Zero-Crossings of a Wavelet Transform
Stephane Mallat

Abstract—Sharp variation points are among the most meaningful features for characterizing transient signals. For a particular class of wavelets, the zero-crossings of a wavelet transform provide the locations of the signal sharp variation points at different scales. The completeness and stability of a signal representation based on zero-crossings of a wavelet transform at the scales $2^j$, for integer $j$, are studied. An alternative projection algorithm is described. It reconstructs a signal from a zero-crossing representation which is stabilized. The reconstruction algorithm has a fast convergence and each iteration requires $O(N \log^2(N))$ computations for a signal of $N$ samples. The zero-crossings of a wavelet transform define a representation which is well adapted for solving pattern recognition problems. As an example, the implementation and workflow of a representation should correspond to a small modification of the original signal. While reviewing some previous work, we shall see that the positions of multiscale zero-crossings may provide a complete representation under certain restrictive assumptions but such a representation is not stable. We show that one can stabilize a zero-crossing representation by adding a complement of information that measures the "size" of the structure between two consecutive zero-crossings. This new signal representation is based on the wavelet transform reformalization of multiscale decompositions. We introduce the most
The norm of \( f(x) \in L^2 \) is given by
\[
\|f\|^2 = \int_{-\infty}^{+\infty} |f(x)|^2 \, dx.
\]

We also denote by \( L^2(\mathbb{L}^2) \) the Hilbert space of all sequences of functions \((g_j(x))_{j \in \mathbb{Z}}\), such that for all \( j \in \mathbb{Z} \), \( g_j(x) \in L^2 \) and
\[
\sum_{j=-\infty}^{+\infty} \|g_j(x)\|^2 < +\infty.
\]

This infinite sum is the norm of the sequence \((g_j(x))_{j \in \mathbb{Z}}\) in \( L^2(\mathbb{L}^2) \).

We denote the convolution of two functions \( f(x) \in L^2 \) and \( g(x) \in L^2 \) by
\[
f \ast g(x) = \int_{-\infty}^{+\infty} f(u)g(x-u) \, du.
\]

The Fourier transform of \( f(x) \in L^2 \) is written \( \hat{f}(\omega) \) and is defined by
\[
\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{\omega x} \, dx.
\]

II. Properties of the Wavelet Transform

The wavelet transform is a linear operation that decomposes a signal into components that appear at different scales. This transform is based on the convolution of the signal with a dilated filter. Such a decomposition has been

transform. The principles of such a dyadic scale decomposition was studied in mathematics by Littlewood and Paley in the 1930’s. The wavelet transform at the scale \( 2^j \) is given by
\[
W_{2^j} f(x) = f \ast \psi_{2^j}(x).
\]

At each scale \( 2^j \), the function \( W_{2^j} f(x) \) is continuous since it is equal to the convolution of two functions in \( L^2 \). The Fourier transform of \( W_{2^j} f(x) \) is
\[
\hat{W}_{2^j}(\omega) = \hat{f}(\omega) \hat{\psi}(2^j \omega).
\]

By imposing that
\[
\sum_{j=-\infty}^{+\infty} |\hat{\psi}(2^j \omega)|^2 = 1,
\]
we ensure that the whole frequency axis is covered by a dilation of \( \hat{\psi}(\omega) \) by the scales factors \( (2^j)_{j \in \mathbb{Z}} \). Any wavelet satisfying equation (5) is called a dyadic wavelet. We also call dyadic wavelet transform the sequence of functions
\[
(W_{2^j} f(x))_{j \in \mathbb{Z}}.
\]

We denote by \( W \) the dyadic wavelet operator defined by \( W = (W_{2^j} f(x))_{j \in \mathbb{Z}} \).

