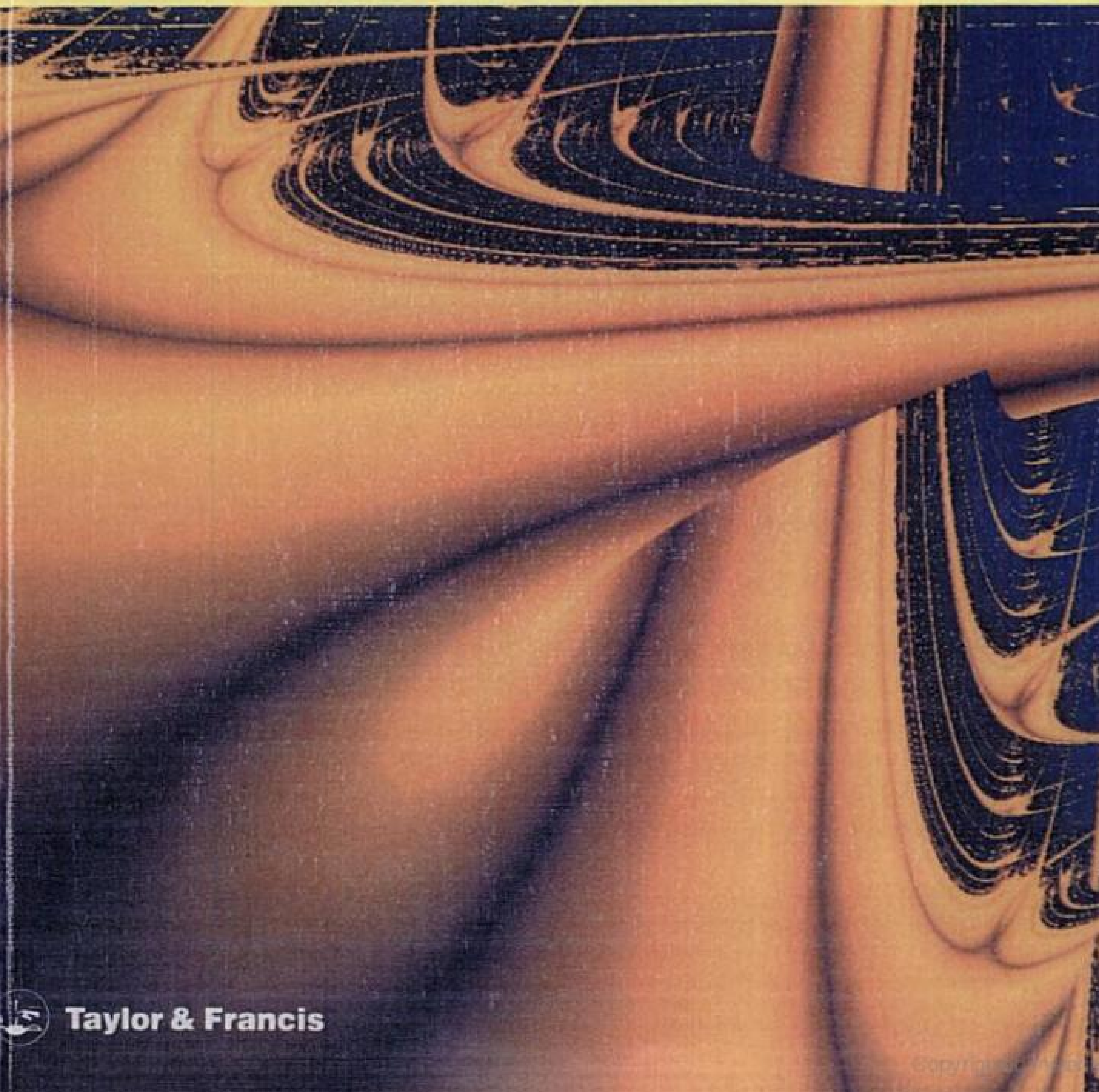
The General Problem of the

STABILITY OF MOTION

A. M. Lyapunov



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A. M. Lyapunov (1857–1918)

INT. J. CONTROL, 1992, VOL. 55, NO. 3, 521–527

Guest Editorial

Lyapunov Centenary Issue

One hundred years ago A. M. Lyapunov's major memoir on the stability of motion was published. To mark the centenary the present issue of the *International Journal of Control* is devoted to Lyapunov's work. Most of the issue is taken up with an English translation (by the present editor) of the 1892 memoir; in addition a bibliography of Lyapunov's published works and a translation of a biography have been prepared by J. F. Barrett.

Another contribution to the study of stability, that of E. J. Routh, was similarly marked with a centenary issue of the *International Journal of Control*; see Vol. 26, No. 2 (1977).

Translation notes

Aleksandr Mikhailovich Lyapunov's *The General Problem of the Stability of Motion* was published in Russian by the Mathematical Society of Kharkov in 1892. A French translation by É. Davaux appeared in the *Annales de la Faculté des Sciences de l'Université de Toulouse*, Vol. 9 (1907), pp. 203–474, and this was reprinted by Princeton University Press in 1949. Lyapunov himself reviewed and corrected the French version, and introduced some additional material.

While the French translation has done excellent service over the past 85 years, there is no doubt that work of this length and depth is much easier to absorb if it is in one's own language. For this reason the present English version has been prepared.

The English is a translation from the French, and to reduce the risk of deviating too far from Lyapunov's original treatment, a rather literal translation of the French is given. Thus the temptation to split up some of Lyapunov's longer and more involved sentences has been resisted. (It seems that the French translation is also a rather literal one.)

Lyapunov's mathematical style is economical, so much so that he sometimes leaves his readers at a temporary loss as to which equations or other relations he has in mind. The present writer has therefore inserted occasional interpretive comments, in square brackets.

Lyapunov numbered his sections but did not provide them with titles. To improve readability, titles have been inserted in the English version; these have been obtained from the brief descriptions which Lyapunov gave in his contents list.

A few minor errors which have been found in the French have been silently corrected in the English. When, however, the errors evidently go back to the Russian original, attention is drawn to them.

Historical notes

One of Lyapunov's main results is to the following effect. Suppose that for a given dynamical system we can find a function of the state coordinates which is positive-definite, and for which the rate of change following a small displacement from equilibrium is always negative or zero. Then the equilibrium is stable. Control scientists are familiar with this stability criterion, which Lyapunov gives as Theorem I in his section 16. Let us look into the historical background to the result.

During the 18th century, astronomers and mathematicians made great efforts to show that the observed deviations of planets and satellites from fixed elliptical orbits were in agreement with Newton's principle of universal gravitation, provided that due account was taken of the disturbing forces exerted by the bodies on one another. The deviations are of two kinds: first, oscillatory motions with relatively short periods, i.e. periods of the order of a few years, and second, residual slow changes in the ellipse parameters, which changes may be non-oscillatory or may be oscillatory with very long periods, perhaps of the order of tens of thousands of years. The first kind are known as *periodic inequalities*, and may be accounted for as the response of a body to the periodic forces exerted on it by its neighbours' continual tracing of their orbits. The second kind are called *secular inequalities*, and for the solar system the question arises as to whether the secular inequalities will build up over the millennia and destroy the system.

Laplace (1784, sections 1–4, 13, 14) investigated secular inequalities, and arrived at the following results. Consider a system of k satellites describing ellipses with varying parameters round a central body with relatively large mass. Suppose that the jth satellite has mass m_j, its ellipse has major semi-axis a_j and eccentricity e_j, and the plane of the ellipse is inclined to a fixed reference plane at an angle i_j. In earlier work Laplace had found that, at least approximately, the a_j are not subject to secular inequalities, and a further study by Lagrange (1776) had arrived at the same conclusion. This property enabled Laplace to treat the a_j as constants; and on allowing the e_j and the i_j to vary with time, and restricting attention to secular inequalities, he claimed that

$$V_1(e_1, e_2, \ldots, e_k) \equiv \sum_{j=1}^{k} m_j(a_j)^{1/2} e_j^2 = \text{const.} \tag{1}$$

$$V_2(i_1, i_2, \ldots, i_k) \equiv \sum_{j=1}^{k} m_j(a_j)^{1/2} \tan^2 i_j = \text{const.} \tag{2}$$

The constancy of these expressions means that if $e_1, e_2, \ldots, e_k$ and $|i_1|, |i_2|, \ldots, |i_k|$ are all initially small, they will remain so, i.e. we have stability. Thus Laplace's results tended to support the hypothesis that the solar system is stable. However, since various approximations were made in his analysis, this conclusion was not established rigorously.

Actually $-V_1$ represents a contribution made by the e's to an expression for the total angular momentum, under the assumption that these eccentricities are all small. It is conservation of angular momentum that requires this contribution to be constant. Similarly $-V_2$ represents a contribution to angular momentum made by the i's, assuming that these inclinations are all small. (Incidentally the last assumption implies that tan i_j in (2) can be replaced by i_j.)

Now the functions V_1 and V_2 in (1) and (2) are each positive-definite and with zero rate of change. Thus they are examples of what we now call Lyapunov functions. Lyapunov was the son of an astronomer, and was himself mainly interested in mathematical problems of astronomy. He would therefore have been familiar with Laplace's application of V_1 and V_2 to the study of stability.

Next let us turn to another part of the background against which Lyapunov's technique was developed. A principle already used in the 17th century was as follows: if a system of interconnected heavy bodies is in equilibrium, the centre of gravity is at the lowest point. This is known as Torricelli's principle, since he was the first to publish material relating to it (Torricelli 1644). But Torricelli was a friend and disciple of Galileo, and according to Gliozzi (1976) the principle was already known to Galileo.

In a more modern formulation, Torricelli's principle amounts to the result that for a mechanical system a configuration with minimum potential energy corresponds to a point of equilibrium. Lagrange (1788) gave a proposition to the effect that for a conservative system an isolated minimum of the potential energy corresponds to a stable point of equilibrium. Lagrange's proof was not convincing, since in effect it involved neglecting nonlinear terms in the differential equations and confining attention to the resulting linear system. Dirichlet (1846) took up the problem of finding a more rigorous demonstration of Lagrange's principle, and supplied what was essentially the following argument.

Let the kinetic and potential energies of the system be T and U respectively. Consider the point of equilibrium as embedded within a small region R of configuration space. Let U_{m} be the minimum value of the potential energy for points on the boundary of this region. Then since the point of equilibrium corresponds to a local minimum of potential energy we may choose the initial position in R so close to this point that the initial potential energy U_0 satisfies

$$U_0 < U_{\mathrm{m}} \tag{3}$$

Let us further choose the initial velocities so small that the initial total energy satisfies

$$T_0 + U_0 < U_{\mathrm{m}} \tag{4}$$

Suppose next that the system subsequently reaches the boundary of region R. At this point we have $U \geqslant U_{\mathrm{m}}$ and $T \geqslant 0$, so that

$$T + U \geqslant U_{\mathrm{m}} \tag{5}$$

But (4) and (5) contradict conservation of energy which requires

$$T + U = T_0 + U_0 \tag{6}$$

Hence the above supposition that the system can reach the boundary of R is false, i.e. stability holds.

Joseph Liouville (1842, 1855) applied Lagrange's principle to the study of equilibrium figures of rotating fluid bodies in which the particles attract each other gravitationally. Actually Liouville used a maximum kinetic energy principle rather than minimum potential energy; but since the total energy is constant the two principles are equivalent (Lagrange had mentioned both principles). Liouville published only enigmatic excerpts from his work on this problem; however, Lützen (1984, 1990) has located much of the missing material in Liouville's manuscripts.

In 1882 Chebyshev proposed to Lyapunov the problem of finding whether for rotating fluid bodies there exist non-ellipsoidal figures of equilibrium which are close to the known ellipsoidal figures. This problem led Lyapunov to the study of the stability of ellipsoidal figures of equilibrium, and he published his master's dissertation on the subject in 1884. His approach was similar to that of Liouville, i.e. he investigated conditions for the figure to have a minimum of potential energy, or rather of potential energy modified to allow for centrifugal force. Presumably Lyapunov learned of Liouville's work on rotating fluids from Chebyshev, who had often travelled to France and was a friend of Liouville.

So some eight years before he published his 1892 memoir on stability (which was his doctoral dissertation) Lyapunov was already fully conversant with Lagrange's principle of minimum potential energy. It no doubt occurred to him that in Dirichlet's proof the fact that the total energy is composed of *two* positive-definite functions T and U is inessential. What is essential is that the *total* energy is positive-definite and with zero rate of change (i.e., as we would now say, the total energy is a Lyapunov function).

Thus suppose that instead of $T+U$ we consider a single positive-definite function V of the state coordinates, such that $V=0$ at a point of equilibrium, and let V_m be the minimum value of V for points on the boundary of a small region S containing the point of equilibrium in its interior. Let us choose the initial position, at time $t=t_0$, so close to the point of equilibrium that the initial value V_0 of V satisfies

$$V_0 < V_m \tag{7}$$

Then if the state reaches the boundary of region S we have there

$$V \geqslant V_m \tag{8}$$

Inequalities (7) and (8) yield

$$V > V_0 \tag{9}$$

which is incompatible with the assumed property

$$\frac{dV}{dt} \leqslant 0 \quad (t > t_0) \tag{10}$$

Hence the boundary of region S cannot actually be reached; in other words we have stability.

This proof of his stability criterion is basically the one given by Lyapunov in his section 16, although he disguises it a little. He acknowledges that it uses the same considerations as in Dirichlet's proof.

The work of Routh and of Poincaré further influenced Lyapunov's treatment of stability problems; and he also made use of the second edition (1879) of Thomson and Tait's *Treatise on Natural Philosophy*.

For further notes and references on the early history of stability theory, see Fuller (1982, section 4). For a discussion of methods of investigating the stability of the solar system see Message (1984).

Lyapunov's first method

Another of Lyapunov's results is as follows. Suppose that for the system

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \tag{11}$$

the f's are analytic functions with

$$f_i(0, 0, \ldots, 0) = 0 \quad (i = 1, 2, \ldots, n) \tag{12}$$

Consider the linear system of the first approximation, obtained by expanding the f's in power series and dropping all except the linear terms. Suppose that the linear system is asymptotically stable, i.e. that the $|x_i|$, starting from small initial values, all approach zero when t approaches infinity. Then the nonlinear system (11) also is asymptotically stable.

In his sections 11–13 Lyapunov proves a more general version of this result by using what he calls his first method, i.e. by finding series solutions for system (11). The proof is involved due to the multiplicity of the terms which arise, and because he allows the f's to depend explicitly on t. However, a preliminary idea of his technique can be obtained by working with a simple example.

Thus suppose we take the scalar nonlinear system

$$\frac{dx}{dt} = -x + x^2 \tag{13}$$

The linear system of the first approximation is

$$\frac{dx}{dt} = -x \tag{14}$$

with solution

$$x(t) = x(0)e^{-t} \tag{15}$$

and is consequently asymptotically stable. We have to show that the same property holds for the nonlinear system (13).

