Estimating Deformations of Stationary Processes

BY MAUREEN CLERC\(^1\) AND STÉPHANE MALLAT\(^2\)

\textit{Ecole Nationale des Ponts et Chaussées, Ecole Polytechnique, and Courant Institute}

This paper studies classes of nonstationary processes, such as warped processes and frequency-modulated processes, that result from the deformation of stationary processes. Estimating deformations can often provide important information about an underlying physical phenomenon. A computational harmonic analysis viewpoint shows that the deformed autocovariance satisfies a transport equation at small scales, with a velocity proportional to a deformation gradient. We derive an estimator of the deformation from a single realization of the deformed process, with a proof of consistency under appropriate assumptions.

1. Introduction

When a nonstationary process \(X\) results from the deformation of a stationary process \(Y\), estimating the deformation can provide important information about an underlying physical process of interest. From one realization of \(X = DY\), we wish to recover the deformation operator \(D\), which is assumed to belong to a specific transformation group \(D\). For example, a Doppler effect produces a warping deformation in time \(X(x) = Y(\theta(x))\), where \(\theta'(x)\) depends upon velocity. The deformation of a stationary texture \(f\) perspective in an image also produces a warping, where \(x \in \mathbb{R}^2\) is a spatial variable; recovering the Jacobian matrix of \(\theta(x)\) characterizes the shape of the three-dimensional surface which is eing viewed \([9]\). The frequency modulation of a stationary process \(X(x) = Y(x) \exp(\text{i} \theta(x))\) corresponds to another class of deformations encountered in signal processing, in transmissions \(f\) frequency modulation, where the message is carried \(f\theta'(x)\).

Estimating the deformation \(D \in D\) from \(X = DY\) is an inverse problem. As we suppose no prior knowledge about the stationary process \(Y\), the deformation \(D\) can only be recovered up to the subgroup \(G\) of \(D\) which leaves the set of stationary processes globally invariant. Rather than the deformation itself, we therefore seek to estimate the equivalence class of \(D\) in \(D/G\). We consider cases where \(G\) is a finite-dimensional Lie group. Under appropriate assumptions, this equivalence class can be represented \(f\) a vector field on \(G\), which corresponds to a deformation gradient. A local analysis of the deformation is performed \(f\) decomposing the autocovariance of \(X\) over an

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appropriate family of localized functions, which are called atoms in the harmonic analysis literature. The deformation gradient is shown to appear as a velocity vector in a transport equation satisfied by a localized autocovariance. This general result is applied to one-dimensional warping and frequency modulation, where the atoms are classical wavelets, and multidimensional warping, where the atoms are called warplets.

Computing the deformation gradient requires estimating the autocovariance of $X$ projected over a family of localized atoms, from a single realization. Under certain conditions on the autocovariance of the stationary process $Y$, one can obtain consistent estimators for one-dimensional warping and frequency modulation. Numerical examples illustrate these results. Let us mention that the stationarity hypothesis on $Y$ can be relaxed supposing only that $Y$ has stationary increments, in which case our estimation of the deformation gradient remains consistent.

The paper is organized in three main sections: after discussing the well-posedness of the inverse problem in Section 2, we establish in Section 3 a transport equation for the localized autocovariance of a deformed process; Section 4 introduces estimators and proves their consistency.

2. Inverse Problem We want to estimate a deformation operator $D$ which belongs to a known group $D$, from a single realization of $X = DY$. The process $Y$, which is not known a priori, is assumed wide-sense stationary. Since we are limited to a single realization, we concentrate on second-order moments. For this reason, stationarity will always be understood in the wide-sense, meaning that

$$E\{Y(x)\} = E\{Y(0)\}$$

and $E\{Y(x)Y^*(y)\} = c_Y(x - y)$ with $c_Y(0) < +\infty$, where $z^*$ denotes the complex conjugate of $z \in \mathbb{C}$. We shall further suppose that $Y(x)$ is stochastically continuous, which means that its covariance $c_Y(x)$ is continuous at $x = 0$. The deformation operators $D$ that we shall consider are defined over distributions $\mathcal{T}$; for simplicity we restrict their domain to functions of $\mathbb{R}^d$. An operator $D$ acts on a stochastic process realization $Y$ realization $\mathcal{T}$.

We shall thus consider processes $Y$ whose realizations are functions of $\mathbb{R}^d$. For example, if $Y$ is a Gaussian process and $|c_Y(x) - c_Y(0)| = O(|\log \|x\||^{-1-\varepsilon})$ for some $\varepsilon > 0$, then one can prove that its realizations are continuous with probability one [2]. This hypothesis will be satisfied by our estimation theorems.

2.1. Class of Solutions Since we only know that the process $Y$ is stationary, the set of solutions to the inverse problem is the set of all operators $\tilde{D} \in \mathcal{D}$ such that $\tilde{D}^{-1}X$ is stationary. In general, this set is larger than $\{D\}$. Let $\mathcal{G}$ be the set of all operators $G \in \mathcal{D}$ such that if $Y$ is a wide-sense stationary process, then $GY$ is also wide-sense stationary. One can verify that $\mathcal{G}$ is a subgroup of $\mathcal{D}$, which we call stationarity invariant group. Clearly, if $D$ is a solution of the inverse problem, any operator $\tilde{D} = DG$ with $G \in \mathcal{G}$ is also a solution. The set of solutions of the inverse problem therefore
contains the equivalence class of \( D \) in the quotient group \( \mathcal{D}/\mathcal{G} \). In order for the set of solutions to the inverse problem to \( \mathcal{T} \)e exactly equal to the equivalence class of \( D \) in \( \mathcal{D}/\mathcal{G} \), we need to impose a condition on the stationary process \( Y \), so that any deformation \( \tilde{D} \in \mathcal{D} \) such that \( \tilde{D}Y \) is wide-sense stationary necessarily \( \mathcal{T} \)elongs to \( \mathcal{G} \). This is not true for all stationary processes \( Y \), \( \mathcal{T} \)ut we give sufficient conditions on the covariance of \( Y \) to guarantee this form of uniqueness. In this paper we concentrate on three deformation groups.

**Example 1** The frequency modulation group modifies signal frequency:

\[
\mathcal{D} = \left\{ D : D f(x) = e^{i \theta(x)} f(x) \text{ where } \theta(x) \text{ is real and } \mathbb{C}^4 \right\}.
\]

In transmissions with frequency modulation, \( \theta'(x) \) is proportional to the signal to \( \mathcal{T} \)ransmitted, and the stationary process \( Y \), which is in this case assumed to have zero mean, is the carrier. The stationarity invariant group is

\[
\mathcal{G} = \left\{ G(\phi, \xi) : G(\phi, \xi) f(x) = e^{i \phi + \xi x} f(x) \text{ with } (\phi, \xi) \in \mathbb{R}^2 \right\}.
\]

Two operators \( D_1 \) and \( D_2 \) such that \( D_1 f(x) = e^{i \theta_1(x)} f(x) \) and \( D_2 f(x) = e^{i \theta_2(x)} f(x) \) are in the same equivalence class in \( \mathcal{D}/\mathcal{G} \) if and only if \( \theta_1(x) = \phi + \xi x + \theta_2(x) \) and hence

\[
\theta_1'(x) = \theta_2'(x).
\]

The following proposition gives a sufficient condition on the covariance \( \gamma_Y(x) \) to identify \( \theta'(x) \) from the covariance of \( X = D Y \). The proof is in Appendix A.1.

**Proposition 2.1.** Let \( X = D Y \), where \( Y \) is a stationary process, and \( D \) belongs to the frequency modulation group \( \mathcal{D} \) in (1). If there exists an \( \varepsilon > 0 \) such that

\[
\forall x \in [-\varepsilon, \varepsilon] \ , \ \gamma_Y(x) > 0
\]

then the equivalence class of \( D \) in \( \mathcal{D}/\mathcal{G} \) is uniquely characterized by the covariance of \( X \).

**Example 2** The one-dimensional warping group is defined \( \mathcal{T}y \)

\[
\mathcal{D} = \left\{ D : D f(x) = f(\theta(x)) \text{ where } \theta(x) \text{ is } \mathbb{C}^3 \text{ and } \theta'(x) > 0 \right\}.
\]

Such time warping appears in many physical phenomena, such as the Doppler effect. We easily verify that the stationarity invariant group is the affine group:

\[
\mathcal{G} = \left\{ G_{(u,s)} : G_{(u,s)} f(x) = f(u + sx) \text{ with } (u, s) \in \mathbb{R} \times \mathbb{R}^+ \right\}.
\]

Note that operators in the stationarity invariant group \( \mathcal{G} \) are not time-invariant: they do not commute with translations.
Two warping operators $D_1$ and $D_2$ are in the same equivalence class in $\mathcal{D}/\mathcal{G}$ if and only if there exists $(u, s)$ such that $\theta_1(x) = u + s \theta_2(x)$, or equivalently
\begin{equation}
\frac{\theta''_1(x)}{\theta'_1(x)} = \frac{\theta''_2(x)}{\theta'_2(x)}.
\end{equation}
The following proposition, whose proof is in Appendix A.2, gives a sufficient condition on $Y$ to characterize the equivalence class of $D$ uniquely. Perrin and Senoussi [13] provide a similar result.

**Proposition 2.2.** Let $X = DY$, where $Y$ is stationary, and $D \in \mathcal{D}$, where $\mathcal{D}$ is the warping group (3). If there exists an $\varepsilon > 0$ such that $c_Y$ is $C^1$ on $(0, \varepsilon]$ and
\begin{equation}
\forall x \in (0, \varepsilon] , \quad c_Y(x) < 0 ,
\end{equation}
then the equivalence class of $D$ in $\mathcal{D}/\mathcal{G}$ is uniquely characterized by the covariance of $X$.

Condition (5) is met by a very wide range of stationary processes, including Poisson pulse processes, and Ornstein-Uhlenbeck processes [16]. White noise, however, violates (5) since $c_Y(x) = 0$ for $x \neq 0$.

**Example 3** The warping problem in two dimensions has an important application in image analysis, particularly in recovering a three-dimensional surface shape $\mathcal{T}$ analyzing texture deformations. Warping deformations are also used in geostatistics [12, 15], to model nonstationary phenomena. Stationarizing the data is suggested as an initial step before applying classical geostatistical methods such as kriging. We study a $d$-dimensional warping problem, specified $\mathcal{T}$ an invertible function $\theta(x)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ with
\begin{equation}
\theta(x_1, \ldots, x_d) = \left(\theta_1(x_1, \ldots, x_d), \ldots, \theta_d(x_1, \ldots, x_d)\right).
\end{equation}
The Jacobian matrix of $\theta$ at position $x \in \mathbb{R}^d$ is written
\begin{equation}
J_\theta(x) = \begin{pmatrix}
\frac{\partial \theta_i(x)}{\partial x_j} \\
1 \leq i, j \leq d
\end{pmatrix}.
\end{equation}
If the Jacobian determinant $\det J_\theta(x)$ does not vanish, it corresponds to a change of metric and $\theta(x)$ is invertible. We consider a group of regular warping operators
\begin{equation}
\mathcal{D} = \left\{ D: \quad Df(x) = f(\theta(x)) \quad \text{where} \quad \theta(x) \text{ is in } C^2(\mathbb{R}^d) \text{ and } \det J_\theta(x) > 0 \right\}.
\end{equation}
Let $GL^+(\mathbb{R}^d)$ be the group of linear operators in $\mathbb{R}^d$ with a strictly positive determinant. We easily verify that the stationarity invariant group is the affine group:
\begin{equation}
\mathcal{G} = \left\{ G_{(u,S)}: \quad G_{(u,S)} f(x) = f(u + S x) \quad \text{with} \quad (u, S) \in \mathbb{R}^d \times GL^+(\mathbb{R}^d) \right\} .
\end{equation}
Two operators $D$ and $\tilde{D}$ such that $Df(x) = f(\theta(x))$ and $\tilde{D}f(x) = f(\tilde{\theta}(x))$ are in the same equivalence class in $\mathcal{D}/\mathcal{G}$ if and only if
\begin{equation}
\exists (u, S) \in \mathbb{R}^d \times GL^+(\mathbb{R}^d) , \quad \theta(x) = u + S \tilde{\theta}(x).
\end{equation}
The partial derivative of the Jacobian matrix in a fixed direction $x_k$ is again a matrix:
\[
\frac{\partial J_\theta(x)}{\partial x_k} = \left( \frac{\partial^2 \theta_i(x)}{\partial x_k \partial x_j} \right)_{1 \leq i,j \leq d}.
\]

We will use the notation $\vec{\nabla} J_\theta(x)$ to denote the set of matrices \( \left\{ \frac{\partial J_\theta(x)}{\partial x_k} \right\}_{k=1,...,d} \).

Condition (8) is equivalent to the following matrix equalities, which generalize (4):
\[
(9) \quad \forall k \in \{1, ..., d\}, \quad J_\theta^{-1}(x) \frac{\partial J_\theta(x)}{\partial x_k} = J_\theta^{-1}(x) \frac{\partial J_\theta(x)}{\partial x_k}.
\]

Proof for this equivalence can be found at the end of Appendix A.3.

There are cases for which the inverse warping problem cannot be solved. For example, consider a stationary process $Y(x) = Y_1(x_1)$ which only depends on the first variable, and a warping deformation which leaves $x_1$ invariant: $\theta(x_1, ..., x_d) = (x_1, \theta_1(x_2, ..., x_d))$. In this case
\[
(10) \quad X(x) = Y(x_1, \theta_1(x_2, ..., x_d)) = Y_1(x_1) = Y(x).
\]

Hence $\theta$ cannot be recovered. The following proposition, whose proof is in Appendix A.3, gives a sufficient condition on $c_Y(x)$ to guarantee that the inverse warping problem has a unique solution in $D/G$.

**Proposition 2.3.** Let $X = DY$, where $Y$ is stationary, and $D \in D$, where $D$ is the multidimensional warping group (7). If the covariance of $Y$ satisfies
\[
(11) \quad c_Y(0) - c_Y(x) = |x|^h \eta(x) \text{ with } h > 0 \text{ and } \eta(0) > 0,
\]

where $\eta(x)$ is $C^2$ in a neighborhood of 0, then the equivalence class of $D$ in $D/G$ is uniquely characterized by the autocovariance of $X$.

The inverse warping problem has been applied to the reconstruction of three-dimensional surfaces from deformations of textures in images [6]. One can model the image of a textured three-dimensional surface as
\[
X(x) = Y(\theta(x)) .
\]

The stationary process $Y$ depends upon the textured reflectance of the surface, and $\theta(x)$ is the two-dimensional warping due to the imaging process, which projects the surface onto the image plane [5].

We showed in (9) that solving the inverse warping problem is equivalent to computing normalized partial derivatives of the Jacobian matrix $J_\theta$:
\[
(12) \quad J_\theta^{-1}(x) \frac{\partial J_\theta(x)}{\partial x_1} \text{ and } J_\theta^{-1}(x) \frac{\partial J_\theta(x)}{\partial x_2} .
\]
Differential geometry derivations by Gårding [9] have proved that these matrices specify the local orientation and curvature of the three-dimensional surface in the scene. Knowing these surface parameters, it is then possible to recover the three-dimensional coordinates of the surface, up to a constant scaling factor. We will see in Section 3.4 that the Jacobi matrices (12) appear as velocity vectors in a transport equation satisfied by the autocovariance of X.

2.2. Stationarity Invariant Group. The stationarity invariant group \( \mathcal{G} \) specifies the class of solutions of the inverse problem \( X = DY \), and Section 3 will show that it is also an important tool to identify the equivalence class of \( D \) in \( \mathcal{D}/\mathcal{G} \). This section examines the properties of operators that belong to such a group. Recall that an operator \( G \) is said to be stationarity invariant if, for any wide-sense stationary and stochastically continuous process \( Y \) the process \( X = GY \) is also wide-sense stationary.

The following theorem characterizes this class of operators. We denote \( \mathcal{T} y \cdot y \) the inner product between two vectors \( x \) and \( y \) of \( \mathbb{R}^d \).

**Theorem 2.1.** An operator \( G \) is stationarity invariant if and only if there exists \( \hat{\rho}(\omega) \) from \( \mathbb{R}^d \) to \( \mathbb{C} \) with \( \text{ess sup}_{\omega \in \mathbb{R}^d} |\hat{\rho}(\omega)| < \infty \), and \( \lambda(\omega) \) from \( \mathbb{R}^d \) to \( \mathbb{R}^d \), such that

\[
G e^{i\omega \cdot x} = \hat{\rho}(\omega) e^{i(\lambda(\omega) \cdot x)} \,.
\]

(13)

The proof is in Appendix A.4. This theorem proves that a stationarity invariant operator acts on a sinusoid \( \mathcal{T} y \) transposing its frequency and modifying its amplitude. The examples given in the previous section correspond to specific classes of such operators, where \( \lambda(\omega) \) is affine in \( \omega \). Suppose that \( \lambda(\omega) = \overline{S} \omega + \xi \) with \( \xi \in \mathbb{R}^d \) and where \( S \) is an invertible linear operator in \( \mathbb{R}^d \), whose adjoint is denoted \( \overline{S} \). In this case, the operator \( G \) in (13) satisfies

\[
Gf(x) = e^{i\xi \cdot x} f \ast \rho(Sx) \,.
\]

(14)

where \( \rho(x) \) is the inverse Fourier transform of \( \hat{\rho}(\omega) \). If \( \rho(x) = e^{i\phi} \delta(x - v) \) then the operator \( G \) defined in (14) consists of \( \mathcal{T} o \) of a frequency modulation and a warping.

Let us define the translation operator \( T_v \), for \( v \in \mathbb{R}^d \), \( \mathcal{T} y \)

\[
T_v f(x) = f(x - v) \,.
\]

The following proposition proves that linear operators of the form (14) are characterized \( \mathcal{T} y \) a weak form of commutativity with \( T_v \).

**Proposition 2.4.** A linear operator \( G \) which is bounded in \( L^2(\mathbb{R}^d) \) is stationarity invariant and satisfies (14) if and only if it satisfies

\[
\exists \xi \in \mathbb{R}^d, \exists S \in GL^+(\mathbb{R}^d), \forall v \in \mathbb{R}^d, \ G T_v S = e^{i\xi \cdot v} T_v G \,.
\]

(15)
This result, which can be viewed as a transport property, is proved in Appendix A.5. In the rest of the paper, we concentrate on deformations for which the stationarity invariant operators satisfy (15).

