between the zero-crossings of $Z_j \tau(r(x))$ and the zero-crossings of $Z_{j'} \tau(r(x))$. When beginning at the coarser scale $2^J$ we do not have any prior estimation of the disparity to constrain the search. This is, however, not a problem since the number of zero-crossings of $Z_j \tau(r(x))$ and $Z_{j'} \tau(r(x))$ is small when $J$ is big enough (see Fig. 8(b)).

The coarse to fine strategy reduces considerably the complexity of the search for a match since we use the matching information at the previous scale to constrain the search at the next scale. This strategy supposes that we have a high confidence in the matches at the coarser scales since any error at a coarse scale might propagate at finer scales. The matching errors are due to the fact that the left and right signals are not only translated from one-another, but also distorted because of the noise and the perspective effect. Most of the distortion appears at the finer scales as shown in Fig. 8(b). We therefore have a better matching confidence at the coarse scales than at the finer scales.

In order to avoid side effects, at each scale, we did not try to match the zero-crossings at the borders. As we can see from the successive matchings shown in Fig. 9, we are getting a dense matching on both signals. There are, some domains where we do not match the zero-crossings because there is too much distortion between $Z_j \tau(r(x))$ and $Z_{j'} \tau(r(x))$. We have included in our algorithm a confidence threshold $C$ in order to eliminate the matches where the minimal distance $d_{min}$ is larger than $1/C$. Fig. 9 shows that in some domains, we find matches at a coarse scale but not at finer scales because there is too much high frequency noise.

The simple stereo matching algorithm can of course be enhanced by using some further property of the disparity function such as a smoothness constraint [14] or a monotonicity constraint [4]. However, our goal here is more to illustrate the simplicity of the implementation of a matching algorithm with this zero-crossing representation, rather than develop a full stereo matching system.

**X. Conclusion**

We study the completeness, stability, and application to pattern recognition of a multiscalar representation based on zero-crossings. The main result of the paper is an iterative algorithm that reconstructs the original signal from its zero-crossing representation. We proved the convergence of the algorithm but did not prove that the reconstruction is independent from the initial start of the iteration. The numerical experiments seem to indicate that the reconstruction is independent from the choice of the initial point which means that the zero-crossing representation is complete and stable. The proof of this result remains an open mathematical problem. In order to illustrate the application of this representation to pattern
matching, we described the implementation of a coarse to
fine stereo-matching algorithm. The simplicity and the efﬁciency of this matching algorithm shows that this represen-
tation is indeed well adapted for pattern recognition
problems.

In a zero-crossing representation, the number of values
to be coded depends upon the irregularity of the signal.
For signals that are mostly smooth with sparse singulari-
ties such as discontinuities, this type of coding can be very
compact. In collaboration with Sifeng Zhong, we have
recently extended this representation in two dimensions
[13], and shown that one reconstruct images from multi-
scale edges with a similar alternative projection algorithm.
This image representation provides a compact reorganiza-
tion of the information for a large class of images.

APPENDIX 1

PROOF OF LEMMA 1

For any ﬁnite energy discrete signal \( D = \{d_n\}_{n \in \mathbb{Z}} \), we
want to ﬁnd \( f(x) \in L^2 \) such that

\[
\forall n \in \mathbb{Z}, \quad S_n f(n) = d_n.
\]  

(44)

Let \( f(x) \in L^2 \), by deﬁnition we have \( S_n f(n) = f * \phi(n) \).
This convolution product can be rewritten as an inner product in \( L^2 \): \( S_n f(n) = \langle f(x), \phi(n - x) \rangle \). Let \( U \) be the vector space generated by the family of functions \( \{\phi(n - x)\}_{n \in \mathbb{Z}} \). If this family is a basis of \( U \) then for any discrete sequence \( \{d_n\}_{n \in \mathbb{Z}} \) of ﬁnite energy, there exists \( f(x) \in L^2 \) satisfying (44). One can show [10] that the family \( \{\phi(x - n)\}_{n \in \mathbb{Z}} \) is a Hilbert basis if and only if for strictly positive constants \( C_1 \) and \( C_2 \), and all real \( \omega \), the Fourier trans-
form \( \hat{\phi}(\omega) \) satisﬁes

\[
C_1 \leq \sum_{n=-\infty}^{+\infty} |\hat{\phi}(\omega + 2n\pi)|^2 \leq C_2.
\]

The values \( \{d_n\}_{n \in \mathbb{Z}} \) characterize the orthogonal projection
of \( f(x) \in L^2 \) on \( U \). This orthogonal projection can be interpreted as an approximation at the resolution 1 of the function \( f(x) \) [11].