Let us try to fit a solution of the form

$$x = x_1 + x_2 + x_3 + \cdots \tag{16}$$

where $|x_1|$ is small and $|x_1|, |x_2|, |x_3|, \ldots$ are assumed to have successively decreasing orders of magnitude. Substitution of (16) in (13) gives

$$\frac{dx_1}{dt} + \frac{dx_2}{dt} + \cdots = -(x_1 + x_2 + \cdots) + (x_1 + x_2 + \cdots)^2 \tag{17}$$

Equating terms with the same order of magnitude, we get

$$\frac{dx_1}{dt} = -x_1 \tag{18}$$

$$\frac{dx_r}{dt} = -x_r + \sum_{j=1}^{r-1} x_j x_{r-j} \quad (r = 2, 3, \ldots) \tag{19}$$

The solution of (18) is

$$x_1 = ae^{-t} \tag{20}$$

where a is an arbitrary constant. Substituting (20) in (19) with $r = 2$ we find

$$\frac{dx_2}{dt} = -x_2 + a^2e^{-2t} \tag{21}$$

This linear equation is easily integrated, and a particular solution is

$$x_2 = -a^2e^{-2t} \tag{22}$$

In the same way equations (19) can be solved with further values of r taken successively, and induction leads to

$$x_r = -(-a)^r e^{-rt} \tag{23}$$

Thus the series solution (16) for the nonlinear system (13) takes the form

$$x = ae^{-t} - a^2e^{-2t} + a^3e^{-3t} - \cdots \tag{24}$$

Putting

$$q = ae^{-t} \tag{25}$$

we have

$$x = q - q^2 + q^3 - q^4 + \cdots \tag{26}$$

This series is convergent if $|q| < 1$. If we choose $|a| < 1$ we shall have $|q(t)| < 1$ for all $t > 0$, and thus $x(t)$ will be finite for all $t > 0$. Furthermore since $q(t) \to 0$ when $t \to \infty$, equation (26) gives

$$x(t) \to 0 \quad \text{for } t \to \infty \tag{27}$$

i.e. the nonlinear system is asymptotically stable.

Note that series (26) sums to

$$x = q(1 + q)^{-1} = a/(a + e^t) \tag{28}$$

and it can be readily checked that this function satisfies the differential equation (13).

Singular cases

When the linear system of the first approximation is on the boundary of stability, the nonlinearities have a decisive influence on whether the actual system is stable or not. Lyapunov devoted considerable effort to exploring the simplest of these *singular cases*. For this purpose he used his second method, i.e. application of Lyapunov functions (see his sections 28–41, 56–64).

In his 1892 memoir he deliberately omitted discussion of one of these cases—that where the linear system of the first approximation has two zero eigenvalues. However, a treatment of this case was published posthumously, and an English translation is available (Lyapunov 1966). It may be noted that the English version, despite its title *Stability of Motion*, is not a translation of the 1892 memoir.

Stability and asymptotic stability

Lyapunov's definition of *stability* allows the system to perform persistent small oscillations about a point of equilibrium, or about a state of motion. Control engineers are more interested in achieving *asymptotic stability*, in which any small oscillations eventually die out, and they may wonder why Lyapunov gives so much consideration to ordinary as opposed to asymptotic stability. The reason is that Lyapunov was, as already mentioned, mainly concerned with astronomical problems. In such problems, as idealized mathematically, there are no dissipative forces, so that asymptotic stability does not occur. The best that can be hoped for is ordinary stability in which the bodies perform small oscillations (called librations by astronomers) about their nominal motions.

For such cases the system is at best on the boundary of stability, and the question of whether nonlinearities in the differential equations will cause the system to move off the boundary into instability is a matter of some delicacy. In fact Lyapunov was unable to resolve the problem of stability of motion for Hamiltonian systems of high enough order to be relevant to astronomical problems, although he obtained partial results in this direction (see his section 45).

When a system has ordinary but not asymptotic stability, if a Lyapunov function can be found its rate of change will be zero, so that the function will be an integral of the system. Thus the search for Lyapunov functions in astronomical problems is equivalent to the search for new integrals of their differential equations—a formidable task indeed.

Lyapunov's failure to make much progress with some of the stability problems of astronomy would have been a disappointment to him, and this is perhaps what induced him to abandon his original plan of writing a more extensive treatise on stability, and to content himself with the still substantial memoir which is now before us. He could scarcely have foreseen the interest in his methods that would eventually arise in the field of control science.

For further discussion of Lyapunov's work see Pressland (1931) and Grigorian (1974).

Acknowledgments

Thanks are due to J. F. Barrett who, besides contributing the two papers mentioned earlier, has supplied helpful criticism and advice during the preparation of the issue.

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INT. J. CONTROL, 1992, VOL. 55, NO. 3, 531–773

The general problem of the stability of motion

A. M. LYAPUNOV

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Translated from French into English by A. T. Fuller.‡

Preface

In this work some methods are expounded for the resolution of questions concerning the properties of motion and, in particular, of equilibrium, which are known by the terms *stability* and *instability*.

The ordinary questions of this kind, those to which this work is devoted, lead to the study of differential equations of the form

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \dots, \quad \frac{dx_n}{dt} = X_n,$$

of which the right-hand sides, depending on time t and on unknown functions $x_1, x_2, \dots, x_n$ of t, may be developed, provided the x_s are sufficiently small in absolute value, in series of positive integer powers of the x_s, and vanish when all these variables are equal to zero.

The problem reduces to finding if it is possible to choose the initial values of the functions x_s so small that, for all time following the initial instant, these functions remain in absolute value less than limits given in advance, which may be as small as one wishes.

When we know how to integrate our differential equations, this problem certainly presents no difficulties. But it will be important to have methods which permit it to be resolved independently of the possibility of this integration.

It is known that there exist cases where the problem considered reduces to a problem of maxima and minima.§ But the range of the questions which can be resolved by this procedure is very limited, and in most cases it is necessary to resort to other methods.

The procedure ordinarily used consists in neglecting, in the differential equations under study, all the terms of higher than first order with respect to the quantities x_s

† Mr Lyapunov has very graciously authorized the publication in French of his memoir *Obshchaya zadacha ob ustoichivosti dvizheniya* printed in 1892 by the Mathematical Society of Kharkov. The [French] translation has been reviewed and corrected by the author [Lyapunov], who has added a note based on an article which appeared in 1893 in *Communications de la Société mathématique de Kharkow*.

‡ [Comments in square brackets are by A.T.F.]

§ We have in mind the cases where there applies the known theorem of Lagrange on the maxima of the force-function [this is minus the potential energy function], relating to the stability of equilibrium; also, the cases where there applies a more general theorem of Routh on the maxima and minima of the integrals of the equations of motion, allowing the resolution of certain questions relative to the stability of motion (see *The advanced part of A Treatise on the Dynamics of a System of Rigid Bodies*, fourth edition, 1884, pp. 52, 53).

and in considering, in place of the given equations, the linear equations thus obtained.

It is in this way that the question is treated in the work of Thomson and Tait, *Treatise on Natural Philosophy* (Vol. I, Part I, 1879), in the works of Routh, *A Treatise on the Stability of a Given State of Motion* (1877) and *A Treatise on the Dynamics of a System of Rigid Bodies* (Part II, fourth edition, 1884) and, finally, in the work of Zhukovski [Joukowsky] 'On the stability of motion', *Memoires Scientifiques de l'Université de Moscou*, Section physico-mathématique (4^{me} Cahier, 1882) [*Uchenie zapiski Moskovskogo universiteta*, otdel fiziko-matematicheskii, 1882, Vip. 4].

The procedure just mentioned certainly involves an important simplification, especially in the case where the coefficients of the differential equations are constants. But the legitimacy of such a simplification is not at all justified *a priori*, because for the problem considered there is then substituted another which might turn out to be totally independent. At least it is obvious that, if the resolution of the simplified problem can answer the original one, it is only under certain conditions, and these last are not usually indicated.

Nevertheless it should be noted that some authors (thus, for example, Routh), recognized that this procedure is not rigorous, not limiting themselves to a first approximation resulting from the integration of the above-mentioned linear equations, but considering equally a second and some further approximations, obtained by the usual methods. But in operating thus one makes little advance, for, in general, by this approach one obtains only a more exact representation of the functions x_s within the limits of a certain interval of time. This certainly does not give new data for obtaining any conclusions on stability.

The only attempt, as far as I know, at a rigorous solution belongs to Poincaré, who, in the remarkable memoir composed of several papers—'Sur les courbes définies par les équations différentielles' (*Journal de Mathématiques*, third series, Vols. VII and VIII; fourth series, Vols. I and II), and, in particular, in the two last parts, considered questions of stability for the case of second order systems of differential equations, and arrived also at some related questions pertaining to systems of third order.

Although Poincaré limited himself to very special cases, the methods he used allow much more general applications and could still lead to many new results. This will be seen in what follows, for, in a large part of my researches, I was guided by the ideas developed in the above-mentioned memoir.

The problem that I set myself in undertaking the present study can be formulated as follows: to indicate the cases where the first approximation really resolves the question of stability, and to give procedures which allow it to be resolved, at least in certain cases, when the first approximation no longer suffices.

To arrive at some results, it will be necessary at the outset to make certain hypotheses, relative to the differential equations considered.

The simplest hypothesis, and at the same time one which will facilitate the more important and interesting applications, consists in this: that the coefficients in the developments of the right-hand sides of these equations are constant quantities. The more general hypothesis that the coefficients are periodic functions of time also corresponds to very numerous questions of interest.

It is under these two hypotheses that I principally treat the question.

For the rest, I touch on the more general case where the said coefficients are arbitrary functions of time which never exceed, in absolute value, certain limits.

It is under this general hypothesis that the question is treated in the first chapter of my work, where I demonstrate a proposition concerning the integration of the relevant differential equations with the aid of certain series,† and where I indicate some conclusions relative to stability which follow from it. Under the same hypothesis some further propositions are demonstrated here, forming the basis for later conclusions.

The first chapter forms only a sort of introduction where I demonstrate some fundamental propositions, while the second and third constitute the main part; and it is there that the cases of constant and periodic coefficients are considered.

I begin in each of these two chapters with some remarks concerning the linear differential equations which correspond to the first approximation, and in the third chapter, where the case of periodic coefficients is treated, I enter in some detail into the subject of what is called the *characteristic equation*.

Passing next to the principal question, I make apparent under what conditions it is resolved with the first approximation, and I then come to the *singular* cases where it is necessary to take account of terms of higher order than the first.

Now the cases of this kind are very varied, and in each of them the problem has its own special character, so that there cannot be any question of general methods which could embrace all cases.

Thus the different possible cases are to be considered separately, and I limit myself here to the simplest ones which present the less serious difficulties. It is their study, and the exposition of the corresponding methods for the resolution of stability questions, which constitute the greater part of the last two chapters.

Without entering into lengthier detail on the content of this work, I will however note that in the second chapter I treat the question of periodic solutions of nonlinear differential equations. This question is found to have a direct relation with the methods which I have proposed for one of the singular cases. Moreover its examination leads to some conclusions on conditional stability for the more interesting cases where the differential equations have the canonical form [Hamiltonian equations]. These conclusions constitute almost all that can be said of a general nature on these important cases.

The reader will not find in the present work a solution of such and such a problem of mechanics. According to the original plan, applications of this kind were to form a fourth chapter. But subsequently I dropped the intention of adding this, having in view the following considerations.

All the more interesting and important questions of mechanics (such as, for example, those which lead to canonical equations) are such that, in the singular cases where the first approximation does not suffice, the problem becomes more difficult, and at present one cannot indicate any method to resolve it. This is why, in the examination of these questions, I would have had to limit myself solely to examples of two kinds: to those where the question reduces to a problem of maxima and minima (by virtue of the theorem of Routh), or indeed to those where it is

† The series in question here have been considered, under more special hypotheses, in my memoir 'Sur les mouvements hélicoidaux permanents d'un corps solide dans un liquide' (*Communications de la Société mathématique de Kharkow*, second series, Vol. I, 1888). I subsequently learnt that Poincaré had considered these series, under the same hypotheses, in his thesis *Sur les propriétés des fonctions définies par les équations aux différences partielles* (1879).

resolved with the first approximation. But these examples, although they present a certain interest, are not relevant to the principal object of my researches which, as already mentioned, consist of the examination of methods relating to singular cases belonging to certain categories. As regards examples relevant to these methods, one would be obliged to choose them from the sphere of those questions of mechanics where the resistances of the medium are considered. One could without doubt cite as many examples of this kind as one would like; but they would not in themselves present a great interest, and would only be of importance for illuminating the said methods. Now if one had in view exclusively this last aim, the examples of an analytic nature that I have given at suitable points in the last two chapters are largely sufficient.

I may remark, in finishing, that my work is not a treatise on stability, where the consideration of problems of mechanics of all kinds would be obligatory. Such a treatise would include many questions which are not even touched on here.

I have had in mind solely to expound in this work that which I have arrived at up to the present moment, and which, perhaps, may serve as a point of departure for other researches of the same kind.

During the printing of this work, which extended over more than two years, there have appeared two very interesting works by Poincaré, treating questions related to many of those which I have considered. I refer to his memoir 'Sur le problème des trois corps et les équations de la dynamique' which appeared in *Acta mathematica*, Vol. XIII, a short time after I had begun to arrange the printing of my work, as well as the first volume to appear of his treatise entitled *Les méthodes nouvelles de la mécanique céleste* (Paris, 1892).

In the first are found certain results analogous to those which I have obtained, which I indicate at suitable points of my work. As for the second, I have not yet had time to study it in detail; but insofar as the questions which I have considered are concerned, it does not seem to contain essential additions to the memoir of *Acta mathematica*.

I should mention an expression of which I often make use, as do French and German mathematicians, for brevity, namely this: a series satisfying *formally* such and such equations.

This expression has a very vague sense; but I judged it superfluous to enter into explanations, since there cannot arise any doubt about its meaning in the cases where I have occasion to use it.

A. LYAPUNOV
Kharkov, 5 April 1892

CHAPTER I. Preliminary analysis

Generalities on the question under study

1. [*The problem of stability from a general point of view. Definition of stability*]

Let us consider a material system with k degrees of freedom. Let

$$q_1, q_2, \ldots, q_k$$

be k independent variables by which we agree to define its position.

We shall suppose that we have taken for these variables quantities which remain real for each position of the system.

In considering these variables as functions of time t, we shall designate their first derivatives with respect to t as

$$q_1', q_2', \ldots, q_k'.$$

In each problem of dynamics in which the forces are given in a determinate way, these functions will satisfy k differential equations of the second order.

Let us suppose as found for these equations a particular solution

$$q_1 = f_1(t), \quad q_2 = f_2(t), \quad \ldots, \quad q_k = f_k(t),$$

in which the quantities q_j are expressed as real functions of t, only giving for the q_j, whatever the value of t, real values.†

To this particular solution will correspond a determinate motion of our system. In comparing it, in a certain respect, with other possible motions for this system under the action of the same forces, we shall call it *the undisturbed motion*, and all the others with which it is compared will be termed *disturbed motions*.

In taking t_0 as a given time instant, let us designate the corresponding values of the quantities q_j, q_j', in an arbitrary motion, as q_{j0}, q_{j0}'.

Let

$$q_{10} = f_1(t_0) + \varepsilon_1, \quad q_{20} = f_2(t_0) + \varepsilon_2, \quad \ldots, \quad q_{k0} = f_k(t_0) + \varepsilon_k,$$
$$q_{10}' = f_1'(t_0) + \varepsilon_1', \quad q_{20}' = f_2'(t_0) + \varepsilon_2', \quad \ldots, \quad q_{k0}' = f_k'(t_0) + \varepsilon_k',$$

where $\varepsilon_j, \varepsilon_j'$ are real constants.

These constants, which we shall call *perturbations*, define a disturbed motion. We shall suppose that we can attribute to them all sufficiently small values.

In speaking of disturbed motions *near* to an undisturbed motion, we shall understand by this motions for which the perturbations are sufficiently small in absolute value.

After these preliminaries, let $Q_1, Q_2, \ldots, Q_n$ be given real and continuous functions of the quantities

$$q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'.$$

† It can happen that for the quantities q_j, depending on the way the latter are chosen, only values between certain limits are real.