3. Conservation and Transport The stationarity of a random process $Y$ is a conservation property of its autocovariance through translation. After deforming $Y$, one obtains a process $X(x) = D Y(x)$ which is no longer stationary and whose autocovariance thus does not satisfy the same conservation property. Yet, we show that the stationarity of $Y$ implies a conservation of the autocovariance of $X$, along characteristic curves in an appropriate parameter space. These characteristic curves, which identify the equivalence class of $D$ in $D/G$, are computed $\mathcal{T}y$ locally approximating $D^{-1} \mathcal{T}y$ a “tangential” operator $G_{\beta(v)} \in G$. Assuming that the stationary invariant operators satisfy the transport property (15), the conservation equation can $\mathcal{T}e$ rewritten as a transport equation whose velocity term, called deformation gradient, is related to $\nabla v \beta(v)$. This deformation gradient characterizes the equivalence class of $D$ in $D/G$. Section 3.1 gives the general transport equation, and Sections 3.2, 3.3 and 3.4 apply this result to one-dimensional warping, frequency modulation, and multidimensional warping.

3.1. Transport in Groups We consider a stationarity invariant group $G$ whose elements satisfy the transport property (15) and can $\mathcal{T}e$ written under a parametric form

$$G_{\beta} f(x) = G(\phi, \xi, S; v) f(x) = e^{i(\xi \cdot x + \phi)} f \ast \rho(Sx - v),$$

where $\rho$ is a tempered distribution. With this assumption, the stationarity invariant group $G$ is a finite-dimensional Lie group. The translation parameter $v$ is isolated because of its particular role, and since the phase $\phi$ has no influence on the autocovariance, we also set it apart, and write

$$G_{\beta} = e^{i\phi} F_\alpha T_v$$

with

$$F_\alpha f(x) = e^{i\xi \cdot x} f \ast \rho(Sx) \quad \text{and} \quad \alpha = (\xi, S).$$

The group product and inverse are denoted

$$F_{a_1} F_{a_2} = F_{a_1 a_2} \quad \text{and} \quad F_{a}^{-1} = F_{a^{-1}}.$$

To identify the tangential deformation $G_{\beta(v)} \in G$ which approximates $D^{-1}$ for functions supported in a neighborhood of $v \in \mathbb{R}^d$, we use a family of test functions constructed from a single function $\psi(x)$ whose support is in $[-1,1]^d$. For $\sigma > 0$, $\psi_\sigma(x) = \psi(x/\sigma)$ has its support in $[-\sigma,\sigma]^d$. Let $F_\alpha$ be the adjoint of $F_\alpha$. An atomic decomposition of a process $X(x)$ is obtained by applying its autocovariance operator to a family of deformed and translated test functions, which are called atoms:

$$A_X^\alpha(u, \alpha) = \mathbb{E}\{|X, T_u F_\alpha \psi_\sigma|^2\}.$$
This atomic decomposition only depends on $X$ through its autocovariance.

Let us now explain how to identify the tangential deformation $G_{\beta(v)}$ from a conservation property of atomic decompositions. If $Y$ is a stationary process, then $A^\psi_X(u, \alpha)$ does not depend upon $u$, therefore $\nabla_u A^\psi_X(u, \alpha) = 0$. This is not the case for the atomic decomposition of the deformed process $X = DY$:

$$A^\psi_X(u, \alpha) = \mathbb{E}\{\langle X, T_u F_\alpha \psi_\sigma \rangle \} = \mathbb{E}\{\langle Y, D T_u F_\alpha \psi_\sigma \rangle \}.$$  

However, we now show that this atomic decomposition satisfies a conservation property along characteristic lines that depend upon $D$. The following proposition proves that if $D^{-1}$ can be approximated by a certain $G_{\bar{\beta}(v)}$, for functions having a support in a neighborhood of $v$, then there exists a function $\gamma$ such that for all $u$ and $\alpha$,

$$\nabla_u A^\psi_X(u, \alpha \ast \gamma(u)) \approx 0 \quad \text{for } \sigma \text{ small},$$

Before stating the proposition, let us set some notation: if $f(x)$ and $g(x)$ are two functions defined in $\mathbb{R}^d$, then $\nabla_x g$ is a vector with $d$ components, and $(f, \nabla_x g)$ is also a vector, whose $d$ components are the inner products $\left\langle f, \frac{\partial g}{\partial x_k} \right\rangle$. We denote $\Re\{f, \nabla_x g\}$ the real part of this vector. We write $c(\sigma) = O(\sigma)$ if there exists a constant $C$ such that for $\sigma$ small, $|c(\sigma)| \leq C \sigma$, without specifying the sign. We define the covariance operator of $X$ by

$$K_X f(x) = \int \mathbb{E}\{X(x)X^*(y)\} f(y) \, dy.$$  

**Proposition 3.1.** Let $X = DY$, with $Y$ stationary. Suppose that for each $v \in \mathbb{R}^d$, there exists $\beta(v)$ such that for each $\alpha$, the function $\psi_{v, \alpha, \sigma} = G_{\bar{\beta}(v)} T_v F_\alpha \psi_\sigma$ satisfies

$$\Re\{K_X \psi_{v, \alpha, \sigma}, D^{-1}(\nabla_v + \nabla_x) D \psi_{v, \alpha, \sigma}\} = O(\sigma) |\Re\{K_X \psi_{v, \alpha, \sigma}, \nabla_u \psi_{v, \alpha, \sigma}\}|.$$  

(16)

If there exists a differentiable invertible map $u(v)$, and two functions $\phi(u)$ and $\gamma(u)$ such that

$$G_{\bar{\beta}(v)} T_v = e^{i\phi(u(v))} T_{u(v)} F_{\gamma(u(v))},$$

then for each $(u, \alpha)$, we have at $t = u$:

$$\left| \nabla_u A^\psi_X(u, \alpha \ast \gamma(t)) + \nabla_t A^\psi_X(u, \alpha \ast \gamma(t)) \right| = O(\sigma) \left| \nabla_u A^\psi_X(u, \alpha \ast \gamma(t)) \right|.$$  

(18)

The norms in (18) are Euclidean norms of $d$-dimensional vectors. The proof of Proposition 3.1 can be found in Appendix B.1.

We have assumed that the stationarity invariant group $G$ is composed of elements of the form $G_{\beta} = e^{i\phi} T_v$.

In order to shed some light on the meaning of (16) and the role of the function $\beta(v)$, we examine the case when the deformation operator $D$ itself is a stationarity invariant operator. In this case, a
function $\beta$ such that $G_{\beta(v)} T_v \mathcal{F}_\alpha \psi_\sigma$ satisfies (16) can be chosen independently of $v$. Indeed, $G_{\beta(v)} = D^{-1}$ yields $D \psi_{v,\alpha,\sigma} = T_v \mathcal{F}_\alpha \psi_\sigma$. For any function $f(x)$, $(\nabla_v + \nabla_x) T_v f(x) = 0$ therefore
\[(\nabla_v + \nabla_x) D \psi_{v,\alpha,\sigma} = (\nabla_v + \nabla_x) T_v T_v \mathcal{F}_\alpha \psi_\sigma = 0,
\]
and the left-hand side of (16) vanishes. When $D$ is not stationarity invariant, $\beta(v)$ must be chosen so that (16) holds when $\sigma \to 0$. This imposes a form of tangency between $G_{\beta(v)}$ and $D^{-1}$, when applied to functions supported in a neighborhood of $v$. In the following three sections, we will identify the $\beta$ which are appropriate for each type of deformation considered.

The partial differential equation (18) which results from the aTove proposition can be written as a transport equation in the $(u; \alpha)$ domain, $\tau_y$ expanding the gradient with respect to $t$:
\[
\nabla_t A^\alpha_X(u, \alpha * \gamma(t)) = \nabla_t (\alpha * \gamma(t)) \cdot \nabla_\alpha A^\alpha_X(u, \alpha * \gamma(t)) ,
\]
where $\nabla_\alpha A^\alpha_X(u, \alpha)$ is a vector of partial derivatives with respect to each component of parameter $\alpha$. Replacing the free variable $\alpha \tau_y \alpha * \gamma^{-1}(u)$ in (18) gives, at $t = u$,
\[
\left| \nabla_\alpha A^\alpha_X(u, \alpha) + \nabla_t (\alpha * \gamma^{-1}(u) * \gamma(t)) \cdot \nabla_\alpha A^\alpha_X(u, \alpha) \right| = O(\sigma) \left| \nabla_\alpha A^\alpha_X(u, \alpha) \right| .
\]

When $\sigma$ is sufficiently small, the right-hand side can be neglected, yielding a transport partial differential equation. This is illustrated in the next three sections, in which we apply this proposition to the warping deformation and the frequency modulation problems. Section 4 will afterwards show how, from a single realization of $X$, we can estimate the partial derivatives of $A^\alpha_X(u, \alpha)$ and compute the deformation gradient.

3.2. Scale Transport. If $D$ is a one-dimensional warping deformation $Df(x) = f(\theta(x))$ with $x \in \mathbb{R}$, then $D^{-1}f(x) = \theta'(x)f(\theta(x))$. The stationarity invariant group is the affine group, whose elements are $G_{\beta}(f(x) = f(u + sx)$ with $\beta = (u, s)$. The adjoint of $G_{\beta}$ is
\[
G_{\beta}^* f(x) = s^{-1} f ((x - u)/s) = T_u \mathcal{F}_s f(x) \quad \text{with} \quad \mathcal{F}_s f(x) = s^{-1} f (x/s) .
\]

Let $\psi \tau_e$ be a function whose integral vanishes: $\int \psi(x) dx = 0$. The function $\psi$ is called a wavelet [11]. Using the aTove expression of the adjoint operator $\mathcal{F}_s$, the atomic decomposition $A^\alpha_X(u, \alpha) = \mathbb{E} \{|X, T_u \mathcal{F}_s \psi_0|^2 \}$ can be written
\[
A^\alpha_X(u, \alpha) = \mathbb{E} \left\{ |\langle X, s^{-1} \psi ((s \sigma)^{-1}(x - u)) \rangle|^2 \right\} .
\]

We reduce the number of parameters $\tau_y$ dividing $A^\alpha_X(u, \alpha) \tau_y \sigma^2$, and replacing the product $\sigma \tau_y$ a single scale parameter $s$. The resulting atomic decomposition
\[
A_X(u, s) = \mathbb{E} \left\{ |\langle X, s^{-1} \psi (s^{-1}(x - u)) \rangle|^2 \right\} .
\]
is called a scalogram, and can be interpreted as the expected value of a squared wavelet transform. Figure 1(a) shows the scalogram $A_Y(u, s)$ of a stationary process $Y$. As expected, its value does not
depend upon \( u \). Figure 1(\( \tau \)) gives \( A_X(u, s) \) for a warped process \( X(x) = DY(x) = Y(\theta(x)) \). The warping causes the values of the scalogram of \( Y \) to migrate in the \((u; \log s)\) plane.

Let us now give the expression of \( \beta(u) \) corresponding to the tangential approximation of Proposition 3.1. A tangential approximation of \( D^{-1} \) can be found by noting that, for a regular function \( f \) supported in a neighborhood of \( v = \theta(u) \),

\[
D^{-1} f(x) \approx \theta(u) f(v + \theta(u)(x - u)) \, .
\]

The right-hand side of (21) can be written as \( G_{\beta(v)} f(x) \), and operators \( D^{-1} \) and \( G_{\beta(v)} \) both translate the support of \( f \) from a neighborhood of \( v \) to a neighborhood of \( u(v) \).

In order to derive a transport equation from Proposition 3.1, we must make some assumptions on the autocovariance of \( Y \), which will guarantee the uniqueness of the inverse warping problem at the same time. Proposition 2.2 shows that it is necessary to specify the behavior of the autocovariance kernel \( c_Y(x) \) in a neighborhood of 0. The following theorem supposes that \( c_Y(x) \) is nearly \( h \)-homogeneous in a neighborhood of 0. We denote \( \frac{\partial f}{\partial a} = \partial_a f \), and \( \partial_{\log s} f = \frac{\partial f}{\partial \log s} = s \frac{\partial f}{\partial x} \).

**Theorem 3.1 Scale Transport.** Let \( Y \) be a stationary process whose covariance satisfies

\[
c_Y(0) - c_Y(x) = |x|^h \eta(x) \quad \text{with} \quad h > 0, \quad \eta(0) > 0 \, ,
\]

and where \( \eta \) is \( C^1 \) in a neighborhood of 0. Let \( \psi(x) \) be a \( C^1 \) function supported in \([-1,1]\), such that

\[
\int \psi(x) \, dx = 0 \quad \text{and} \quad \text{Re} \int \int |x - y|^h \psi^*(x) \psi(y) \, dx \, dy \neq 0 \, .
\]
If

\[ X(x) = Y(\theta(x)) , \]

where \( \theta(x) \) is \( C^3 \) and \( \theta'(x) > 0 \), then for each \( u \in \mathbb{R} \) such that \( \theta'(u) \neq 0 \), when \( s \) tends to zero

(24) \[ \partial_s A_X(u, s) - (\log \theta)'(u) \partial_{\log \theta} A_X(u, s) = O(s) \partial_s A_X(u, s) . \]

The proof is in Appendix B.2. The conditions imposed on \( c_Y \) and \( \psi \) in this theorem guarantee that \( \partial_{\log \theta} A_X(u, s) \) does not vanish for \( s > 0 \). The deformation gradient \((\log \theta)'(u)\), which specifies the equivalence class of \( D \) in \( D/G \), can thus \( \mathcal{T} \) be computed from (24) \( \mathcal{T} \) by letting \( s \) go to zero. It is therefore not surprising that (22) imposes a stronger condition on \( c_Y \) than the uniqueness condition (5) of Proposition 2.2. The estimation of \((\log \theta)'(u)\) from a single realization of \( X \) will \( \mathcal{T} \) be studied in Section 4.1.

**Remark 3.1.** Since \( \psi(x) \) has a zero integral, one can verify that \( A_X(u, s) \) can be expressed from the covariance of the increments of \( X(x) \), which itself depend upon the covariance of the increments of \( Y(x) \). It is therefore possible to extend this theorem by supposing only that \( Y(x) \) has stationary increments and by replacing (22) by a similar condition on the autocovariance of the increments. Fractional Brownian motions [1, 7] are examples of processes \( Y(x) \) with stationary increments whose deformations by warping satisfy (24).

### 3.3. Frequency Transport

If the deformation operator \( D \) is a frequency modulation, \( Df(x) = e^{i\theta(x)} f(x) \), the stationarity invariant subgroup \( G \) is composed of operators \( G_\beta \) such that

\[ G_\beta f(x) = e^{i(\phi+\xi x)} f(x) . \]

In this case \( F_\xi f(x) = e^{i\xi x} f(x) \). Let us choose an even, positive window function \( \psi(x) \geq 0 \), with a support equal to \([-1, 1]\). The atomic decomposition of process \( X \) is the well-known spectrogram:

\[ A^\psi_X(u, \xi) = E\{ |\langle X(x), \psi_\sigma(x-u)e^{-i\xi(x-u)} \rangle|^2 \} = E\{ |\langle X(x), \psi_\sigma(x-u)e^{-i\xi x} \rangle|^2 \} . \]

Figure 2(a) shows a spectrogram \( A^\psi_X(u, \xi) \), whose values do not depend upon \( u \) because \( Y \) is stationary. Figure 2(\( \mathcal{T} \)) depicts \( A^\psi_X(u, \xi) \) for \( X(x) = DY(x) = e^{i\theta(x)} Y(x) \), with \( \theta(x) = \lambda_1 \cos(\lambda_2 x) \) where \( \lambda_1 \) and \( \lambda_2 \) are two constants. The frequency modulation translates the spectrogram of \( Y \) non-uniformly along the frequency axis.

Let us now give the expression of \( \beta(v) \) corresponding to the tangential approximation of Proposition 3.1, when \( D \) is a frequency modulation. A tangential approximation of \( D^{-1} \) can \( \mathcal{T} \) be found \( \mathcal{T} \) by noting that if \( f \) is supported in a neighborhood of \( v \),

\[ D^{-1} f(x) = e^{i\theta(x)} f(x) \approx e^{i(\theta(v)+\theta'(v)(x-v))} f(x) \]
and one can define a stationarity invariant approximation of $D^{-1} \tau_y G_{\hat{\rho}(v)}$ such that

$$
\overline{G_{\hat{\rho}(v)}} f(x) = e^{i \theta(v) + \theta'(v)(x-v)} f(x).
$$

The following theorem uses this tangential approximation to derive from Proposition 3.1 a transport equation, satisfied $\tau_y$ the spectrogram $A_X(u, \xi)$ in the $(u; \xi)$ plane, when the window width $\sigma$ decreases to 0. The frequency $\xi$ is chosen large enough so that the period of $e^{i \xi x}$ is smaller than the support size $\sigma$ of $\psi_\sigma$. We set $\xi = \xi_0 / \sigma$ and select $\xi_0$ so that $\hat{\psi}(\omega)$ and its first $[h] + 2$ derivatives vanish at $\omega = \xi_0$, where $[h]$ denotes the smallest integer larger or equal to $h$.

**Theorem 3.2 Frequency transport.** Let $Y$ be a stationary process such that there exists $h > 0$ with

$$
c_Y(0) - c_Y(x) = |x|^h \eta(x)
$$

where $\eta$ is continuous in a neighborhood of 0, and $\eta(0) > 0$. Let $\psi$ be an even, positive, $C^1$ function supported in $[-1,1]$ and let $\xi_0$ be such that $\hat{\psi}(\omega)$ and its first $[h] + 2$ derivatives vanish at $\omega = \xi_0$ but

$$
\int \int |x - y|^h (x-y) \sin[\xi_0(x-y)] \psi(x) \psi(y) \, dx \, dy \neq 0.
$$

If

$$
X(x) = e^{i \theta(x)} Y(x) \text{ where } \theta(x) \text{ is } C^{[h]+4},
$$

is a stationary process such that $c_Y(x) = \int c_X(u, \xi) \, du$, then $Y$ is the unique solution to the equation

$$
\tau_y A_Y(u, \xi) = \overline{G_{\hat{\rho}(v)}} A_X(u, \xi) \text{ for all } (u; \xi),
$$

Furthermore, the solutions can be constructed by taking the Fourier transform of $A_Y(u, \xi)$ in the $u$-variable with respect to $\xi$.
then for each \( u \in \mathbb{R} \) such that \( \theta''(u) \neq 0 \) and for \( \xi = \frac{\xi_0}{\sigma} \), when \( \sigma \to 0 \)

\[
\partial_u A^\omega_X(u, \xi) - \theta''(u) \partial_\xi A^\omega_X(u, \xi) = O(\sigma^2) \partial_u A^\omega_X(u, \xi) .
\]

The proof is in Appendix B.3. To satisfy the theorem hypothesis, one may choose \( \psi(x) \) to \( \mathcal{T} \) be a \( \mathcal{T}_0 \)x-spline \( \mathcal{T} \) taken \( \mathcal{T} \) convolving the indicator function \( 1_{[-1/2m,1/2m]} \) with itself \( m \) times: let \( m \geq [h] + 3 \) and

\[
\hat{\psi}(\omega) = \left( \frac{\sin(\omega/(2m))}{\omega/(2m)} \right)^m \exp \left( \frac{-i \varepsilon \omega}{2m} \right) ,
\]

where \( \varepsilon = 1 \) if \( m \) is odd and \( \varepsilon = 0 \) if \( m \) is even. With \( \xi_0 = 2m \pi, \psi \) satisfies the theorem hypotheses. The deformation gradient \( \theta''(u) \) can \( \mathcal{T} \)e characterized from equation (27) \( \mathcal{T} \) by letting \( \sigma \) go to zero, and we proved in (2) that \( \theta''(u) \) specifies the equivalence class of \( \mathcal{D} \) in \( \mathcal{D}/\mathcal{G} \). Section 4.2 will impose additional conditions on \( \mathcal{C}_Y \) and \( \theta \) to \( \mathcal{T} \nabla \) a consistent estimation of \( \theta''(u) \) from one realization of the frequency modulated process \( X \).