From (4) and (5) and by applying the Parseval theorem, we obtain an energy conservations equation
\[
\|f\|^2 = \sum_{j=-\infty}^{+\infty} \|W_{2^j} f(x)\|^2.
\]
Let us now study in more detail the properties of the wavelet transform zero-crossings. We call smoothing function the impulse response of a low-pass filter. The convolution of a function \( f(x) \) with a smoothing function attenuates part of its high frequencies without modifying the lowest frequencies and hence smoothes \( f(x) \). Let us show that if the wavelet is the second derivative of a smoothing function, the zero-crossings of a wavelet transform indicate the location of the signal sharper variation points. Let \( \theta(x) \) be a smoothing function, and

\[
\psi(x) = \frac{d^2 \theta(x)}{dx^2}.
\]  

(14)

We denote \( \theta(x) = 1/s \theta(x/s) \) the dilatation of \( \theta(x) \) by a factor \( s \). Since

\[
W_s f(x) = f * \psi_s(x),
\]  

(15)

we derive that

\[
W_s f(x) = f * \left( s^2 \frac{d^2 \theta_s}{dx^2} \right)(x) = s^2 \frac{d^2}{dx^2} (f * \theta_s)(x).
\]  

(16)

Hence, \( W_s f(x) \) is proportional to the second derivative of \( f(x) \) smoothed by \( \theta_s(x) \). The zero-crossings of \( W_s f(x) \) correspond to the inflection points of \( f * \theta_s(x) \). When the smoothing function \( \theta(x) \) is a Gaussian, detecting the zero-crossings of a wavelet transform is equivalent to a Marr-Hildreth edge detector [14].

III. REVIEW OF COMPLETENESS AND STABILITY
RESULTS FROM ZERO-CROSSINGS
lytic functions. The zero-crossing characterization as explained by Logan is not stable: "the problem of actually recovering (the signal) from its sign changes appears to be Laplacian of a Gaussian satisfies the heat differential equation.
the corresponding reconstruction algorithm has a fast convergence.

IV. STABILIZED ZERO-CROSSING REPRESENTATION

Instead of considering the zero-crossings of a wavelet transform on a continuum of scales \( s \), we restrict ourselves to dyadic scales \( 2^{j} \in \mathbb{Z} \). In order to stabilize the zero-crossing representation, we also record the value of the wavelet transform integral between two zero-crossings. We compute an integral measure instead of a gradient value because it will then enable us to define a simple \( L^2 \) norm on the zero-crossing representation. This is particularly important for pattern recognition applications, as explained in Sections VIII and IX.

Let \( f(x) \in L^2 \) and \( (W_{2j}f(x))_{j \in \mathbb{Z}} \) be its dyadic wavelet transform. For any pair of consecutive zero-crossings of \( W_{2j}f(x) \) whose abscissa are respectively \( z_{n-1}, z_{n} \), we record the value of the integral

\[
e_{n} = \int_{z_{n-1}}^{z_{n}} W_{2j}f(x) \, dx.
\]

(19)

Equation (16) proves that

\[
W_{2j}f(x) = 2^{2j} \frac{d^2}{dx^2} (f * \theta_{2j})(x).
\]

(20)

Since \( z_{n-1} \) and \( z_{n} \) are two zero-crossings of \( W_{2j}f(x) \), these abscissa correspond to two consecutive extrema of \( (d/dx)(f * \theta_{2j})(x) \). Equations (19) and (20) yield

\[
e_{n} = 2^{2j} \left( \frac{d}{dx} (f * \theta_{2j})(z_{n}) - \frac{d}{dx} (f * \theta_{2j})(z_{n-1}) \right).
\]

The integral \( e_{n} \) is proportional to the difference between two consecutive extrema of the derivative of \( f(x) \) smoothed at the scale \( 2^{j} \). This gives an estimate of the size of the structure which is between the two “edges” located at \( z_{n-1} \) and \( z_{n} \). If \( W_{2j}f(x) \) has a zero-crossing \( z_{0} \) of minimum abscissa, then we consider that \( -\infty \) is also a zero-crossing and we record the integral of \( W_{2j}f(x) \) between \( -\infty \) and \( z_{0} \). The equivalent is done if there exists a zero-crossing of maximum abscissa. In order to make sure that these integrals are finite, we suppose that \( f(x) \) is

\[Z_{2j}f(x) \]

satisfy the constraints

\[
\int_{-\infty}^{z_{0}} Z_{2j}f(x) \, dx = \int_{z_{0}}^{z_{n}} W_{2j}f(x) \, dx,
\]