For the undisturbed motion they will become known functions of t, which we shall designate by $F_1, F_2, ..., F_n$ respectively. For a disturbed motion they will be functions of the quantities

$$t, \varepsilon_1, \varepsilon_2, ..., \varepsilon_k, \varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_k.$$

When all the $\varepsilon_j, \varepsilon'_j$ are equal to zero, the quantities

$$Q_1 - F_1, \quad Q_2 - F_2, \quad ..., \quad Q_n - F_n$$

will be zero for each value of t. But if, without making the constants $\varepsilon_j, \varepsilon'_j$ zero, we suppose them infinitely small, the question arises of knowing whether it is possible to assign to the quantities $Q_s - F_s$ infinitely small limits, such that these quantities never surpass them in absolute value.

The solution of this question, which constitutes the object of our researches, depends on the character of the undisturbed motion under consideration, as well as the choice of the functions $Q_1, Q_2, ..., Q_n$ and of the time instant t_0. Thus, this choice being fixed, the answer to this question will characterize in a certain respect the undisturbed motion, and it is this answer which will express for it the property which we call *stability*, or the opposite property, which will be called *instability*.

We shall concern ourselves exclusively with cases where the solution of the question considered does not depend on the choice of the instant t_0 at which the perturbations are produced. This enables us to adopt here the following definition.

Let $L_1, L_2, ..., L_n$ be given positive numbers. If for all values of these numbers, no matter how small, we can choose positive numbers

$$E_1, E_2, ..., E_k, E'_1, E'_2, ..., E'_k$$

such that, the inequalities†

$$|\varepsilon_j| \leqslant E_j, \quad |\varepsilon'_j| \leqslant E'_j \quad (j = 1, 2, ..., k),$$

being satisfied, we have

$$|Q_1 - F_1| < L_1, \quad |Q_2 - F_2| < L_2, \quad ..., \quad |Q_n - F_n| < L_n,$$

for all values of t greater than t_0, the undisturbed motion will be called stable WITH RESPECT TO THE QUANTITIES $Q_1, Q_2, ..., Q_n$; in the contrary case, it will be called, with respect to the same quantities, unstable.

Let us cite some examples.

If a particle, attracted to a fixed centre in proportion to the inverse square of the distance, describes a circular trajectory, its motion, with respect to the radius vector drawn from the centre of attraction, and equally with respect to its speed, is stable. The same motion, with respect to the rectangular coordinates of the particle, is unstable.

If the same particle describes an elliptical trajectory its motion is unstable, not only with respect to the rectangular coordinates, but also with respect to the radius vector and the speed. But it is stable, for example, with respect to the quantity‡

$$r - \frac{p}{1 + e \cos \varphi},$$

† In general, we agree to understand by $|x|$ the absolute value of the quantity x, or its modulus when x is complex.

‡ [This quantity equated to zero gives the equation of the ellipse in polar coordinates. p is the latus rectum of the ellipse.]

where p and e are a parameter and the eccentricity of the ellipse described by the particle in the disturbed motion, and r and ϕ are the radius vector of the particle in the disturbed motion and the angle made by this radius vector with the smallest radius vector in the undisturbed motion.

When a solid body with one fixed point, and not subjected to any force, rotates about the greatest or least of the axes of the ellipsoid of inertia relative to this point, its motion is stable with respect to the angular speed and to the angles made by the instantaneous axis with fixed axes or with axes attached to the body. In contrast, when it turns about the mean axis of the ellipsoid of inertia, its motion with respect to these same quantities is unstable.

It can happen that it is impossible to find limits E_j, E_j' satisfying the preceding definition, when the perturbations are arbitrary, but it is possible to do so as soon as the perturbations are subject to conditions of the form

$$f=0 \quad \text{or} \quad f \geqq 0$$

where f is a function of the quantities

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \quad \varepsilon_1', \varepsilon_2', \ldots, \varepsilon_k',$$

becoming zero when all these quantities are assumed equal to zero.

In such cases we shall say that the undisturbed motion is stable for perturbations subject to such and such conditions.

Thus, in the preceding example, the elliptical motion of the particle is stable with respect to its rectangular coordinates or to any other coordinates, for perturbations satisfying the condition of constancy of total energy or, according to the terminology of Thomson and Tait, for *conservative* perturbations.

In this way, for unstable motions, one will be able to speak of *conditional stability*.

2. [*General form of the differential equations studied for disturbed motion*]

The resolution of our question depends on the study of the differential equations of the disturbed motion or, in other words, on the study of the differential equations satisfied by the functions

$$Q_1 - F_1 = x_1, \quad Q_2 - F_2 = x_2, \quad \ldots, \quad Q_n - F_n = x_n.$$

The order of the system of these last equations will be, in general, the same, that is to say $2k$; but in certain cases it can be lower.

We shall suppose the number n and the functions Q_s to be such that the order of this system is n and that the system reduces to the normal form:

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \ldots, \quad \frac{dx_n}{dt} = X_n, \tag{1}$$

and throughout what follows we shall work with these last equations, calling them the differential equations of the disturbed motion.

All the X_s in equations (1) are known functions of the quantities

$$x_1, x_2, \ldots, x_n, t,$$

becoming zero for

$$x_1 = x_2 = \dots = x_n = 0.$$

We shall now make some hypotheses regarding these functions, and throughout what follows we shall treat equations (1) exclusively under these hypotheses.

We shall allow that the functions X_s are given not only for real values, but also for complex values of the quantities $x_1, x_2, \dots, x_n$ for which the moduli are sufficiently small, and that, at least for each value of t real and greater than or equal to t_0, these functions may be developed in positive integer powers of the quantities $x_1, x_2, \dots, x_n$ as absolutely convergent series for all values of the x_s satisfying the conditions

$$|x_1| \leqslant A_1, \quad |x_2| \leqslant A_2, \quad \dots, \quad |x_n| \leqslant A_n,$$

where $A_1, A_2, \dots, A_n$ are either non-zero constants or functions of t which never become zero.

In this manner all the X_s will be *holomorphic*† functions [analytic functions] of the quantities $x_1, x_2, \dots, x_n$, at least for t real and greater than t_0.

Let

$$X_s = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \sum P_s^{(m_1, m_2, \dots, m_n)} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n},$$

where the summation extends over all the non-negative values of the integers $m_1, m_2, \dots, m_n$ satisfying the condition

$$m_1 + m_2 + \dots + m_n > 1.$$

In these developments all the coefficients $p_{s\sigma}$, $P_s^{(m_1, m_2, \dots, m_n)}$ are functions of t, which, in accordance with our hypothesis, must remain finite and, by the very nature of the problem, real for every real value of t greater than or equal to t_0. We shall suppose, moreover, that for all these values of t these are continuous functions.

In attributing to t any one of the said values and in considering, in the development of X_s, the ensemble of terms of dimension higher than the first, for all complex values of the quantities $x_1, x_2, \dots, x_n$ of which the moduli are respectively equal to $A_1, A_2, \dots, A_n$, let us designate by M_s an upper bound for the modulus of this ensemble. Then we shall have, according to a known theorem [compare Goursat *Cours d'analyse mathématique*, Vol. II (1905), p. 273],

$$|P_s^{(m_1, m_2, \dots, m_n)}| < \frac{M_s}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}}. \tag{2}$$

† In making use of this expression for brevity in all that follows, we believe it necessary to state in a precise manner what we understand by the term. In considering a function of the variables $x_1, x_2, \dots, x_n$, we shall call it *holomorphic* with respect to these variables whenever it can be presented in the form of a multiple series of order n, ordered according to positive integer powers of the quantities x_s, at least for values of these last for which the moduli do not exceed certain non-zero limits.

In general, in what follows we shall consider only real values of t, not less than t_0, and if in such and such a case the need arises to consider other values of t, we shall always say so expressly.

Let us note that if, in the place of time, we take for the independent variable any real continuous function of time, increasing indefinitely with it, this function will be able to play the same role as time in questions of stability. For this reason the independent variable t in equations (1) need not designate time; but, in every case it will be a function of time satisfying the condition just enunciated.

Let us make further the following remark.

Let $a_1, a_2, ..., a_n$ be the values of the functions $x_1, x_2, ..., x_n$ for $t = t_0$. Then, to each system of real and sufficiently small† values of the quantities

$$\varepsilon_1, \varepsilon_2, ..., \varepsilon_k, \quad \varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_k, \tag{3}$$

there will correspond a system of real values of the quantities

$$a_1, a_2, ..., a_n. \tag{4}$$

Moreover, however small a given positive number A, we can always make the quantities (4) smaller than A by subjecting the quantities (3) to the condition of being, in absolute value, below a sufficiently small limit E.

We shall now assume that, however small the given positive number E, it will always be possible to find a positive number A such that to each system of real values of the quantities (4) which are smaller than A there corresponds one or more systems of real values of the quantities (3) smaller than E.

Under this condition, the quantities (4) can play the same role in the question of stability as the quantities (3), provided that the functions x_s satisfying equations (1) are entirely determined by specifying the quantities (4). This last condition, because of the hypotheses which we shall make later relative to equations (1) (Section 4), will always be satisfied. This is why in what follows we shall consider the quantities (4) in place of the quantities (3).

3. [*Integration by means of series ordered according to the powers of arbitrary constants*]

For the integration of equations (1) in the problem we are concerned with, there presents itself naturally the method of successive approximations, based on the assumption that the initial values (i.e. corresponding to $t = t_0$) of the functions being sought are sufficiently small.

This method, in its simplest form, leads to series which can be obtained in the following manner.

On putting

$$x_s = x_s^{(1)} + x_s^{(2)} + x_s^{(3)} + ... \quad (s = 1, 2, ..., n) \tag{5}$$

and on considering the quantities $x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)}$, as well as their derivatives with respect to t, as being of mth order, let us substitute these expressions for the

† In saying that a quantity is small, we shall always suppose that its absolute value is meant.

functions x_s in equations (1), and in each of these last let us equate terms of the same order on the two sides of the equality. In this way we shall obtain the following systems of differential equations:

$$\frac{dx_s^{(1)}}{dt} = p_{s1}x_1^{(1)} + p_{s2}x_2^{(1)} + \ldots + p_{sn}x_n^{(1)} \quad (s = 1, 2, \ldots, n), \tag{6}$$

$$\frac{dx_s^{(m)}}{dt} = p_{s1}x_1^{(m)} + p_{s2}x_2^{(m)} + \ldots + p_{sn}x_n^{(m)} + R_s^{(m)}$$

$$(m > 1, s = 1, 2, \ldots, n). \tag{7}$$

The $R_s^{(m)}$ here are entire rational functions [polynomial functions] of the quantities $x_\sigma^{(\mu)}$ with coefficients representing sums of products which are the functions $P_s^{(m_1, m_2, \ldots, m_n)}$ multiplied by positive integers. Moreover, the $R_s^{(m)}$ corresponding to a given value of m only depend on the $x_\sigma^{(\mu)}$ for which $\mu < m$.

As a consequence, the functions $x_s^{(m)}$ which we have introduced will be calculable on giving m successively the values $1, 2, 3, \ldots$

The first problem we shall have to concern ourselves with will thus consist of integrating the system of linear homogeneous equations (6).

On taking account of the continuity admitted for the coefficients $p_{s\sigma}$, it is not difficult to show that there will always exist a set of n^2 functions, finite and continuous for all the values of t considered,† this set representing a system of n independent solutions for the system of equations (6).

This proposition can be proved by forming in effect certain expressions for the functions $x_s^{(1)}$, satisfying the equations considered for every value of t greater than t_0, and taking the given values for $t = t_0$. Such expressions can be obtained in the form of series, on considering, for example, the equations which may be deduced from equations (6) by multiplying the right-hand sides by a parameter ε, and seeking to satisfy these new equations by series ordered according to positive integer powers of ε. If these series are formed under the hypothesis that the values of the functions sought for $t = t_0$ do not depend on ε, they will be absolutely convergent for all the values considered for t, *whatever the value of* ε. On putting $\varepsilon = 1$, we shall obtain the above-mentioned expressions for the $x_s^{(1)}$.

Let us suppose, then, that we have succeeded in finding by some means a system of n independent particular solutions for equations (6).

Let

$$x_{s1}, x_{s2}, \ldots, x_{sn}$$

be the functions of t, representing the function $x_s^{(1)}$, in these solutions.

Then the general integral of system (6) will be expressed by the equations

$$x_s^{(1)} = a_1 x_{s1} + a_2 x_{s2} + \ldots + a_n x_{sn} \quad (s = 1, 2, \ldots, n), \tag{8}$$

where $a_1, a_2, \ldots, a_n$ are arbitrary constants.

† In speaking of values of t, we shall always have in mind definite numbers. Thus we shall never consider infinity as a value of t.

After having found the functions $x_s^{(1)}$, we shall be able to determine the other $x_s^{(m)}$ by successive integration of the systems of non-homogeneous linear equations (7), corresponding to $m = 2, 3, \dots$

Each of these integrations may be effected by means of quadratures. Moreover, each of them will introduce n arbitrary contants and, to determine the latter, we shall be able to fix on any hypothesis, provided that the series obtained are convergent, at least within certain limits.

These constants will be entirely determined if we introduce the condition that all the $x_s^{(m)}$ for $m > 1$ become zero for $t = t_0$.

Let us seek, under this hypothesis, formulae for determining the functions $x_s^{(m)}$ when all the $x_\sigma^{(\mu)}$ for $\mu < m$ are already found.

Let us put

$$\begin{vmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \dots & \dots & \dots & \dots \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{vmatrix} = \Delta.$$

This determinant will be a function of t, not vanishing for any of the values considered for t, for, according to a known theorem [see e.g. E. Goursat, E. R. Hedrick and O. Dunkel *Differential Equations*, Boston, 1917, pp. 152–154]

$$\Delta = Ce^{\int \sum_1^n p_{ss}\, dt},$$

where C is a constant different from zero.

Let us designate the minor of this determinant, corresponding to the element x_{ij}, by Δ_{ij}.

Then the required formulae may be written thus:

$$x_s^{(m)} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{sj} \int_{t_0}^{t} \frac{\Delta_{ij}}{\Delta} R_i^{(m)}\, dt \quad (s = 1, 2, \dots, n). \tag{9}$$

The functions $x_s^{(m)}$ defined by these formulae remain finite and continuous for all the values considered for t.

Relative to the constants $a_1, a_2, \dots, a_n$ these are entire and homogeneous functions [polynomials] of the mth degree.

Moreover, if the chosen system of particular solutions of equations (6) is such that for $t = t_0$ all the x_{ij} take real values, the coefficients in these functions remain real for all the values considered for t.

After having obtained in this way the functions $x_s^{(m)}$, let us come to the question of the convergence of the series (5), which will present themselves as ordered according to positive integer powers of the constants a_s.

4. [*Study of the convergence of these series in the case where the arbitrary constants are taken as the initial values of the functions sought*]

We have already made some hypotheses relating to the coefficients in the expansions of the right-hand sides of equations (1). Now we add one more.

We shall assume that we can take for the quantities $A_1, A_2, \ldots, A_n$, $M_1, M_2, \ldots, M_n$ functions of t such that for every value of T greater than t_0, t varying between the limits t_0 and T, there exists for each of the functions A_s a non-zero lower bound, and for each of the functions M_s an upper bound.

Under this assumption we are going to demonstrate that for all values of t between t_0 and T, no matter how large the given number T, the preceding series (considered as ordered according to the powers of the quantities a_s) will be absolutely convergent, as long as the moduli of the a_s do not exceed a certain limit dependent on T.

We shall prove this, as for other similar theorems which we shall meet later, with the help of the method commonly used in such cases, which is due to Cauchy.