APPENDIX 2

A PARTICULAR CLASS OF ONE-DIMENSIONAL
DYADIC WAVELETS

This appendix deﬁnes the class of wavelets used for
implementation of discrete algorithms. From \( \langle S_n f(n) \rangle_{n \in \mathbb{Z}} \)
we want to be able to compute

\[
\{Z_{i,j} f(n)\}_{n \in \mathbb{Z}}, (W_{i,j} f(n))_{n \in \mathbb{Z}} \}_{1 \leq i \geq j}.
\]

with discrete convolutions. If \( J = 1 \), this implies that we can compute \( \langle S_j f(n) \rangle_{n \in \mathbb{Z}} \) by convolving \( \{S_k f(n)\}_{n \in \mathbb{Z}} \) with
a discrete ﬁlter \( H \). In other words, the Fourier series of
\( \langle S_j f(n) \rangle_{n \in \mathbb{Z}} \) is equal to the Fourier series of \( \{S_k f(n)\}_{n \in \mathbb{Z}} \)
multiplied by a 2\pi periodic function \( H(\omega) \). The Fourier

<table>
<thead>
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<th>( n )</th>
<th>( h_n )</th>
<th>( g_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0.7118</td>
</tr>
<tr>
<td>1</td>
<td>0.2864</td>
<td>0.2309</td>
</tr>
<tr>
<td>2</td>
<td>0.0450</td>
<td>-0.1120</td>
</tr>
<tr>
<td>3</td>
<td>0.0393</td>
<td>-0.0226</td>
</tr>
<tr>
<td>4</td>
<td>0.0132</td>
<td>0.0862</td>
</tr>
<tr>
<td>5</td>
<td>0.0032</td>
<td>0.0939</td>
</tr>
</tbody>
</table>

series of these two signals are respectively

\[
\sum_{n=-\infty}^{+\infty} f * \phi(n) e^{-in\omega} \quad \text{and} \quad \sum_{n=-\infty}^{+\infty} f * \phi_2(n) e^{-in\omega}.
\]  

(45)

By applying the Poisson formula, we can rewrite these
two series as

\[
\sum_{n=-\infty}^{+\infty} \hat{f}(\omega + 2n\pi) \hat{\phi}(\omega + 2n\pi)
\]

and

\[
\sum_{n=-\infty}^{+\infty} \hat{f}(\omega + 2n\pi) \hat{\phi}(2\omega + 2n\pi).
\]  

(46)

The left series is equal to the right series multiplied by
\( H(\omega) \) for all \( f(\omega) \) if and only if

\[
\hat{\phi}(2\omega) = H(\omega) \hat{\phi}(\omega).
\]  

(47)

Since \( |\hat{\phi}(0)| = 1 \), we must have \( |H(0)| = 1 \). If we cascade
(47), we obtain a necessary condition on \( \hat{\phi}(\omega) \),

\[
\hat{\phi}(\omega) = \prod_{n=1}^{+\infty} H(2^{-n}\omega).
\]  

(48)

Conversely, if the 2\pi periodic function \( H(\omega) \) satis-
ﬁes

\[
|H(\omega)|^2 + |H(\omega + \pi)|^2 \leq 1,
\]  

(49)

then one can show [10] that the function \( \phi(x) \) whose
Fourier transform is deﬁned by (49) is a function in \( L^2 \).
The function \( H(\omega) \) can be interpreted as the transfer
function of a discrete low-pass ﬁlter.

Let us now characterize the corresponding wavelet
\( \phi(x) \). As a consequence of equation (29), we have

\[
|\hat{\phi}(2\omega)|^2 = |\hat{\phi}(\omega)|^2 - |\hat{\phi}(2\omega)|^2.
\]  

(50)

Substituting (47) in (50) yields

\[
\hat{\phi}(2\omega) = G(\omega) \hat{\phi}(\omega),
\]  

(51)

with

\[
|G(\omega)|^2 + |H(\omega)|^2 = 1.
\]  

(52)

The function \( G(\omega) \) is chosen 2\pi periodic and can be
interpreted as the transfer function of a high-pass ﬁlter.