3.4. Multidimensional scale transport

For a multidimensional warping where \( Df(x) = f(\theta(x)) \) with \( x \in \mathbb{R}^d \), the adjoint of \( D^{-1} \) is \( \overline{D^{-1}} f(x) = \det J_\theta(x) f(\theta(x)) \). The matrix \( J_\theta(x) \) is the Jacobi\( \mathcal{T} \)ian matrix of \( \theta \) at position \( x \), as defined in (6). The stationarity invariant group \( \mathcal{G} \) is the affine group, composed of operators \( G_\beta \) with \( \beta = (u, S) \in \mathbb{R}^d \times GL^+(\mathbb{R}^d) \), such that

\[
G_\beta f(x) = f(u + S x) .
\]

The adjoint of \( G_\beta \) is

\[
\overline{G}_\beta f(x) = \det S^{-1} f(S^{-1}(x - u)) = T_u \overline{F}_S f(x)
\]

where \( \overline{F}_S f(x) = \det S^{-1} f(S^{-1} x) \) and

\[
S = \left( s_{i,m} \right)_{1 \leq i, m \leq d} .
\]

Similarly to (21), we find a tangential approximation to \( \overline{D^{-1}} \) \( \mathcal{T} \) noting that for a regular function \( f \) supported in a neighborhood of \( v = \theta(u) \),

\[
\overline{D^{-1}} f(x) \approx \det J_\theta(x) \left( \theta(u) + J_\theta(u) (x - u) \right) = \overline{G}_{\beta(v)} f(x) .
\]

The operators \( \overline{D^{-1}} \) and \( \overline{G}_{\beta(v)} \) Toth translate the support of \( f \) from a neighborhood of \( v \) to a neighborhood of \( u(v) = \theta^{-1}(v) \).

Let \( \psi \) \( \mathcal{T} \) a function such that \( \int_{\mathbb{R}^d} \psi(x) dx = 0 \). A multidimensional extension of the scalogram is given \( \mathcal{T} \)

\[
A^\omega_X(u, S) = E \left\{ |(X(x) \cdot \det S^{-1} \psi_\sigma (S^{-1}(x - u))|^2 \right\}
\]

\[
= E \left\{ |(X(x) \cdot \det S^{-1} \psi (\sigma^{-1} S^{-1}(x - u))|^2 \right\} .
\]
As in the one-dimensional case, we divide \( A_X(u, s) \) \( \tau_y \sigma^{2d} \) and replace the product \( \sigma S \tau_y \) a matrix which we still denote \( S \). The resulting atomic decomposition

\[
A_X(u, S) = E \{(X(x), \det S^{-1} \psi (S^{-1}(x - u))) \}^2
\]

is similar to the scalogram (20) \( \tau_y \) since the scale parameter \( s \) is replaced \( \tau_y \) by a warping matrix \( S \), we call it a warpgram.

For a one-dimensional warping, the velocity term of transport equation (24) is \( (\log \theta)'(u) = \theta'(u)/\theta(u) \). In two dimensions it becomes a set of matrices, indexed \( \tau_y \) the direction \( k \) of spatial differentiation:

\[
\text{for } 1 \leq k \leq d, \quad J_0^{-1}(u) \frac{\partial J_0(u)}{\partial u_k} = \left( \gamma^i_{l,m}(u) \right)_{1 \leq i, m \leq d}.
\]

This set of matrices has been shown in (9) to specify the equivalence class of \( D \) in \( D/G \). For each \((l, m)\) we introduce the vector

\[
\vec{\gamma}_{l,m}(u) = (\gamma^i_{l,m}(u))_{1 \leq i \leq d}.
\]

The partial derivative \( \partial_{\log \sigma} A_X(u, s) = s \partial_s A_X(u, s) \) which appears in the one-dimensional transport equation (24) now becomes a matrix composed of partial derivatives with respect to the scale parameters, and the transpose \( S^t \) of \( S \):

\[
\left( \frac{\partial A_X(u, S)}{\partial s_{i,j}} \right)_{1 \leq i, j \leq d} S^t = \left( a_{i,j}(u, S) \right)_{1 \leq i, j \leq d}.
\]

The following theorem isolates \( \sigma = (\det S)^{-1/d} \tau_y \) writing \( S = \sigma \tilde{S} \) with \( \det \tilde{S} = 1 \), and gives a \( d \)-dimensional transport equation when \( \sigma \) goes to zero.

**Theorem 3.3.** Suppose that \( X(x) = Y(\theta(x)) \), where \( Y \) is stationary, \( \theta(x) \) is \( C^3 \) and \( \det J_0(x) > 0 \). Suppose that the autocovariance kernel \( c_Y \) of \( Y \) satisfies

\[
c_Y(0) - c_Y(x) = |x|^h \eta(x),
\]

with \( \eta(0) > 0 \) and \( \eta \in C^2 \) in a neighborhood of 0. For each \( u \in \mathbb{R}^d \) and for each \( \tilde{S} \) with \( \det \tilde{S} = 1 \), if there exists \( C(u, \tilde{S}) > 0 \) such that, for \( S = \sigma \tilde{S} \) and \( \sigma \) small enough,

\[
\left| Re \int \int \nabla c_Y(S(x - y)) \nabla J_0(u)J_0^{-1}(u)S(x - y) \psi^*(x) \psi(y) \, dx \, dy \right| \geq C(u, \tilde{S}) \sigma^h,
\]

then when \( \sigma \) goes to zero

\[
\left| \nabla_u A_X(u, S) - \sum_{l,m=1}^d \gamma_{l,m}(u) a_{l,m}(u, S) \right| = O(\sigma) \left| \nabla_u A_X(u, S) \right|.
\]
The proof of this Theorem is in Appendix B.4.

The purpose of condition (34) is to ensure that $|\nabla u A_X(u, S)|$ on the right-hand side of (35) is not too small. If $\theta(x)$ is a separable warping function of the form $\theta(x_1, \ldots, x_d) = (\theta_1(x_1), \ldots, \theta_d(x_d))$, then one can verify (see Appendix B.5) that (34) holds, for all functions $\theta_i$ such that $\theta_i^p$ does not vanish, if

$$
Re \int \int |\tilde{S}(x - y)|^p \psi^*(x) \psi(y) \, dx \, dy \neq 0.
$$

The above condition is similar to the second part of condition (23) in Theorem 3.1. Condition (34) is more involved however, because in general, coupling occurs between different directions.

For $\sigma$ sufficiently small, neglecting the right-hand side of (35) yields $d$ scalar equations:

$$
\text{for } 1 \leq k \leq d, \quad \partial_{u_k} A_X(u, S) - \sum_{i,m=1}^{d} \gamma^k_{i,m}(u) a_{i,m}(u, S) = 0.
$$

For any $(u, S)$, the values $\partial_{u_k} A_X(u, S)$ and $a_{i,m}(u, S)$ defined in (32) depend upon the autocovariance of $X$, and have to be estimated. For each direction $k$, there are a total of $d^2$ unknown coefficients $\gamma^k_{i,m}(u)$, equal to the $d^2$ matrix components of $J^{-1}_\theta(u) \partial_{u_k} J_\theta(u)$. To compute them we need to select $d^2$ warping matrices $\{S_i\}_{i=1}^{d^2}$ and invert the linear system:

$$
\begin{bmatrix}
 a_{1,1}(u, S_1) & a_{1,2}(u, S_1) & \cdots & a_{d,1}(u, S_1) \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{1,1}(u, S_{d^2}) & a_{1,2}(u, S_{d^2}) & \cdots & a_{d,1}(u, S_{d^2})
\end{bmatrix}
\begin{bmatrix}
 \gamma^k_{1,1}(u) \\
 \gamma^k_{1,2}(u) \\
 \vdots \\
 \gamma^k_{d,1}(u) \\
 \gamma^k_{d,2}(u) \\
 \vdots \\
 \gamma^k_{d,d}(u)
\end{bmatrix}
= 
\begin{bmatrix}
 \partial_{u_k} A_X(u, S_1) \\
 \partial_{u_k} A_X(u, S_2) \\
 \vdots \\
 \partial_{u_k} A_X(u, S_{d^2})
\end{bmatrix}.
$$

Changing the direction index $k$ only modifies the right-hand side of (37). Note that in order for the system to be invertible, the matrix on the left-hand side of (37) must have full rank. The matrices $S_k$ must therefore be appropriately chosen, and the inverse warping problem must have a unique solution. This is not always the case, as shown by the example in (10).

![Image](image.png)

**Fig. 3.** Shape-from-texture: examples of objects whose shape is inferred by analyzing texture (i.e., pattern) variations.
The system (37) has been used in Computer Vision, for relief reconstruction from photographs of surfaces with regular patterns, a problem called shape-from-texture [6]. Examples of such “textured surfaces” are provided in Figure 3. Perceptually, these analyzing the variations of size, shape and density of the patterns within the 2D images, we can infer the 3D shape of the objects. This cue to 3D shape is monocular (i.e., it uses a single image), unlike stereo vision, in which two images of a scene taken from a different viewing point are compared. Mathematically, one can model the pattern variations across the image $\tau y$ the two deformation gradient vectors

$$\begin{pmatrix}
\gamma_{1,1}^1(u) \\
\gamma_{1,2}^1(u) \\
\gamma_{2,1}^1(u) \\
\gamma_{2,2}^1(u)
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\gamma_{1,1}^2(u) \\
\gamma_{1,2}^2(u) \\
\gamma_{2,1}^2(u) \\
\gamma_{2,2}^2(u)
\end{pmatrix}.$$ 

These two vectors represent the elements of the two matrices defined in (12), and as mentioned in Section 2.1, one can recover from them the normal vector field of the surface $\tau e$ being viewed, and hence its shape.

4. Estimation of Deformations The deformation gradient appears as a velocity vector in the transport (19). To recover it from a single realization of $X$, the derivatives $\vec{\nabla}_\alpha A^X(u,\alpha)$ and $\vec{\nabla}_\alpha A^X(u,\alpha)$ of the atomic decomposition of $X$ have to be estimated. With a single realization, a sample mean estimator has a variance of the same order of magnitude as the term it estimates. This variance can be reduced with a spatial smoothing, while the bias, which is proportional to the width of the smoothing kernel, is controlled. The next two sections study the consistency of such smoothed estimators for the one-dimensional warping problem and the frequency modulation problem. Lastly, Section 4.3 discusses multidimensional warping estimation.

4.1. Warping in one dimension The scalogram of $X$ is an expected value

$$A_X(u,s) = E\{|\langle X, \psi_{u,s}\rangle|^2\},$$

with $\psi_{u,s}(x) = s^{-1}\psi((x-u)/s)$. If $X(x) = Y(\theta(x))$ then Theorem 3.1 proves that

$$\partial_u A_X(u,s) - (\log \theta)'(u) \partial_{\log \theta} A_X(u,s) = O(s) \partial_u A_X(u,s).$$

From $X(x)$ approximated at a resolution $N$, one can compute the empirical scalogram $|\langle X, \psi_{u,s}\rangle|^2$ at scales $s \geq N^{-1}$ and locations $u = k/N$ with $k \in \mathbb{Z}$ [11]. We introduce a kernel estimator $A_X(u,s)$, using the averaging kernel

$$g(x) = \begin{cases} 
\Delta^{-1}(1 - |x/\Delta|) & \text{if } |x| \leq \Delta \\
0 & \text{if } |x| > \Delta
\end{cases}.$$
Let
\[ \bar{\partial}_u A_X(u, s) = 2N^{-1} \sum_{|k/N - u| \leq \Delta} g(u - k/N) \text{Re} \left[ (X, \psi_{k/N,u}) (X, \partial_u \psi_{k/N,u})^* \right] \]
\[ \bar{\partial}_{\log \theta} A_X(u, s) = 2N^{-1} \sum_{|k/N - u| \leq \Delta} g(u - k/N) \text{Re} \left[ (X, \psi_{k/N,u}) (X, \partial_{\log \theta} \psi_{k/N,u})^* \right]. \]

In view of equation (38), we suggest the following estimator for \((\log \theta)'(u)\):
\[ \left( \frac{\text{log} \theta' \right| (u)}{\partial_{\log \theta} A_X(u, N^{-1})} \]

One must guarantee that when \(s = N^{-1}\) and \(N\) increases, \(\bar{\partial}_u A_X(u, s)\) and \(\bar{\partial}_{\log \theta} A_X(u, s)\) are close to \(\partial_u A_X(u, s)\) and \(\partial_{\log \theta} A_X(u, s)\), respectively. For this, we introduce a Gaussianity assumption on the underlying stationary process \(Y\). This assumption allows to derive the consistency of the estimators from the fast spatial decorrelation (in \(k\)) of \((X, \psi_{k/N,u})\) and \((X, \partial_{\log \theta} \psi_{k/N,u})\). Fast decorrelation of these two random variables will be ensured by adjusting the choice of wavelet \(\psi\) to the behavior of the autocovariance function \(c_Y\) in a neighborhood of \(0\).

The following theorem proves the weak consistency of the proposed estimator \((\text{log} \theta')'(u)\) of \((\log \theta)'(u)\), by selecting the averaging interval \(\Delta\) according to the scale \(s = N^{-1}\). A wavelet \(\psi(x)\) is said to have \(p\) vanishing moments if
\[ \int x^k \psi(x) dx = 0 \quad \text{for } 0 \leq k < p. \]

**Theorem 4.1 Consistency, Warping.** Let \(X(x) = Y(\theta(x))\), where \(Y\) is a stationary Gaussian process whose covariance satisfies
\[ c_Y(0) = c_Y(x) = |x|^h \eta(x) \quad \text{with } h > 0 \text{ and } \eta(0) > 0. \]

Let \(\psi\) be a \(C^2\) wavelet supported in \([-1, 1]\) with \(p\) vanishing moments, such that
\[ 2p - h > 1/2 \quad \text{and} \quad \int \int |x - y|^h \psi^*(x) \psi(y) dx dy \neq 0. \]

Let \(\Delta = N^{-1/5}\). If \(\eta(x)\) is \(C^{2p}\) in a neighborhood of \(0\), and if \(\theta(x) \in C^3 \cap C^{2p}\), then for each \(u \in \mathbb{R}\) such that \(\theta''(u) \neq 0\)
\[ \text{Pro} \left\{ \left| (\log \theta)'(u) - (\log \theta)'(u) \right| \leq 2 (\log N) N^{-1/5} \right\} \xrightarrow{N \to \infty} 1. \]

The proof is in Appendix C.1. Since all estimations are based on wavelet coefficients, one can easily verify that the results still hold if \(Y\) is not stationary. Thus has stationary increments. In particular, it applies to fractional Brownian motion \([1, 7]\), for which \(\eta(x) = 1\).

Figure 4 displays a numerical experiment conducted on a single realization of a warped process. The signal \(X\) in Figure 4(\(\tau\)) is obtained by warping a stationary signal \(Y\), depicted in Figure 4(a).
Figure 4. (a) Stationary signal $Y$ and its empirical scalogram $\|\langle Y, \psi_{u,x} \rangle\|^2$. (b) Warped signal $X(x) = Y(\theta(x))$ and its empirical scalogram. (c) $\log \theta(x) = \lambda_1 + \lambda_2 \text{sign}(1/2 - x)|x - 1/2|^2$, where $\lambda_1$ and $\lambda_2$ are two constants (full line) and its estimation from $X$ (dashed line). (d) Stationarized signal and its empirical scalogram.

Figure 4(c) shows in dotted line the estimate $\hat{\log \theta}$ of $\log \theta'$ of $\hat{\log \theta'}(u)$, and choosing the additive integration constant so that $\int_0^1 \exp(\hat{\log \theta'}) = \int_0^1 \theta$. An estimate $\hat{\theta}$ for the warping function can be obtained up to an additive constant $\tilde{\gamma}$ by integrating $\exp(\hat{\log \theta'})$. It is then possible to stationarize the deformed signal $X \tilde{\gamma}$ computing $X \circ (\hat{\theta})^{-1}$. Figure 4(d) displays such a stationarized signal. We refer the reader to [6] for details on the numerical implementation of this method.

4.2. Frequency Modulation. For a frequency modulated process, $X(x) = Y(x)e^{\theta(x)}$, Theorem 3.2 shows that the deformation gradient $\theta'(u)$ can be computed from the spectrogram

$$A_X^u(u, \xi) = E \{|\langle X(x), \psi_\sigma(x - u)e^{\xi(x-u)} \rangle|^2\}$$
with the equation
\begin{equation}
\partial_u A_X^\sigma(u, \xi) - \theta''(u) \partial_\xi A_X^\sigma(u, \xi) = O(\sigma^2) \partial_\xi A_X^\sigma(u, \xi)
\end{equation}
evaluated at a frequency \( \xi = \xi_0/\sigma \).

