Wavelets for a Vision

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Invited Paper

Early on, computer vision researchers have realized that multiscale transforms are important to analyze the information content of images. The wavelet theory gives a stable mathematical foundation to understand the properties of such multiscale algorithms. This tutorial describes major applications to multiresolution search, multiscale edge detection, and texture discrimination.

I. INTRODUCTION

Multiscale processing is hardly avoidable to develop efficient image recognition algorithms. Before wavelets were called "wavelets," researchers such as Burt and Adelson [7], Koenderink [18], Mm [24], Witkin [36], and Rosenfeld [30] had established the necessity to extract multiscale image information. Some of these ideas have later been formalized and refined by the wavelet theory. In parallel, psychophysics, and physiological experiments [11] have shown that multiscale transforms seem to appear in the visual cortex of mammals. This was an important motivation to further study the application of such transforms to image analysis. To explain the impact of wavelets for low-level vision, we concentrate on three major applications: multiresolution processing, multiscale edge detection, and texture discrimination.

Multiresolution algorithms modify the image resolution to process as little data as possible, for any particular visual task. Coarse to fine searches process first a low resolution image and zoom selectively into finer scale information, if necessary. Applications to stereo vision and optical flow measurements are described.

Local image contrasts are often more informative than light intensity values. A wavelet transform measures gray level image variations at different scales. Contours of image structures correspond to sharp contrasts and can be detected from the local maxima of a wavelet transform. Their importance is illustrated by our ability to recognize complex scenes from a drawing that outlines edges. The wavelet theory relates the behavior of multiscale edges to local image properties. It also opens the door to reconstruction algorithms which recover images from multiscale edges.

Among low-level vision problems, texture discrimination is certainly one of the most difficult. Despite the fact that textures are quickly preattentively discriminated by a human observer [21], there is still no appropriate model for textures. The perception of textures as opposed to edges depends upon local but not pointwise properties. However, there is no predefined neighborhood size over which textures can be analyzed. This has motivated the use of
The high resolution visual center is called fovea. It is responsible for high acuity tasks such as reading or recognition. A retina with a uniform resolution equal to the highest fovea resolution would require about 10,000 times more photoreceptors. Such a uniform resolution retina would increase considerably the size of the optic nerve that transmits the retina information to the visual cortex and the size of the visual cortex that processes this data.

Active vision strategies compensate the nonuniformity of visual resolution by moving the fovea with eye saccades. Regions of a scene with a high information content are scanned successively. These saccades are partly guided by the lower resolution information gathered at the periphery of the retina. This multiresolution sensor has the advantage to provide high resolution information at selected locations and a large field of view with relatively little data.

Multiresolution algorithms implement in software the search for important high resolution data. A uniform high resolution image is measured by a camera but a small part of this information is processed. The high resolution information is selectively considered depending upon lower resolution processing.

Such algorithms are efficiently implemented with multiresolution pyramids introduced by Burt and Adelson. Let us normalize the image resolution to one. A multiresolution pyramid computes the image approximation at lower resolutions for \( j < 0 \). As explained in the background article, an approximation of \( f(s, t) \) at a resolution \( 2^j \) is defined as an orthogonal projection on a space \( V_j \). It has been proved that such multiresolution spaces admit orthogonal basis of \( V_j \) of dilated separable scaling functions \( \{ f_{j, m} \} \).

The approximation at a resolution \( 2^j \) is thus characterized by the inner products \( \langle f_{j, m}, \phi \rangle \). We suppose that \( f_0[n, m] \) is the discrete image at the resolution one measured by the camera. One can prove that image approximations \( f_j[n, m] \) at smaller resolutions are computed with a succession of low-pass filterings and subsamplings. Let \( h[n] \) be the Conjugate Mirror Filter associated to the scaling function \( \phi \) and \( h_2[n, m] = h[-n]h[-m] \). An image \( f_j[n, m] \) at a resolution \( 2^j \) is obtained from a higher resolution image \( f_{j+1}[n, m] \) with a low-pass filtering with \( h_2[n, m] \) and a subsampling by two along the rows and columns.

