PC2 : Notions de base théoriques

Systèmes Dissipatifs / Conservatifs

1 Introduction

The Petite Classe is divided into three parts. First we come back on some fundamentals of the theory we have studied during the Class, in particular as regards the duality autonomous/non autonomous systems of ordinary differential equations. Then we study classical mechanical systems, for which the friction-less pendulum is a simple but interesting example. Finally, we will focus on population dynamics in order to tackle the question of the fundamental differences between conservative and dissipative systems, which can be found in the same modeling hierarchy of a given phenomenon. For the people interested in the the field of population dynamics, the books of Mark Kot and James Dickson Murray [1, 3] offer a broad perspective on the field and connect with areas we will cover in the rest of the course.

2 Fundamentals

2.1 Solution, integral curve and orbit

Consider the dynamical system on the pair $(u(t), v(t)) \in \mathbb{R}^2$:

$$\mathrm{d}_t u = v, \qquad \mathrm{d}_t v = -u. \tag{1}$$

2.1.1 Find an invariant of the system (1), also called a first integral of the motion.

2.1.2 Find a solution passing through the point $(u_0, v_0) = (1, 0)$ at $t = 0$.

2.1.3 Identify the solution, the integral curve, as well as the orbit and propose a graphical interpretation allowing to clarify the difference between these various notions.

2.2 Autonomous versus non-autonomous dynamical systems

The purpose of the present exercice is to investigate the implication of non-autonomous character of the system on the structure of critical points. Whereas the notion of critical point is well-understood in the framework of autonomous systems, we point out the difficulties associated with the non-autonomous case. Let us consider a general dynamical system

$$\mathrm{d}_t x = f(x, t), \tag{2}$$

with proper initial conditions $x(0) = x_0$. A first way of defining an equilibrium or critical point of (2) is to consider a frozen time equilibrium, that is there exists $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$f(\bar{x}, \bar{t}) = 0. \tag{3}$$

2.2.1 Suppose that there exists such an equilibrium and assume that $D_x f(\bar{x}, \bar{t}) \neq 0$. Show that there exists $\bar{x}_0(t)$ a function defined on a neighborhood V of $\bar{t}$ such that, of course, $\bar{x}_0(\bar{t}) = \bar{x}$, and such that $f(\bar{x}_0(t), t) = 0$, for $t \in V$.

2.2.2 As an example, consider the dynamical system

$$\mathrm{d}_t x = -x + t, \tag{4}$$

with $x(0) = x_0$, for which a solution is easily found to be $x(t) = t - 1 + \exp(-t)(x_0 + 1)$. Describe the behavior of the solution and focus in particular on the asymptotic behavior. Propose a graph of the dynamics of the solution in the (x, t) plane of (4). Describe what are the frozen or instantaneous equilibrium points. Are they solutions of the original system ?

2.2.3 Prove that in general, a frozen equilibrium point is not a solution of the original dynamical system, except if $\bar{x}_0$ does not depend on time. What is the implication on f ?

3 Mechanical systems - Friction-less pendulum

In this part, we will consider a mechanical system obtained through the Lagrange equation and formalism as explained in Class and focus on the friction-less pendulum.

The system is taken as $x = (x_1)$, $x \in \mathbb{R}$, a single variable denoting the angle of the pendulum compared to the downward position. The usual equation for the non-dimensional pendulum is given by :

$$\ddot{x} = -\sin(x). \tag{5}$$

3.1 Recast the previous system into the usual Lagrange formalism and identify the potential as well as the kinetic energy.

3.2 Identify the energy and the Lagrangian of the system.

3.3 Explain how the initial condition is setting up the energy level of the solution.

3.4 Recast the previous system into a system of first order ordinary differential equations.

3.5 From the invariance of energy, describe the partition of the phase space into various orbits as in Figure 1. Describe the orbits that are observable and the ones that are limit cases. Identify the α-limit and ω-limit sets in the various configurations.

3.6 Using the notebook, integrate numerically the dynamics for various initial points related to Figure 1 in the case without friction and describe the dynamics. Do the ω-limits / α-limits sets depend on the initial conditions ? Explain. Can you recover the whole set of ω-limits / α-limits sets using numerical integration ?

3.7 Reproduce the same study adding friction and switching from a conservative to a dissipative dynamical system. What about the set of ω-limits / α-limits sets ?