To compute an estimator of the smoothed partial derivatives of the spectrogram, we relate the spectrogram coefficients to a particular wavelet transform. Observe that
\begin{equation}
\psi_\delta(x - u) \exp \left( i \xi_0 \frac{x - u}{\sigma} \right) = \psi^1 \left( \frac{x - u}{\sigma} \right)
\end{equation}
where
\begin{equation}
\psi^1(x) = \psi(x) e^{i \xi_0 x}.
\end{equation}
Since \( \psi \) is real, \( \hat{\psi}(\omega) \) is even, and if \( \hat{\psi}(\omega) \) has a zero of order \( [h] + 3 \) at \( \omega = \xi_0 \), then
\[
\int x^k \psi^1(x) dx = (-i)^k \frac{d^k \psi^1}{d \omega^k} (-\xi_0) = 0 \quad \text{for } k \leq [h] + 2.
\]
This means that \( \psi^1 \) is a wavelet with \( [h] + 3 \) vanishing moments [11]. We write \( \psi_{\delta, \sigma}(x) = \sigma^{-1} \psi^1(\sigma^{-1}(x - u)) \). The scalogram associated to this wavelet is defined \( \mathbb{T}y_A \chi(u, \sigma) = \mathbb{E}\{ |\langle X, \psi_{\delta, \sigma} \rangle|^2 \} \). It results from (45) that
\[
A_X^\sigma(u, \xi_0/\sigma) = \mathbb{E} \left\{ |\langle X(x), \psi^1(\sigma^{-1}(x - u)) \rangle|^2 \right\} = \sigma^2 A_X(u, \sigma),
\]
and hence
\[
\partial_u A_X^\sigma(u, \xi_0/\sigma) = \sigma^2 \partial_u A_X(u, \sigma).
\]
Let \( \partial_u A_X(u, \sigma) \mathbb{T}e \) the estimator calculated in (40) at a scale \( \sigma = N^{-1} \) for a certain averaging width \( \Delta \) from a single realization of \( X(x) \) sampled at a resolution \( N \). We introduce the estimator
\[
\partial_u \hat{A}_X^\sigma(u, \xi_0/\sigma) = \sigma^2 \partial_u \hat{A}_X(u, \sigma).
\]

To compute an empirical estimator of the other partial derivative, \( \partial_\xi A_X^\sigma(u, \xi_0/\sigma) \), observe that
\[
\partial_\xi A_X^\sigma(u, \xi) = 2 \Re \left\{ \langle X(x), \psi_{\sigma}(x - u) e^{i \xi(x - u)} \rangle \langle X(x), \partial_\xi \psi_{\sigma}(x - u) e^{i \xi(x - u)} \rangle^* \right\}.
\]
Introducing a new wavelet
\begin{equation}
\psi^2(x) = x \psi^1(x) = x \psi(x) e^{i \xi_0 x},
\end{equation}
and \( \psi_{\sigma}(x) = \sigma^{-1} \psi^2(\sigma^{-1}(x - u)) \), this partial derivative can be rewritten, for \( \xi = \xi_0/\sigma \):
\[
\partial_\xi A_X^\sigma(u, \xi_0/\sigma) = 2 \sigma^3 \Im \left\{ \langle X, \psi_{\sigma} \rangle \langle X, \psi_{\sigma}^2 \rangle^* \right\}.
\]
Similarly to (41), for \( \sigma = N^{-1} \) we define the averaging kernel estimator
\begin{equation}
\partial_\xi \hat{A}_X^\sigma(u, \xi_0/\sigma) = 2 \sigma^3 N^{-1} \sum_{|k| N^{-1} \leq \Delta} g(u - k/N) \Im \left\{ \langle X, \psi_{k/N, \sigma}^1 \rangle \langle X, \psi_{k/N, \sigma}^2 \rangle^* \right\}.
\end{equation}
The following theorem proves that for \( \sigma = N^{-1} \) and an appropriate choice of the averaging interval \( \Delta \)
\[
\hat{\theta}'(u) = \frac{\partial_u \hat{A}_X^\sigma(u, N\xi_0)}{\partial_x \hat{A}_X^\sigma(u, N\xi_0)}
\]
is a weakly consistent estimator of \( \theta'(u) \) when \( N \to \infty \). The proof resides in the spatial decorrelation of wavelet coefficients of \( X \), and for the same reasons as in Theorem 4.1, we introduce a Gaussianity assumption on the underlying stationary process \( Y \).

**Theorem 4.2 Consistency, frequency modulation.** Let \( X(x) = Y(x) e^{i\theta(x)} \), where \( Y \) is a stationary Gaussian process such that there exists \( h > 0 \) with
\[
c_Y(0) - c_Y(x) = |x|^h \eta(x) \quad \text{and} \quad \eta(0) > 0 .
\]
Suppose that \( \psi^1(x) = \psi(x) e^{i\xi_0 x} \) is a compactly supported wavelet with \( p \geq |h| + 3 \) vanishing moments, such that
\[
\int \int |x - y|^h (x - y) \sin[\xi_0(x - y)] \psi(x) \psi(y) \, dx \, dy \neq 0 .
\]
Let \( \Delta = N^{-1/5} \). If \( \eta \in C^{2p} \) in a neighborhood of 0 and if \( \theta \in C^{2p} \), then for each \( u \in \mathbb{R} \)
\[
\Pr \left\{ \left| \hat{\theta}'(u) - \theta'(u) \right| \leq 2 (\log N) N^{-1/5} \right\} \xrightarrow{N \to \infty} 1 .
\]

The proof is in Appendix C.2. The numerical example in Figure 5 shows the estimation of a frequency modulation, with a \( \tau_{10} \) spline window \( \psi(x) \) defined in (28). We explained that the empirical estimator \( \hat{\theta}'(u) \) is in fact computed from wavelet coefficients associated to the two wavelets \( \psi^1 \) and \( \psi^2 \) defined in (46) and (47). Figure 5(a) shows a realization of a stationary signal \( Y(x) \) and the corresponding empirical scalogram \( \hat{A}_Y(u, s) = |\langle Y, \psi^1_s \rangle|^2 \). The frequency modulated signal \( X(x) = X(x) \exp(i\theta(x)) \) and its empirical scalogram are in Figure 5(7). The derivative \( \theta' \) of the frequency modulation is plotted in Figure 5(c) (full line). An estimate \( \hat{\theta}' \) of \( \theta' \) is obtained by integrating \( \hat{\theta}' \), and choosing the additive integration constant so that \( \int_0^1 \hat{\theta}' = \int_0^1 \theta' \). Figure 5(c) plots \( \hat{\theta}' \) (dashed line), superposed on the theoretical function \( \theta' \) (full line). Lastly, Figure 5(d) represents the stationarized process \( X(x) \exp(-i\hat{\theta}(x)) \) and its empirical scalogram.

4.3. Warping in higher dimension For a multidimensional warping, at each position \( u \), the deformation gradient corresponds to a set of \( d \) matrices \( \tilde{\gamma}_{h,m}(u) = (\gamma_{h,m}(u))_{1 \leq k \leq d} \) defined in (31). Theorem 3.3 shows that these coefficients appear in the velocity term of the transport equation (35) satisfied \( \tilde{\gamma} \) the warpsogram of \( X \):
\[
A_X(u, S) = E\{|\langle X, \psi_{u,s} \rangle|^2\} ,
\]
with
\[
\psi_{u,s}(x) = (\det S^{-1}) \psi(S^{-1}(x - u)) .
\]
If the process $X$ is measured at a resolution $N$, we compute the warpgram with functions $\psi_{u,S}$ whose support in any direction is larger than $N^{-1}$. We therefore require that $S = \sigma \hat{S}$ where $\sigma \geq KN^{-1}$, and all the eigenvalues of $\hat{S}$ are greater than $K^{-1}$. The position parameter is also restricted to a uniform grid $u = N^{-1} k$ with $k \in \mathbb{Z}^d$. For $a = u$ or $s_{i,j}$, we have $\partial_a A_X(u,S) = 2 \text{Re}\{\langle X, \psi_{u,S} \rangle \langle X, \partial_a \psi_{u,S} \rangle^* \}$, and we introduce the kernel estimator

$$
\widehat{\partial_a A_X}(u,S) = 2N^{-d} \sum_{|N^{-1} k - u| \leq \Delta} g_2(u - N^{-1} k) \text{Re}\{\langle X, \psi_{N^{-1} k,S} \rangle \langle X, \partial_a \psi_{N^{-1} k,S} \rangle^* \},
$$

where $g_2(u_1, u_2) = g(u_1) g(u_2)$ is the separable product of two window functions defined in (39).

We also define

$$
\left( \widehat{a_{i,m}(u,S)} \right)_{1 \leq i, m \leq d} = \left( \frac{\partial_a A_X(u,S)}{\partial_{a_{i,m}} A_X(u,S)} \right)_{1 \leq i, j \leq d} S^t.
$$

The $k^{th}$ component of the deformation gradient, $(\gamma^k_{i,m}(u))$, can then be estimated by

$$
\left( \begin{array}{c}
\gamma^k_{1,1}(u) \\
\vdots \\
\gamma^k_{d,d}(u)
\end{array} \right) = \left( \begin{array}{ccc}
\widehat{a_{1,1}(u,S_1)} & \ldots & \widehat{a_{d,d}(u,S_1)} \\
\vdots & \ddots & \vdots \\
\widehat{a_{1,1}(u,S_d)} & \ldots & \widehat{a_{d,d}(u,S_d)}
\end{array} \right)^{-1} \left( \begin{array}{c}
\partial_{a_k} A_X(u,S_1) \\
\vdots \\
\partial_{a_k} A_X(u,S_d)
\end{array} \right).
$$

Extending the consistency Theorem 4.1 to more than one dimension is possible, but requires technical hypotheses that are not yet well understood.

In Section 2.1, we mentioned that the warping of textures in images specify the three-dimensional shapes of the textured objects appearing in the scene. The estimator defined in (52) is used in [5, 6] to compute shape from texture, and provides a good estimation of three-dimensional surfaces from two-dimensional images.
A. Proofs of Section 2

A.1. Proof of Proposition 2.1 Let \( Y \) be a stationary process, and suppose that there exists an \( \varepsilon > 0 \) such that \( c_Y(x) > 0 \) for \( |x| < \varepsilon \). Let \( \tilde{Y} \) be another stationary process. We want to show that if the autocovariance kernels of \( Y(x) \exp[i\theta(x)] \) and of \( \tilde{Y}(x) \exp[i\tilde{\theta}(x)] \) are equal, i.e. if

\[
c_Y(x - y) \exp(i[\theta(x) - \theta(y)]) = c_{\tilde{Y}}(x - y) \exp(i[\tilde{\theta}(x) - \tilde{\theta}(y)]),
\]

then \( \theta''(x) = \tilde{\theta}''(x) \). The functions \( \theta \) and \( \tilde{\theta} \) are assumed \( C^1 \), therefore \( \mu = \theta - \tilde{\theta} \) is also \( C^1 \). Let us fix \( x \in \mathbb{R} \); our goal is to prove that \( \mu''(x) = 0 \). We choose \( y \in \mathbb{R} \) such that \(|x - y| < \varepsilon \). After dividing both sides of (53) \( \mathbb{T} \) by \( c_{\gamma}(x - y) > 0 \), it appears that \( e^{i[\mu(x) - \mu(y)]} \) is a function of \( x - y \). Therefore \( \mu(x) - \mu(y) \) is also a function of \( x - y \), and in particular for all \( z \),

\[
\mu(x) - \mu(y) = \mu(x + z) - \mu(y + z).
\]

Differentiating this expression with respect to \( x \) shows that \( \mu(x) = \mu'(x + z) \), thus \( \mu''(x) = 0 \).

A.2. Proof of Proposition 2.2 Let \( Y \) be a stationary process and let \( \varepsilon > 0 \) such that \( c_Y(x) \) is \( C^1 \) for \( 0 < |x| < \varepsilon \), with \( c_Y'(x) < 0 \). Let \( \tilde{Y} \) denote another stationary process, and let us suppose that the autocovariance kernels of \( Y(\theta(x)) \) and of \( \tilde{Y}(\tilde{\theta}(x)) \) are equal. The functions \( \theta \) and \( \tilde{\theta} \) are assumed \( C^3 \), therefore \( \mu = \theta \circ \tilde{\theta}^{-1} \) is also \( C^3 \). Proving the proposition amounts to proving that \( \mu \) is linear, or equivalently, that \( \mu'' \) vanishes everywhere. By definition of \( \mu \),

\[
c_{\gamma}(x - y) = c_Y(\mu(x) - \mu(y)).
\]

Let us fix \( x \in \mathbb{R} \), and choose \( y \neq x \), \( \varepsilon \) sufficiently close to \( x \) so that \(|\mu(x) - \mu(y)| < \varepsilon \). Differentiating (54) with respect to \( x \) and \( y \) shows that

\[
c_{\gamma}'(\mu(x) - \mu(y)) \mu'(y) = c_{\gamma}'(\mu(x) - \mu(y)) \mu'(x).
\]

Since \( c_{\gamma}'(\mu(x) - \mu(y)) < 0 \), we obtain \( \mu'(x) = \mu'(y) \), therefore \( \mu''(x) = 0 \).

A.3. Proof of Proposition 2.3 Let \( Y \) be a stationary process such that \( c_Y \) satisfies (11). Let \( \tilde{Y} \) denote another stationary process, and suppose that the autocovariance kernels of \( Y(\theta(x)) \) and of \( \tilde{Y}(\tilde{\theta}(x)) \) are equal. Let \( \mu = \theta \circ \tilde{\theta}^{-1} \); \( \mathbb{T} \) definition of \( \mu \),

\[
c_Y(\mu(x) - \mu(y)) = c_{\gamma}(x - y).
\]

Differentiating this expression with respect to \( x \) and \( y \), for \( x \neq y \), shows that

\[
\nabla c_Y(\mu(x) - \mu(y)) J_{\mu}(y) = \nabla c_Y(\mu(x) - \mu(y)) J_{\mu}(x).
\]
Let us fix \( x \in \mathbb{R}^d \), and prove that \( \nabla J_\mu(x) = 0 \). Let \( \varepsilon > 0 \) such that \( \eta(z) \) is \( C^2 \) for \( |z| < \varepsilon \). Let us choose \( y \in \mathbb{R}^d \) such that \( 0 < |\mu(x) - \mu(y)| < \varepsilon \) and let \( z = \mu(x) - \mu(y) \):

\[
\nabla_{cY} (z) = -|z|h^{-2} (h \eta(z) z + |z|^2 \nabla \eta(z)).
\]

Replacing this expression in (55) and dividing each sides by \(-h |z|h^{-2} \eta(z)\) proves that

\[
(z + h^{-1}|z|^2 \nabla \log \eta(z)) J_\mu(y) = (z + h^{-1}|z|^2 \nabla \log \eta(z)) J_\mu^{-1}(z + \mu(y)),
\]

so

\[
(z + h^{-1}|z|^2 \nabla \log \eta(z)) J_\mu(y) J_\mu^{-1}(z + \mu(y)) = z + h^{-1}|z|^2 \nabla \log \eta(z).
\]

Introducing a function \( \tilde{\mu} \) such that

\[
\tilde{\mu}(z) = J_\mu(y) \mu^{-1}(z + \mu(y)),
\]

this can be rewritten

\[
(z + h^{-1}|z|^2 \nabla \log \eta(z)) J_\tilde{\mu}(z) = (z + h^{-1}|z|^2 \nabla \log \eta(z)).
\]

Noticing that \( z J_\tilde{\mu}(\lambda z) = \frac{d}{d\lambda} \tilde{\mu}(\lambda z) \), we have, for \( \lambda \geq 0 \),

\[
\frac{d}{d\lambda} \tilde{\mu}(\lambda z) = z + h^{-1}|z|^2 \lambda \nabla \log \eta(\lambda z)(Id - J_\tilde{\mu}(\lambda z)),
\]

which, when integrated between \( \lambda = 0 \) and \( \lambda = 1 \), gives

\[
\tilde{\mu}(z) - \tilde{\mu}(0) = z + h^{-1}|z|^2 \int_0^1 \lambda \nabla \log \eta(\lambda z)(Id - J_\tilde{\mu}(\lambda z)) d\lambda.
\]

After replacing \( \tilde{\mu}(0) = J_\mu(y) \mu^{-1}(z + \mu(y)) \), we obtain

\[
\mu^{-1}(z + \mu(y)) = J_\mu^{-1}(y) z + y + J_\mu^{-1}(y) h^{-1}|z|^2 \int_0^1 \lambda \nabla \log \eta(\lambda z)(Id - J_\mu(\lambda z)) d\lambda.
\]

Since \( c_Y \) is even, \( \nabla \eta(0) = 0 \) and so \( \nabla \log \eta(0) = 0 \). Let us denote \( \nabla \log \eta(z) = |z| \bar{a}(z) \). Recalling that \( \eta \) is twice continuously differentiable in a neighborhood of \( z \) for \( 0 < |z| < \varepsilon \), the function \( \bar{a}(z) \) is differentiable for \( 0 < |z| < \varepsilon \), and its gradient (the matrix \( (\partial_j a_i)_{ij} \)) is uniformly bounded for \( 0 < |z| < \varepsilon \). Differentiating (57) with respect to \( z \) shows that

\[
J_\mu^{-1}(\mu^{-1}(z + \mu(y))) = J_\mu^{-1}(y) (Id + |z|^2 A(z)),
\]

where \( A(z) \) is a matrix defined by

\[
A(z) = |z|^{-2} \frac{\partial}{\partial z} \left( h^{-1}|z|^2 \int_0^1 \lambda |\lambda z| \bar{a}(\lambda z)(Id - J_\mu(\lambda z)) d\lambda \right).
\]

Let us calculate \( A(z) \) explicitly: \( |z|^2 A(z) \) is the sum of three terms:

\[
\frac{\partial}{\partial z} \left( h^{-1}|z|^2 \int_0^1 \lambda |\lambda z| \bar{a}(\lambda z)(Id - J_\mu(\lambda z)) d\lambda \right) = (I) + (II) + (III).
\]
If $a$ and $b$ are two vectors, we denote $a \otimes b$ the matrix whose elements are $(a_i b_j)_{ij}$. Then

\[
(I) = 2h^{-1} \left( \int_0^1 \lambda |\lambda z| \tilde{a}(\lambda z)(Id - J_\mu(\lambda z)) d\lambda \right) \otimes z
\]

(60)

\[
(II) = h^{-1} |z|^2 \int_0^1 |\lambda|^3 \left( \tilde{a}(\lambda z) \otimes \frac{\partial}{\partial \lambda} + |\lambda z| \tilde{V}_{\lambda}(\lambda z) \right)(Id - J_\mu(\lambda z)) d\lambda
\]

(III) = $-h^{-1} |z|^3 \int_0^1 |\lambda|^3 \tilde{a}(\lambda z) \tilde{V}_{\lambda}^T J_\mu(\lambda z) d\lambda$.

More precisely for (III), the $i, j$ element of $\tilde{a}(\lambda z) \tilde{V}_{\lambda} J_\mu(\lambda z)$ is $\sum_k a_k(\lambda z) \frac{\partial}{\partial \lambda} (J_\mu)_{i k}(\lambda z)$. Because $\tilde{\mu}$ is in $\mathcal{C}^2$, and $\mathcal{T}$ cause $a(\lambda z)$ as well as $\tilde{V} a(\lambda z)$ are uniformly bounded for $0 < |\lambda z| < \varepsilon$, the matrix $A(z)$ resulting from the division of (I) + (II) + (III) $\mathcal{T} y |z|^2$ is uniformly bounded for $0 < |z| < \varepsilon$.