If \( f_0[n, m] \) has \( N^2 \) nonzero samples, with appropriate border treatments, \( f_j[n, m] \) has \( 2^{2j} N^2 \) nonzero pixels.

Fig. 1 shows an example of multiresolution image pyramid over five octaves.

Fig. 1. Multiresolution image pyramid. From top to bottom the resolution decreases by two from one image to the next. These images are obtained with a cascade of low-pass filtering and subsampling.

B. Coarse to Fine Multiresolution Processing

Coarse to fine multiresolution search reduces the computational complexity by beginning at low-resolution and adaptively increasing the resolution to gather the necessary details. We describe applications to the estimation of optical flow from time image sequences and depth from stereo images.

The optical flow is computed from a sequence of images at time intervals \( A \). Let \( I_k[n, m] \) be the gray level image intensity at time \( kA \). A pixel \( (n_0, m_0) \) gives the light intensity reflected by a point \( P \) in the 3-D scene. If the image gray level is not constant in the neighborhood of \( (n_0, m_0) \), a change between the relative position of \( P \) and \( M \) is indicated.
the camera creates an intensity displacement. The velocity of this gray level displacement in the image plane is called the optical flow. If there is no change of lightning, it can be related to the 3-D velocity of \( P \). There are several approaches to compute the optical flow, including the use of local differential operators \([9]\). A simple technique is to estimate the gray level displacement from frame to frame by finding a correspondence between the pixels of successive images \( I_k[n, m] \) and \( I_{k+1}[n, m] \).

A similar point matching problem appears in stereo vision \([12]\). A point \( P \) in a 3-D scene is projected at different locations in the image planes of two stereo cameras. From the difference between the position of this projection in the images, \( I_1[n, m] \) and \( I_2[n, m] \), the 3-D coordinates of \( P \) in the scene can be estimated. The main difficulty of stereo vision is to match the projections on the left and right camera planes. The same problem appears if the neighborhood of \( n, m \) is too large there might not be any appropriate match in the image.

We search for a point \( K \) in the neighborhood of \( (n, m) \) that maximizes the normalized correlation (1). A point \( (n', m') \) is a candidate if and only if the neighborhood of \( (n', m') \) includes objects whose displacement projection then induces important distortions between the images and thus does not match well any other domain of the original image. The coarse information is correlated on the original image. The low resolution estimates are used to constrain the correlation search to a limited area. This correlation is always smaller than one and is equal to 0.25 if the neighborhood of \( n, m \) includes smaller components having different displacements than the neighborhood \( n', m' \). A point \( (n, m) \) is correlated over small neighborhoods. Varying this size allows us to disambiguate several potential matches in the image. The center location \( (n, m) \) of the neighborhood maximizes the normalized correlation (1). A point \( (n, m) \) is correlated over large neighborhoods whereas the fine information is correlated on the original image at the resolution 1. Let us compute the multiresolution pyramids of \( I_1[n, m] \) and \( I_2[n, m] \) with a maximum depth \( -J \leq \log_2 N \). We first correlate the points of the lower resolution images \( I_1[n, m] \) and \( I_2[n, m] \) over neighborhoods of size \( K \). We first correlate the points of the lower resolution images \( I_1[n, m] \) and \( I_2[n, m] \) over neighborhoods of size \( K \), which is typically equal to three or five. For any point \( (n', m') \) of \( I_1[n, m] \) we find \( (n, m) \) in \( I_2[n, m] \) whose

A multiresolution matching algorithm correlates first low resolution approximations of \( I_1[n, m] \) and \( I_2[n, m] \) and refines the match at high resolution guided by the lower resolution estimates \([31]\). Let us compute the multiresolution pyramids \( \{I_1[n, m]\}_{J \leq -1} \) and \( \{I_2[n, m]\}_{J \leq -1} \) of \( I_1[n, m] \) and \( I_2[n, m] \), with a maximum depth \( -J \leq \log_2 N \). We first correlate the points of the lower resolution images \( I_1[n, m] \) and \( I_2[n, m] \) over neighborhoods of size \( K \).
locally warped. When estimating an optical flow at a coarse resolution $2^j$, we mentioned that there might be regions of size $2^j K$ in $I_1[n,m]$ which include components with different velocities. These structures are translated at different positions of $I_2[n,m]$, which modifies the properties of gray level neighborhoods. In this case, a correlation of the coarse resolution images $I_1[n,m]$ and $I_2[n,m]$ produces a wrong estimate of the flow. This error can be detected at a higher resolution $2^l$ where the smaller correlation length $2^{-l} K$ can resolve the regions having different motions. It is thus necessary to use verification strategies that detect misleading coarse resolution information from the fine-resolution images.