Let us note at the outset that, t being comprised between the limits t_0 and T, we can assign constant upper bounds to the moduli of all the x_{ij} and Δ_{ij}/Δ, or equivalently to the moduli of all the

$$x_{ii}-1, \quad x_{ij} \quad (i \lessgtr j), \tag{10}$$

$$\frac{\Delta_{ii}}{\Delta}-1, \quad \frac{\Delta_{ij}}{\Delta} \quad (i \lessgtr j). \tag{11}$$

Let K be such an upper bound for the quantities (10), and K' one for the quantities (11).

If the considered system of particular solutions of equations (6) is defined by the condition that, for $t = t_0$,

$$x_{ii}=1, \quad x_{ij}=0 \quad (i \lessgtr j),$$

we can take for K and K' continuous functions of T which become zero for $T = t_0$.

Let us designate, in a general manner, by $\{u\}$ the result of the replacement, in an arbitrary function u of the quantities $a_1, a_2, \ldots, a_n$, of all the terms by their moduli.

Then, on designating by a the greatest of the quantities $|a_s|$, we obtain from (8) and (9) the following inequalities:

$$\{x_s^{(1)}\} < (1+nK)a,$$

$$\{x_s^{(m)}\} < \int_{t_0}^{T} \{R_s^{(m)}\}\, dt + (K+K'+nKK') \sum_{i=1}^{n} \int_{t_0}^{T} \{R_i^{(m)}\}\, dt.$$

These inequalities will hold for every value of t between t_0 and T.

We note further that, by the nature of the original expression for $R_i^{(m)}$ as a function of the quantities $x_s^{(\mu)}$, $P_i^{(m_1, \ldots, m_n)}$, on replacing in it these last by upper bounds for the quantities

$$\{x_s^{(\mu)}\}, \quad |P_i^{(m_1, m_2, \ldots, m_n)}|,$$

we shall have an upper bound for the quantity $\{R_i^{(m)}\}$.

If then we designate by $x^{(\mu)}$ a common upper bound for quantities

$$\{x_1^{(\mu)}\}, \quad \{x_2^{(\mu)}\}, \quad \ldots, \quad \{x_n^{(\mu)}\}$$

over the limits considered for t, and by $R^{(m)}$ what each of the functions

$$R_1^{(m)}, R_2^{(m)}, \ldots, R_n^{(m)}$$

becomes when we replace in them the $x_s^{(\mu)}$ by the $x^{(\mu)}$ and the $P_i^{(m_1, m_2, \ldots, m_n)}$ by upper bounds $P^{(m_1, m_2, \ldots, m_n)}$, independent of i, for their absolute values over the same limits of t, we obtain

$$\{x_s^{(m)}\} < (1+nK)(1+nK')(T-t_0)R^{(m)}.$$

We see from this that we can take

$$x^{(1)} = (1+nK)a,$$

$$x^{(m)} = (1+nK)(1+nK')(T-t_0)R^{(m)} \quad (m = 2, 3, \ldots).$$

On the other hand, in conformity with inequalities (2), we can take for the $P^{(m_1, \ldots, m_n)}$ the following expressions:

$$P^{(m_1, m_2, \ldots, m_n)} = \frac{M}{A^{m_1+m_2+\ldots+m_n}},$$

where M is a common upper bound for all functions M_s over the specified limits for t, and A is a common lower bound for all the functions A_s over the same limits for t.

Now if we replace the coefficients $P_s^{(m_1, \ldots, m_n)}$ by these expressions, the ensembles of terms of degree higher than the first in the functions X_s become identical with the expansion of the function†

$$M\left\{\frac{1}{\left(1-\frac{x_1}{A}\right)\left(1-\frac{x_2}{A}\right)\ldots\left(1-\frac{x_n}{A}\right)} - 1 - \frac{x_1+x_2+\ldots+x_n}{A}\right\}.$$

In consequence, for the choice made for the quantities $P^{(m_1, \ldots, m_n)}$, the quantity $R^{(m)}$ will represent the ensemble of terms of the mth dimension relative to the indices of the quantities $x^{(s)}$ in the expansion of the expression

$$M\left\{\left(1-\frac{1}{A}\sum_{s=1}^{\infty} x^{(s)}\right)^{-n} - 1 - \frac{n}{A}\sum_{s=1}^{\infty} x^{(s)}\right\}.$$

From this it results that, if we consider the equation

$$x = (1+nK)a + Ah\left\{\left(1-\frac{x}{A}\right)^{-n} - 1 - n\frac{x}{A}\right\}, \tag{12}$$

where

$$h = (1+nK)(1+nK')\frac{M(T-t_0)}{A},$$

the series

$$x^{(1)} + x^{(2)} + x^{(3)} + \ldots$$

† [Use is made here of the following identity:

$$(1+y_1+y_1^2+\ldots)(1+y_2+y_2^2+\ldots)\ldots(1+y_n+y_n^2+\ldots) = \sum y_1^{m_1}y_2^{m_2}\ldots y_n^{m_n}$$

the sum being over all non-negative integer values of $m_1, m_2, \ldots, m_n$.]

will represent the expansion in positive integer powers of a of the root x of this equation, vanishing for $a = 0$. Hence this series will certainly converge if a is less than the quantity

$$g = \frac{A}{1+nK}\left\{1-(n+1)h\left[\left(\frac{1}{nh}+1\right)^{\frac{n}{n+1}}-1\right]\right\},$$

representing the smallest of the moduli of all the values of a for which equation (12) has multiple roots. This series will moreover be convergent even for $a = g$, for it has positive coefficients, and on the other hand, for the root in question, when a approaches g, there exists a limit.†

Now, because of the very definition of the quantities $x^{(m)}$, the convergence of the series considered implies the absolute convergence of the series (5) for all t between t_0 and T.

We can thus conclude that, for these values of t, the series (5) will be absolutely convergent if the moduli of the constants a_s do not exceed the quantity g.

We obtain at the same time an upper bound for the moduli of the sums of these series, under the conditions

$$t_0 \leqslant t \leqslant T, \quad |a_s| \leqslant g \quad (s = 1, 2, \ldots, n). \tag{13}$$

This limit is represented by the value, corresponding to $a = g$, of the root in question of equation (12) and, as is easy to convince ourselves, it does not exceed A.

It results from this last circumstance that if we substitute series (5) in the functions X_s, we can represent these functions by series ordered according to positive integer powers of the quantities a_s.

We may thus write, under these conditions, the equalities

$$X_s = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + R_s^{(2)} + R_s^{(3)} + \ldots \quad (s = 1, 2, \ldots, n),$$

which by virtue of equations (6) and (7) can be presented in the form

$$X_s = \frac{dx_s^{(1)}}{dt} + \frac{dx_s^{(2)}}{dt} + \frac{dx_s^{(3)}}{dt} + \ldots \quad (s = 1, 2, \ldots, n).$$

Now the series which appear on the right-hand sides are, under the conditions considered, uniformly convergent for all values of t between t_0 and T, and as a consequence they represent the derivatives of the functions defined by the series (5).

Finally then, under conditions (13) the series (5) represent functions which really do satisfy equations (1).

On the subject of the number g it is to be noted that, for $T = t_0$, it takes the value of the quantity

$$\frac{A}{1+nK},$$

† [In this paragraph Lyapunov is using (12), which may be written $f(x) = 0$, to replace the inequality $f(x) < 0$. This is justifiable since consideration of the graph of $f(x)$ shows that $f(x) = 0$ can have two real roots with $0 < x < A$, and the smallest of these is greater than a corresponding x satisfying $f(x) < 0$. Lyapunov then finds a condition for $f(x) = 0$ to have such a real root.]

for the same T. Now this value, conforming to what we have remarked above, can be supposed equal to the corresponding value of the quantity A, if the system that we have chosen of particular solutions to equations (6) is such that, for $t = t_0$,

$$x_{ii} = 1, \quad x_{ij} = 0 \quad (i \lessgtr j).$$

Under this last hypothesis the constants a_s are the values of the functions x_s for $t = t_0$. [Use (8) and recall that $x_s^{(m)}(t_0) = 0$ $(m > 1)$.]

We can therefore affirm that, A_0 being the smallest of the values taken by the functions A_s for $t = t_0$, and the a_s representing arbitrary given numbers of which the absolute values are below A_0, we can find a limit T greater than t_0, such that the functions x_s, satisfying equations (1) and taking the values a_s for $t = t_0$, are susceptible of being represented by absolutely convergent series ordered according to increasing powers of the a_s, for every value of t between t_0 and T.

Remark

One can certainly obtain, for representing the functions x_s over the same limits of variation of t, an infinity of other absolutely convergent series, ordered according to positive integer powers of arbitrary constants.

All the series of this kind can be deduced from the preceding ones by means of substitutions of the form

$$a_s = f_s(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad (s = 1, 2, \ldots, n), \tag{14}$$

the f_s being holomorphic functions of the quantities α_σ that one may adopt as new arbitrary constants.

In considering such series, let us take it that all the functions f_s become zero for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, but that the functional determinant [Jacobian] of these functions with respect to the quantities α_σ does not then become zero.

Then, if we take, in the series in question, the ensembles of terms of degree not greater than the mth relative to the constants α_σ, these ensembles will represent what we shall call the expressions for the functions x_s at the mth approximation.

It is known that, under the hypotheses made relative to the functions f_s, we can always satisfy equations (14), on taking for the α_σ certain holomorphic functions of the quantities a_s, vanishing for $a_1 = a_2 = \ldots = a_n = 0$, and that, the quantities $|\alpha_\sigma|$, $|a_s|$ being subject to the condition of not exceeding sufficiently small limits, this solution will be the only one possible.

By consequence, the different mth approximations furnished by the various series of the kind considered, being expressed in terms of the constants a_s, will be developable in series ordered according to positive integer powers of the a_s, and these series will only differ from one another in the terms of degree higher than the mth.

5. [*Two principal hypotheses under which the question will be studied. Steady motion and periodic motion. Two classes of method in the study of stability*]

From the general point of view with which we have considered the problem up to now, we have had in mind only to establish that there always exist, at least for t not going outside certain limits, functions satisfying equations (1) and taking at a given instant given sufficiently small values, and that the method of successive approximations yields series which, under certain conditions, can serve to determine these functions. But, as we move on to procedures for the resolution of questions of stability, we shall be obliged to abandon this point of view, limiting ourselves in our study to more precise hypotheses relating to the differential equations of the disturbed motion.

We shall consider principally the following two cases:

(1) when all the coefficients $p_{s\sigma}$, $P_s^{(m_1, \ldots, m_n)}$ are constant quantities, and

(2) when these are periodic functions of t with one and the same real period.

The first case could be considered as a special case of the second. We prefer however to examine it separately, for a number of reasons.

In the first case, following the example of Routh, we shall call the undisturbed motion (for the quantities with respect to which the stability is studied) *steady*; in the second case, we shall call it *periodic*.

In considering these two cases we shall see that, for our problem, the study of the first approximation will be of great importance.

We shall show under what conditions this study suffices to resolve completely the question of stability, and under what conditions it becomes, in general, insufficient. At the same time we shall give methods to resolve the question in certain cases in this last category.

But, before passing to the detailed examination of the question, we shall pause to consider some general propositions which will serve as points of departure for our researches.

All the procedures that we can indicate for resolving the question which concerns us can be divided into two categories.

In one, we shall gather together all those which reduce to the direct study of the disturbed motion, and which, as a result, depend on the search for general or particular solutions of the relevant differential equations.

We shall have, in general, to seek these solutions in the form of infinite series, of which the simplest type is furnished by the series considered in the preceding section. These are series ordered according to the positive integer powers of arbitrary constants. But we shall also meet in what follows certain series of another nature.

The collection of all the procedures for the study of stability, belonging to this category, will be called *the first method*.

In the other category, we shall collect every kind of procedure which is independent of the search for solutions of the differential equations of the disturbed motion.

Such is, for example, the known procedure for the examination of stability of equilibrium in the case where there exists a force-function [potential energy function times (-1)].

These procedures can reduce to the search for and study of integrals of equations (1), and in general all those which we shall meet in what follows will be based on the search for functions of the variables $x_1, x_2, \ldots, x_n, t$, for which the total derivative with respect to t, formed under the hypothesis that $x_1, x_2, \ldots, x_n$ are functions of t satisfying equations (1), has to satisfy such and such given conditions.

The collection of all the procedures in this category will be called *the second method*.

The principles of the latter, expressed in some general theorems, will be expounded at the end of this chapter. For the present we shall give some attention to the application of the first method to a rather general case of the differential equations of the disturbed motion, which embraces the case of steady motion as well as that of periodic motion.

This is the case where we can assume that for $t \geqq t_0$ there exist for the functions A_s a non-zero lower bound A and for the functions M_s an upper bound M, and where we can assign, for the same values of t, an upper bound for the absolute values of all the coefficients $p_{s\sigma}$.

We shall begin with the study of the linear differential equations corresponding to the first approximation.

On certain systems of linear differential equations

6. [*Characteristic numbers of functions*]

Let us agree to begin with on some expressions, and let us demonstrate some auxiliary propositions.

We are going to consider functions of a real variable t, taking completely determined values for every value of t which is greater than or equal to a certain limit t_0. Moreover, we shall only consider functions of which the moduli have upper bounds, as long as t is subjected to remaining in the interval (t_0, T), T being an arbitrary number greater than t_0.

If the modulus of such a function admits an upper bound under the sole condition $t > t_0$, we shall say that it is a *bounded* function. If, on the contrary, for a suitable choice of values of t greater than t_0, the modulus of the function considered can become greater than any given number, however great it may be, this function will be called *unbounded.* Finally, every bounded function which tends to zero when t increases indefinitely will be said to be a *vanishing* function.

When we have to consider at the same time as the function x the function $1/x$, we shall assume that, T being any number greater than t_0, the greatest lower bound of the function x in the interval (t_0, T) is different from zero.

With these definitions, we have the following propositions.

LEMMA I. *If x is a bounded function of t, $xe^{-\lambda t}$ will be a vanishing function, whatever the positive constant λ.*

This lemma follows immediately from the preceding definitions.

LEMMA II. *If x is not a vanishing function of t, $xe^{\lambda t}$ will be an unbounded function, whatever the positive constant λ.*

In fact, if x is not a vanishing function, we can always find a positive constant a such that, for a suitable choice of values of t greater than a limit T given arbitrarily, however great this may be, the modulus of the function x can be made greater than a. Thus, in considering only values of t chosen in this way, we shall have

$$|xe^{\lambda t}| > ae^{\lambda T}.$$

The lemma is proved by this, since the right-hand side of the inequality can be made as great as we wish on choosing T sufficiently great.

LEMMA III. *On taking x as a function of t, and λ_1 and λ' as real constants, let us assume that the function $z = xe^{\lambda t}$ is vanishing for $\lambda = \lambda_1$, and is unbounded for $\lambda = \lambda'$. Then we can find a real number λ_0 such that the function z for $\lambda = \lambda_0 + \varepsilon$ is unbounded or vanishing according as ε is a positive or a negative constant, and this will hold no matter how small ε may be.*

In fact it results from the preceding lemmas that, if there exists a constant value of λ for which the function z is bounded and non-vanishing, this value will be the value sought.

In the contrary case, on inserting between the numbers λ_1 and λ' a series of intermediate numbers and on passing successively in this series from the smallest to the largest, starting from λ_1 (for λ_1 is necessarily less than λ') we shall, to begin with, meet only numbers for which the function z is vanishing, and then only numbers for which it is unbounded.

