PC6 : Dynamics around critical points, hyperbolicity, continuation and bifurcations

1 Introduction

The Petite Classe is divided into three parts. The first part is devoted to toy models in order to understand some situations, we have encountered during the course, and to apply the ideas of theoretical part of the course. The second part is devoted to a full study of the Brusselator system of equations [6] and the van der Pol oscillator [8, 9, 3], but from a different perpective compared to what we have done previously in terms of integrating the system in time. We will focus in this PC on the study of branches of equilibria. The third part of the PC is devoted to the introduction of the notion of numerical continuation of equilibria along branches parametrized by one of several parameters. Without resorting to a numerical integration in time, which could be difficult near bifurcation points, or even impossible or misleading near limit points or unstable equilibria, we want to follow branches of equilibria and detect bifurcations, even unstable branches of solutions. We apply the simplest methods (natural continuation and pseudo-arclength continuation) to a series of system we have already investigated and follow several branches of critical points.

2 Some toy dynamical systems and course application

2.1 A vector field with a singular germ non $\mathcal{C}^0$-conjugated with the one related to its linerarization

We consider the following vector field:

$$X(x_1, x_2) = \Big(-x_2 - x_1(x_1^2 + x_2^2), x_1 - x_2(x_1^2 + x_2^2)\Big), \tag{1}$$

which admits a singular germ $(X, 0)$.
2.1.1 Show that the orbits of the linearized vector field X_1 are circles.

2.1.2 Taking the scalar product of the equation of a trajectory $\phi(t)$ by itself, prove that the evolution of the norm $||\phi(t)||^2$ is strictly decreasing in time.

2.1.3 Explain why the two germs $(X, 0)$ and $(X_1, 0)$ can not be $\mathcal{C}^0$-conjugated.

2.2 Local / global stable/unstable manifold

We consider the following vector field:

$$X(x_1, x_2) = (x_2, 1 - x_1^2), \tag{2}$$

which admits a singular hyperbolic germ at $a = (-1, 0)$.
2.2.1 Show that $f(x_1, x_2) = \frac{x_2^2}{2} - x_1 + \frac{x_1^3}{3}$ is a first integral of X.

2.2.2 Represent the phase portrait of X.

2.2.3 Identify the unstable manifold at a and comment on the local and global structures.

2.3 Stable/unstable manifold

We consider the following vector field:

$$X(x_1, x_2) = (-x_1, \alpha x_2^3), \tag{3}$$

which admits a singular germ at $O = (0, 0)$.
2.3.1 Solve for the exact solution of the system.

2.3.2 Identify the stable and unstable manifold depending on the sign of α.

2.3.3 Depending on the sign of α, where is the center manifold associated with the eigenvector of the zero eigenvalue of the Jacobian matrix?

3 Study of the ω-limit sets of the Brusselator model

The dynamics of the oscillating reaction discovered by Belousov and Zhabotinsky [2, 1], can be modeled through the so-called Brusselator model [6], which we have already integrated through many numerical methods:

$$\begin{cases} \mathrm{d}_t y_1 = a - (b+1) y_1 + y_1^2 y_2, \\ \mathrm{d}_t y_2 = b\, y_1 - y_1^2 y_2, \\ y_1(0) = y_1^0, \\ y_2(0) = y_2^0, \end{cases} \tag{4}$$

where a and b are two positive parameters. The purpose here is to focus the study on ω-limit sets, among which equilibria, and on their stability. The reader is encouraged to make the link with the previous system studied in the previous PCs.

3.1 Study of equilibria

3.1.1 Identify the critical points or equilibria of the dynamics. Propose a parametrization by a single parameter (a for example - for a fixed $b = 3$) of the 1d sub-manifold of equilibrium points.

3.1.2 Study the stability of such points. Identify the set of parameters for which the equilibria are hyperbolic.

3.1.3 What is the value of a for which the equilibrium point in not hyperbolic for $b = 3$. What happens in terms of the eigenvalues of the system at this point? Identify the set of parameters for which there is a change in stability of the system. What happens to the eigenvalues? Illustrate a typical behavior of the system on each side of the Hopf bifurcation point.

3.2 Proof of the existence of a limit Cycle - Poincaré-Bendixon Theorem

We focus in this part on the case $b = 3$ and we consider the branch of points for which the equilibrium point is unstable
3.2.1 What range of a parameters are we considering here?

3.2.2 Identify a invariant compact set through the dynamical system for $a = 1$.

3.2.3 Using the Poincaré-Bendixon theorem, prove the existence of a limit cycle for the dynamical system.

3.2.4 Reuse the same arguments for the van der Pol oscillator studied previously in this subsection with $\varepsilon = 1$.

3.3 Dynamics around a Hopf bifurcation point

The purpose of this subsection is to identify to what extent numerical simulation can be efficient at predicting the dynamics of a system at non-hyperbolic critical points. In particular, we envision the study of the Brusselator system with $b = 3$, at the Hopf bifurcation point $a^H(b = 3) = \sqrt{2}$.

3.2.1 What happens in terms of the Jacobian matrix at that point? What are the eigenvalues and what is the dynamics of the linearized system?

3.2.2 By simulating the system in the neighborhood of that point, describe what you observe.

3.2.3 Can you draw from the numerical simulations any firm conclusions on the dynamics of the system at the Hopf bifurcation point, as well as on the stability of the critical point?