For the zero-crossing model, we want to build a wavelet
\( \psi(x) \) equal to a second-order derivative of a smoothing
function \( \theta(x) \). This implies that \( \phi(\omega) \) must have a zero of
order 2 in \( \omega = \theta \). Since \( |\hat{\phi}(0)| = 1 \), (51) yields that \( G(\omega) \)
must have a zero of order 2 in \( \omega = \theta \). Table 1 gives the
first coefﬁcients of the impulse response of ﬁlters \( H = \langle h_n \rangle_{n \in \mathbb{Z}} \) and \( G = \langle g_n \rangle_{n \in \mathbb{Z}} \) that satisfy these properties.

The impulse response of these ﬁlters is exponentially
decreasing and here we only give the first five coefficients. Both filters are symmetrical with respect to 0. The numerical experiments given in this paper are computed with these filters. For high precision computations, one needs to include more coefficients. The corresponding wavelet $\psi(x)$ is shown in Fig. 4. This wavelet has one small ripple on each side that can produce a few spurious zero-crossings. This ripple cannot be totally removed for the class of dyadic wavelet that we described in this appendix.

**APPENDIX 3**

**FAST WAVELET ALGORITHMS FOR ONE-DIMENSIONAL SIGNALS**

This appendix describes an algorithm for computing a discrete wavelet transform and the inverse algorithm that reconstructs the original signal from its wavelet transform. We suppose that the wavelet $\psi(x)$ is characterized by the two discrete filters $H$ and $G$ described in Appendix II. We denote $H_j$ and $G_j$ the discrete filters obtained by putting $2^j - 1$ zeros between each coefficients of the filters $H$ and $G$. The transfer function of these filters is respectively $H(2^j \omega)$ and $G(2^j \omega)$. We also denote by $\hat{H}_j$ and $\hat{G}_j$ the filters whose transfer functions are respectively $\hat{H}(2^j \omega)$ and $\hat{G}(2^j \omega)$ (complex conjugates of $H(2^j \omega)$ and $G(2^j \omega)$). We denote by $A * B$ the convolution of two discrete signals $A$ and $B$.

The following algorithm computes the discrete wavelet transform of the discrete signal $S_j^f$. At each scale $2^i$, it decomposes $S_j^f$ into $S_j^{f_1} \ldots f$ and $W_j^{f_2} \ldots f$.

\[
\begin{align*}
  j & = 0, \\
  \text{while} \ (j < J), \quad & W_j^{2^j}f = S_j^{2^j}f * G_j, \\
  S_j^{2^j}f = S_j^{2^j}f * H_j, \\
  j & = j + 1, \\
  \text{end of while}.
\end{align*}
\]

The proof of this algorithm is based on the properties of the wavelet $\psi(x)$ described in Appendix II. If the original signal $(S_j f(n))_{j \in \mathbb{Z}}$ has $N$ nonzero samples, then each signal $S_j^{f_1} \ldots f_2$ and $W_j^{f_2} \ldots f$ has $N$ nonzero samples. Since there are at most $\log(N)$ scales, the complexity of the algorithm is $O(N \log(N))$. The constant depends upon the number of nonzero coefficients in the filters $H$ and $G$.

The inverse wavelet transform algorithm reconstructs $S_j f$ from the discrete dyadic wavelet transform. At each scale $2^i$, it reconstructs $S_j^{f_1} \ldots f$ from $S_j^{f_1} \ldots f$ and $W_j^{f_2} \ldots f$. The complexity of this reconstruction algorithm is also $O(N \log(N))$.

\[
\begin{align*}
  j & = J, \\
  \text{while} \ (j > 0), \quad & S_j^{f_1} \ldots f = W_j^{f_1} \ldots f * \hat{G}_{j-1}, + S_j^{f_1} \ldots f * \hat{H}_{j-1}, \\
  j & = j - 1, \\
  \text{end of for}.
\end{align*}
\]

**APPENDIX 4**

**PROJECTION OPERATOR ON $\Gamma$**

In this appendix, we describe more precisely the projection operators $P_x$ defined in Section V. In order to define properly the set $\Gamma$, we first define the notion of zero-crossings for functions in $L^2$. We shall say that a function $g(x)$ is strictly positive on an interval $[a, b]$ if

\[
\forall (x, y) \in [a, b]^2, \quad \int_y^x g(u) \, du \geq 0
\]

and

\[
\exists (x, y) \in [a, b]^2, \quad \int_y^x g(u) \, du > 0. \tag{53}
\]