Consequently, in the last case we can always obtain, by successive insertions of intermediate numbers according to some law chosen in a suitable way, two infinite series of numbers: non-decreasing

$$\lambda_1, \lambda_2, \lambda_3, \ldots$$

and non-increasing

$$\lambda', \lambda'', \lambda''', \ldots$$

such that every number of the first series is less than every number of the second series, that the difference

$$\lambda^{(n)} - \lambda_n$$

can be made as small as we wish by choosing n sufficiently large, and that, for every value of n, the function

$$xe^{\lambda_n t}$$

is vanishing and the function

$$xe^{\lambda^{(n)} t}$$

is unbounded.

These two series define a number λ_0, not less than any of the numbers of the first series and not greater than any of the numbers of the second series, which will be the required number.

We shall call the number λ_0 the *characteristic number* of the function x. [Thus the characteristic number of a function $x(t)$ is a measure of the rate of exponential decay of $x(t)$ for large t.]

Remark

The function x, for which the product $xe^{\lambda t}$ is a vanishing function for every value of λ or unbounded for every value of λ, does not have a characteristic number. But we can agree to say that in the first case the characteristic number is $+\infty$, and in the second case $-\infty$. With this convention, every function will have a characteristic number, finite or infinite.

Let us cite some examples.

For every constant different from zero the characteristic number is zero, and for zero it is $+\infty$.

For the function t^m (m constant) the characteristic number equals 0.

For the function $e^{t\cos(1/t)}$ the characteristic number equals -1.

For the function $e^{-t\cos(1/t)}$ the characteristic number equals $+1$.

For the function $e^{\pm t\sin t}$ the characteristic number equals -1.

For the function $e^{te^{\sin t}}$ the characteristic number equals $-e$.

For the function $e^{-te^{\sin t}}$ the characteristic number equals $+\frac{1}{e}$.

For the function t^t the characteristic number equals $-\infty$.

For the function t^{-t} the characteristic number equals $+\infty$.

[Thus, taking the last function, we have

$$t^{-t} = e^{-(\log t)t}$$

In the exponent here t is multiplied by $(-\log t)$, which approaches $-\infty$ when t increases. Hence the characteristic number is $+\infty$.]

Remark

In general, if $f(t)$ is a real function and λ is a real constant, such that we can make as small as we like the quantity

$$|\lambda - f(t)|$$

by suitable choice of the values of t greater than a given arbitrary limit, and if, further, for every positive constant ε no matter how small it may be, we can find a limit T such that we have

$$\lambda - f(t) < \varepsilon$$

t being greater than T, then λ will be the characteristic number of the function

$$e^{-tf(t)}.$$

We shall confine ourselves, in the propositions which follow, to cases where the characteristic numbers are finite. But Lemmas IV, V and VIII will also be true in all the cases of infinite characteristic numbers where they retain a definite meaning.

LEMMA IV. *The characteristic number of the sum of two functions is equal to the least of the characteristic numbers of the functions when these numbers are different, and is not less than these numbers when they are equal.*

Thus, let λ_1 and λ_2 be the characteristic numbers of the functions x_1 and x_2, and let $\lambda_1 \leqslant \lambda_2$.

The functions

$$x_1 e^{(\lambda_1+\varepsilon)t}, \quad x_2 e^{(\lambda_1+\varepsilon)t}$$

will then be vanishing for every negative value of ε. The same will then hold for their sum. On the other hand, if we have $\lambda_1 < \lambda_2$, and if ε is subject to the inequality

$$0 < \varepsilon < \lambda_2 - \lambda_1$$

the first of these functions will be unbounded, the second vanishing; hence their sum will be unbounded. Now the latter will thus be unbounded for every positive value of ε.

As a consequence the characteristic number of the function $x_1 + x_2$, never being less than λ_1, becomes equal to λ_1 if $\lambda_1 < \lambda_2$.

Remark

When the component functions having equal characteristic numbers are such that their ratio is a purely imaginary quantity, or, in general, a complex quantity with a constant argument differing from an odd multiple of π, the characteristic number of the sum is always equal to the characteristic number of the component functions.

LEMMA V. *The characteristic number of the product of two functions is not less than the sum of their characteristic numbers.*

In fact, if λ_1 and λ_2 are the characteristic numbers of the functions x_1 and x_2, the function

$$x_1 x_2 e^{(\lambda_1 + \lambda_2 + \varepsilon)t} = x_1 e^{\left(\lambda_1 + \frac{\varepsilon}{2}\right)t} x_2 e^{\left(\lambda_2 + \frac{\varepsilon}{2}\right)t}$$

is vanishing for every negative value of ε.

That the characteristic number of the product can be greater than the sum of the characteristic numbers of the factors appears clearly enough from the examples cited above. [E.g. the characteristic numbers of $e^{t \sin t}$ and $e^{-t \sin t}$ are both -1, but the characteristic number of their product is 0, not -2.]

COROLLARY. *The sum of the characteristic numbers of the functions x and $1/x$ is not greater than zero.*

LEMMA VI. *If*

$$x = e^{-t(f + i\varphi)},$$

where $i = \sqrt{-1}$, f and φ being real functions of t, for the sum of the characteristic numbers of the functions x and $1/x$ to be equal to zero, it is necessary and sufficient that the function f should have a limit when t increases indefinitely.

Indeed, if with t increasing indefinitely the function f tends to a number λ, the latter will obviously represent the characteristic number of the function x, and $-\lambda$ will be the characteristic number of the function $1/x$. The indicated condition is thus sufficient.

As for the necessity of the same condition, it results from this: that if λ and $-\lambda$ are the characteristic numbers of the functions x and $1/x$, the two functions

$$e^{-t(\varepsilon - \lambda + f)} \quad \text{and} \quad e^{-t(\varepsilon + \lambda - f)}$$

will be vanishing for every given positive value of ε, however small it may be; and this last condition is possible only if we have

$$|\lambda - f| < \varepsilon$$

for all values of t greater than a sufficiently large limit.

LEMMA VII. *If the sum of the characteristic numbers of the functions x and $1/x$ is equal to zero, the characteristic number of the product z of the function x and an arbitrary function y is equal to the sum of the characteristic numbers of these latter.*

Thus, let λ, μ, S be the characteristic numbers of the functions x, y, z, and let us suppose that the characteristic number of the function $1/x$ is equal to $-\lambda$.

Then, on applying Lemma V to each of the two inequalities

$$z = xy, \quad y = z\frac{1}{x},$$

we shall have

$$S \geqslant \lambda + \mu, \quad \mu \geqslant S - \lambda,$$

whence

$$S = \lambda + \mu.$$

Let x be an integrable function of t.

Designating by t_1 a given number not less than t_0, let us consider the integral

$$u = \int_{t_1}^{t} x\, dt$$

if the characteristic number of the function x is negative or equal to zero, and the integral

$$u = \int_{t}^{\infty} x\, dt$$

if this characteristic number is positive.

Then we shall demonstrate the following proposition.

LEMMA VIII. *The characteristic number of an integral is not less than the characteristic number of the function to be integrated.*

Let λ be the characteristic number of the function x. The function

$$xe^{(\lambda - \eta)t}$$

will then be vanishing, and consequently bounded, whenever η is a positive constant. Let us designate by M an upper bound for its modulus for $t \geqslant t_0$.

If $\lambda > 0$, we have, on assuming that $\eta < \lambda$,

$$|u| < M \int_{t}^{\infty} e^{-(\lambda - \eta)t}\, dt = \frac{M}{\lambda - \eta} e^{-(\lambda - \eta)t},$$

from which there results that

$$ue^{(\lambda - \varepsilon)t}$$

is a vanishing function for every value of ε greater than η. Now, we may suppose that η is as small as we like. By consequence, the preceding function is vanishing for every positive value of ε.

If $\lambda \leqslant 0$, we have

$$|u| < M \int_{t_1}^{t} e^{-(\lambda - \eta)t}\, dt = \frac{M}{\eta - \lambda} e^{-(\lambda - \eta)t} + \text{const.},$$

from which it results that

$$ue^{(\lambda - \varepsilon)t}$$

is a vanishing function for every value of ε greater than η, and hence for every positive value of ε.

In what follows, we shall have to consider sets† [vectors] composed of several functions, and we shall then use the expression *characteristic number of a set*, on naming thus the least of the characteristic numbers of the functions comprising the set.

7. [*Characteristic numbers of solutions of linear differential equations*]

Let us consider the system of linear differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n \quad (s = 1, 2, \ldots, n), \tag{15}$$

assuming that all the coefficients $p_{s\sigma}$ are given in a determinate way at least for all values of t not less than a certain limit t_0, and that they represent continuous, real and bounded functions of t.

In speaking of a solution of this system of equations, we understand that it means a set [vector] of n functions

$$x_1, x_2, \ldots, x_n,$$

simultaneously satisfying these equations (and, as a consequence, finite and continuous) for each value of t not less than t_0. Such sets of functions, as has already been remarked above, can always be found. Moreover, we can obtain n sets such that a system of n independent solutions can be deduced from them.

THEOREM I. *Every solution of the system of differential equations* (15), *other than the obvious solution*

$$x_1 = x_2 = \ldots = x_n = 0,$$

has a finite characteristic number.

† [Here and elsewhere the Russian and French words for 'group' have been translated as 'set', to avoid confusion arising from the algebraical meaning of 'group'.]

Only speaking of solutions where the functions x_s are not all identically zero, let us first consider real solutions, i.e. such that all the x_s are real functions of t.

Taking λ as a real constant, let us put

$$z_s = x_s e^{\lambda t} \quad (s = 1, 2, ..., n). \tag{16}$$

Then equations (15) are transformed into the following:

$$\frac{dz_s}{dt} = p_{s1}z_1 + p_{s2}z_2 + ... + (p_{ss} + \lambda)z_s + ... + p_{sn}z_n \quad (s = 1, 2, ..., n),$$

from which we deduce

$$\frac{1}{2}\frac{d}{dt}\sum_{s=1}^{n} z_s^2 = \sum_{s=1}^{n} (p_{ss} + \lambda)z_s^2 + \sum (p_{s\sigma} + p_{\sigma s})z_s z_\sigma,$$

supposing that the second summation on the right-hand side extends over all possible combinations of different numbers s and σ taken from the sequence $1, 2, ..., n$.

The right-hand side of this equality is a quadratic form in the quantities $z_1, z_2, ..., z_n$, in which the coefficients depend on λ and t.

Now, the functions $p_{\sigma s}$ being bounded, it is clear that we can always find values of λ such that this form is positive-definite for all the considered values of t, while remaining moreover greater than the form

$$\tfrac{1}{2}N(z_1^2 + z_2^2 + ... + z_n^2) \tag{17}$$

N being a positive number chosen arbitrarily. Similarly, it is obvious that we can also find values of λ such that, for the above-mentioned values of t, this form is negative, while remaining always in absolute value greater than form (17).

For each value of λ of the first kind, we shall have the inequality

$$\frac{d}{dt}\sum z_s^2 > N\sum z_s^2,$$

from which, on designating by C a positive constant, we obtain [by separating the variables and integrating both sides of the resulting inequality]

$$\sum z_s^2 > Ce^{Nt}$$

for every value of t greater than a certain limit.

For values of λ of the second kind, we shall have

$$\frac{d}{dt}\sum z_s^2 < -N\sum z_s^2,$$

whence (if C, as before, represents a positive constant)

$$\sum z_s^2 < Ce^{-Nt}$$

also holding for every value of t greater than a certain limit.

Thus, in the first case, the quantity $\sum z_s^2$ will increase indefinitely with t; in the second, it will tend towards zero.

In this manner we see that there will always be values of λ such that, in the set of functions (16), some will be found unbounded, and that, on the other hand, there will be values of λ such that all these functions are vanishing.

From this we conclude that, in every real solution

$$x_1, x_2, \ldots, x_n, \tag{18}$$

other than the obvious solution $x_1 = x_2 = \ldots = x_n = 0$, we shall always find functions with finite characteristic numbers, but that we shall never find any with the characteristic number $-\infty$. Hence, the characteristic number of the set of functions (18) is always finite.

Next, to extend the theorem to the case of complex solutions, it suffices to note that such a solution

$$x_1 = u_1 + \sqrt{-1}v_1, \quad x_2 = u_2 + \sqrt{-1}v_2, \quad \ldots, \quad x_n = u_n + \sqrt{-1}v_n \tag{19}$$

of the system of equations (15) will be formed of two real solutions

$$\left.\begin{matrix} u_1, u_2, \ldots, u_n, \\ v_1, v_2, \ldots, v_n \end{matrix}\right\} \tag{20}$$

of the same system, and that, according to Lemma IV and the Remark made on its subject, the characteristic number of the set of functions (19) will be equal to the characteristic number of the set of functions (20).

Remark

We have assumed that all the coefficients $p_{s\sigma}$ in equations (15) are real. But, having proved the theorem under this hypothesis, it is obviously easy to extend it to the case of complex coefficients, provided that these are continuous and bounded functions of t. This is why the propositions which we shall demonstrate later, relating to equations (15), will be true also in the case of complex coefficients.

For equations (15) let there be found k solutions

$$\left.\begin{matrix} x_{11}, x_{21}, \ldots, x_{n1}, \\ x_{12}, x_{22}, \ldots, x_{n2}, \\ \ldots\ldots. \\ x_{1k}, x_{2k}, \ldots, x_{nk}. \end{matrix}\right\} \tag{21}$$

On putting

$$x_s = C_1 x_{s1} + C_2 x_{s2} + \ldots + C_k x_{sk} \quad (s = 1, 2, \ldots, n),$$

where $C_1, C_2, \ldots, C_k$ are constants, of which *none is zero*, we shall say that the solution

$$x_1, x_2, \ldots, x_n$$

is a *linear combination* of solutions (21).

From Lemma IV it results that the characteristic number of a solution representing a linear combination of several solutions is not less than the characteristic number of the system of combined solutions (that is to say, not less than the characteristic number of the set of functions comprising the system of solutions), and that it is equal to this number when the characteristic numbers of all the solutions in the combination are different.

We conclude from this that, if we have several solutions of which the characteristic numbers are distinct, these solutions will be independent. Thus [bearing in mind that system (15) has just n linearly independent solutions], we have the following proposition.

THEOREM II. *The system of equations* (15) *cannot have more than n solutions, other than the obvious solution*

$$x_1 = x_2 = \ldots = x_n = 0,$$

for which the characteristic numbers are all distinct.

In what follows, without saying explicitly that the solution where all the x_s are zero must be excluded, this will always be understood.

8. [*Normal systems of solutions*]

Suppose that for the system of equations (15) we have found a system of n independent [vector] solutions. In forming with these solutions all possible linear combinations, we can deduce from them every other complete system of independent solutions.

Let us suppose that every system of n independent solutions found is transformed into another according to the following rule: every time that there can be formed with solutions of this [first] system a linear combination, of which the characteristic number is greater than the characteristic number of the set of [the linear combination's] component† solutions, one of the latter, and specifically one of those for which the characteristic numbers are equal to the characteristic number of the set, is replaced, in the [second] system considered, by this linear combination.

As the number of different characteristic numbers that the solutions of the system of equations (15) can possess is limited, we shall arrive, on operating in this way, at a system of n solutions such that *every linear combination of the solutions of which it is composed will have a characteristic number equal to the characteristic number of the set of* [*the linear combination's*] *component solutions.*

We shall call such a system of n solutions (which will evidently be independent) a *normal* system.

The coefficients $p_{s\sigma}$ in equations (15) being supposed real, we can find for these equations a system of n independent real solutions. Starting from such a system and only using, in the formation of linear combinations, real coefficients, we shall be able to obtain a system of n solutions satisfying the preceding condition for all linear combinations with real coefficients. But then this system will satisfy this condition equally for linear combinations with complex coefficients (Lemma IV, Remark). This system will consequently be a normal system.