The evocative power of drawings clearly shows that edges are among the most important features for pattern recognition. But what is an edge? When looking at a brick wall, we may decide that the edges are the contours of the wall whereas the bricks define a texture. We may also include the contours of each brick among the set of edges and consider the irregular surface of each brick as a texture. The discrimination of edges variations versus textures depends upon the scale of analysis. This has motivated computer vision researchers to detect sharp image variations at different scales [24], [30], [36]. The next section describes a multiscale Canny [31] edge detector that is most often used in vision algorithms. This edge detector is equivalent to the detection of wavelet transform local maxima. The wavelet theory allows one to understand how to combine multiscale edge information to characterize different types of edges. It also provides the mathematical grounds to implement an algorithm that reconstructs images from edges.

A. Wavelet Maxima

Canny's algorithm [31] detects sharp variation points of an image $f(x,y)$ from the modulus of the gradient vector $\nabla f$. However, the sampled coefficients $\{T f_r(2^j, 2^j n) = T f(2^j, 2^j n - \tau)\} \forall n \in \mathbb{Z}$ are not equal to a translation of the values $\{T f(2^j, 2^j n)\} \forall n \in \mathbb{Z}$, when $\tau$ is not proportional to $2^j$. As a result, the wavelet coefficients of a translated function $f_r(t)$ may be very different from the wavelet coefficients of $f(t)$. It is difficult to characterize a pattern from the wavelet coefficients in a basis since these wavelet descriptors depend upon the pattern location.

III. Multiscale Edge Detection

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A. Wavelet Maxima

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The partial derivative of \( f(x, y) \) in a direction \( \vec{n} \) of the \((x, y)\) plane is equal to the inner product

\[
\frac{\partial f(x, y)}{\partial n} = \nabla f(x, y) \cdot \vec{n}.
\]

The absolute value of this partial derivative is maximum if \( \vec{n} \) is parallel to \( \nabla f \). This proves that the gradient vector points locally in the direction of maximum change of the surface. A point \((x_0, y_0)\) is defined to be an edge point if the modulus of \( \nabla f(x, y) \) is locally maximum at \((x_0, y_0)\), when \((x, y)\) varies in a 1-D neighborhood of \((x_0, y_0)\) that is collinear to the direction of \( \nabla f(x_0, y_0) \). These edge points are locations where the surface has locally a maximum rate of change. They are inflection points of \( f(x, y) \).

Inserting (5) in (3) and putting the partial derivative outside the convolution products proves that

\[
\begin{align*}
(T^1 f(2^j, u, v)) & = -2^j \left( \frac{\partial}{\partial u} f(\hat{f}(u, v)) \right) \\
(T^2 f(2^j, u, v)) & = -2^j \left( \frac{\partial}{\partial v} f(\hat{f}(u, v)) \right)
\end{align*}
\]

The two components of the wavelet transform are proportional to the coordinates of the gradient vector of \( f(x, y) \) smoothed by \( \hat{f}(u, v) \). The modulus of the gradient vector \( \nabla f(\hat{f}(u, v)) \) is thus proportional to the wavelet transform modulus

\[
Mf(2^j, u, v) = \sqrt{(T^1 f(2^j, u, v))^2 + (T^2 f(2^j, u, v))^2} \tag{7}
\]

and its angle is...
Fig. 3. The original image is at the top left of Fig. 4. The first and second columns display the wavelet modulus $M_f(23, U, v)$ and angles $A_f(23, U, v)$ at scales $2^3$ for $1 \leq j \leq 4$. The pixels darkness are proportional to the amplitude of $M_f(23, U, v)$ and $A_f(23, U, v)$ (which goes from zero to 2~). The black pixels of the images on the third column are the modulus maxima of $M_f(23, U, v)$ along the direction specified by $A_f(23, U, 0)$. These maxima are chained together. The fourth columns displays the longer edge chains with higher average modulus values.