The negative sign is defined by reversing the inequalities. A function $g(x) \in L^2$ is said to have a zero-crossing in $x_0$ if there exists $\epsilon > 0$ such that $g(x)$ is strictly positive (respectively negative) on the interval $[x_0 - \epsilon, x_0]$ and strictly negative (respectively positive) on the interval $[x_0, x_0 + \epsilon]$. Let us observe that if $f(x)$ is strictly positive on $[a, b]$, equal to zero on $[b, c]$ and strictly negative on $[c, d]$ then any point on the interval $[b, c]$ is a zero-crossing. In this case, we shall say that there exists only 1 zero-crossing, but this zero-crossing is unlocalized in the interval $[b, c]$. If a function $g(x)$ has one zero-crossing unlocalized in an interval $[b, c]$ and $g(x)$ has one zero-crossing in $x_0 \in [b, c]$, we say that the position of the zero-crossing of $g(x)$ and $g_0(x)$ is the same. This definition is necessary in order to insure that the set $\Gamma$ is closed.

Let us suppose that we record all the zero-crossings and integral values of the wavelet transform $(W_{j/2} f(x))_{j \in \mathbb{Z}}$. The corresponding set $\Gamma$ regroups all sequences of functions $(g_{j}(x))_{j \in \mathbb{Z}} \in \ell^2(L^2)$ such that $g_{j}(x)$ has the same zero-crossings and integral values than $W_{j/2} f(x)$ for all $j \in \mathbb{Z}$. Given our definition of zero-crossing, one can prove without major difficulty that the set $\Gamma$ is a closed convex in $L^2(F^2)$. Let us now define the operator $P_x$ that transforms any sequence $(g_{j}(x))_{j \in \mathbb{Z}} \in \ell^2(L^2)$ into the closest sequence $(h_{j}(x))_{j \in \mathbb{Z}} \in \Gamma$, with respect to the norm of $L^2(F^2)$. Let $\varepsilon_{j}(x) = h_{j}(x) - g_{j}(x)$. Each function $h_{j}(x)$ is chosen such that

\[
\| \varepsilon_{j}(x) \| = \int_{-\infty}^{+\infty} |\varepsilon_{j}(x)|^2 \, dx \text{ is minimum.} \tag{54}
\]

Let $x_{n-1}$ and $x_n$ be respectively the abscissas of two consecutive zero-crossings of $W_{j/2} f(x)$ and $e_n$ be the corresponding integral value. Let us suppose that $e_n > 0$, the following conditions must be satisfied

\[
\begin{align*}
  \int_{x_{n-1}}^{x_n} h_{j}(x) \, dx & = \int_{x_{n-1}}^{x_n} (g_{j}(x) + \varepsilon_{j}(x)) \, dx = e_n, \\
  h_{j}(x) & = g_{j}(x) + \varepsilon_{j}(x) \geq 0, \quad \text{for } x \in [x_{n-1}, x_n].
\end{align*}
\]

The global minimization of $\| \varepsilon_{j}(x) \|$ is equivalent to the
minimization of \( \int_{z_{n-1}}^{z_n} e_i(x) dx \) for each pair of consecutive zero-crossings \((z_{n-1}, z_n)\), with the two constraints
\[
\begin{align*}
\int_{z_{n-1}}^{z_n} e_i(x) dx &= e_n - \int_{z_{n-1}}^{z_n} g_i(x) dx, \\
e_i(x) &\geq -g_i(x), \quad \text{for } x \in [z_{n-1}, z_n].
\end{align*}
\] (56)

This minimization problem is solved by using the Lagrange multipliers. One can prove that there exists a lagrange multiplier \( \lambda \) such that
\[
e_i(x) = \begin{cases} 
\lambda, & \text{if } -g_i(x) < \lambda, \\
-g_i(x), & \text{if } -g_i(x) \geq \lambda.
\end{cases}
\] (57)

The value of \( \lambda \) is specified by the fact that
\[
\int_{z_{n-1}}^{z_n} e_i(x) dx = e_n - \int_{z_{n-1}}^{z_n} g_i(x) dx.
\] (58)