Replacing $z \mathcal{T} y \mu(x) - \mu(y)$ in (58) gives

\[
J^{-1}_\mu(x) = J^{-1}_\mu(y) \left( Id + |\mu(x) - \mu(y)|^2 A(\mu(x) - \mu(y)) \right),
\]

therefore for any unit-length vector $x_k \in \mathbb{R}^d$,

\[
\frac{\partial}{\partial x_k} J^{-1}_\mu(x) = \lim_{\lambda \to 0} \frac{J^{-1}_\mu(x + \lambda x_k) - J^{-1}_\mu(x)}{\lambda}
\]

\[
= \lim_{\lambda \to 0} \frac{J^{-1}_\mu(x + \lambda x_k)|\mu(x) - \mu(x + \lambda x_k)|^2}{\lambda} A(\mu(x) - \mu(x + \lambda x_k))
\]

\[
= 0.
\]

This proves that $\tilde{V} J^{-1}_\mu(x) = 0$, and therefore $\tilde{V} J_\mu(x) = 0$. As a consequence, for each direction $x_k$,

\[
\frac{\partial}{\partial x_k} J_\mu(\theta(x)) = 0.
\]

Since $J_\mu(x) = J_\theta(\theta^{-1}(x)) J^{-1}_\theta(\theta^{-1}(x))$, we obtain

\[
\frac{\partial}{\partial x_k} (J_\theta(x) J^{-1}_\theta(x)) = 0,
\]

and expanding the above differential expression then gives

\[
\frac{\partial J_\theta(x)}{\partial x_k} J^{-1}_\theta(x) - J_\theta(x) J^{-1}_\theta(x) \frac{\partial J_\theta(x)}{\partial x_k} J^{-1}_\theta(x) = 0,
\]

which is equivalent to (9).

Let us prove the equivalence $\mathcal{T}$ between (8) and (9).

If $\theta$ and $\hat{\theta}$ satisfy (8), then $J_\theta(x) = S J_{\hat{\theta}}(x)$, which implies that $J_\theta(x) J^{-1}_{\hat{\theta}}(x)$ is independent of $x$.

The preceding calculations then prove (9).

Conversely, assuming that $\theta$ and $\hat{\theta}$ satisfy (9), then for each direction $x_k$, $\frac{\partial}{\partial x_k} (J_\theta(x) J^{-1}_{\hat{\theta}}(x)) = 0$,

which proves that $J_\theta(x) J^{-1}_{\hat{\theta}}(x)$ is a constant matrix, $\mathcal{T}$ elonging to $GL^+(\mathbb{R}^d)$ as the product of two elements of $GL^+(\mathbb{R}^d)$. The partial differential system $J_\theta(x) = S J_{\hat{\theta}}(x)$ can $\mathcal{T}$ be integrated to prove that $\theta(x) = u + S \hat{\theta}(x)$. 

A.4. Proof of Theorem 2.1. Let us consider a specific family of zero-mean wide-sense stationary processes defined $\mathcal{Y}$

$$Y_{\omega}(x) = Y e^{i \omega \cdot x}$$

where $Y$ is a zero-mean random variable with variance $\sigma^2$. Then

$$E\{Y_{\omega}(x) Y^*_\omega(y)\} = \sigma^2 \exp(i \omega \cdot (x - y)) = c_{Y_{\omega}}(x - y) .$$

Let $G \mathcal{T}$ be a stationarity invariant operator. If $X_{\omega}(x) = G Y_{\omega}(x)$ then

$$E\{X_{\omega}(x) X^*_\omega(y)\} = \sigma^2 f_{\omega}(x) f^*_\omega(y)$$

where $f_{\omega}(x) = G e^{i \omega \cdot x}$. But since $G$ is stationarity invariant, $X_{\omega}$ is stationary, therefore

$$E\{X_{\omega}(x) X^*_\omega(y)\}$$

is a function of $x - y$. This implies that for any $(x, y)$ the product $f_{\omega}(x) f^*_\omega(y)$ is a function of $x - y$. Therefore there exists $\hat{\rho}(\omega) \in \mathbb{C}$ and $\lambda(\omega) \in \mathbb{R}^d$ such that

$$f_{\omega}(x) = G e^{i \omega \cdot x} = \hat{\rho}(\omega) e^{i \lambda(\omega) \cdot x} .$$

Let us now prove that an operator which satisfies (61) is indeed stationarity invariant if and only if $\text{ess sup}_{\omega \in \mathbb{R}^d} |\hat{\rho}(\omega)| < \infty$. Let $Y \mathcal{T}$ be a zero-mean, stochastically continuous wide-sense stationary process. It therefore admits a spectral representation:

$$Y(x) = \int_{\mathbb{R}^d} e^{i \omega \cdot x} dZ(\omega) ,$$

where $Z(\omega)$ is an orthogonal process $[14]$. Let $dH(\omega) = E\{|dZ(\omega)|^2\}$, we have

$$c_{Y}(0) = \int_{\mathbb{R}^d} dH(\omega) < +\infty .$$

Since $\text{ess sup}_{\omega \in \mathbb{R}^d} |\hat{\rho}(\omega)| < \infty$ ,

$$\int_{\mathbb{R}^d} e^{i \lambda(\omega) \cdot x} \hat{\rho}(\omega) dZ(\omega)$$

is convergent in the mean-squared sense. Therefore $GY(x) = G \int_{\mathbb{R}^d} e^{i \omega \cdot x} dZ(\omega)$ is equal to $\int_{\mathbb{R}^d} e^{i \lambda(\omega) \cdot x} \hat{\rho}(\omega) dZ(\omega)$. This shows that $GY$ is wide-sense stationary, since

$$E\{GY(x) GY^*(y)\} = \int_{\mathbb{R}^d} e^{i \lambda(\omega) \cdot (x - y)} |\hat{\rho}(\omega)|^2 dH(\omega)$$

is a function of $x - y$ and $E\{|GY(x)|^2\} < \infty$. For any wide-sense stationary process $Y$, one can write

$$GY(x) = G E\{Y(0)\} + G (Y(x) - E\{Y(0)\}) .$$

Since $Y(x) - E\{Y(0)\}$ is zero-mean and wide-sense stationary, $G(Y(x) - E\{Y(0)\})$ is wide-sense stationary, therefore so is $GY(x)$.

In order for $G$ to $\mathcal{T}$ be stationary invariant, for any positive integrable measure $dH(\omega)$ one must have $\int_{\mathbb{R}^d} |\hat{\rho}(\omega)|^2 dH(\omega) < +\infty$. One can verify that a necessary and sufficient condition is that $\text{ess sup}_{\omega \in \mathbb{R}^d} |\hat{\rho}(\omega)| < \infty$.
A.5. Proof of Proposition 2.4 The autocovariance operator of a process $Z$ is defined by

$$K_Z f(x) = \int \mathbb{E}\{Z(x)Z^*(y)\} f(y) \, dy.$$ 

Let $Y$ be a stationary process, let $G$ be a bounded linear operator satisfying (15) and let $X = GY$. The autocovariance operators of $X$ and $Y$ satisfy

$$K_X = G K_Y G.$$ 

Since $Y$ is stationary, $K_Y$ commutes with the translation operator $T_v$ for any $v \in \mathbb{R}^d$. We derive from (15) that $K_X$ also commutes with $T_v$ and hence $X$ is wide-sense stationary. The operator $G$ is therefore stationarity invariant and Theorem 2.1 proves that

$$G e^{i\omega \cdot x} = \hat{\rho}(\omega) e^{i\lambda(\omega) \cdot x}.$$ 

Inserting this expression in the equality $G T_v f(x) = e^{iS_v} T_v G f(x)$ for $f(x) = e^{i\omega \cdot x}$ implies

$$\hat{\rho}(\omega) e^{i\lambda(\omega) \cdot x} e^{-iS_v \omega} = \hat{\rho}(\omega) e^{i\lambda(\omega) \cdot (x-v)} e^{-i\xi v},$$

from which we derive that $\lambda(\omega) = \bar{S} \omega + \xi$ for all $\omega$ where $\hat{\rho}(\omega) \neq 0$. For $\omega$ such that $\hat{\rho}(\omega) = 0$, (62) clearly holds with $\lambda(\omega) = \bar{S} \omega + \xi$. So $G$ can indeed be written as in (14).

Conversely, if $G$ satisfies (14) then a direct calculation shows that (15) holds.

B. Proofs of Section 3

B.1. Proof of Proposition 3.1 (Transport) The autocovariance operator of $X = DY$ satisfies $K_X = D K_Y D$, therefore

$$\langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle = \langle K_Y \overline{D} \psi_{v,\alpha,\sigma}, \overline{D} \psi_{v,\alpha,\sigma} \rangle.$$ 

Let us compute

$$\nabla_v \langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle = 2 \Re \langle K_Y \overline{D} \psi_{v,\alpha,\sigma}, \nabla_v \overline{D} \psi_{v,\alpha,\sigma} \rangle$$

$$= 2 \Re \langle K_Y \overline{D} \psi_{v,\alpha,\sigma}, (\nabla_v + \nabla_x) \overline{D} \psi_{v,\alpha,\sigma} \rangle - 2 \Re \langle K_Y \overline{D} \psi_{v,\alpha,\sigma}, \nabla_x \overline{D} \psi_{v,\alpha,\sigma} \rangle.$$ 

Since $Y$ is stationary, for any $g$ we have $\langle K_Y g, \nabla_x g \rangle = 0$, so

$$\nabla_v \langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle = 2 \Re \langle K_X \psi_{v,\alpha,\sigma}, \overline{D}^{-1} (\nabla_v + \nabla_x) \overline{D} \psi_{v,\alpha,\sigma} \rangle.$$ 

Hypothesis (16) thus implies that

$$|\nabla_v \langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle| = O(\sigma) \left| \Re \langle K_X \psi_{v,\alpha,\sigma}, \nabla_x \psi_{v,\alpha,\sigma} \rangle \right|.$$ 

Since $\psi_{v,\alpha,\sigma} = T_{\Phi(v)} T_{\alpha} \psi_{v}$, the transport property (17) shows that

$$\psi_{v,\alpha,\sigma} = e^{i\phi(v)} T_{\alpha} \mathcal{T}_{\gamma(v)} \psi_{v} = e^{i\phi(v)} T_{\alpha} \mathcal{T}_{\gamma(v)} \psi_{v} = e^{i\phi(v)} T_{\alpha \gamma(v)} \psi_{v}.$$ 

The phase $e^{i\phi(u)}$ disappears from $\langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle$. Indeed,

$$\langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle = \langle K_X T_{u(v)} \hat{F}_{\alpha*\gamma(u(v))} T_{u(v)} \hat{F}_{\alpha*\gamma(u(v))} \rangle = A_X^\sigma(u(v), \alpha*\gamma(u(v)))$$

\(\nabla_y \) definition of \(A_X^\sigma\). Since \(\nabla_x f = \nabla_u f J_v^{-1}(u)\),

$$\nabla_x \langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle = \nabla_u A_X^\sigma(u(v), \alpha*\gamma(u(v))) = \nabla_u A_X^\sigma(u(v), \alpha*\gamma(u(v))) J_v^{-1}(u)$$

This implies that

$$\left| \nabla_u A_X^\sigma(u(v), \alpha*\gamma(u(v))) \right| \leq \| J_v(u) \| \left| \nabla_v \langle K_X \psi_{v,\alpha,\sigma}, \psi_{v,\alpha,\sigma} \rangle \right|,$$

where \(\| J_v(u) \|\) is the operator sup norm of \(J_v(u)\). Using (63) shows that for \(u\) fixed

(64)

$$\left| \nabla_u A_X^\sigma(u, \alpha*\gamma(u)) \right| = O(\sigma) \left| \text{Re} \langle K_X \psi_{v,\alpha,\sigma}, \nabla_x \psi_{v,\alpha,\sigma} \rangle \right|.$$

Note that the gradient with respect to \(u\) on the left-hand side of the above expression involves partial derivatives of \(A_X^\sigma\) with respect to \(u\) and \(\alpha\) since the variable \(u\) appears in \(\alpha*\gamma(u)\).

Since \(\nabla_x T_{u} f(x) = -\nabla_x T_{u} f(x)\), using the symmetry of \(K_X\) we get

(65)

$$2\text{Re} \langle K_X \psi_{v,\alpha,\sigma}, \nabla_x \psi_{v,\alpha,\sigma} \rangle = -\nabla_x A_X^\sigma(u, \alpha*\gamma(t)) \text{ at } t = u.$$

Inserting (65) in (64) finally proves that

$$\left| \nabla_u A_X^\sigma(u, \alpha*\gamma(u)) \right| = O(\sigma) \left| \nabla_u A_X^\sigma(u, \alpha*\gamma(t)) \right| \text{ at } t = u,$$

which implies (18).

B.2. Proof of Theorem 3.1 (Scale Transport). This theorem is proved as a consequence of Proposition 3.1. Operator \(G_{\beta(v)}\) is given \(\nabla_y\) (21)

$$G_{\beta(v)} f(x) = \theta'(u) f(v + \theta'(u)(x - u))$$

where \(u = \theta^{-1}(v)\) is a differentiable invertible map. Notice that

$$G_{\beta(v)} T_v = T_{u(v)} \hat{F}_{\alpha(u(v))},$$

with \(\alpha(u) = 1/\theta'(u)\), therefore transport property (17) holds.

Let us now verify hypothesis (16) concerning

$$\psi_{v,\alpha,\sigma} = G_{\beta(v)} T_v \hat{F}_{\alpha} \psi_{\sigma}$$

with \(T_s f(x) = 1/s f(x/s)\). The scalogram renormalization (20) is equivalent to dividing \(\psi_{\sigma}(x)\) \(\nabla_y\) \(\sigma\), which yields \(\psi_{\sigma}(x) = 1/\sigma \psi(x/\sigma)\), and replacing \(\sigma\) \(\nabla_y\) \(s\) which gives

$$\psi_{v,\alpha,\sigma}(x) = \varphi_{v,\alpha}(x) = \frac{\theta'(u)}{s} \psi \left( \frac{\theta'(u)}{s}(x - u) \right).$$
If we can prove that

\[ |Re\{K_X \varphi_{v, s}, D^{-1}(\partial_v + \partial_u) \overline{\varphi_{v, s}}\}| = O(s)|\partial_v A_X(u, s)| \]

then Proposition 3.1 can be applied: we obtain a transport equation (19) with \( \alpha = s, \gamma(u) = 1/\theta'(u) \) and \( \alpha * \gamma(t) = s/\theta'(t) \):

\[
\left| \partial_v A_X(u, s) - s \theta'(u) \theta'(t) \theta'(u)^2 \partial_u A_X(u, s) \right| = O(s) \left| \partial_v A_X(u, s) \right| \text{ at } t = u,
\]

which proves (24). There now only remains to prove (66).

Let us compute

\[
\varphi_{v, s} = D^{-1}(\partial_v + \partial_u) \overline{\varphi_{v, s}}.
\]

Since \( Df(x) = (\theta'(\theta^{-1}(x)))^{-1} f(\theta^{-1}(x)) \) and \( D^{-1}f(x) = \theta'(x)f(\theta(x)) \), a direct calculation gives

\[
D^{-1}\partial_x \overline{\varphi_{v, s}}(x) = -\frac{\theta'(u)}{s \theta'(x)^2} \psi \left( \frac{\theta'(u)}{s} (x - u) \right) + \frac{\theta'(u)^2}{s^2 \theta'(x)} \psi' \left( \frac{\theta'(u)}{s} (x - u) \right)
\]

and

\[
D^{-1}\partial_u \overline{\varphi_{v, s}}(x) = \frac{\theta'(u)}{s \theta'(u)} \psi \left( \frac{\theta'(u)}{s} (x - u) \right) + \frac{1}{s^2} (x - u) \theta'(u) - \theta'(u) \psi' \left( \frac{\theta'(u)}{s} (x - u) \right),
\]

therefore

\[
\varphi_{v, s}(x) = \frac{1}{s \theta'(u)} \left( \theta''(u) - \left( \frac{\theta'(u)}{\theta'(x)} \right)^2 \theta''(x) \right) \psi \left( \frac{\theta'(u)}{s} (x - u) \right)
\]

\[
+ \frac{1}{s^2} \left( \frac{\theta'(u)}{\theta'(x)} \left( \theta'(u) - \theta'(x) \right) - (u - x) \theta'(u) \right) \psi' \left( \frac{\theta'(u)}{s} (x - u) \right).
\]

Since \( \psi \) is supported in \([-1, 1]\), \( \varphi_{v, s} \) is supported in \([u - s/\theta'(u), u + s/\theta'(u)]\). Since \( \theta \in C^3 \), Taylor series expansions of \( \theta'(x) \) and of \( \theta''(x) \) around position \( u \) prove that, for \( x \) close to position \( u \),

\[
\left| \varphi_{v, s}(x) - \left( 2 \frac{\theta''(u)}{\theta'(u)^2} - \frac{\theta'''(u)}{\theta'(u)^2} \right) \frac{\theta'(u)(x - u)}{s} \psi \left( \frac{\theta'(u)}{s} (x - u) \right) \right.
\]

\[
\left. + \frac{1}{2} \left( \frac{\theta'(u)(x - u)}{s} \right)^2 \psi' \left( \frac{\theta'(u)}{s} (x - u) \right) \right| = O(x - u).
\]

The autocovariance kernel of \( X(x) \) is \( c_X(x, y) = c_Y(\theta(x) - \theta(y)) \), hence

\[
\langle K_X \varphi_{v, s}, \varphi_{v, s} \rangle = \int \int c_Y(\theta(x) - \theta(y)) \varphi_{v, s}(x) \varphi_{v, s}(y) dx dy.
\]

Since \( \int \varphi_{v, s}(x) dx = \int \psi(x) dx = 0 \),

\[
\langle K_X \varphi_{v, s}, \varphi_{v, s} \rangle = - \int \left( c_Y(0) - c_Y(\theta(x) - \theta(y)) \right) \varphi_{v, s}(x) \varphi_{v, s}(y) dx dy.
\]
The supports of $\varphi_{v,s}$ and $\phi_{v,s}$ are in $[u - s/\theta'(u), u + s/\theta'(u)]$ and for $z$ in a neighborhood of 0, the continuity of $\eta$ implies that $c_V(0) - c_Y(z) = \eta(0) |z|^h + o(|z|^h)$. Since $\theta'$ is continuous at $u$, a Taylor series expansion of $\theta$ around $u$ combined with a change of variables $x' = (x - u)\theta'(u)/s$ and $y' = (y - u)\theta'(u)/s$ yield, for $s$ sufficiently small,

\begin{equation}
\langle K_X \varphi_{v,s}, \phi_{v,s} \rangle + \int \int \eta(0) |s(x' - y')|^{h}\psi^*(x') \left( 2 \frac{\theta'(u)^2}{\theta'(u)^3} - \frac{\theta''(u)}{\theta'(u)^2} \right) \times \\
\times \left[ y' \psi(y') + \frac{y^2}{2} \psi'(y') \right] \frac{s}{\theta'(u)} dx' dy' = o(s^{h+1})
\end{equation}

and therefore

\begin{equation}
|\text{Re} \langle K_X \varphi_{v,s}, \phi_{v,s} \rangle | = O(s^{h+1}).
\end{equation}