By virtue of this remark, we may assume if need arises that all the functions entering into the composition of a normal system are real.

† [By the components of a linear combination are meant, not the scalar elements of this vector, but rather the terms which are linearly combined.]

From the definition of a normal system it results that, if we can find a system of n solutions for which the characteristic numbers are all different, this system is a normal system.

From the same definition there follows the next proposition.

THEOREM I. *Suppose found a system of n independent solutions*

$$\begin{gathered} x_{11}, x_{21}, \ldots, x_{n1}, \\ x_{12}, x_{22}, \ldots, x_{n2}, \\ \cdots\cdots \\ x_{1n}, x_{2n}, \ldots, x_{nn}. \end{gathered}$$

Consider a new system

$$\left.\begin{gathered} z_{11}, z_{21}, \ldots, z_{n1}, \\ z_{12}, z_{22}, \ldots, z_{n2}, \\ \cdots\cdots \\ z_{1n}, z_{2n}, \ldots, z_{nn}, \end{gathered}\right\} \tag{22}$$

on putting

$$z_{sk} = x_{sk} + \alpha_{k1} x_{s,k+1} + \alpha_{k2} x_{s,k+2} + \ldots + \alpha_{k,n-k} x_{sn},$$

and on taking for $\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{k,n-k}$ *constants, such that the characteristic number of the solution*

$$x_1, x_2, \ldots, x_n,$$

where

$$x_s = x_{sk} + \beta_1 x_{s,k+1} + \beta_2 x_{s,k+2} + \ldots + \beta_{n-k} x_{sn},$$

$\beta_1, \beta_2, \ldots, \beta_{n-k}$ *being arbitrary constants, is not greater than the characteristic number of the solution*

$$z_{1k}, z_{2k}, \ldots, z_{nk}.$$

Then the system of solutions (22) *is normal.*†

To prove this, we note that if system (22) were not a normal system we could find among its solutions a set of solutions possessing a common characteristic number λ, and such that we could deduce from it linear combinations with characteristic number greater than λ. Now, by the very definition of the quantities z_{sk}, there obviously do not exist such solutions in system (22).

Let k be the number of all the *distinct* characteristic numbers which can belong to solutions of equations (15), and let

$$\lambda_1, \lambda_2, \ldots, \lambda_k$$

be these numbers.

† [This theorem is to the following effect. In testing a set of solutions for normality, we may test each solution successively, and in forming a test linear combination we may omit from its component solutions any previously tested solution.]

Let us designate by n_s the number of solutions with the characteristic number λ_s in an arbitrary system of n independent solutions. Some of the numbers n_s can be zero. But they will be in every case such that

$$n_1 + n_2 + \dots + n_k = n.$$

Assuming that

$$\lambda_1 < \lambda_2 < \dots < \lambda_k,$$

let us further designate by N_s the exact upper limit of the number of independent solutions with the characteristic number λ_s. We shall evidently have

$$N_1 > N_2 > \dots > N_k,$$

$$N_1 = n, \quad n_s + n_{s+1} + \dots + n_k \leqslant N_s \quad (s = 1, 2, \dots, k).$$

With these definitions we have the following propositions.

THEOREM II. *For every normal system of solutions*

$$n_1 = n - N_2, \quad n_2 = N_2 - N_3, \quad \dots, \quad n_{k-1} = N_{k-1} - N_k, \quad n_k = N_k.$$

In fact, each solution is a linear combination of certain solutions of the normal system. Also, according to the property of this system, to deduce from it a solution possessing a characteristic number λ_s, we have to consider linear combinations of solutions for which the characteristic numbers are not less than λ_s. Therefore the number of independent solutions with the characteristic number λ_s cannot be greater than the quantity

$$n_s + n_{s+1} + \dots + n_k,$$

corresponding to a normal system. Then, for the latter,

$$n_s + n_{s+1} + \dots + n_k = N_s,$$

which yields the theorem.

THEOREM III. *The sum*

$$S = n_1\lambda_1 + n_2\lambda_2 + \dots + n_k\lambda_k$$

of the characteristic numbers of all the solutions constituting a system of n independent solutions attains its upper limit for a normal system.

Thus, on putting

$$n_s + n_{s+1} + \dots + n_k = N'_s,$$

we have [substituting $n_1 = n - N'_2$, $n_2 = N'_2 - N'_3$, ..., in S]

$$S = n\lambda_1 + N'_2(\lambda_2 - \lambda_1) + N'_3(\lambda_3 - \lambda_2) + \dots + N'_k(\lambda_k - \lambda_{k-1}).$$

Now we have just seen that, for a normal system, each of the numbers N'_s attains its upper limit N_s. Thus, on noting that in the expression for S the coefficients of the numbers $N'_2, N'_3, \dots, N'_k$ are all positive, we conclude that S attains its maximum for a normal system.

THEOREM IV. *Each system of n independent solutions, for which the sum of the characteristic numbers of all the solutions which compose it attains its upper limit, is a normal system.*

This theorem results from the very definition of a normal system, for, if it were possible to form with the solutions of the system considered a linear combination with characteristic number greater than the characteristic number of the set of component solutions, we could find a system of n independent solutions for which the sum of all the characteristic numbers would be greater than that of the system considered.

THEOREM V. *The sum of the characteristic numbers of independent solutions of the system of equations* (15) *in no case exceeds the characteristic number of the function*

$$e^{\int \sum_{s=1}^{n} p_{ss}\, dt}.$$

In fact, if Δ is the determinant formed with n independent solutions, we have [compare the equation before (9)]

$$e^{\int \sum p_{ss}\, dt} = C\, \Delta,$$

where C is a constant, and, by virtue of Lemmas IV and V, the characteristic number of Δ is not less than†

$$n_1 \lambda_1 + n_2 \lambda_2 + \quad \dots \quad + n_k \lambda_k.$$

COROLLARY. *Each system of n independent solutions for which the sum of the characteristic numbers of all the solutions is equal to the characteristic number of the function*

$$e^{\int \sum p_{ss}\, dt},$$

is a normal system.

[This corollary follows from theorems IV and V.]

It must nevertheless be noted that it is not always possible to obtain a system of n independent solutions such that this equality holds.

Thus, if we have the system of equations

$$\frac{dx_1}{dt} = x_1 \cos \log t + x_2 \sin \log t,$$

$$\frac{dx_2}{dt} = x_1 \sin \log t + x_2 \cos \log t,$$

† [The determinant is a sum of products, in each of which the factors can be split into sets such that the sth set consists of n_s factors with characteristic numbers not less than λ_s.]

we shall get, on suitably determining the arbitrary constant,

$$e^{\int \sum p_{ss}\, dt} = e^{t(\sin \log t + \cos \log t)},$$

which represents a function for which the characteristic number is $-\sqrt{2}$. Now, our equations admit the following system of solutions:

$$\begin{array}{cc} x_1 & x_2 \\ e^{t \sin \log t}, & e^{t \sin \log t}, \\ e^{t \cos \log t}, & -e^{t \cos \log t}, \end{array}$$

and it is easy to convince ourselves that this system is normal. However, the sum of the characteristic numbers which corresponds to it (and which is equal to -2) is less than the preceding number.

9. [*Regular and irregular systems of equations*]

We know (Lemma V, corollary) that the sum of the characteristic numbers of the functions

$$e^{\int \sum p_{ss}\, dt} \quad \text{and} \quad e^{-\int \sum p_{ss}\, dt}$$

is never greater than zero.

Therefore, if μ is the characteristic number of the second of these functions, the sum S of the characteristic numbers of the solutions of a normal system cannot exceed the number $-\mu$. Moreover, the equality $S = -\mu$ is possible only if the sum of the characteristic numbers of the two functions under consideration is zero.†

This equality

$$S + \mu = 0$$

for equations with constant or periodic coefficients, does actually hold. But it can also hold in many other cases.

In general, if we have $S + \mu = 0$, the system of linear differential equations will be said to be *regular*. In the contrary case, it will be called *irregular*.

Thus, for example, the system of equations‡

$$\frac{dx_1}{dt} = x_1 \cos at + x_2 \sin bt,$$

$$\frac{dx_2}{dt} = x_1 \sin bt + x_2 \cos at$$

is regular, whatever the real constants a and b.

At the end of the previous section there was cited an example of an irregular system of equations.

---

† [If β is the characteristic number of the first function, we have with use of Theorem V that $S \leqslant \beta \leqslant -\mu$. Hence $S = -\mu$ only if $\beta + \mu = 0$.]

‡ [Two solutions of this system are (for $a \neq 0, b \neq 0$) $x_1 = -x_2 = e^{(\sin at)/a + (\cos bt)/b}$ and $x_1 = x_2 = e^{(\sin at)/a - (\cos bt)/b}$.]

To give an example of a more general character, let us consider the following system:

$$
\begin{aligned}
\frac{dx_1}{dt} &= p_{11}x_1, \\
\frac{dx_2}{dt} &= p_{21}x_1 + p_{22}x_2, \\
&\cdots\cdots\cdots\cdots \\
\frac{dx_n}{dt} &= p_{n1}x_1 + p_{n2}x_2 + \dots + p_{nn}x_n,
\end{aligned} \tag{23}
$$

in which the equation for dx_s/dt does not contain the functions $x_{s'}$ for $s' > s$.

Concerning systems of equations with this form, we can establish the following proposition.

THEOREM. *For the system of equations* (23) *to be regular, it is necessary and sufficient that the sum of the characteristic numbers of the functions*

$$e^{\int p_{ss}\,dt} \quad \text{and} \quad e^{-\int p_{ss}\,dt}$$

should be equal to zero for every value of s.

First let us show that this condition is necessary.

We have, for equations (23), the following system of n independent solutions:

$$(1) \quad x_1 = e^{\int p_{11}\,dt}, \quad x_s = e^{\int p_{ss}\,dt} \int \sum_{i=1}^{s-1} p_{si}x_i e^{-\int p_{ss}\,dt}\,dt \quad (s = 2, 3, \dots, n),$$

$$(2) \quad x_1 = 0, \quad x_2 = e^{\int p_{22}\,dt}, \quad x_s = e^{\int p_{ss}\,dt} \int \sum_{i=2}^{s-1} p_{si}x_i e^{-\int p_{ss}\,dt}\,dt \quad (s = 3, 4, \dots, n),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(n) \quad x_1 = x_2 = \dots = x_{n-1} = 0, \quad x_n = e^{\int p_{nn}\,dt}.$$

To fix ideas, we shall suppose that all the integrals

$$\int p_{ii}\,dt,$$

which appear in exponents become zero for $t = t_0$. As for the other integrals, we suppose them to be such that, in the kth solution, the functions

$$x_{k+1}, x_{k+2}, \dots, x_n$$

reduce, for $t = t_0$, to given constants

$$\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{k,n-k}.$$

Then if

$$x_{1k}, x_{2k}, \dots, x_{nk}$$

is the kth solution of the system considered *under the hypothesis that all the α are equal to zero*, for the kth solution with the α arbitrary it will become

$$x_s = x_{sk} + \alpha_{k1} x_{s,k+1} + \alpha_{k2} x_{s,k+2} + \dots + \alpha_{k,n-k} x_{sn} \quad (s = 1, 2, \dots, n).$$

From this, in view of Theorem I of the previous section, we conclude that for a suitable choice of the constants α the considered system of solutions will be normal.

Assuming these constants to be chosen in this manner, let us designate the respective characteristic numbers of the considered solutions by

$$\mu_1, \mu_2, \dots, \mu_n.$$

Let us further designate

$$\left.\begin{array}{l} \text{the characteristic number of the function } e^{\int p_{ss}\,dt} \text{ by } \lambda_s, \\ \text{the characteristic number of the function } e^{-\int p_{ss}\,dt} \text{ by } \lambda'_s, \end{array}\right\} (s = 1, 2, \dots, n),$$

the characteristic number of the function $e^{\int \sum p_{ss}\,dt}$ by S,

the characteristic number of the function $e^{-\int \sum p_{ss}\,dt}$ by S'.

We obviously have [because inspection of the sth solution shows that one of its scalar elements has characteristic number λ_s]

$$\mu_s \leqslant \lambda_s \quad (s = 1, 2, \dots, n).$$

Thus, if we assume that system (23) is regular, which brings in the equality

$$\sum \mu_s = S,$$

and if we note that because of Lemma V the sum $\sum \lambda_s$ cannot be greater than S, we must have

$$\sum \lambda_s = S.$$

Now, under the same assumption, we have

$$S + S' = 0.$$

Hence, on referring to Lemma VII, we conclude that the characteristic number of the function

$$e^{\int \sum p_{ss}\,dt}\, e^{-\int p_{kk}\,dt}$$

is equal to $S + \lambda'_k$. We therefore get (Lemma V)

$$S + \lambda'_k \geqslant \sum \lambda_s - \lambda_k,$$

and from this, by virtue of the above-mentioned equality, there results

$$\lambda_k + \lambda'_k \geqslant 0.$$

Now the sum $\lambda_k + \lambda'_k$ cannot be positive [Lemma V, Corollary]. We must therefore have

$$\lambda_k + \lambda'_k = 0,$$

which shows the necessity of the condition in the theorem.

To prove that this condition is sufficient, we shall consider another determination of the integrals, assuming that every integral of the form

$$\int \sum_{i=1}^{s-1} p_{si} x_i e^{-\int p_{ss}\, dt} \, dt,$$

where the characteristic number of the function to be integrated is positive, tends to zero as t increases indefinitely. Then, in the considered system of solutions, every integral of this form will possess a characteristic number not less than the characteristic number of the function to be integrated (Lemma VIII).

Hence, if we assume that

$$\lambda_s + \lambda'_s = 0 \quad (s = 1, 2, \ldots, n),$$

and if, in considering the kth solution (in which $x_1, x_2, \ldots, x_{k-1}$ are equal to zero), we note that the function x_k has for a characteristic number the value λ_k, we easily reach the conclusion† that the characteristic numbers of all the other functions which constitute this solution will be not less than λ_k.

It follows from this that λ_k is the characteristic number of the kth solution.

Now we have, in any case [using Lemma V and its corollary]

$$\sum \lambda_s \leqslant S \leqslant -S' \leqslant -\sum \lambda'_s,$$

and, as a consequence of what we have assumed,

$$\sum \lambda_s + \sum \lambda'_s = 0.$$

[Hence the above inequality chain becomes an equality chain.]

We therefore obtain the equality

$$\sum \lambda_s + S' = 0,$$

and we conclude: (1) that the system of equations (23) is regular, and (2) that the system of solutions found is normal. [See the corollary to Theorem V.]

† [An inductive proof can be constructed, on applying Lemmas IV, V, VIII and recalling that the p_{ij} were assumed bounded in Section 7.]

Remark

Because of Lemma VI, the condition expressed in the theorem is equivalent to the following: *each of the functions*

$$\frac{1}{t}\int_{t_0}^{t} p_{ss}\,dt \quad (s = 1, 2, \ldots, n)$$

(and if the coefficients p_{ss} are complex quantities, the real parts of these functions) *must tend towards a finite limit when t increases indefinitely.*

10. [*Reducible systems of equations*]

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be all the distinct characteristic numbers of the solutions of equations (15), and let n_s be the number of solutions possessing the characteristic number λ_s *in a normal system.* We agree to say that *the system of these equations possesses*

n_1 characteristic numbers equal to λ_1,

n_2 characteristic numbers equal to λ_2,

. . . .

n_k characteristic numbers equal to λ_k.

In this way, to each system of n linear differential equations of the nature under consideration will correspond a set of n characteristic numbers, among which some can be equal.