Within a discrete model, \( \Gamma \) is defined as the set of all discrete signals \( \{g_i^m\}_{m \in Z} \) such that each signal \( g_i^m = (g_i(m))_{m \in Z} \) has the same zero-crossing position and integral values as the discrete signal \( (W_{2f}(m))_{m \in Z} \). The set \( \Gamma \) is a closed convex. One can easily derivable from our continuous model that the discretization of the non-expansive projector on \( \Gamma \) consists in computing a discrete signal \( e_i^d = (e_i(m))_{m \in Z} \) such that for any pair of consecutive zero-crossings \((z_{n-1}, z_n)\) and integer \( m \in [z_{n-1}, z_n] \),
\[
e_i(m) = \begin{cases} 
\lambda, & \text{if } -g_i(m) < \lambda, \\
-g_i(m), & \text{if } -g_i(m) \geq \lambda.
\end{cases}
\] (59)

and \( \lambda \) must be such that
\[
\sum_{m < z_n} \epsilon_i(m) = e_n - \sum_{m \geq z_n} g_i(m) = e_n.
\] (60)

The most difficult to compute is the value of \( \lambda \). Let \( K \) be the number of integers in the interval \([z_{n-1}, z_n]\). We first sort the values of \( -g_i(m) \) for \( m \in [z_{n-1}, z_n] \) so that \( -g_i(m_k) \geq -g_i(m_{k-1}) \geq \cdots \geq -g_i(m_1) \). One can prove that \( \lambda \) is computed by the following algorithm.

\[
\begin{align*}
\lambda &= c_n, \\
k &= K, \\
\text{while } (\lambda < -g_i(m_k)) \\
\lambda &= \frac{k\lambda + g_i(m_k)}{k + 1}, \\
k &= k - 1, \\
\text{end of while}.
\end{align*}
\]

The total complexity for computing \( \lambda \) is \( O(K \log(K)) \) because of the first sorting step. To compute \( e_i(m) \) once we know \( \lambda \) is done with (59) in \( O(K) \) computations. If the original discrete signal \( D = \{S_n(f(m))\}_{m \in Z} \) has \( N \) nonzero samples, each signal \( g_i^d \) has also \( N \) samples so the computation of \( e_i^d \) requires \( O(N \log(N)) \) operations. Since there are at most \( \log(N) \) scales \( 2^j \), the total number of computations to implement to discrete projector \( P_{2f} \) is \( O(N \log^2(N)) \).

**APPENDIX 5**

**DISTANCE BETWEEN ZERO-CROSSING REPRESENTATIONS**

In this appendix, we prove that \( \|Z_{2f} - f\| \leq \|W_{2f} - f\| \) and derive that
\[
d(Z, Z_{2f}) \leq \|f\|^2 + \|g\|^2.
\]

One can easily prove that among all functions that have an integral equal to a given value \( \epsilon \) on an interval \([a, b]\), the function which is constant on this interval has the minimum \( L^2([a, b]) \) norm. Between two consecutive zero-crossings \( z_{n-1} \) and \( z_n \)
\[
\int_{z_{n-1}}^{z_n} Z_{2f}(x) dx = \int_{z_{n-1}}^{z_n} W_{2f}(x) dx.
\] (61)

Since \( Z_{2f}(x) \) is constant on the interval \([z_{n-1}, z_n]\),
\[
\int_{z_{n-1}}^{z_n} |Z_{2f}(x)| dx \leq \int_{z_{n-1}}^{z_n} |W_{2f}(x)| dx.
\] (62)

If there is a first zero-crossing \( z_0 \), we define \( Z_{2f}(x) \) so that
\[
\int_{z_0}^{z_n} |Z_{2f}(x)| dx \leq \int_{z_0}^{z_n} |W_{2f}(x)| dx.
\]

The equivalent is true if there is a last zero-crossing between this last zero-crossing and \( +\infty \). We therefore derive that
\[
\|Z_{2f} - f\|^2 = \int_{-\infty}^{+\infty} |Z_{2f}(x) - f(x)|^2 dx \leq \int_{-\infty}^{+\infty} |W_{2f}(x) - f(x)|^2 dx = \|W_{2f} - f\|^2.
\] (63)

Hence, we obtain
\[
\sum_{j=-\infty}^{+\infty} \|Z_{2f}(x)\|^2 \leq \sum_{j=-\infty}^{+\infty} \|W_{2f}(x)\|^2 = \|f\|^2.
\] (64)

This proves that \( (Z_{2f}(x))_{x \in \mathbb{R}} \in L^2(L^2) \). Since \( d(Z, Z_{2f})^2 = \sum_{j=-\infty}^{+\infty} \|Z_{2f}(x) - Z_{2f}(x)\|^2 \), (64) yields
\[
d(Z, Z_{2f}) \leq \|f\|^2 + \|g\|^2.
\]

**REFERENCES**