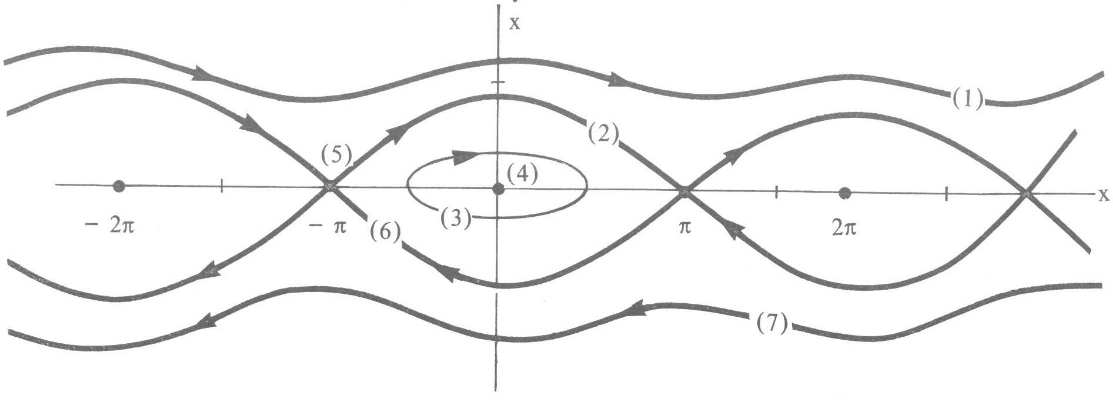

FIGURE 1 – Phase portrait of the friction-less pendulum ((1) and (7) the pendulum is going around in a periodic motion, (4) and (5) are stationary motion stable and unstable respectively, (3) follows a periodic behavior of the oscillation with an amplitude lower than π, (2) and (6) are heteroclinic or homoclinic orbits depending on the manifold we work on (periodicity or not).

4 Population dynamics

In the final part of the Petite Classe, we will tackle the interesting field of population dynamics through three levels of modeling in order to clarify the difference between conservative systems such as the pendulum case or the Lotka-Volterra [1] system [2, 5], for which the system admits invariant quantities [2] and dissipative systems such as the Rosenzweig-MacArthur model [4], where we can also observe periodic orbits and limit cycles, but where the structure of the dynamics are completely different, thus helping to figure out the key differences between conservative and dissipative systems.

4.1 The most elementary model

A.J. Lotka published in 1925 his book entitled "Elements of Physical Biology" [2]. He proposed to represent the kinetics of populations living in community by systems of differential equations. In one of the chapters, he considers the example of a population of herbivorous animals that feed on plants. By analogy with the equations used for chemical kinetics, by representing by $u_1(t)$ the total mass of plants and by $u_2(t)$ the total mass of herbivores at the moment t, Lotka introduces the following model :

$$\begin{cases} \mathrm{d}_t u_1 = u_1\,(1 - u_2) \\ \mathrm{d}_t u_2 = u_2\,(-k + u_1) \end{cases} \tag{6}$$

Vito Volterra's interest in the problems of equilibrium between animal species in ecosystems was initiated by the zoologist Umberto D'Ancona (1896-1964). D'Ancona had been working for some years on fishing statistics in the northern Adriatic Sea. These data related to the percentage of predator fish (selachians - such as rays or sharks) caught in three Italian ports. D'Ancona found that the share of these fishes was greater during the First World War when fishing is less intense. As predators feed on other fish that in turn feed on plankton, it appears that a reduction in fishing effort favors predatory species. Volterra, without any link or knowledge of Lotka's work, proposed the same model to explain the dynamics. He notes, like Lotka, that this system oscillates periodically, "with a period that depends on the initial condition". Volterra's approach is faithful to his mechanistic conceptions. He schematizes the populations by two systems of particles moving at random in a closed container which represents the ecosystem, here the sea. It is the well-known physical model of the perfect gas where particles move and collide at random in a closed container. In the Volterra model, each collision corresponds to a "meeting" between a "prey-particle" and a "predator-particle", thus giving the predator the opportunity to devour a prey. Volterra published his work in an article in Italian in 1926, then published in 1931 a book entitled "Leçons sur la théorie mathématique de la lutte pour la vie", in which he studied other models.

4.1.1 The objective of this question is to obtain information about the qualitative behavior of the solution of the problem (6).

> **4.1.1.a** Show that the system (6) admit an invariant. Express it as a function of variables u_1 and u_2 as well as parameter k.
>
> **4.1.1.b** Explain how the invariant impacts the dynamics. What are the ω-limits and α-limits sets of a solution ? Do they depend on the initial datum ?
>
> **4.1.1.c** Does the numerical integration allow a proper resolution of the dynamics ? Using a different scale for the representation of the invariant, can you conjecture why any solver cannot be used for conservatives systems [3] ?

4.2 Adding intraspecific competition - change of mathematical behavior

The Lotka-Volterra model is the simplest one imaginable. It has necessarily a number of imperfections. One of the major flaws is that in the absence of predators, $(y = 0)$, the equations are reduced

1. Alfred James Lotka (1880 - 1949) et Vito Volterra (1860 - 1940).
2. The book of Alfred J. Lotka and the original article of Vito Volterra can be consulted on demand, as well as the books of Murray and Kot.
3. The last course of MAP551 will be dedicated to the symplectic integrators, which have been designed in order to preserve invariants.

to a single equation : $d_t x = x$, for which we obtain easily a solution : $x(t) = x(0)\exp(t)$ and the population of prey " explodes " exponentially with time.