4 Continuation of equilibria - limit points / Hopf / pitchfork

In this section, we envision a completely different approach in order to study the equilibria branches of a system. Starting from an equilibrium point of the system, we will investigate a numerical method in order to follow a branch of such equilibria, without resorting to the simulation of the time dynamics of the system. The reason for that is two-fold. First we want to focus on the behavior of such branches without having to conduct several costly simulations; second, we want to be able to visit some branches, which are unstable and for which the time simulation would not lead to any piece of information. We also want to follow such branches and identify bifurcation point, as well as potentially follow new branches issued from the bifurcation point as in the pitchfork bifurcation.

The description of the continuation method used in the notebook is provided in the appendix and is inspired from [5, 4] as well as used at SANDIA at the beginning of the years 2000[1]. The idea of this section is to illustrate the principles of pseudo-arclength continuation in order to identify critical point branches on some very simple systems, knowing perfectly that there are some more involved algorithms in order to treat more complex cases, which are generalization of the proposed algorithms in this PC.

4.1 Brusselator model - Hopf bifurcation - change of stability

For the Brusselator model, we will propose a continuation of the equilibria branch using a natural continuation.

4.1.1 Describe the two algorithms based on the appendix for the continuation of equilibria of the system we can use. Considering we have fixed $b = 3$, identify the branch of equilibria and recall the stability analysis of such points, and the location of the Hopf bifurcation.

4.1.2 Use the natural algorithm in order to reproduce the branch for several b parameters and check that we can reproduce the expected branch of equilibria as well as the switch of stability through the Hopf bifurcation.

4.1.3 Comment on the notion of accuracy for such an algorithm.

4.2 Thermal explosion - limit point - unstable branch

We focus in this first subsection on the thermal explosion equation with heat loses. Compared to the study we have conducted in PC1, we rather use another form of the system:

$$\mathrm{d}_t\theta = \mathrm{F_k}\ \exp(\theta) - \theta, \tag{5}$$

where the Frank-Kamenetskii parameter is the inverse of the α parameter we have used.

4.2.1 Describe the branches of equilibria we have studied in the first PC for $\mathrm{F_k} < 1/e$. What happens for $\mathrm{F_k} = 1/e$? We call such a point a limit point.

4.2.2 Study the stability of the equilibria on the two branches. Propose a figure with the graph of equilibria in θ as a function of $\mathrm{F_k}$. What happens at that limit point in terms of branches of equilibria?

4.2.3 Use the continuation algorithm starting from the point $(\theta = 0, \mathrm{F_k} = 0)$ in order to represent the continuation of the equilibria and explain how we can go "through" the limit point using pseudo arclength. Is such a point a bifurcation point?

[1] LOCA: Library Of Continuation Algorithms – http://www.cs.sandia.gov/loca/

4.3 Bead on a hoop configuration - pitchfork bifurcation - new branches of equilibria

A circular wire hoop rotates with constant angular velocity ω about a vertical diameter. A small bead moves, with or without friction, along the hoop, as presented in Figure 1, where the dynamics of the bead is described through the θ angle. The equation of motion, using the standard notation in classical mechanics, can be shown to be [7]:

$$\ddot{\theta} = -\omega_c^2 \sin\theta + \omega^2 \sin\theta\,\cos\theta - \alpha\theta \tag{6}$$

with $\omega_c = \sqrt{g/R}$, where the gravity acceleration is denoted by g and the radius of the hoop is denoted R. The coefficient α is related to the friction in the system and can be idealized to be zero in the frictionless configuration.

In order to recast the system into our more mathematical notations, we introduce the following notation. Let $y_1 = \theta$ and $y_2 = \dot{\theta}$, its time derivative. Then, we can switch to a first order system of differential equations, when there is no friction:

$$\begin{cases} \mathrm{d}_t y_1 = y_2 \\ \mathrm{d}_t y_2 = \sin y_1\,(\omega^2 \cos y_1 - \omega_c^2) \end{cases} \tag{7}$$

and with friction:

$$\begin{cases} \mathrm{d}_t y_1 = y_2 \\ \mathrm{d}_t y_2 = \sin y_1\,(\omega^2 \cos y_1 - \omega_c^2) - \alpha\, y_2 \end{cases} \tag{8}$$

4.3.1 Depending on the rotation velocity of the hoop, identify the number of the equilibria in the system and give their analytic expression in the frictionless configuration. Do we have to work in the (y_1, y_2) plane, or is it sufficient to work with y_1 alone? Explain.

4.3.2 Explain what is the influence of friction on the equilibria.

4.3.3 Conduct a stability analysis on the various branches of equilibria in both the frictionless and with friction configurations. Are the equilibria hyperbolic ones?

4.3.4 Explain how the case with friction is the right framework in order to study the pitchfork bifurcation. When does this bifurcation take place? Is the equilibrium point hyperbolic at that location?

4.3.5 Use the continuation tool provided in the notebook in order to get the various branches of equilibria. Explain how you will choose the initial vector in the two-dimensional plane (ω, y_1) at the bifurcation point in order to detect the proper branches of solutions for $\omega > \omega_c$.

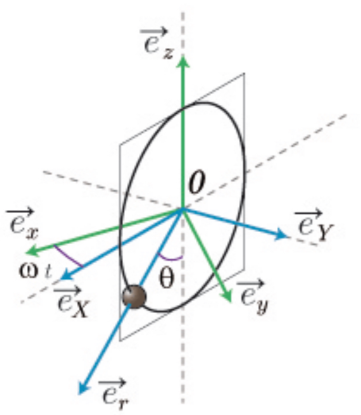

Figure 1: Various referentials for the study of the bead on a rotating hoop.

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