Let us now compute

\begin{equation}
\langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle = \int \int c_Y(\theta(x) - \theta(y)) \varphi^*_{v,s}(x) \frac{d}{dy} \varphi_{v,s}(y) dx dy.
\end{equation}

With an integration by parts,

\begin{equation}
\langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle = \int \int \theta'(y) c_Y(\theta(x) - \theta(y)) \varphi^*_{v,s}(x) \varphi_{v,s}(y) dx dy,
\end{equation}

and since $c_Y'(z)$ is antisymmetric

\begin{equation}
\text{Re} \langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle = -\frac{1}{2} \int \int \left( \theta'(x) - \theta'(y) \right) c_Y(\theta(x) - \theta(y)) \text{Re} \left( \varphi^*_{v,s}(x) \varphi_{v,s}(y) \right) dx dy.
\end{equation}

A change of variable $x' = (x - u)\theta'(u)/s$ and $y' = (y - u)\theta'(u)/s$ gives

\begin{equation}
\text{Re} \langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle = -\frac{1}{2} \int \int \left( \theta'(u + sx'/\theta'(u)) - \theta'(u + sy'/\theta'(u)) \right) c_Y \left( \theta(u + sx'/\theta'(u)) - \theta(u + sy'/\theta'(u)) \right) \text{Re} \left( \psi^*(x') \psi(y') \right) dx' dy'.
\end{equation}

Because of assumption (22), since $\eta$ is $C^1$ in a neighborhood of 0, $c_Y'(z) = h \eta(0) \text{sign}(z) |z|^{h-1} + o(|z|^{h-1})$. With a Taylor expansion for $\theta$, we get, for $s$ small enough,

\begin{equation}
\text{Re} \langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle + \frac{s}{2} \frac{\theta'(u)}{\theta(u)} \int \int h \eta(0) s^h |x - y|^h \text{Re} \left( \psi^*(x) \psi(y) \right) dx dy = o(s^h).
\end{equation}

Since $\text{Re} \int |x - y|^h \psi^*(x) \psi(y) dx dy \neq 0$ and $\theta'(u) \neq 0$, there exists $a(u) > 0$ such that

\begin{equation}
|\text{Re} \langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle | \geq a(u) s^h + o(s^h),
\end{equation}

and (68) implies that

\begin{equation}
|\text{Re} \langle K_X \varphi_{v,s}, \phi_{v,s} \rangle | = O(s) |\text{Re} \langle K_X \varphi_{v,s}, \partial_x \varphi_{v,s} \rangle |.
\end{equation}

We have therefore proved (66).
B.3. *Proof of Theorem 3.2 (Frequency Transport).* This theorem is proved as a consequence of Proposition 3.1. The operator $G_{\beta(v)}$ is defined in (25) $T_y$:

$$
(70) \quad G_{\beta(v)} f(x) = e^{i(\theta(v) + \theta'(v)(x - v)} f(x).
$$

Let $u(v) = v$, which is clearly a differentiable and invertible map. We have

$$
(71) \quad G_{\beta(v)} T_v = e^{i(\theta(u))} T_{u(v)} \overline{f}_v
$$

with $\overline{f}_v f(x) = e^{-2\pi x} f(x)$ and $\alpha(u) = -\theta(u)$, therefore transport property (17) holds.

Let us now verify hypothesis (16):

$$
(72) \quad \psi_{\nu, \xi, \sigma}(x) = G_{\beta(v)} T_v \overline{f}_v \psi_{\sigma}(x) =
$$

$$
\exp[i(\theta(v) + \theta'(v)(x - v))] \exp[-i\xi(x - v)] \psi \left( \frac{x - v}{\sigma} \right).
$$

A direct calculation shows that

$$
(73) \quad \left< K \psi_{\nu, \xi, \sigma}, \overline{D}^{-1} (\partial_v + \partial_x) \overline{D} \psi_{\nu, \xi, \sigma} \right> =
$$

$$
\sigma^2 \int c_Y (\sigma(x' - y')) \exp[i(\theta(v + \sigma x') - \theta(v + \sigma y') - \theta'(v)(x' - y'))] \times
$$

$$
\times i(\theta'(v + \sigma y') - \theta'(v) - \theta''(v)(x' - y')) \exp[i \xi(x' - y')] \psi(x') \psi(y') dx' dy',
$$

and with a change of variables $x' = (x - v)/\sigma$ and $y' = (y - v)/\sigma$, introducing $\xi_0 = \sigma \xi$,

$$
(74) \quad \sigma^2 \int c_Y (\sigma(x' - y')) \exp[i(\theta(v + \sigma x') - \theta(v + \sigma y') - \theta'(v)(x' - y'))] \times
$$

$$
\times i(\theta'(v + \sigma y') - \theta'(v) - \theta''(v)(x' - y')) \exp[i \xi_0(x' - y')] \psi(x') \psi(y') dx' dy' = 0.
$$

The function $\psi$ is real, therefore $\hat{\psi}$ is even, and since it vanishes at $\xi_0$, it also vanishes at $-\xi_0$. Therefore

$$
(74) \quad \sigma^2 \int c_Y (0) i(\theta(v + \sigma y) - \theta'(v) - \theta''(v)(\sigma y) \exp[i \xi_0(x - y)] \psi(x) \psi(y) dx dy = 0.
$$
After subtracting (74) from (73),

\[
\langle K_X \psi_{v,\xi,\sigma}, \overline{D^{\tau - \Gamma}} (\partial_v + \partial_x) \overline{D} \psi_{v,\xi,\sigma} \rangle =
\]

\[
\sigma^2 \int c_Y(\sigma(x - y)) (\exp[i(\theta(v + \sigma x) - \theta(v + \sigma y) - \sigma \theta'(v)(x - y))] - 1) \times
\]

\[
\times i(\theta'(v + \sigma y) - \theta'(v) - \theta'(v)\sigma y) e^{i \xi_0(x - y)} \psi(x) \psi(y) \, dx \, dy +
\]

\[
+ \sigma^2 \int (c_Y(\sigma(x - y)) - c_Y(0)) i(\theta'(v + \sigma y) - \theta'(v) - \theta'(v)\sigma y) e^{i \xi_0(x - y)} \psi(x) \psi(y) \, dx \, dy .
\]

Since \( \theta \in C^{4+[h]} \), we can perform the following Taylor expansions, where \( a_k, b_k \) and \( c_k \) are real parameters which depend on the derivatives \( \theta^{(k)}(v) \), for \((x, y) \in [0,1]^2\):

\[
\exp[i(\theta(v + \sigma x) - \theta(v + \sigma y) - \sigma \theta'(v)(x - y))] =
\]

\[
1 + i \sum_{k=2}^{2+}[h] a_k \sigma^k (x - y)^k + \sum_{k=4}^{2+}[h] b_k \sigma^k (x - y)^k + O(\sigma^{3+[h]})
\]

\[
\theta'(v + \sigma y) - \theta'(v) = \sum_{k=1}^{2+}[h] c_k \sigma^k y^k + O(\sigma^{3+[h]}) .
\]

In particular, \( a_2 = \theta''(v)/2 \) and \( c_2 = \theta^{(2)}(v)/(k - 1)! \). Replacing these Taylor expansions in (75), we obtain

\[
\langle K_X \psi_{v,\xi,\sigma}, \overline{D^{\tau - \Gamma}} (\partial_v + \partial_x) \overline{D} \psi_{v,\xi,\sigma} \rangle =
\]

\[
- \sigma^2 \int \int c_Y(\sigma(x - y)) \left[ i \sum_{k=2}^{2+}[h] a_k \sigma^k (x - y)^k + \sum_{k=4}^{2+}[h] b_k \sigma^k (x - y)^k \right] \times
\]

\[
\times \sum_{k=2}^{2+}[h] c_k \sigma^k y^k e^{i \xi_0(x - y)} \psi(x) \psi(y) \, dx \, dy
\]

\[
- i\sigma^2 \int \sigma^h |x - y|^h \eta(\sigma(x - y)) \left[ \sum_{k=2}^{2+}[h] c_k \sigma^k y^k \right] e^{i \xi_0(x - y)} \psi(x) \psi(y) \, dx \, dy
\]

\[
+ O(\sigma^{5+[h]}) .
\]

In the first of these two integrals, one can replace \( c_Y(\sigma(x - y)) \) \( \overline{\tau} c_Y(0) - \sigma^h |x - y|^h \eta(\sigma(x - y)) \).

Since \( \psi \) and its first \([h] + 2\) derivatives vanish at \( \xi_0 \), we derive that \( e^{i \xi_0(t)} \psi(t) \) is a function with \([h] + 3\) vanishing moments [11], so the first integral is of the order of \( O(\sigma^{\theta+h}) \).

Consider the real part of the second integral; since \( \eta \) is even, exchanging \( x \) and \( y \) shows that

\[
\int \int |x - y|^h \eta(\sigma(x - y)) y^2 \sin(\xi_0(y - x)) \psi(x) \psi(y) \, dx \, dy = 0 .
\]
Since $2 + \lceil h \rceil \geq 3$, the real part of the second integral is equal to

$$
\sigma^{5 + h} \tilde{\eta}(0) \frac{\theta^{(4)}(v)}{6} \int \sin(\xi_o(x - y)) |x - y|^h y^3 \psi(x) \psi(y) \, dx \, dy + o(\sigma^{5 + h}) .
$$

As a consequence,

$$
(78) \quad \Re \left\{ K_{X \psi_v, \xi, \sigma} \tilde{D}^{-1} (\partial_v + \partial_x) \tilde{D} \psi_v, \xi, \sigma \right\} = 
\sigma^{5 + h} \tilde{\eta}(0) \frac{\theta^{(4)}(v)}{6} \int \sin(\xi_o(x - y)) |x - y|^h y^3 \psi(x) \psi(y) \, dx \, dy + o(\sigma^{5 + h}) .
$$

Let us now compute the leading term of $|\Re \left\{ K_{X \psi_v, \xi, \sigma}, \partial_x \right\}|$. After a change of variables,

$$
\langle K_{X \psi_v, \xi, \sigma}, \partial_x \rangle = 
\sigma^2 \iint c_Y(\sigma(x - y)) \exp[i(\theta(v + \sigma x) - \theta(v + \sigma y) - \sigma \theta'(v)(x - y))] \times
\exp[i\xi_o(x - y)] i(\partial'(v) + \xi_o/\sigma) \psi(x) \psi(y) \, dx \, dy +
\sigma^2 \iint c_Y(\sigma(x - y)) \exp[i(\theta(v + \sigma x) - \theta(v + \sigma y) - \sigma \theta'(v)(x - y))] \times
\exp[i\xi_o(x - y)] \psi(x) \frac{1}{\sigma} \psi'(y) \, dx \, dy .
$$

Using Taylor expansions (76) and (77), we obtain

$$
(79) \quad \langle K_{X \psi_v, \xi, \sigma}, \partial_x \rangle =
\sigma^2 \iint c_Y(\sigma(x - y)) \left[ 1 + i \sum_{k=2}^{2 + \lceil h \rceil} a_k \sigma^k (x - y)^k + \sum_{k=4}^{2 + \lceil h \rceil} b_k \sigma^k (x - y)^k \right] \times
\exp[i\xi_o(x - y)] i(\partial'(v) + \xi_o/\sigma) \psi(x) \psi(y) \, dx \, dy
\sigma^2 \iint c_Y(\sigma(x - y)) \left[ 1 + i \sum_{k=2}^{2 + \lceil h \rceil} a_k \sigma^k (x - y)^k + \sum_{k=4}^{2 + \lceil h \rceil} b_k \sigma^k (x - y)^k \right] \times
\exp[i\xi_o(x - y)] \psi(x) \frac{1}{\sigma} \psi'(y) \, dx \, dy + O(\sigma^{4 + \lceil h \rceil}) .
$$

Exchanging $x$ and $y$ shows that

$$
\iint c_Y(\sigma(x - y)) \sin[\xi_o(y - x)] \psi(x) \psi(y) \, dx \, dy = 0 ;
$$

and since $\psi$ is even and $\psi'$ is odd, changing $x$ to $-x$ and $y$ to $-y$ shows that

$$
\iint c_Y(\sigma(x - y)) \cos[\xi_o(y - x)] \psi(x) \psi'(y) \, dx \, dy = 0 .
$$
Writing $c_{\gamma}(\sigma(x-y)) = c_{\gamma}(0) - \sigma^h|x-y|^h\eta(\sigma(x-y))$, and noticing that $e^{i\phi t} \psi(t)$ is a function with $[h] + 3$ vanishing moments, the first integral in (79) has a real part equal to

$$\sigma^{3+h} \frac{\eta'(v)}{2} \int |x - y|^{2+h} \xi_0 \cos[\xi_0(y - x)] \psi(x) \psi(y) dx dy + o(\sigma^{3+h}) .$$

Because $\psi$ is even, the second integral in (79) has a real part equal to

$$-\sigma^{3+h} \frac{\eta'(v)}{2} \int |x - y|^{2+h} \sin[\xi_0(y - x)] \psi(x) \psi'(y) dx dy + o(\sigma^{1+h}) .$$

An integration by parts with respect to $y$ shows that

$$\int \int |x - y|^{2+h} \sin[\xi_0(y - x)] \psi(x) \psi'(y) dx dy =$$

$$-\xi_0 \int \int |x - y|^{2+h} \cos[\xi_0(y - x)] \psi(x) \psi(y) dx dy +$$

$$+ (2 + h) \int \int |x - y|^{1+h} \text{sign}(x - y) \sin[\xi_0(y - x)] \psi(x) \psi(y) dx dy .$$

Summing up the two contributions, we see that

$$\sigma^{3+h} \eta(0) (1 + h/2) \theta''(v) \int |x - y|^h(x - y) \sin[\xi_0(x - y)] \psi(x) \psi(y) dx dy + o(\sigma^{3+h}) .$$

Because of the hypothesis that

$$\int \int |x - y|^h(x - y) \sin[\xi_0(x - y)] \psi(x) \psi(y) dx dy \neq 0 ,$$

comparing (80) and (78) proves a result which is stronger than (16), because the right-hand side has an $O(\sigma^2)$ instead of $O(\sigma)$:

$$|\text{Re} \left< K_X \psi_{v,\xi,\sigma} , \partial_x + \partial_x \right| \text{Re} \left< K_X \psi_{v,\xi,\sigma} , \partial_x \psi_{v,\xi,\sigma} \right> | = O(\sigma^2) |\text{Re} \left< K_X \psi_{v,\xi,\sigma} , \partial_x \psi_{v,\xi,\sigma} \right> | .$$

Adapting Proposition 3.1 to account for the $O(\sigma^2)$ term, we obtain a transport equation (19) with $\alpha = \xi$, $\gamma(u) = -\theta'(u)$ and $\alpha_1 + \alpha_2 = \alpha_1 + \alpha_2$: for $u$ such that $\theta'(u) \neq 0$,

$$|\partial_x A^\alpha_X(u,\xi) - \theta'(u) \partial_x A^\alpha_X(u,\xi)| = O(\sigma^2) |\partial_x A^\alpha_X(u,\xi)| ,$$

which proves (27).

### B.4. Proof of Theorem 3.3 (Multidimensional Scale Transport)

The proof of this theorem follows the same lines as the proof of Theorem 3.1. The hypotheses of Proposition 3.1 are verified in order to apply (19) in $d$ dimensions.

Let us verify hypothesis (16) concerning

$$\psi_{v,\xi,\sigma} = \overrightarrow{\partial \beta(v)} T_{\xi} \overrightarrow{\partial S} \psi_{\sigma}$$
with $F_S f(x) = \det S^{-1} f(S^{-1} x)$ and where $G_{\beta(v)}$ has $T$een defined in (29). Note that the transport property (17) clearly holds. The warprogram renormalization (30) is equivalent to dividing $\psi_S(x) \, \bar{T}y \, \sigma^d$ and replacing $\sigma S \, \bar{T}y \, S$. We replace $\psi_{S, \sigma}(x) \, \bar{T}y$

$$\varphi_{S}(x) = \det(S^{-1} J_\theta(u)) \psi(S^{-1} J_\theta(u)(x - u)) .$$

Let us define the vector of functions

$$\bar{\varphi}_{S} = \bar{D}^{-1}(\bar{\nabla}_v + \bar{\nabla}_x)D \varphi_{v,S} .$$

We now prove that for any fixed $u$ and $\bar{S}$ such that $\det \bar{S} = 1$, if $S = \sigma \bar{S}$ then

$$\left| \Re \langle K_X \varphi_{S} , \bar{\varphi}_{S} \rangle \right| = O(\sigma) \left| \Re \langle K_X \varphi_{S} , \bar{\nabla}_x \varphi_{S} \rangle \right| .$$

Let us first compute an upper bound for $\left| \Re \langle K_X \varphi_{S} , \bar{\varphi}_{S} \rangle \right|$. Since

$$\bar{D}^{-1} f(x) = \det(J_\theta(x)) f(\theta(x))$$

and

$$\bar{D} f(x) = \det(J_\theta^{-1}(\theta^{-1}(x))) f(\theta^{-1}(x)) ,$$

we have

$$\bar{D}^{-1} \bar{\nabla}_x \bar{D} \varphi_{v,S}(x) = \sigma^{-d} \left[ - \frac{\det J_\theta(u)}{\det J_\theta(x)} \nabla \det J_\theta(x) J_\theta^{-1}(x) \psi(S^{-1} J_\theta(u)(x - u)) \right. \left. + \det J_\theta(u) \nabla \psi(S^{-1} J_\theta(u)(x - u)) S^{-1} J_\theta(u) J_\theta^{-1}(x) \right]$$

and

$$\bar{D}^{-1} \nabla_v \bar{D} \varphi_{v,S}(x) = \sigma^{-d} \left[ \nabla \det J_\theta(u) \psi(S^{-1} J_\theta(u)(x - u)) J_\theta^{-1}(u) \right. \left. + \det J_\theta(u) \nabla \psi(S^{-1} J_\theta(u)(x - u)) S^{-1} (\nabla J_\theta(u)(x - u) - J_\theta(u)) J_\theta^{-1}(u) \right] .$$

After summing these two expressions, a Taylor expansion of $\det J_\theta$, $J_\theta^{-1}$ and of $\nabla \det J_\theta$ in the vicinity of position $u$ shows that for $S = \sigma \bar{S}$ and $\sigma$ small, there exists $C(u, \bar{S})$ such that

$$| \varphi_{v,S} | \leq C(u, \bar{S}) \sigma^{1-d} .$$

By definition of $K_X$,

$$\langle K_X \varphi_{v,S} , \bar{\varphi}_{v,S} \rangle = \iint c_Y(\theta(x) - \theta(y)) \varphi_{v,S}(x) \bar{\varphi}_{v,S}(y) \, dx \, dy .$$