Let us suppose that equations (15) are transformed by means of a linear substitution

$$z_s = q_{s1}x_1 + q_{s2}x_2 + \ldots + q_{sn}x_n \quad (s = 1, 2, \ldots, n),$$

possessing the following properties: (1) all the coefficients $q_{s\sigma}$ are continuous and bounded functions of t; (2) their first derivatives are functions of the same character; (3) the reciprocal of the determinant formed by means of these coefficients is a bounded function of t.

After such a transformation, the coefficients in the transformed equations will enjoy the same fundamental properties as in the original equations.

It is easy to prove that *the set of characteristic numbers of the transformed system of equations will always be identical with the set of characteristic numbers of the original system.*

In fact, by the nature of the substitution considered, not only its coefficients but also the coefficients of the inverse substitution are bounded functions of t. Consequently, if, on starting from an arbitrary solution of one system of equations, we deduce from it a solution of the other, these two solutions will have the same characteristic number. From this (because of the very notion of a normal system of solutions) it results that each number which repeats a certain number of times in the set of characteristic numbers of the one system, will necessarily repeat the same number of times in the set of characteristic numbers of the other.†

† [For clarification of this proof, and for proofs of further regularity properties mentioned in this section, see J. G. Malkin, *Theorie der Stabilität einer Bewegung*, Munich, 1959, pp. 290–293. For an English version of the Russian original of Malkin's book see United States Atomic Energy Commission Translation AEC-tr-3352.]

In this way, the characteristic numbers of a system of linear differential equations possess, in relation to the transformations considered, the properties of invariants. Also, the same properties belong to the characteristic numbers of the functions

$$e^{\int \Sigma p_{ss}\, dt} \quad \text{and} \quad e^{-\int \Sigma p_{ss}\, dt}.$$

It follows that the transformed system of equations will always be of the same class (i.e. regular or irregular) as the original system.

The system of equations under consideration can be such that, by a suitable choice of transformations having the character considered, we can transform it into a system with constant coefficients.

In this case we shall call it a *reducible* system of equations.

From what we have just noted, it results that only regular systems of equations can be reducible.

We shall see later (Chapter III [Section 47]) that every system of equations in which the coefficients are periodic functions of t with the same real period is a reducible system.

Let us consider an arbitrary system of equations.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all the characteristic numbers (among which some can be equal) and let

$$\begin{array}{c} x_{11}, x_{21}, \ldots, x_{n1}, \\ x_{12}, x_{22}, \ldots, x_{n2}, \\ \cdots\cdots\cdots\cdots \\ x_{1n}, x_{2n}, \ldots, x_{nn}, \end{array}$$

be a normal system of solutions, in which the jth solution has λ_j for a characteristic number.

On designating by Δ the determinant formed with the functions x_{ij}, let us assume that all the functions

$$\frac{e^{-\sum\limits_{s=1}^{n} \lambda_s t}}{\Delta}, \quad x_{ij} e^{\lambda_j t} \quad (i, j = 1, 2, \ldots, n)$$

are bounded.

We can demonstrate that under this condition the considered system of equations is reducible.

Thus, on designating the minor of the determinant Δ corresponding to the element x_{ij} by Δ_{ij}, we conclude from the preceding conditions that the functions

$$\frac{\Delta_{ij}}{\Delta} e^{-\lambda_j t} \quad (i, j = 1, 2, \ldots, n)$$

are bounded. [To show this, start by expressing the minor as a sum of signed products of its elements.] It will also be the same for their first derivatives with respect to t, for we know [see e.g. Malkin *loc. cit.* p. 295] that the functions

$$\frac{\Delta_{1j}}{\Delta}, \frac{\Delta_{2j}}{\Delta}, \ldots, \frac{\Delta_{nj}}{\Delta}$$

for each value of j, satisfy a system of linear differential equations adjoint with respect to the considered system.

Consequently, the substitution

$$z_s = \frac{\Delta_{1s}}{\Delta} e^{-\lambda_s t} x_1 + \frac{\Delta_{2s}}{\Delta} e^{-\lambda_s t} x_2 + \dots + \frac{\Delta_{ns}}{\Delta} e^{-\lambda_s t} x_n$$

$$(s = 1, 2, \dots, n)$$

possesses all the properties of the substitutions considered, and on applying it we shall transform the system under consideration into the system of equations

$$\frac{dz_s}{dt} + \lambda_s z_s = 0 \quad (s = 1, 2, \dots, n)$$

with constant coefficients. [In the above expression for z_s substitute $x_i = x_{is}$ $(i = 1, 2, \dots, n)$. This will give $z_s = e^{-\lambda_s t}$, differentiation of which yields the differential equations just stated.]

On a general case of the differential equations of disturbed motion

11. [*A new type of series ordered according to the powers of the arbitrary constants*]

Let us now go back to equations (1).

In only considering, as before, real values of t not less than a certain limit t_0, we shall suppose that all the coefficients $P_s^{(m_1, m_2, \dots, m_n)}$ are real, continuous and bounded functions of t. We shall further assume that we can find positive constants M and A such that the inequalities

$$|P_s^{(m_1, m_2, \dots, m_n)}| < \frac{M}{A^{m_1 + m_2 + \dots + m_n}}.$$

are satisfied for all the considered values of t.

Let us assume that the system of linear differential equations corresponding to the first approximation is regular [as defined in Section 9], and let us designate by

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

the characteristic numbers of this system.

We are going to show that, on choosing from these numbers any k of them

$$\lambda_1, \lambda_2, \dots, \lambda_k, \tag{24}$$

we can formally satisfy equations (1) by series containing k arbitrary constants

$$\alpha_1, \alpha_2, \dots, \alpha_k$$

and having the following form:

$$x_s = \sum L_s^{(m_1, m_2, \dots, m_k)} \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_k^{m_k} e^{-\sum_{i=1}^{k} m_i \lambda_i t}$$

$$(s = 1, 2, \dots, n), \tag{25}$$

where $L_s^{(m_1, m_2, \dots, m_k)}$ are continuous functions of t and are independent of α_i, for which the characteristic numbers are positive or zero, and where the summation extends over all non-negative integer values of the numbers $m_1, m_2, \dots, m_k$ subject to the condition

$$m_1 + m_2 + \dots + m_k > 0.$$

In what follows we shall consider exclusively the case where the chosen characteristic numbers (24) are all positive, and, under this hypothesis, we are going to show that if the moduli of $\alpha_1, \alpha_2, ..., \alpha_k$ do not exceed a certain limit the series (25) will be absolutely convergent and will represent functions which really do satisfy equations (1), for all values of t greater than t_0.

Let us turn to the formulae of Section 3.

Let us take it that the system of particular solutions of equations (6) with which we are concerned is normal, and that the solution

$$x_{1s}, s_{2s}, ..., x_{ns}$$

possesses the characteristic number λ_s $(s = 1, 2, ..., n)$.

Let us put

$$x_s^{(1)} = \alpha_1 x_{s1} + \alpha_2 x_{s2} + ... + \alpha_k x_{sk} \quad (s = 1, 2, ..., n)$$

and let us integrate next the systems of equations (7) corresponding to $m = 2, 3, ...$

In Section 3 we assumed that all the functions $x_s^{(m)}$ for $m > 1$ had to become zero for $t = t_0$.

Here we shall no longer maintain this assumption, but will replace it by another which we are going to indicate immediately.

Let us assume that all the functions $x_s^{(\mu)}$ for $\mu < m$ have been found, and represent with respect to the constants α_i entire and homogeneous functions of the μth degree. Then the functions $R_i^{(m)}$, because of their expressions in terms of the quantities $x_s^{(\mu)}$, will present themselves relative to these same constants in the form of entire and homogeneous functions of the mth degree.

Let

$$\frac{\Delta_{ij}}{\Delta} R_i^{(m)} = \sum T_{ij}^{(m_1, m_2, ..., m_k)} \alpha_1^{m_1} \alpha_2^{m_2} ... \alpha_k^{m_k},$$

the T being functions of t which are independent of the constants α_s.

Then, on writing [compare (9)]

$$x_s^{(m)} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{sj} \int \frac{\Delta_{ij}}{\Delta} R_i^{(m)} dt,$$

$$\int \frac{\Delta_{ij}}{\Delta} R_i^{(m)} dt = \sum \alpha_1^{m_1} \alpha_2^{m_2} ... \alpha_k^{m_k} \int T_{ij}^{(m_1, m_2, ..., m_k)} dt,$$

we shall assume that those of the integrals

$$\int T_{ij}^{(m_1, m_2, ..., m_k)} dt$$

where the function to be integrated possesses a positive characteristic number are to be taken between the limits $+\infty$ to t. As for the integrals where the function to be integrated has a negative or zero characteristic number, we shall only assume that we have

$$\int T_{ij}^{(m_1, m_2, ..., m_k)} dt = \int_{t_0}^{t} T_{ij}^{(m_1, m_2, ..., m_k)} dt + C_{ij}^{(m_1, m_2, ..., m_k)},$$

the C being constants independent of the α_s.

The integrals in question will then have characteristic numbers not less than those of the functions to be integrated (Lemma VIII).

In proceeding thus, starting from $m = 2$, we shall have for all the $x_s^{(m)}$ expressions which are entire and homogeneous with respect to the constants $\alpha_1, \alpha_2, ..., \alpha_k$.

Let

$$x_s = x_s^{(1)} + x_s^{(2)} + ... \quad (s = 1, 2, ..., n)$$

be the series obtained under this assumption.

To give them the form (25), we should put

$$L_s^{(m_1, m_2, ..., m_k)} = e^{\sum_{i=1}^{k} m_i \lambda_i t} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{sj} \int T_{ij}^{(m_1, m_2, ..., m_k)} \, dt,$$

and from this it is easy to conclude that the characteristic numbers of the functions $L_s^{(m_1, m_2, ..., m_k)}$ will be not less than zero.

In fact, the system of equations (6) being assumed to be regular, the characteristic number of the function $1/\Delta$ will be equal to [in view of the equation preceding (9)]

$$-(\lambda_1 + \lambda_2 + ... + \lambda_n),$$

and, by consequence, the characteristic number of the functions Δ_{ij}/Δ will be not less than $-\lambda_j$. [Consider the minor as a sum of signed products of its elements.]

Therefore, if we agree that what was said about the functions L is true when we have

$$m_1 + m_2 + ... + m_k < m,$$

[and if we keep in mind that, in equation (9), the expression $R_i^{(m)}$ is a homogeneous polynomial of degree m in the $x_r^{(\mu)}$ with $\mu < m$] we shall be able to conclude (Lemmas IV, V) that the characteristic number of the function $T_{ij}^{(m_1, m_2, ..., m_k)}$ for which

$$m_1 + m_2 + ... + m_k = m,$$

and by consequence also that of the integral

$$\int T_{ij}^{(m_1, m_2, ..., m_k)} \, dt$$

are not less than

$$m_1 \lambda_1 + m_2 \lambda_2 + ... + m_k \lambda_k - \lambda_j.$$

Now it follows from this that the characteristic number of each function L, for which the sum of indices m_i is equal to m, is not less than zero.

Thus, that the property in question belongs to the functions L, which is true in the case where $\sum m_i = 1$, is true in general.

Remark

To arrive at this result, it is not necessary to integrate between the limits $+\infty$ to t each of the functions $T_{ij}^{(m_1, m_2, ..., m_k)}$ with a positive characteristic number. It suffices to do so only when we have

$$m_1 \lambda_1 + m_2 \lambda_2 + ... + m_k \lambda_k - \lambda_j > 0.$$

12. [*Theorem on the convergence of these series*]

Passing now to the question of the convergence of series (25), we shall assume that the characteristic numbers (24), which have been adopted for forming these series, are all positive.

Under this hypothesis, on agreeing for simplicity that $t_0 = 0$, we are going to demonstrate the following proposition.

THEOREM. *If, on taking ε to be a positive constant and putting*

$$\alpha_s e^{-(\lambda_s - \varepsilon)t} = q_s \quad (s = 1, 2, \ldots, k)$$

we replace the α_s in series (25) by their expressions in terms of the q_s, the new series

$$x_s = \sum Q_s^{(m_1, m_2, \ldots, m_k)} q_1^{m_1} q_2^{m_2} \ldots q_k^{m_k} \quad (s = 1, 2, \ldots, n), \tag{26}$$

which will be ordered in increasing powers of the q_s, will enjoy the property that, for every value of ε however small it may be, we shall be able to find positive constants $Q^{(m_1, m_2, \ldots, m_k)}$ such that, t being positive, we always have

$$\left|Q_s^{(m_1, m_2, \ldots, m_k)}\right| < Q^{(m_1, m_2, \ldots, m_k)},$$

and such that the series

$$\sum Q^{(m_1, m_2, \ldots, m_k)} q_1^{m_1} q_2^{m_2} \ldots q_k^{m_k} \tag{27}$$

is convergent, as long as the moduli of the quantities q_s do not exceed a certain limit q different from zero.

Let us only consider values of ε less than each of the numbers

$$\lambda_1, \lambda_2, \ldots, \lambda_k.$$

Then we shall be able to find a positive integer l, such that all the expressions

$$m_1(\lambda_1 - \varepsilon) + m_2(\lambda_2 - \varepsilon) + \ldots + m_k(\lambda_k - \varepsilon) - \lambda_j + \varepsilon$$
$$(j = 1, 2, \ldots, n)$$

where $m_1, m_2, \ldots, m_k$ satisfy the condition

$$m_1 + m_2 + \ldots + m_k \geqslant l,$$

will be greater than a positive number H given arbitrarily.

Let η be a positive constant less than ε.

The functions

$$Q_s^{(m_1, m_2, \ldots, m_k)} e^{\eta t} = L_s^{(m_1, m_2, \ldots, m_k)} e^{-[(m_1 + m_2 + \ldots + m_k)\varepsilon - \eta]t} \tag{28}$$

will be vanishing [as defined in Section 6]. We can thus assign to the modulus of each of these functions a constant upper limit, valid for all positive values of t.

Let us suppose that we have found such limits for all those of these function for which

$$m_1 + m_2 + \ldots + m_k < l$$

and, on assuming that these limits are independent of s, let us designate them by

$$Q^{(m_1, m_2, \ldots, m_k)}.$$

Among these functions there will be, together with others [other terms], the following:

$$x_{ij}e^{(\lambda_j - \varepsilon + \eta)t}.$$

Now if we suppose further that $\eta > \varepsilon/2$, similar upper limits will be obtained also for the moduli of the functions

$$\frac{\Delta_{ij}}{\Delta} e^{-(\lambda_j + 2\eta - \varepsilon)t}.$$

Let K and K' be constants such that we have

$$|x_{ij}|e^{(\lambda_j - \varepsilon + \eta)t} < K, \quad \left|\frac{\Delta_{ij}}{\Delta}\right| e^{-(\lambda_j + 2\eta - \varepsilon)t} < K'$$

for all values of i and j taken from the sequence $1, 2, ..., n$, and for every positive value of t.

To obtain upper limits for the moduli of those of quantities (28) for which the sum of the indices $m_1, m_2, ..., m_k$ is not less than l, let us resort to the formulae

$$x_s^{(m)} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{sj} \int_{\infty}^{t} \frac{\Delta_{ij}}{\Delta} R_i^{(m)}\, dt \quad (s = 1, 2, ..., n), \tag{29}$$

in which, conforming to what we have agreed, all the integrals are taken between limits from $+\infty$ to t, since for $m \geqslant l$ the characteristic numbers of all the functions to be integrated will be positive.