(22)

\[
\int_{-\infty}^{z_{0}} |Z_{2j}f(x)|^2 \, dx \leq \int_{-\infty}^{z_{n}} |W_{2j}f(x)|^2 \, dx.
\]

(23)

If there exists a zero-crossing of maximum abscissa, \( Z_{2j}f(x) \) is defined similarly between this zero-crossing and \( +\infty \). Equation (23) enables us to prove in Appendix V that \( \|Z_{2j}f\| \leq \|W_{2j}f\| \) and that \( (Z_{2j}f(x))_{j \in \mathbb{Z}} \in L^{2}(L^2) \).

The sequence of piece-wise constant functions \( Zf = (Z_{2j}f(x))_{j \in \mathbb{Z}} \) is called a zero-crossing representation of \( f(x) \). Fig. 5(c) shows the zero-crossing representation of the signal in Fig. 5(a). As expected, the zero-crossings indicate the position of the sharper variation points of \( f(x) \) smoothed at different scales.

V. RECONSTRUCTION FROM A ZERO-CROSSING REPRESENTATION

Let us now study the reconstruction of a function from its zero-crossing representation. We reformalize the completeness problem within the wavelet framework and then derive an algorithm to perform the reconstruction. Let \( f(x) \in L^2 \) and \( (W_{2j}f(x))_{j \in \mathbb{Z}} \) be its dyadic wavelet transform. Since \( f(x) \) can be recovered from its dyadic wavelet transform, we first try to reconstruct \( (W_{2j}f(x))_{j \in \mathbb{Z}} \) given the zero-crossings and integral values of each function \( W_{2j}f(x) \), \( j \in \mathbb{Z} \). Clearly, for any scale \( 2^{j} \), there exists an infinite number of functions \( g_{j}(x) \) that have the same zero-crossings and integral values as \( W_{2j}f(x) \). The piece-
wavelet transform of \( f(x) \), for all initial sequences \( (g_j(x))_{j \in \mathbb{Z}} \), we must prove that the intersection of \( \Gamma \) and \( V \) is reduced to one element. We as yet have no mathematical proof of this uniqueness; however, the numerical experiments described in Section VII show that the algorithm does reconstruct the wavelet transform of \( f(x) \) for any initial sequence.

VI. DISCRETE DYADIC WAVELET TRANSFORM

A proper implementation of the zero-crossing representation and of the reconstruction algorithm raises several important questions. The input signal is generally measured with a finite resolution that imposes a finer scale when computing the wavelet transform. In practice, the scale parameter must also vary on a finite range. This section explains how to interpret mathematically a dyadic wavelet transform on a finite range of scales. In all
and

$\hat{W}_2 f(\omega) = \hat{\phi}(2^{\prime} \omega) \hat{f}(\omega). \quad (28)$

Equation (25) yields

$|\hat{\phi}(\omega)|^2 = \sum_{j=0}^{\infty} |\hat{\phi}(2^{j} \omega)|^2 + |\hat{\phi}(2^{j} \omega)|^2. \quad (29)$

Using Parseval's theorem, we derive from (27)–(29) the following energy conservation equation

$||S_1 f(x)||^2 = \sum_{j=0}^{\infty} ||W_2 f(x)||^2 + ||S_2 f(x)||^2. \quad (30)$

This equation proves that the higher frequencies of $S_1 f(x)$ that have disappeared in $S_2 f(x)$ can be recovered from the dyadic wavelet transform $W_2 f(x)$. \hfill \Box$
sinusoidal waves, Brownian processes, image scan-lines, etc. We observed that the reconstruction was independent from the initial sequence that we chose, which seems to indicate that the intersection of $\Gamma$ and $\mathcal{V}$ is reduced to the wavelet transform of $f(x)$. This would mean that the zero-crossing representation is complete. The numerical stability of the iterative algorithm also indicates that the reconstruction is stable. We have tested the reconstruction from a zero-crossing representation with another wavelet that is much less regular. The same numerical results are obtained with this other wavelet. We therefore conjecture that for a large class of dyadic wavelets, the zero-crossings plus the integral values of $(W_2, f(x))_{x \in \mathbb{R}}$ provide a complete and stable representation of $f(x)$.