The wavelet $\psi$ has one vanishing moment, so $\int \varphi_{v,S}(x) \, dx = 0$, and therefore

$$\langle K_X \varphi_{v,S} , \bar{\varphi}_{v,S} \rangle = \iint [c_Y(\theta(x) - \theta(y)) - c_Y(0)] \varphi_{v,S}(x) \bar{\varphi}_{v,S}(y) \, dx \, dy ,$$
which implies that
\[
|\langle K_X \varphi_v, S, \tilde{\varphi}_v, S \rangle \rangle | \leq \int \int |c_Y(\theta(x) - \theta(y)) - c_Y(0)| |\varphi_v, S(x)| |\tilde{\varphi}_v, S(y)| \, dx \, dy.
\]
Substituting (83) and (81) in the above inequality, and using condition (33) on \( c_Y \), after a change of variable and a Taylor expansion of \( \theta \) around \( u \), we obtain
\[
|\langle K_X \varphi_v, S, \tilde{\varphi}_v, S \rangle \rangle | = O(\sigma^{h+1}).
\]
To prove (82), we now show that there exists \( C'(u, \tilde{S}) > 0 \) such that
\[
|\text{Re}\langle K_X \varphi_v, S, \tilde{\nabla}_x \varphi_v, S \rangle \rangle | \geq C'(u, \tilde{S}) \sigma^h.
\]
With an integration by parts,
\[
\langle K_X \varphi_v, S, \tilde{\nabla}_x \varphi_v, S \rangle = \int \int \tilde{\nabla}_x c_Y(\theta(x) - \theta(y)) J_\theta(y) \psi_v^*(x) \psi_v S(y) \, dx \, dy
\]
and using the fact that \( \tilde{\nabla}_x c_Y(x) \) is antisymmetric,
\[
\langle K_X \varphi_v, S, \tilde{\nabla}_x \varphi_v, S \rangle = -\frac{1}{2} \int \int \tilde{\nabla}_x c_Y(\theta(x) - \theta(y)) (J_\theta(x) - J_\theta(y)) \text{Re} (\psi_v^*(x) \psi_v S(y)) \, dx \, dy.
\]
Therefore
\[
\text{Re}\langle K_X \varphi_v, S, \tilde{\nabla}_x \varphi_v, S \rangle = \frac{1}{2} \int \int \tilde{\nabla}_x c_Y(S(x-y)) \tilde{\nabla}_x J_\theta(u) J_\theta^{-1}(u) S(x-y) \text{Re} (\psi^*(x) \psi(y)) \, dx \, dy
\]
\[
= -\frac{1}{2} \int \int \left( \tilde{\nabla}_x c_Y(\theta(u) + J_\theta^{-1}(u) Sx) - \theta(u + J_\theta^{-1}(u) Sy) \right) - \tilde{\nabla}_x c_Y(S(x-y)) \times
\]
\[
\times (J_\theta(u) + J_\theta^{-1}(u) Sx) - J_\theta(u + J_\theta^{-1}(u) Sy)) \text{Re} (\psi^*(x) \psi(y)) \, dx \, dy
\]
\[
- \frac{1}{2} \int \int \tilde{\nabla}_x c_Y(S(x-y)) (J_\theta(u) + J_\theta^{-1}(u) Sx) - J_\theta(u + J_\theta^{-1}(u) Sy) - \tilde{\nabla}_x J_\theta(u) J_\theta^{-1}(u) S(x-y)) \times
\]
\[
\times \text{Re} (\psi^*(x) \psi(y)) \, dx \, dy
\]
Because \( \tilde{\nabla}_x c_Y \) is \( C^1 \) in a neighborhood of 0 excluding 0, for small \( \sigma \), second-order Taylor series expansions for \( \theta \) and for \( J_\theta \) around position \( u \) prove that
\[
\text{Re}\langle K_X \varphi_v, S, \tilde{\nabla}_x \varphi_v, S \rangle = \frac{1}{2} \int \int \tilde{\nabla}_x c_Y(S(x-y)) \tilde{\nabla}_x J_\theta(u) J_\theta^{-1}(u) S(x-y) \text{Re} (\psi^*(x) \psi(y)) \, dx \, dy = o(\sigma^h)
\]
Hypothesis (34) guarantees that (84) holds, and therefore (82) is satisfied. Now that conditions (16) and (17) of Proposition 3.1 have been verified, the resulting transport equation (19) can be applied, with \( \alpha = S, \gamma(u) = J_\theta^{-1}(u) \) and \( S_1 * S_2 = S_2 S_1 \). This yields:
\[
|\tilde{\nabla}_u A_X(u, S) + [J_\theta(u)^{-1} \tilde{\nabla}_u J_\theta(u) S] \cdot \tilde{\nabla}_x A_X(u, S)| = O(\sigma) \left| \tilde{\nabla}_u A_X(u, S) \right|.
\]
The final result (35) is derived from this equation by noting that
\[
[J_\theta^{-1}(u) \tilde{\nabla}_u J_\theta(u) S] \cdot \tilde{\nabla}_x A_X(u, S) = \left[ J_\theta^{-1}(u) \tilde{\nabla}_u J_\theta(u) \right] \cdot \left[ \tilde{\nabla}_x A_X(u, S) S^t \right].
\]
B.5. Proof that (36) implies (34) for a separable warping function Using the fact that \( \theta \) is assumed separable, we obtain that \( J_\theta(u)^{-1} \tilde{\nabla}_u J_\theta(u) S(x - y) \) is a diagonal matrix whose \( i \)th element along the diagonal is

\[
\sigma \frac{\theta'_i(u)}{\theta_i'(u_i)} (\tilde{S}(x - y))_i .
\]

Using (33), the leading term of \( \tilde{\nabla}_C Y(S(x - y)) \) when \( \sigma \to 0 \) is

\[
\eta(0) h \sigma^{-1} (\tilde{S}(x - y))|\tilde{S}(x - y)|^{h-2} .
\]

Therefore, when \( \sigma \to 0 \), the \( i \)th component of

\[
Re \int \int \tilde{\nabla}_C Y(S(x - y)) \tilde{\nabla}_u J_\theta(u)^{-1}(u) S(x - y) \psi^*(x) \psi(y) \, dx \, dy
\]

is equivalent to

\[
\eta(0)h \sigma^{-1} \frac{\theta_i'(u_i)}{\theta_i'(u)} Re \int \int |\tilde{S}(x - y)|^{h-2} (\tilde{S}(x - y))_i^2 \psi^*(x) \psi(y) \, dx \, dy .
\]

Using the fact that \( (\sum_{i=1}^d |k_i|^2)^{1/2} \geq \frac{1}{\sqrt{d}} \sum_{i=1}^d |k_i| \), we obtain

\[
Re \int \int \tilde{\nabla}_C Y(S(x - y)) \tilde{\nabla}_u J_\theta(u)^{-1}(u) S(x - y) \psi^*(x) \psi(y) \, dx \, dy \geq \eta(0) h \sigma^{-1} \min_i \left| \frac{\theta_i'(u_i)}{\theta_i'(u)} \right| Re \int \int |\tilde{S}(x - y)|^h \psi^*(x) \psi(y) \, dx \, dy
\]

This shows that if

\[
Re \int \int |\tilde{S}(x - y)|^h \psi^*(x) \psi(y) \, dx \, dy \neq 0
\]

and if none of the \( \theta_i' \) vanish, then (34) is satisfied with \( C(u, \tilde{S}) > 0 \).

C. Proofs of Section 4

C.1. Proof of Theorem 4.1 With a slight modification of the proof of Theorem 3.1, one can prove a stronger result than (24), which is stated in the following lemma:

**Lemma C.1.** Under the hypotheses of Theorem 3.1,

\[
\partial_a A_X(u, s) - (\log \theta)'(u) \partial_{\log a} A_X(u, s) = s(C(u) + o(1)) \partial_a A_X(u, s) ,
\]

where \( C \) is continuous.

Let \( \partial \) be a generic variable denoting either \( u \) or \( s \), and let us introduce

\[
\overline{\partial_a A_X(u, s)} = \int g(u - v) \partial_a A_X(v, s) \, dv .
\]
If \( u \) is such that \( \theta'''(u) \neq 0 \) and \( \Delta \) small enough, then \( \partial_u A_X(v, s) \) keeps a constant sign over \([u - \Delta, u + \Delta]\). Therefore, by continuity of \( C \), convolving the right-hand side of (85) with \( g \) gives

\[
O(s) \partial_u A_X(u, s) .
\]

Convolving the left-hand side of (85) with \( g \) gives:

\[
\partial_u \overline{A_X(u, s)} - \int g(u - v) (\log \theta')'(v) \partial_{\log^* A X}(v, s) \, dv
\]

The hypotheses of Theorem 3.1 imply that \( \partial_{\log^* A X}(u, s) \) does not vanish. By continuity, \( \partial_{\log^* A X}(v, s) \) therefore keeps a constant sign for \( v \) in \([u - \Delta, u + \Delta]\). Because \( (\log \theta')'' \) is bounded over \([u - \Delta, u + \Delta]\),

\[
\int g(u - v) (\log \theta')'(v) \partial_{\log^* A X}(v, s) \, dv = (\log \theta')'(u) \partial_{\log^* A X}(u, s) + O(\Delta) \partial_{\log^* A X}(u, s) .
\]

Regrouping the left and right-hand sides we obtain

\[
(87) \quad \partial_u \overline{A_X(u, s)} - (\log \theta')'(u) \partial_{\log^* A X}(u, s) = O(s) \partial_u \overline{A_X(u, s)} + O(\Delta) \partial_{\log^* A X}(u, s) ,
\]

which can be viewed as an averaged transport equation.

The following lemma, whose proof is in Appendix C.4, shows that the two estimators \( \partial_u \overline{A_X(u, s)} \) and \( \partial_{\log^* A X}(u, s) \) are consistent.

**Lemma C.2.** Let \( X, Y \) and \( \psi \) satisfy the hypotheses of Theorem 4.1. For each \( u \), for \( s \) small enough,

\[
\begin{align*}
\text{Pro}^T \left\{ \left| \partial_{\log^* A X}(u, s) - \partial_{\log^* A X}(u, s) \right| \geq C \left| \partial_{\log^* A X}(u, s) \right| \right\} & \leq \varepsilon_1 , \\
\text{Pro}^T \left\{ \left| \partial_u \overline{A_X(u, s)} - \partial_u \overline{A_X(u, s)} \right| \geq C \left| \partial_u \overline{A_X(u, s)} \right| \right\} & \leq \varepsilon_2 ,
\end{align*}
\]

where \( C = \frac{\log(N \Delta)}{2 \sqrt{N \Delta}} \), \( \varepsilon_1 = \frac{C_1(u) \Delta^2}{(\log(N \Delta))^2} \) and \( \varepsilon_2 = 6 (N \Delta)^{-1/2} C_2(u) \).

The parameters \( C_1(u) \) and \( C_2(u) \), which are defined in the proof of the lemma, are \( \text{T} \)-th order positive. The weak consistency of

\[
\frac{\partial_u \overline{A_X(u, N^{-1})}}{\partial_{\log^* A X(u, N^{-1})}}
\]

as an estimator of \( (\log \theta')'(u) \) then results from the following lemma, whose proof is straightforward:

**Lemma C.3.** If \( X_1 \) and \( X_2 \) are two random variables, and \( C < 1 \) a constant such that

\[
\text{Pro}^T \{ |X_1 - \mathbb{E}\{X_1\}| \leq C |\mathbb{E}\{X_1\}| \} \geq 1 - \varepsilon_1 ,
\]

\[
\text{Pro}^T \{ |X_2 - \mathbb{E}\{X_2\}| \leq C |\mathbb{E}\{X_2\}| \} \geq 1 - \varepsilon_2 ,
\]

\[
\text{Pro}^T \{ |X_1 - \mathbb{E}\{X_2\}| \leq C |\mathbb{E}\{X_2\}| \} \geq 1 - \varepsilon_3 ,
\]

where \( \varepsilon_3 = 2 \varepsilon_1 + \varepsilon_2 \).
then

\[
\Pr^\mathbb{F} \left( \left| \frac{X_2}{X_1} - \frac{E\{X_2\}}{E\{X_1\}} \right| \leq \frac{2C}{1-C} \left| \frac{E\{X_2\}}{E\{X_1\}} \right| \right) \geq 1 - \varepsilon_1 - \varepsilon_2 .
\]

In view of Lemma C.2, one can apply Lemma C.3 to \( X_1 = \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} \) and \( X_2 = \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} \) with \( C = \log \frac{\Delta \sqrt{\Delta}}{\sqrt{\Delta}} \) yielding

\[
\Pr^\mathbb{F} \left( \left| \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} - \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} \right| \leq \frac{2\log(N\Delta)}{\Delta \sqrt{\Delta} - \log(N\Delta)} \left| \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} \right| \right) \geq 1 - \varepsilon_1 - \varepsilon_2 .
\]

Because of the averaged transport equation (57),

\[
(\log \theta)'(u) = O(\Delta) + \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} (1 + O(N^{-1})).
\]

Since \( \Delta > N^{-1} \) and \( (\log \theta)'(u) \) is Tounded, we derive

\[
(\log \theta)'(u) = O(\Delta) + \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})}
\]

therefore

\[
\Pr^\mathbb{F} \left( \left| \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} - (\log \theta)'(u) \right| \leq \frac{2\log(N\Delta)}{\Delta \sqrt{\Delta} - \log(N\Delta)} (\log \theta)'(u) + O(\Delta) \right) \geq 1 - \varepsilon_1 - \varepsilon_2 .
\]

We choose \( \Delta \) such that \( \Delta^{-1}(N\Delta)^{-1/2} = \Delta \), i.e. \( \Delta = N^{-1/5} \). When \( N \to \infty \), \( \varepsilon_1 \) and \( \varepsilon_2 \), whose expressions are given in Lemma C.2, Toth tend to 0. Moreover, for \( N \) large enough,

\[
\frac{|(\log \theta)'(u)|}{2 \log(N\Delta)} \leq \frac{2\log(N\Delta)}{\Delta \sqrt{\Delta} - \log(N\Delta)} + O(\Delta) \leq 2(\log N) N^{-1/5} .
\]

Therefore

\[
\lim_{N \to \infty} \Pr^\mathbb{F} \left( \left| \frac{\partial_x A_X(u, N^{-1})}{\partial_x A_X(u, N^{-1})} - (\log \theta)'(u) \right| \leq 2(\log N) N^{-1/5} \right) = 1 .
\]

C.2. Proof of Theorem 4.2

**Lemma C.4.** Under the hypotheses of Theorem 3.2,

\[
\partial_u A^z_X(u, \xi_0/\sigma) - \theta''(u) \partial_{\xi} A^z_X(u, \xi_0/\sigma) = \sigma^2 (C(u) + o(1)) \partial_{\xi} A^z_X(u, \xi_0/\sigma) ,
\]

where \( C \) is continuous.

**Proof of Lemma C.4.** The proof mimicks the proof of Lemma C.1. In the proof of Theorem 3.2, we showed in (78) that

\[
Re \left\langle K^\mathbb{F}_{\psi_x, \xi, \sigma}, D^{-1} (\partial_v + \partial_x) D^\mathbb{F}_{\psi_x, \xi, \sigma} \right\rangle = \sigma^{3/4} (A(v) + o(1)) ,
\]
and in (80) that

\begin{equation}
(92) \quad Re \left\langle K_X \psi_{\nu, \xi, \sigma}, \partial_u \psi_{\nu, \xi, \sigma} \right\rangle = \sigma^{1+h} (B(v) + o(1)) ,
\end{equation}

with \( B(v) \) continuous. Comparing (91) and (92) shows that

\begin{equation}
Re \left\langle K_X \psi_{\nu, \xi, \sigma}, \overline{D^{-1}} (\partial_u + \partial_v) D\psi_{\nu, \xi, \sigma} \right\rangle = \sigma^2 (C(v) + o(1)) Re \left\langle K_X \psi_{\nu, \xi, \sigma}, \partial_u \psi_{\nu, \xi, \sigma} \right\rangle ,
\end{equation}

with \( C(v) \) continuous. This implies, \( \forall \gamma \) repeating the argument of Lemma C.1, that (90) is satisfied.

Using Lemma C.4 it is easy to see that

\begin{equation}
(93) \quad \overline{\partial_u A^X_X(u, \xi_0 / \sigma)} - \overline{\partial_k A^X_X(u, \xi_0 / \sigma)} = O(\sigma^2) \overline{\partial_u A^X_X(u, \xi_0 / \sigma)} + O(\Delta) \overline{\partial_k A^X_X(u, \xi_0 / \sigma)} .
\end{equation}

As in the proof of Theorem 4.1, one can complete the following lemma with Lemma C.3 to prove the weak consistency result (50).

**Lemma C.5.** Let \( X, Y \) and \( \psi \) satisfy the hypotheses of Theorem 4.2. Then for each \( u \), for \( N \) large enough,

\begin{align*}
Pr \left\{ |\overline{\partial_k A^X_X(u, N \xi_0)} - \overline{\partial_k A^X_X(u, N \xi_0)}| \geq C |\overline{\partial_k A^X_X(u, N \xi_0)}| \right\} \leq \varepsilon_1 , \\
Pr \left\{ |\overline{\partial_u A^X_X(u, N \xi_0)} - \overline{\partial_u A^X_X(u, N \xi_0)}| \geq C |\overline{\partial_u A^X_X(u, N \xi_0)}| \right\} \leq \varepsilon_2 ,
\end{align*}

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are defined in Lemma C.2.

The proof of Lemma C.5 is almost identical to the one of Lemma C.2; the only difference is that \( \psi^2 \) has \( p - 1 \) vanishing moments instead of \( p \), so that Lemma C.7 must \( \forall \varepsilon \) replaced with the following lemma, which is proved \( \forall \varepsilon \) using the same method.