B. Multiscale Edge Processing of $(X_0, Y_0)$

Once edges are detected at several scales, we must understand how to integrate this multiscale information for pattern recognition. One might be tempted to look for a "best" scale where the edges are well discriminated from noises and textures. The wavelet theory shows that much finer properties are derived by analyzing edge behaviors across scales. The multiscale edge information is in fact rich enough to recover close image approximations.

The background article [10] explains that the decay of a wavelet transform depends upon the local regularity of the signal. This regularity is quantified by Lipschitz exponents. A function $f(z, y)$ is said to be Lipschitz $\alpha$ at $(x_0, y_0)$, with $0 \leq \alpha \leq 1$, if for all points $(z, y)$ in a 2-D neighborhood

$$\frac{|f(z, y) - f(x_0, y_0)|}{|z - x_0|} \leq C$$

The larger $\alpha$, the more regular the function. At a discontinuity, the function is Lipschitz $\alpha = 0$. If $1 > \alpha > 0$ the image is continuous but not differentiable and $C^\alpha$ characterizes the type of singularity at that location. The Lipschitz regularity of a function $f(z, y)$ is related to the asymptotic decay of the two wavelet components $|T_1 f(2^j, u, v)|$ and $|T_2 f(2^j, u, v)|$ when the scale $2^j$ decreases. This decay is controlled by the modulus $M_f(2^j, U, v)$ and one can prove [25] that a necessary condition for $f(z, y)$ to be Lipschitz $\alpha$ at $(x_0, y_0)$ is the existence of $C > 0$ such that

$$M_f(2^j, U, v) \leq C$$

where $M_f(2^j, U, v)$ is the modulus maxima along the direction specified by $A_f(23, U, v)$. These maxima are chained together.
Fig. 4. The image at the top left is the original image. The image at the top right is reconstructed from the multiscale wavelet maxima shown in the third column of Fig. 3, plus the lower scale information. The image at the lower right is reconstructed from the thresholded edges shown in the fourth column of Fig. 3.

Suppose that the image has an isolated edge curve along which \( f(z, y) \) has singularities which are Lipschitz. By measuring the decay of the modulus maxima across scales, we derive from (9) an estimate of the Lipschitz regularity along the edge. At some edge locations, the image is not singular but has a smooth transition that is locally sharper. For example, the diffraction effect creates smooth edges at the borders of shadows. An analysis of the decay of wavelet maxima can also provide an estimate of the local smoothness of edges [23].

Multiscale edges give a rich description of the image information and one may wonder whether it is possible to reconstruct the whole image from these edges. This issue was raised by Marr [24] and studied by several researchers in computer vision [14], [37]. The wavelet theory allows one to express the nonlinear constraints derived from the knowledge of the modulus maxima locations \( (U, v) \) as well as the values of \( M_f(2^n U, v) \) and \( A_f(2^n U, v) \) at these locations. An alternate projection algorithm recovers an image which belongs to the set of functions whose wavelet transform satisfies these maxima constraints [23].

The upper right image of Fig. 4 shows the reconstructed image from the multiscale edges displayed in Fig. 3. Since edges are computed up to the scale \( 2^4 \), the image low-frequencies at scales larger than \( 2^4 \) are used to complement the edge information in the reconstruction. When edges are computed up to the coarser scale \( \log_2 N \), this complement of information is reduced to the average value of the image intensity. Extensive numerical experiments show that the reconstructed images are visually identical to...
the original ones, although Meyer [26] and Berman [4]
proved that an image is not uniquely characterized by its
multiscale edges. There are no visual distortions because
reconstruction errors remain below the visual sensitivity
threshold. The mathematical problem is still open and
despite further studies [35], we do not understand why
these reconstruction algorithms work so well and under
what condition multiscale edges do provide a complete
and stable signal representation. Since we can reconstruct
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The energy of the wavelet transform \( \mathcal{U}(\mathbf{x}, \omega) \) is shown along four orientations at three scales \( \{2^k : k < 3\} \). From left to right, the orientations are 0, 45, 90, and 135 degrees, respectively. The scale increases from top to bottom. The energy value is large when the orientation and scale match the texture structures.

