Mini-Projet : Turing Patterns

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1 Introduction

In this mini-projet, we focus on the analysis and simulation of Turing patterns in the continuity of what has been done in PC7. The objectives of the project are threefold : 1- to conduct analysis close to what we have done for the PC in terms of appearance of stationnary modes depending on the size of the domain, 2- to analyze the dynamics of the system in order parameter range, when several dynamics are observed (Hopf bifurcation), 3- to study, depending on time, the dynamics in 2D of Turing patterns and related literature.

2 Simulation of Turing pattern in 1D

In this section, we focus on the simulation of Turing patterns [6, 1, 3] using the Lengyel-Epstein model [2] and use an article where some simulation were performed some time ago [4].

The system of reaction diffusion is the following one:

$$\begin{cases} \partial_t u = D_u \, \partial_x^2 u + f(u,v), \qquad f(u,v) = a - u - \dfrac{4\,u\,v}{1+u^2}, \\ \partial_t v = \delta \left[D_v \, \partial_x^2 v + g(u,v) \right], \qquad g(u,v) = b \left(u - \frac{u\,v}{1+u^2} \right), \end{cases} \tag{1}$$

where $x \in \Omega \subset \mathbb{R}$, $D_u = 1$, $D_v = 1.5$, $\delta = 8$, and where a and b remain the control parameters.

2.1 Conduct and analysis of the source term and identify the only equilibrium when the diffusion in not present as a function of a. Study the stability of such a state in terms of a and b and show that there can be a Hopf bifurcation for some values of b as a function of a.

2.2 Investigate the critical value of b_T such that, for a given a, we have a stable equilibrium in the previous question and the development of a Turing pattern (instability of the equilibrium with diffusion) and provide the interval of b we can work with in order to get a Turing pattern.

2.3 We take $a = 30$ and $b = 2.8$. Identify the eigenvalue related to the instability and plot the stability curve as a function of n^2. What is the most amplified mode, knowing that we work on a domain $[0, L]$, $L = 50$? Conduct a numerical simulation by taking a perturbed initial solution of the equilibrium state and observe the evolution with time of the solution. Propose an analysis of the results. Identify the integer such that $n_*^2 L/2\pi$ is the most amplified mode and check that this mode is the one emerging from the simulation. Make the link with what is commonly know as the "stripes" pattern [3, 5, 2]. Propose several size of the domain and analyze the impact on the development of the pattern depending on the parameters.

3 Turing Patterns in another parameter range, different dynamics

Following what you can find in the proposed articles, propose and analyze several potential dynamics of the system, where for example an oscillating constant in space in an interval state is connected to a stationary solution in another domain. In order to do so, you will have to change the initial profile and break the symmetry....

4 Turing Patterns in 2D

Following, what has been done in [4], propose a study of the 2D case in terms of simulation and dynamics.

References

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