**Lemma C.6.** Let \( Y(x) = X(x) e^{\theta(x)} \), let \( W_k = \langle X, \psi^1_{k/N, \sigma} \rangle \) and \( Z_k = \langle X, \psi^2_{k/N, \sigma} \rangle \). Under the hypotheses of Lemma C.5, for \( \sigma \) small enough, there exist two continuous functions \( M_1 \) and \( M_2 \) such that for \( |k - l| \leq 2 \),

\begin{align*}
|E \{ W_k W_l^* \} | & \leq M_1 (\sigma k) \sigma^h , \\
|E \{ W_k Z_l^* \} | & \leq M_1 (\sigma k) \sigma^h , \\
|E \{ Z_k Z_l^* \} | & \leq M_1 (\sigma k) \sigma^h ,
\end{align*}
and for $|k - l| > 2$,

\[
\begin{align*}
|E \{ W_k W_l^* \} | & \leq M_2(\sigma k) \frac{\sigma^{2p}}{\sigma(|k - l| - 2)^{2p-h}}, \\
|E \{ W_k Z_l^* \} | & \leq M_2(\sigma k) \frac{\sigma^{2p-1}}{\sigma(|k - l| - 2)^{2p-1-h}}, \\
|E \{ Z_k Z_l^* \} | & \leq M_2(\sigma k) \frac{\sigma^{2p-2}}{\sigma(|k - l| - 2)^{2p-2-h}}.
\end{align*}
\]

Since $p \geq \lceil h \rceil + 3$, we have $2(2p - 2 - h) > 1$, therefore the variance term

\[
E\{ [\partial_\xi A^\sigma_X(u, \xi) - \partial_\xi A^\sigma_X(u, \xi)]^2 \}
\]

can be controlled as in the proof of Lemma C.2 (cf Appendix C.4).

C.3. Proof of Lemma C.1  In one dimension, the proof of Proposition 3.1 can be adapted to show that, if (16) is replaced by

\[
Re\{ K_X \psi_v, \tilde{\beta}, \sigma \} \big( \partial_v + \partial_s \big) \overline{D^\sigma} \psi_v, \tilde{\beta}, \sigma \} = c(u, \sigma) Re\{ K_X \psi_v, \tilde{\beta}, \sigma \} \partial_v \psi_v, \tilde{\beta}, \sigma \},
\]

and if (17) holds, then the resulting transport equation (19) is replaced by

\[
\partial_v A^\sigma_X(u, \alpha) + \partial_t (\alpha * \gamma^{-1}(u) * \gamma(t)) \partial_v A^\sigma_X(u, \alpha) = c(u, \sigma) \partial_v A^\sigma_X(u, \alpha).
\]

Recall that in the proof of Theorem 3.1, (67) proves that

\[
Re\{ K_X \varphi_{v, \tau} , \varphi_{v, \nu} \} = \frac{\sigma^h}{\vartheta(u)} \big( B(u) + o(1) \big),
\]

where $B$ is continuous. On the other hand, (69) proves that

\[
Re\{ K_X \varphi_{v, \nu} , \varphi_{v, \nu} \} = \frac{\sigma^h}{\vartheta(u)} \big( B(u) + o(1) \big) \eta(0) \int \int |x - y|^h \psi^*(x) \psi(y) dx dy,
\]

where $\vartheta(u)/\vartheta(u)$ is continuous in $u$.

Comparing (95) and (96), and recalling that

\[
\begin{align*}
\psi_{v, \tau} & = \psi_v, \tilde{\beta}, \sigma \\
\phi_{v, \nu} & = \overline{D^\sigma} (\partial_v + \partial_s) \overline{D^\sigma} \psi_v, \tilde{\beta}, \sigma
\end{align*}
\]

we see that (94) holds, with

\[
c(u, \sigma) = s(C(u) + o(1))
\]

and $C$ continuous. This proves that (85) is indeed satisfied.
C.A. Proof of Lemma C.2 We start by proving (88). Let $n = N\Delta$ and let us choose $u = 0$
without loss of generality. We seek an upper bound for the variance of $\overline{\log_s A_X(0, s)}$
$$V_{\log s} = E\left\{ |\overline{\log_s A_X(0, s)} - \overline{\log_s A_X(0, s)}|^2 \right\}. $$
One can see that
$$\left| \frac{\partial^2}{\partial u \partial \log s} A_X(u, s) \right| = O(s^h)$$
and a Riemann series approximation shows that
$$\int g(v) \overline{\log_s A_X(v, s)} \, dv - N^{-1} \sum_{k=-n}^{n} g(k/N) \overline{\log_s A_X(k/N, s)} = O(s^h/N).$$
Replacing $\overline{\log_s A_X(0, s)}$ by its expression (41), and noticing that the real part is smaller than the
modulus, we obtain
$$V_{\log s} \leq \frac{4}{N^2} E\left\{ \sum_{|k| \leq n} g_k W_k Z_k - g_k E\{W_k Z_k\} \right\}^2 + O(s^{2h}/N^2)$$
where $g_k$, $W_k$ and $Z_k$ respectively denote $g(k/n)$, $(X, \psi_k/N, \nu)$ and $(X, \overline{\log_s \psi_k/N, \nu})^\ast$. Expanding
$|\sum_{|k| \leq n}|^2$ into $\left(\sum_{|k| \leq n}\right) \cdot \left(\sum_{|l| \leq n}\right)^\ast$,
$$V_{\log s} \leq \frac{4}{N^2} E\left\{ \sum_{|k| \leq n} g_k W_k Z_k - g_k E\{W_k Z_k\} \right\} \sum_{|l| \leq n} |g_l W_l Z_l - g_l E\{W_l Z_l\}|^\ast + O(s^{2h}/N^2)$$
$$\leq \frac{4}{N^2} \sum_{|k| \leq n} \sum_{|l| \leq n} |g_k g_l E\{W_k Z_k W_l^\ast Z_l^\ast\} - g_k g_l E\{W_k Z_k E\{W_l^\ast Z_l^\ast\}\}| + O(s^{2h}/N^2).$$
Since $Y$ is Gaussian, so is $X$, as well as the random variables $W_k$ and $Z_k$. Hence
$$E\{W_k Z_k W_l^\ast Z_l^\ast\} = E\{W_k Z_k\} E\{W_l^\ast Z_l^\ast\} + E\{W_k W_l^\ast\} E\{Z_k Z_l^\ast\} + E\{W_k Z_l^\ast\} E\{Z_k W_l^\ast\}.$$
Therefore
$$V_{\log s} \leq \frac{4}{N^2} \sum_{|k| \leq n |l| \leq n} g_k g_l |E\{W_k W_l^\ast\} E\{Z_k Z_l^\ast\}| + E\{W_k Z_l^\ast\} E\{Z_k W_l^\ast\}| + O(s^{2h}/N^2)$$
$$\leq \frac{4}{n^2} \sum_{|k| \leq n |l| \leq n} |E\{W_k W_l^\ast\}| |E\{Z_k Z_l^\ast\}| + |E\{W_k Z_l^\ast\}| |E\{Z_k W_l^\ast\}| + O(s^{2h}/N^2).$$
Each of the terms appearing in the sum above is now $O$bounded thanks to the following decorrelation lemma:
lemma C.7. Let \( X(x) = Y(\theta(x)) \), let \( W_k = \langle X, \psi_{k/N, s} \rangle \) and \( Z_k = \langle X, \partial_\log * \psi_{k/N, s} \rangle \). Under the hypotheses of Theorem 4.1, for \( s \) small enough, there exist continuous functions \( M_1 \) and \( M_2 \) such that, for \( |k - l| \leq 2 \),

\[
(98a) \quad |E \{ W_k W_l^* \} | \leq M_1(s) s^h,
\]

\[
(98\upu) \quad |E \{ W_k Z_l^* \} | \leq M_1(s) s^h,
\]

\[
(98c) \quad |E \{ Z_k Z_l^* \} | \leq M_1(s) s^h,
\]

and for \( |k - l| > 2 \),

\[
(99a) \quad |E \{ W_k W_l^* \} | \leq M_2(s) \frac{s^{2p}}{(s(|k - l| - 2))^{2p-h}},
\]

\[
(99\upu) \quad |E \{ W_k Z_l^* \} | \leq M_2(s) \frac{s^{2p}}{(s(|k - l| - 2))^{2p-h}},
\]

\[
(99c) \quad |E \{ Z_k Z_l^* \} | \leq M_2(s) \frac{s^{2p}}{(s(|k - l| - 2))^{2p-h}}.
\]

The proof of the above lemma is in Appendix C.5.

Replacing (98) and (99) in (97), we see that, since \( M_1 \) and \( M_2 \) are continuous and since \( k/N = \Delta \to 0 \) when \( N \to \infty \),

\[
(100) \quad V_{\log s} \leq \frac{4}{n^2} \sum_{|k-l| \leq 2} \frac{n^2}{|k|} \frac{2(M_1(0)^2 + o(1)) s^{2h}}{s^h} + \frac{4}{n^2} \sum_{|k-l| > 2} \frac{n^2}{|k|} \frac{2(M_2(0)^2 + o(1)) s^{4p}}{(s(|k - l| - 2))^{4p-2h}} + O(s^{2h}/N^2).
\]

Since \( 4p - 2h > 1 \),

\[
(101) \quad \sum_{|k-l| > 2} \frac{n^2}{|k|} = K_p n.
\]

Replacing (101) in (100), we obtain

\[
(102) \quad V_{\log s} \leq 8C^2 \frac{s^{2h}}{n} (3M_1(0)^2 + K_p M_2(0)^2) + o(s^{2h}/n).
\]

In the proof of Theorem 3.1, (69) proves that there exists \( a(u) > 0 \) such that

\[
|\partial_u A_X(u, s)| \geq a(u) s^h + o(s^h).
\]

For \( \Delta \) small enough, \( \partial_u A_X(v, s) \) does not change sign for \( |v| \leq \Delta \) thus, after convolution with \( g \),

\[
|\partial_u A_X(0, s)| \geq A_s s^h + o(s^h).
\]

Because of (87), the same applies to \( \partial_\log * A_X(0, s) \), therefore there exists a constant \( C_1 \) such that

\[
V_{\log s} \leq C_1 \left[ \frac{|\partial_\log * A_X(0, s)|}{\sqrt{n}} \right]^2.
\]
Applying Chebyshev’s Lemma [4] then proves that, for all \( \varepsilon > 0 \),
\[
\Pr \left\{ \left| \partial_{\log A} A_X(0, s) - \frac{\partial_{\log A} A_X(0, s)}{\sqrt{n}} \right| \geq \frac{\sqrt{C_1 \varepsilon}}{\varepsilon \sqrt{n}} \right\} \leq \varepsilon^2,
\]
and (88) follows by choosing \( \varepsilon = \frac{\sqrt{C_1 \Delta}}{\log n} \) and \( \varepsilon_1 = \frac{C_1 \Delta^2}{\sqrt{\log n}} \).

Let us now prove (89). We denote \( D_u = |\partial_u A_X(0, s) - \partial_u A_X(s)| \), Recall that
\[
\partial_u A_X(0, s) = \int_{-\Delta}^{\Delta} g(-v) \partial_u A_X(v, s) \, ds.
\]
An integration by parts shows that
\[
\partial_u A_X(0, s) = \frac{1}{\Delta^2} \left[ \int_{0}^{\Delta} A_X(v, s) \, dv - \int_{-\Delta}^{0} A_X(v, s) \, dv \right]
\]
and since \( |\partial_u A_X(u, s)| = O(s^h) \), with a Riemann series approximation,
\[
\Delta^{-2} \int_{0}^{\Delta} A_X(v, s) \, dv - \Delta^{-1} n^{-1} \sum_{k=1}^{n} A_X(k/N, s) = O(s^h / n).
\]
On the other hand, an integration by parts also shows that \( \partial_u A_X(0, s) \) defined in (40) satisfies
\[
\partial_u A_X(0, s) = \frac{1}{n \Delta^2} \sum_{0 < k/N - \Delta \leq \Delta} \sum_{-\Delta \leq k/N - \Delta \leq 0} \langle X, \psi_{k/N} \rangle^2 + O(s^h / N).
\]
Therefore, using once again the notation \( W_k = \langle X, \psi_{k/N} \rangle \),
\[
D_u = \frac{1}{n \Delta} \sum_{k=1}^{n} (|W_k|^2 - \mathbb{E}(|W_k|^2)) - \sum_{k=-n+1}^{0} (|W_k|^2 - \mathbb{E}(|W_k|^2)) + O(s^h / n).
\]
Denoting \( \tilde{W}^- = \sum_{k=-n+1}^{0} |W_k|^2 \) and \( \tilde{W}^+ = \sum_{k=1}^{n} |W_k|^2 \), we have
\[
D_u \leq \frac{1}{n \Delta} \left( |\tilde{W}^+ - \mathbb{E}(\tilde{W}^+)| + |\tilde{W}^- - \mathbb{E}(\tilde{W}^-)| \right) + O(s^h / n).
\]
We are now going to prove that there exists a strictly positive constant \( C_2 \) such that
\[
\forall y, \quad \Pr \{ D_u > y C_2 \frac{s^h}{\Delta \sqrt{n}} \} \leq 6 e^{-y^2 / 2}
\]
and since \( |\partial_u A_X(0, s)| \geq A s^h + o(s^h) \) with \( A > 0 \), choosing \( y = \log n / C_2 \) will then imply (89).

Let us consider the random vector \( W = (W_1, W_2, \ldots, W_n) \), let \( K_W \) denote the covariance operator of \( W \), and \( (\epsilon_j)_{j=1, \ldots, n} \) its Karhunen-Loève Tasis. If \( (\alpha_j)_{j=1, \ldots, n} \) are the eigenvalues of \( K_W \) corresponding to the eigenvectors \( (\epsilon_j)_{j=1, \ldots, n} \), then
\[
W = \sum_{j=1}^{n} \sqrt{\alpha_j} \tilde{W}_j \epsilon_j
\]
where \( \tilde{W}_j \) are independent random variables with variance 1. As a consequence,
\[
\tilde{W}^+ = \| W \|^2 = \sum_{j=1}^{n} \alpha_j \tilde{W}_j^2.
\]
The following lemma, which is proved in [8], relies on a theorem by Bakirov [3].

**Lemma C.8.** If \( \widetilde{W} = \sum_j \beta_j \widetilde{W}_j^2 \) where \( \widetilde{W}_j \) are independent Gaussian random variables with variance one, and \( \sum_j \beta_j^2 = 1 \), then
\[
\forall y, \quad \text{Pro}^\ast\{ |\widetilde{W} - \mathbb{E}\{\widetilde{W}\} | > y \} \leq 6 e^{-y/2} .
\]

The random variable \( \widetilde{W}^+ = (\sum_j \alpha_j^2)^{-1/2} \widetilde{W}^+ \) satisfies the requirements of Lemma C.8, therefore
\[
\forall y, \quad \text{Pro}^\ast\{ |\widetilde{W}^+ - \mathbb{E}\{\widetilde{W}^+\} | > y (\sum_j \alpha_j^2)^{1/2} \} \leq 6 e^{-y/2}
\]

Thus \( \sum_j \alpha_j^2 \) is equal to the Hilbert-Schmidt norm of \( K_W \):
\[
\sum_j \alpha_j^2 = \sum_{j,k} \mathbb{E}\{W_j W_k^*\},
\]
which is bounded by \( B s^{2h/n} \) because of (98a) and (99a). Hence
\[
\forall y, \quad \text{Pro}^\ast\{ |\widetilde{W}^+ - \mathbb{E}\{\widetilde{W}^+\} | > y \sqrt{B s^h \sqrt{n}} \} \leq 6 e^{-y/2} .
\]
The same applies to \( \widetilde{W}^- \), and \( \text{Pro}^\ast \) containing the two and using (103) we attain (104).

**C.5. Proof of Lemma C.7** The three terms \( \mathbb{E}\{W_k W_k^*\} \), \( \mathbb{E}\{W_k Z_k^*\} \) and \( \mathbb{E}\{Z_k Z_k^*\} \) can be written as
\[
I = \iint c_Y(\theta(u+s)+x) - \theta(v+s)y) \psi(x) \psi(y) dx dy,
\]
where \((u,v) = (sk,sl)\), and \( \psi \) and \( \bar{\psi} \) are two wavelets with \( p \) vanishing moments. Clearly,
\[
(105) \quad I = \iint [c_Y(\theta(u+s)+x) - \theta(v+s)y)] \psi(x) \bar{\psi}(y) dx dy .
\]

For \( |u-v| \leq \Delta, |x| \leq 1 \) and \( |y| \leq 1 \), we have
\[
|\theta(u+s)-\theta(v+s)| \leq (\Delta+2s) \sup_{|x-v| \leq \Delta+2s} |\theta'(x)| \leq (\Delta+2s) C_u
\]

because \( \theta \) is continuously differentiable. For \( \Delta \) small enough, \( |\theta(u+s)-\theta(v+s)| \) is therefore in a neighborhood of 0. Since \( \eta \) is assumed continuous in a neighborhood of 0,
\[
|\eta(\theta(u+s)-\theta(v+s))| \leq B
\]

for \( |u-v| \leq \Delta, |x| \leq 1 \) and \( |y| \leq 1 \).

Hence
\[
|I| \leq \iint |\theta(u+s)-\theta(v+s)|^h B |\psi(x)| |\bar{\psi}(y)| dx dy
\]
\[
\leq C(u) s^h
\]
where $C(u)$ is continuous. This proves (98a), (98c) and (98c).

Let us now prove (99). Since $\psi$ and $\tilde{\psi}$ in (105) are compactly supported and have $p$ vanishing moments, there exist two compactly supported functions $\beta$ and $\tilde{\beta}$ such that $\psi(x) = \beta^{(\varphi)}(x)$ and $\tilde{\psi}(y) = \tilde{\beta}^{(\varphi)}(y)$. Integrating (105) $T_\alpha$ parts $p$ times with respect to $x$ and to $y$ gives

$$I = \int \int \frac{\partial^p}{\partial x^p} \frac{\partial^p}{\partial y^p} \{ |\theta(u + sx) - \theta(v + sy)|^h \eta(\theta(u + sx) - \theta(v + sy)) \} \beta(x) \tilde{\beta}(y) \, dx \, dy.$$ But for $|u - v| > 2s$, one can show that

$$\left| \frac{\partial^p}{\partial x^p} \frac{\partial^p}{\partial y^p} \{ |\theta(u + sx) - \theta(v + sy)|^h \eta(\theta(u + sx) - \theta(v + sy)) \} \right| \leq \frac{M(u)}{|u - v| - 2s} s^{2p} \left( |u - v| - 2s \right)^{2p - \kappa} ,$$

where $M(u)$ depends on $h$, on derivatives of $\theta$ up to order $2p$ in a neighborhood of $u$, and on derivatives of $\eta$ up to order $2p$ in a neighborhood of 0. Therefore there exists a continuous $M_2(u)$ such that

$$|I| \leq M_2(u) \frac{s^{2p}}{(s(|u - v| - 2))^{2p - \kappa}} ,$$

which proves (99a), (99c), and (99c).
REFERENCES


CERMICS,
Ecole Nationale des Ponts et Chaussées
INRIA, 06992 Sophia-Antipolis, France.
Maureen.Clerc@cermics.enpc.fr

CMAP, Ecole Polytechnique,
Palaiseau, France,
Courant Institute of Mathematical Sciences,
New York University, NY 10012.
Stephane.Mallat@polytechnique.fr