This paper describes the construction of second generation bandelet orthogonal bases. The decomposition on a bandelet basis is computed using a wavelet filter bank followed by adaptive geometric orthogonal filters, that require $O(N)$ operations. The resulting geometry is multiscale and calculated with a fast procedure that minimizes a Lagrangian cost at each scale. Image compression with the resulting bandelet transform code gives significantly better results than a wavelet transform code.

1. INTRODUCTION

1.1. Geometry is Discrete and Multiscale

Working with digital data means working in a discrete setting. A wavelet transform [1] can be applied on the discrete data to obtain a multiscale representation of the original data. In order to go one step further, today image processing algorithms try to exploit some geometrical regularity of the underlying function. But we must keep in mind that this geometry has to be defined in a discrete setting. In this paper we give a theoretic and algorithmic approach to compute such a geometric representation in the discrete wavelet domain. An application to image compression will be shown.

1.2. Geometrically Regular Images

Functions with geometric regularity are modeled as piecewise $C^\alpha$-regular functions outside a set of edges which are themselves regular. However, natural images often do not have sharp discontinuities, so the model also includes some smoothing by an unknown kernel. The resulting functions can be written as a convolution $f = \tilde{f} \ast h$ where $\tilde{f}$ is a function with sharp features (regular outside a set of edges) and $h$ is the unknown smoothing kernel. We call this class of functions $C^\alpha$-geometrically regular functions (see figure 1, (a)).

Standard wavelet bases [1] are optimal to represent functions with pointwise singularities. However they fail to capture the geometric regularity along the singularities of a surfaces, because of their isotropic support. For $C^\alpha$-geometrically regular functions, the distortion-rate of a wavelet image transform code with $R$ bits satisfies

$$\|f - f_R\|^2 \leq C R^{-1} \log(R).$$

To exploit the anisotropic regularity of a surface along edges, the basis must include elongated functions that are nearly parallel to the edges. Our goal is to build a new class of orthogonal bases for which the distortion rate has an optimal decay.

1.3. Previous Works

Several image representations have been proposed to capture geometric image regularity, and in particular curvelet frames [2] and first generation bandelet bases [3]. However, these constructions are not built directly in the discrete domain, and they do not provide a multiresolution representation of the geometry. In consequence, the implementation and the mathematical analysis is more involved and less efficient.

Contourlets [4] are also bases constructed with elongated basis functions using the combination of a multiscale and a directional filter bank. Our approach is different because it constructs a basis with a multiscale geometry that is adapted to the function that is represented. Asymptotically, the resulting bandelets are regular functions with a compact support, which is not the case of contourlets. Wedgeprints of [5] also provide a discrete compression scheme that is built on top of a wavelet decomposition. However, it does not define a new orthogonal basis but rather a scheme to perform a vector quantization of orthogonal wavelet coefficients, using a model of step edges.

2. CONSTRUCTION OF A BANDELET BASIS

The bandelet decomposition is computed with a geometric orthogonal transform that is applied on orthogonal wavelet coefficients. It is thus computed with a wavelet filter bank followed by directional orthogonal filters. Each geometric direction leads to a different transform, and one can find the optimal set of filters using a best basis algorithm.

To describe the construction of the geometric orthogonal filters, we consider a wavelet transform at a fixed scale
and we perform a zoom on the wavelet transform (by selecting a sub-square $S$), near a singularity (see figure 1). Observe that wavelet coefficients $\langle f, \psi_{jn} \rangle$ are samples of an underlying regularized function

$$\langle f, \psi_{jn} \rangle = f * \psi_{j/2^n} \text{ where } \psi_j(x) = \frac{1}{2^j} \psi(-2^{-j}x).$$

As a result, although the original function may be singular at edge locations, the wavelet coefficients are samples of a regularized function, obtained by convolving the original function $f$ with the “burring” kernel $\psi_j$ of width $2^j$. 
the geometry are also quantized to obtain a precision of $(2^j)/4$. The total number of bits $R$ is decomposed into $R = \sum R_j = \sum (R_{Sj} + R_{Gj} + R_{Bj})$, where, for each scale $2^j$:

- $R_{Sj}$ is the number of bits to code the dyadic segmentation.
- $R_{Gj}$ is the number of bits to code the quantized polynomial geometric flow in each square.
- $R_{Bj}$ is the number of bits to code the quantized bandelet coefficients.

The image restored from its quantized bandelet coefficients is written $f_R$.

For a quantification step $T$, we would like to find the best basis $\mathcal{B}$ that minimizes the resulting distortion-rate. According to the approach in [3], we minimize a Lagrangian which can be shown to approximate the Lagrangian of the true distortion-rate:

$$L(f, R, \mathcal{B}) = \|f - f_R\|^2 + \frac{1}{2} T^2 \sum_j (R_{Sj} + R_{Gj} + R_{Bj}).$$

Thanks to the additivity of this Lagrangian and to the quadtree structure, the minimization of $L$ can be performed using a fast CART-like bottom-up algorithm [3].

Even though we make an exhaustive search for the optimal geometry, for an image of $N$ pixels, the complexity of this best bandelet basis algorithm is $O(N^{3/2})$ for a linear geometry (i.e. $p = 2$). Some heuristics can also be used to reach a linear time complexity $O(N)$. The following theorem computes the asymptotic decay rate of the distortion-rate obtained with this best bandelet basis transform code.

**Theorem:** Given $f$ a $C^\alpha$-geometrically regular function, the transform coding $f_R$ with $R$ bits in the bandelet basis $\mathcal{B}$ minimizing $L(f, R, \mathcal{B})$, with $R = R_S + R_G + R_B$, satisfies

$$\|f - f_R\|^2 \lesssim C \log^{\alpha}(R) R^{-\alpha},$$
with $C$ a constant that depends only on the function $f$.

We note the following important points:

- This exponent $\alpha$ is a priori unknown. The algorithm adapts itself to the best possible $\alpha$, and it is optimal.
- The reconstructed function is as regular as the original function (bandelet functions are regular, see figure 5).
- There is no blocking artifact due to the segmentation (in the spatial domain, bandelet functions do overlap with each other).

### 4. NUMERICAL COMPRESSION RESULTS

The bandelet transform is implemented with the 7/9 CDF biorthogonal wavelet basis. This best bandelet compression procedure was tested on various natural images. The PSNR improvement with respect to an equivalent wavelet transform coder is about 2dB for Barbara and 0.6dB for Lena (see figure 6). On figure 7 one can see, for the finest scale of the wavelet transform, the quadtree selected by the best basis algorithm, and a zoom on the orientation of the linear flow on each dyadic square. Figure 4 (upper row) shows a zoom on fine geometric structure of Barbara image, and one can see that Bandelets perform very well in these areas. The flow-based approach improves the encoding of these geometrically regular patterns.

![Wavelet transform and Segmentation and flow](image)

**Fig. 7.** Wavelet coefficients and bandelets orientations at finest scale (quadtree and flow computed at 0.4 bit/pixel).

### 5. CONCLUSION

We introduced second generation bandelet bases that are entirely discrete and orthogonal and are calculated from a discrete wavelet basis. The construction uses a multi-scale description of the underlying geometry and the corresponding bandelets are regular. Image compression in a best bandelet basis provide a clear improvement over an equivalent transform coder in a wavelet basis.

### 6. REFERENCES