Such behavior is probably correct for a short period of time. But the limitation of resources means that the population can not exceed a certain threshold, called "carrying capacity" by ecologists. The simplest way to model this effect is to set $d_t x = x(1-x)$ in this case.

It is not difficult to convince oneself that solutions are here limited. If the initial population is smaller than the carrying capacity (which is worth 1 here), it begins to grow exponentially before undergoing a downturn and moving towards 1. If the initial population is above the carrying capacity, it exponentially tends to 1.

If we go back to the Lotka-Volterra model and modify it to take into account this competition between prey, we obtain, after a suitable change of units, the following model :

$$\begin{cases} d_t u_1 = u_1\,(1 - u_1 - u_2) \\ d_t u_2 = \beta(u_1 - \alpha)u_2 \end{cases} \tag{7}$$

4.2.1 The objective of this question is again to obtain information about the qualitative behavior of the solution of the problem and to make the with the previously obtained results. (6).

- **4.2.1.a** Describe the dynamics of the system for $\alpha \in [0.1, 1]$ and $\beta \in [0.1, 2]$ using the notebook. What are the critical/equilibrium points of the system ? How do they depend on the parameters ?
- **4.2.1.b** What are the ω-limits sets of the solutions ? Do they depend on the initial datum ?
- **4.2.1.c** Does the numerical integration allow a proper resolution of the dynamics ? Compare the long-term dynamics of the previous and the present systems. Comment.

The conclusion is that by taking into account the term of intraspecific competition, which prevents them from proliferating without limit in the absence of predators, the periodic oscillatory character of the solutions of the original model has been completely destroyed. To retrieve oscillations, it is necessary to modify the interaction term : this is the object of Rosenzweig-McArthur's model.

4.3 Periodic solutions of the Rosenzweig-MacArthur model - limit cycle(s) ?

Soon after the appearance of the Lotka-Volterra model, various modifications were proposed for the predation term. In fact, the number of preys killed by predators is in this model proportional to the product of the number of individuals in each population, i.e. proportional to $x(t)y(t)$. In other words, the number of prey killed per predator increases proportionally to the number of prey itself, so there is no "saturation" or "satiety" effect. This is qualitatively what we observe for certain populations of bacteria.

For mammals feeding on insects or other mammals, a truly different behavior is expected. Indeed, the predator's time will be divided into a search time of his prey followed by a time to "treat" it.

The American ecologist Buzz Holling proposed in 1959 three major types of modeling for the number of prey killed by predator : the first is the Lotka-Volterra model (type I) and the other two introduce a saturation effect when the number of prey exceeds a certain threshold (types II and III). Types II and III differ when the number of prey is very small and distinguish 'generalist' predators from "specialists" predators.

It was in 1963 that American ecologists Robert MacArthur and Michael L. Rosenzweig [4] studied the following prey-predator model :

$$\begin{cases} d_t u_1 = u_1\left(1 - \dfrac{u_1}{\gamma}\right) - \dfrac{u_1 u_2}{1+u_1} \\ d_t u_2 = \beta u_2\Big(\dfrac{u_1}{1+u_1} - \alpha\Big) \end{cases} \tag{8}$$

Le terme de prédation est de type II.

4.3.1 The objective of this question is again to obtain information about the qualitative behavior of the solution of the problem and to make the with the previously obtained results. (6).

4. Robert MacArthur (1930-1972) and Michael L. Rosenzweig (born in 1941).

4.3.1.a Describe the dynamics of the system for $\alpha \in [0.34, 0.7]$ using the notebook. What are the critical/equilibrium points of the system ? How do they depend on the parameters ? Identify numerically α_{cr}, a critical value of α for which there is a "topological change" (called a "bifurcation") of the behavior of the system.
4.3.1.b What are the ω-limits sets of the solutions for $\alpha \in [0.34, \alpha_{cr}]$? and for $\alpha \in [\alpha_{cr}, 0.7]$? Do they depend on the initial datum ? Compare the long-term dynamics of original Lotka-Volterra system, which is a periodic orbit and the present system dynamics. In what do they essentially differ ?
4.3.1.c Does the numerical integration allow a proper resolution of the dynamics in comparison to subsection 4.1.1. ? Comment.

Références

[1] M. Kot. *Elements of mathematical ecology.* Cambridge University Press, Cambridge, 2001.

[2] A.J. Lotka. *Elements of Physical Biology.* Williams & Wilkins Company, 1925.

[3] J. D. Murray. *Mathematical biology. I*, volume 17 of *Interdisciplinary Applied Mathematics.* Springer-Verlag, New York, third edition, 2002. An introduction.

[4] M. L. Rosenzweig and R. H. MacArthur. Graphical representation and stability conditions of predator-prey interactions. *The American Naturalist*, 97(895) :209–223, 1963.

[5] V. Volterra. Fluctuations in the abundance of a species considered mathematically. *Nature*, 118 :558–560, 1926.