The class of wavelet for which this is true remains to be defined. We want to stress that this is only a conjecture based on numerical results, but no proof is given in the paper.

The performance of the reconstruction algorithm is

The distance $d$ makes a global comparison of two zero-crossing representations over the entire spatial domain. A pattern is often a local feature embedded in the signal. For pattern matching purposes, we need to define a local distance which compares locally two zero-crossing representations. In order to derive such a distance from $d$, we study the decomposition at all scales of a local feature such as a Dirac delta function $\delta_\sigma(x)$ centered at $u$.

$W_2_\sigma(x) = \delta_\sigma(x) * \phi_\sigma(x) = \phi_\sigma(x - u).$ (39)

Let $2\sigma$ be the size of an interval where the energy of $\psi(x)$ is mostly concentrated,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx.$$ (40)
Fig. 7. Example of horizontal epipolar geometry of a pair of stereo images. Point $P$ of the scene appears respectively in $P_l$ and $P_r$ in the left and right images, on the corresponding pair of epipolar lines. Disparity $d$ is the difference of positioning of $P_l$ and $P_r$ in each of the image. Disparity is inversely proportional to the distance.
and a zero-crossing of $Z_{2^i}r(x)$ gives a local estimate of the disparity $\tau$. At the next finer scale $2^{i-1}$, we use this local estimate of the disparity in order to constrain the search when trying to find the correspondence between the zero-crossings of $Z_{2^{i-1}}l(x)$ and the zero-crossings of $Z_{2^{i-1}}r(x)$. When beginning at the coarser scale $2^{J}$ we do not have any prior estimation of the disparity to constrain the search. This number of zero-crossings is small when $J$ is big enough (see Fig. 8(b)).

The coarse to fine strategy reduces considerably the complexity of the search for a match since we use the matching information at the previous scale to constrain the search at the next scale. This strategy supposes that
matching, we described the implementation of a coarse to fine stereo-matching algorithm. The simplicity and the efficiency of this matching algorithm shows that this representation is indeed well adapted for pattern recognition problems.

In a zero-crossing representation, the number of values to be coded depends upon the irregularity of the signal. For signals that are mostly smooth with sparse singularities such as discontinuities, this type of coding can be very compact. In collaboration with Sifen Zhong, we have recently extended this representation in two dimensions [13], and shown that one reconstruct images from multi-series of these two signals are respectively

<table>
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</tr>
<tr>
<td>5</td>
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decreasing and here we only give the first five coefficients. Both filters are symmetrical with respect to 0. The
minimization of \( \int_{z_{n-1}}^{z_n} |e_f(x)|^2 \, dx \) for each pair of consecutive zero-crossings \((z_{n-1}, z_n)\) with the two constraints:

\[
\begin{cases}
  \int_{z_{n-1}}^{z_n} \epsilon_f(x) \, dx = e_n - \int_{z_{n-1}}^{z_n} g_f(x) \, dx, \\
  \epsilon_f(x) \geq -g_f(x), & \text{for } x \in [z_{n-1}, z_n].
\end{cases}
\] (56)

This minimization problem is solved by using the Lagrange multipliers. One can prove that there exists a Lagrange multiplier \(\lambda\) such that

\[
\int_{z_{n-1}}^{z_n} \epsilon_f(x) \, dx = e_n - \int_{z_{n-1}}^{z_n} g_f(x) \, dx = \lambda.
\]

In this appendix, we prove that \(\|Zf/f\| \leq \|W_f/f\|\) and derive that

\[
d(Zf, Zg)^2 \leq \|f\|^2 + \|g\|^2.
\]

One can easily prove that among all functions that have an integral equal to a given value \(e\) on an interval \([a, b]\), the function which is constant on this interval has the minimum \(L^2([a, b])\) norm. Between two consecutive zero-crossings \(z_{n-1}\) and \(z_n\).