The distribution of wavelet energy across scales and orientations is different for the two textures. The wavelet representation is complete and stable if there exists two parts of a complex wavelet \( \mathcal{V}(\mathbf{x}, \omega) \) in \( \mathbb{R}^2 \) and \( A > 0 \) and \( B \) such that the Fourier transforms satisfy:

\[
\mathcal{F}[\mathcal{V}(\mathbf{x}, \omega)] = A \mathcal{F}[\mathbf{g}_0(\mathbf{x}, \omega)] + B \mathcal{F}[\mathbf{g}_1(\mathbf{x}, \omega)].
\]

Wavelets with quadrature phases are the real and imaginary parts of a complex wavelet \( \mathcal{V}(\mathbf{x}, \omega) \).

Let \( \mathbf{g}_0(\mathbf{x}, \omega) \) be the Fourier transform of \( \mathcal{V}(\mathbf{x}, \omega) \). The Fourier transform of \( \mathbf{g}_0(\mathbf{x}, \omega) \) is

\[
\mathcal{F}[\mathbf{g}_0(\mathbf{x}, \omega)] = \mathcal{F}_x(\mathbf{g}_0(\mathbf{x}, \omega)) = \mathcal{F}_y(\mathbf{g}_0(\mathbf{x}, \omega)) = 0
\]

since the wavelet has a frequency resolution of the order of one octave. We can restrict the scales to \( \{2^k : k < 3\} \) and define the wavelet transform in each direction by:

\[
\mathcal{V}(\mathbf{x}, \omega) = \mathcal{F}_x(\mathbf{g}_0(\mathbf{x}, \omega)) = \mathcal{F}_y(\mathbf{g}_0(\mathbf{x}, \omega)).
\]
imental results, these algorithms are quite simple and may have a relatively large degree of variability as shown in Fig. 6. Most algorithms attenuate these variations with increasing scale and orientation, the wavelet energy along the orientation of the main texture components and diagonal orientations. On the other hand, the texture at the center has its energy mostly concentrated in the neighborhood of Varying the scale of these structures. In Fig. 7, the filtering of a single texture, we see only one realization of a particular "texture process," i.e., wood. These texture classes of images having the same texture as the realizations are thus particularly difficult to analyze. Moreover, when the process is Gaussian, in which case it is characterized by its covariance, a reliable estimation of the covariance from a single match is generally very difficult. Even if we suppose that the covariance is known, the filtering cannot remove certain image details and keep other components. The filtering and resolution of images. Other diffusion algorithms can be used to modify the scale and resolution of images, but wavelets are not the only tools to modify the scale and resolution of images.

The formalization of texture recognition problems is often easier in a stochastic framework. It does not mean that textures are supposed to be created by some random process but that aggregates the wavelet responses at all scales and orientations in order to find the boundaries of homogeneous textures. Despite their good experimental results, these algorithms are quite simple and may have a relatively large degree of variability as shown in Fig. 6. Most algorithms attenuate these variations with increasing scale and orientation, the wavelet energy along the orientation of the main texture components and diagonal orientations. On the other hand, the texture at the center has its energy mostly concentrated in the neighborhood of Varying the scale of these structures. In Fig. 7, the filtering of a single texture, we see only one realization of a particular "texture process," i.e., wood. These texture classes of images having the same texture as the realizations are thus particularly difficult to analyze. Moreover, when the process is Gaussian, in which case it is characterized by its covariance, a reliable estimation of the covariance from a single match is generally very difficult. Even if we suppose that the covariance is known, the filtering cannot remove certain image details and keep other components. The filtering and resolution of images. Other diffusion algorithms can be used to modify the scale and resolution of images, but wavelets are not the only tools to modify the scale and resolution of images. The filtering and resolution of images. Other diffusion algorithms can be used to modify the scale and resolution of images, but wavelets are not the only tools to modify the scale and resolution of images.


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