Let

$$R_i^{(m)} = \sum R_i^{(m_1, m_2, ..., m_k)} q_1^{m_1} q_2^{m_2} ... q_k^{m_k}$$

$$(m_1 + m_2 + ... + m_k = m),$$

where the $R_i^{(m_1, m_2, ..., m_k)}$ are quantities independent of the constants α_s. Then, on putting, for brevity,

$$m_1\lambda_1 + m_2\lambda_2 + ... + m_k\lambda_k - m\varepsilon = N,$$

we deduce from (29) [and (26)] that

$$Q_s^{(m_1, m_2, ..., m_k)} = -e^{Nt} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{sj} \int_{t}^{\infty} \frac{\Delta_{ij}}{\Delta} e^{-Nt} R_i^{(m_1, m_2, ..., m_k)}\, dt. \tag{30}$$

Let us suppose that on using these formulae we have found upper limits, valid for all positive values of t, for the moduli of the quantities

$$Q_s^{(\mu_1, \mu_2, ..., \mu_k)}, \tag{31}$$

where the sum of the indices $\mu_1, \mu_2, ..., \mu_k$ is less than m, and that these upper limits, in the case where

$$\mu_1 + \mu_2 + ... + \mu_k \geqslant l,$$

are again obtained in the form

$$e^{-\eta t} Q^{(\mu_1, \mu_2, ..., \mu_k)},$$

the $Q^{(\mu_1, \mu_2, ..., \mu_k)}$ being constants.

With the aid of these limits let us form upper limits for the moduli of all the $R_i^{(m_1, m_2, \ldots, m_k)}$ which appear in formulae (30).

For this we note that by the nature of the expressions $R_i^{(m)}$ the quantity $R_i^{(m_1, m_2, \ldots, m_k)}$ represents an entire function of mth degree of those of quantities (31) for which the sum of the indices μ_s is less than m, and that the coefficients of this entire function are linear forms with positive coefficients in those of the quantities

$$P_i^{(\mu_1, \mu_2, \ldots, \mu_n)} \tag{32}$$

for which the sum of the indices $\mu_1, \mu_2, \ldots, \mu_n$ is not greater than m. Moreover, with respect to quantities (31), the degrees of the terms of this function are not below the second.

This agreed, if $R^{(m_1, m_2, \ldots, m_k)}$ is the constant which each of the functions

$$R_1^{(m_1, m_2, \ldots, m_k)}, \quad R_2^{(m_1, m_2, \ldots, m_k)}, \quad \ldots \quad R_n^{(m_1, m_2, \ldots, m_k)}$$

becomes when we replace the quantities (31) by the $Q^{(\mu_1, \mu_2, \ldots, \mu_k)}$ and the quantities (32) by certain upper limits (independent of i and t) of their absolute values, we shall have, for all positive values of t, these inequalities:

$$\left|R_i^{(m_1, m_2, \ldots, m_k)}\right| < e^{-2\eta t} R^{(m_1, m_2, \ldots, m_k)}$$

and the second members [right-hand sides] will represent the upper limits sought.

Now, in using the upper limits obtained, we extract from formula (30) this inequality:

$$\left|Q_s^{(m_1, m_2, \ldots, m_k)}\right| < nKK' R^{(m_1, m_2, \ldots, m_k)} e^{Nt} \sum_{j=1}^{n} e^{-(\lambda_j - \varepsilon + \eta)t} \int_t^\infty e^{-(N - \lambda_j + \varepsilon)t}\, dt,$$

which will be satisfied for every positive value of t. Moreover, on noting that

$$N - \lambda_j + \varepsilon = m_1(\lambda_1 - \varepsilon) + m_2(\lambda_2 - \varepsilon) + \ldots + m_k(\lambda_k - \varepsilon) - \lambda_j + \varepsilon > H,$$

and hence that

$$\int_t^\infty e^{-(N - \lambda_j + \varepsilon)t}\, dt < \frac{1}{H} e^{-(N - \lambda_j + \varepsilon)t},$$

we can replace it by the following:

$$\left|Q_s^{(m_1, m_2, \ldots, m_k)}\right| < \frac{n^2 KK'}{H} R^{(m_1, m_2, \ldots, m_k)} e^{-\eta t}.$$

We conclude from this that we can put

$$Q^{(m_1, m_2, \ldots, m_k)} = \frac{n^2 KK'}{H} R^{(m_1, m_2, \ldots, m_k)}$$

for all the values of $m_1, m_2, \ldots, m_k$ of which the sum is not less than l.

Now, on choosing for K and K' sufficiently large quantities, or for H a sufficiently small quantity, we can arrange for the values supplied by this formula for the Q, in the case where

$$1 < m_1 + m_2 + \ldots + m_k < l,$$

to be not less than those which have been obtained by direct means in this case.

Consequently, on designating by G a sufficiently large positive constant, we can take

$$Q^{(m_1, m_2, \ldots, m_k)} = GR^{(m_1, m_2, \ldots, m_k)} \tag{33}$$

whenever the sum $m_1 + m_2 + \ldots + m_k$ is greater than 1; and when this sum is equal to 1, we can put

$$Q^{(m_1, m_2, \ldots, m_k)} = K.$$

In doing this, let us designate by $x^{(m)}$ the sum

$$\sum Q^{(m_1, m_2, \ldots, m_k)} q_1^{m_1} q_2^{m_2} \ldots q_k^{m_k},$$

extended over all the non-negative values of the integers $m_1, m_2, \ldots, m_k$ which satisfy the condition

$$m_1 + m_2 + \ldots + m_k = m.$$

Then, for $m > 1$, equality (33) will give

$$x^{(m)} = GR^{(m)},$$

where $R^{(m)}$ is what $R_i^{(m)}$ becomes when we replace the $x_s^{(\mu)}$ by the $x^{(\mu)}$ and the quantities (32) by the upper limits adopted above.

This agreed, the series

$$x^{(1)} + x^{(2)} + x^{(3)} + \ldots, \tag{34}$$

ordered in increasing powers of the quantities q_s, will possess terms of which the moduli will be greater than those of the corresponding terms of series (26), for all positive values of t (they will even be greater than these moduli multiplied by $e^{\eta t}$).

Now series (34) can be considered as ordered in increasing powers of the quantity†

$$q_1 + q_2 + \ldots + q_k,$$

and if, conforming to what has been noted in the preceding section, we take the following for an upper limit of quantities (32):

$$\frac{M}{A^{\mu_1 + \mu_2 + \ldots + \mu_n}},$$

our series‡ will not differ essentially from that which we arrived at in Section 4.

Thus, if we restrict ourselves to this hypothesis, we shall certainly be able to find a positive number q, such that $q_1, q_2, \ldots, q_k$ satisfying the conditions

$$|q_s| \leqslant q \quad (s = 1, 2, \ldots, k),$$

series (34) will be absolutely convergent.

The theorem is consequently proved.

---

† [Lyapunov's meaning here may be as follows. Taking e.g. $k = 2$ and with c_{rs} as binomial coefficients, let us write $x^{(m)}$ as

$$x^{(m)} = a_{m0} c_{m0} q_1^m + a_{m-1,1} c_{m-1,1} q_1^{m-1} q_2 + \ldots + a_{0m} c_{0m} q_2^m.$$

Then if a is the a_{rs} with the greatest modulus we have

$$|x^{(m)}| \leqslant |a|(c_{m0}|q_1|^m + c_{m-1,1}|q_1|^{m-1}|q_2| + \ldots) = |a|(|q_1| + |q_2|)^m,$$

i.e. the same inequality as is obtained if we replace $x^{(m)}$ by $a(q_1 + q_2)^m$.]

‡ [Here we have $x^{(m)} = GR^{(m)}$, and in Section 4 we had $x^{(m)} = (1 + nK)(1 + nK')(T - t_0)R^{(m)}$. Hence the series $x^{(1)} + x^{(2)} \ldots$ in the two cases only differ by a multiplicative constant.]

COROLLARY. *There exists a positive constant α such that, $\alpha_1, \alpha_2, \ldots, \alpha_k$ satisfying the conditions*

$$|\alpha_s| \leqslant \alpha \quad (s = 1, 2, \ldots, k),$$

series (25) will be absolutely convergent for all positive values of t, while representing continuous functions of t satisfying equations (1). These functions, t increasing indefinitely, tend towards zero.

[Note that from (28)

$$L_s^{(m_1, m_2, \ldots, m_k)} = Q_s^{(m_1, m_2, \ldots, m_k)} e^{-(m_1+m_2+\ldots+m_k)\varepsilon t}.]$$

Remark

If the system of differential equations of the first approximation is not regular, then, on designating by S the sum of all the characteristic numbers and by μ the characteristic number of the function $1/\Delta$, we shall have

$$S + \mu = -\sigma,$$

where σ is a positive number.

In this case the characteristic number of the function

$$\frac{\Delta_{ij}}{\Delta}$$

is not less than $-\lambda_j - \sigma$. And on considering this, we may easily show that if in the case considered we form according to the rule expounded in the preceding section series similar to those of (25), the characteristic number of the function

$$L_s^{(m_1, m_2, \ldots, m_k)}$$

will be not less than

$$-(m_1 + m_2 + \ldots + m_k - 1)\sigma.$$

Let us suppose that σ is less than each of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then, with a suitable choice of the numbers ε and η, we shall be able to satisfy all the inequalities

$$\lambda_s > \varepsilon > \eta > \frac{\varepsilon + \sigma}{2} \quad (s = 1, 2, \ldots, k).$$

And, these last being satisfied, all the conditions of the preceding proof will be equally satisfied, of which it is easy to convince oneself on taking account of what was said on the subject of the functions L.

Therefore the theorem will not cease to be true when the system of differential equations of the first approximation is not regular, provided that each of the characteristic numbers chosen for the formation of series (25) is greater than σ, and that the condition $\varepsilon > 0$ is replaced by $\varepsilon > \sigma$.

13. [*Ensuing consequences relevant to stability*]

We can obtain from what has been proved the following theorems.

THEOREM I. *If the system of differential equations of the first approximation is regular, and if all the characteristic numbers are positive, the undisturbed motion is stable.*

Under the indicated condition we may take $k = n$.

Then, on designating the values of the functions x_s for $t = 0$ by a_s and on setting $t = 0$ in equations (25), we shall have

$$a_s = f_s(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad (s = 1, 2, \ldots, n),$$

where the f_s are holomorphic functions of the quantities α_j, becoming zero for

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

and moreover such that their functional determinant [Jacobian] with respect to the quantities α_j does not vanish when all the α_j vanish (for it then takes the value of the determinant Δ for $t = 0$).

Consequently the preceding equations are solvable with respect to the quantities α_j, and when the quantities a_s are sufficiently small in absolute value we can obtain

$$\alpha_s = \varphi_s(a_1, a_2, \ldots, a_n) \quad (s = 1, 2, \ldots, n), \tag{35}$$

where the φ_s are holomorphic functions of the quantities a_j, becoming zero for

$$a_1 = a_2 = \ldots = a_n = 0.$$

Let x be a positive and arbitrarily small quantity.

We can find a positive quantity r such that, for all the values of the variables $q_1, q_2, \ldots, q_n$ satisfying the conditions

$$|q_s| \leqslant r \quad (s = 1, 2, \ldots, n),$$

the series (27) (subject to the hypothesis that ε is less than each of the characteristic numbers) is absolutely convergent, and that the modulus of its sum is less than x.

Next, we can find a positive quantity a such that, for all the values of the quantities $a_1, a_2, \ldots, a_n$ satisfying the conditions

$$|a_s| \leqslant a \quad (s = 1, 2, \ldots, n), \tag{36}$$

the moduli of the quantities α_s, defined by equations (35), are not greater than the quantity r.

Thus we can be sure that, if the initial circumstances of the disturbed motion are chosen in agreement with conditions (36), the inequalities

$$|x_s| < x \quad (s = 1, 2, \ldots, n),$$

will be satisfied throughout the duration of the motion which ensues. [The equation before (26) implies that if $|q_s| \leqslant 0$ holds for $t = 0$ then it also holds for all $t > 0$.]

And this proves the theorem.

Remark

Under the conditions of the preceding theorem, in every disturbed motion sufficiently near the undisturbed motion the functions x_s, with t increasing indefinitely, all tend towards zero. We shall express this circumstance by saying that *the disturbed motion* (insofar as it is defined by the expressions of the quantities x_s as functions of t) *approaches asymptotically the undisturbed motion.*

In the same sense, we shall also speak of motions approaching asymptotically an arbitrary given motion.

THEOREM II. *If the system of differential equations of the first approximation is regular, and if among its characteristic numbers there exist positive members, the undisturbed motion will always enjoy a certain conditional stability. Thus, if the number of positive characteristic numbers is k, it will suffice, for it to have stability, that the initial values $a_1, a_2, ..., a_n$ of the unknown functions should satisfy a certain set of $n-k$ equations of the form*

$$F_j(a_1, a_2, ..., a_n) = 0 \quad (j = 1, 2, ..., n-k),$$

where the F_j are holomorphic functions of the quantities a_s, vanishing for $a_1 = a_2 = ... = a_n = 0$. These equations are moreover such that we can obtain from them all the a_s as real holomorphic functions of a certain set of k real independent quantities.

Let us suppose that, to form series (25), we have taken for equations (6) a normal system of *real* solutions.

Then the calculations can be arranged in such a way that all the coefficients L in these series are real functions, and that consequently, the α_s being real, equations (25) define a real solution of the system of equations (1).

This agreed, and on putting $t = 0$ in equations (25), we shall have

$$a_s = f_s(\alpha_1, \alpha_2, ..., \alpha_k) \quad (s = 1, 2, ..., n),$$

the f_s being real holomorphic functions of the quantities α_j, vanishing when all the α_j vanish. These functions will moreover be such that, among the functional determinants [Jacobians] that can be deduced from them on combining these functions in sets of k, there will be found at least one which does not vanish when we put

$$\alpha_1 = \alpha_2 = ... = \alpha_k = 0,$$

for these determinants then reduce to values, corresponding to $t = 0$, of minors of the determinant Δ, formed from the elements of the first k rows. [Δ is not zero—see the equation preceding (9).]

Thus, the $|a_s|$ being sufficiently small, we can derive from the preceding equations the following:

$$\left.\begin{aligned} \alpha_j &= \varphi_j(a_1, a_2, ..., a_n) \quad (j = 1, 2, ..., k), \\ F_s(a_1, a_2, ..., a_n) &= 0 \quad (s = 1, 2, ..., n-k), \end{aligned}\right\} \tag{37}$$

where φ_j, F_s are holomorphic functions of the quantities $a_1, a_2, ..., a_n$, becoming zero when these quantities vanish.

To push the proof further, the procedure will be the same as for the preceding theorem, with the sole difference that we must have in view here $n-k$ equations (37) relating the a_s.

It may be noted that, the perturbations being sufficiently small, the disturbed motions corresponding to equations (37) approach asymptotically the undisturbed motion.

Remark

If the system of differential equations of the first approximation is not regular, but possesses k characteristic numbers greater than the quantity σ (Section 12, Remark), there will be found $n-k$ conditions similar to the preceding, under which the undisturbed motion will be stable.

Some general propositions

14. [*General remarks on the functions defined by the differential equations of the disturbed motion*]

Passing now to the exposition of the principles of the second method, let us at the outset call attention to some general conclusions which can be drawn from what has been shown in Sections 3 and 4.

As in the preceding section, we are going to consider equations (1) exclusively under the hypothesis that for the functions A_s, which were mentioned in Sections 2 and 4, t being greater than its initial value t_0, there can be assigned a non-zero lower limit A.

Under this hypothesis, on designating by $a_1, a_2, ..., a_n$ constants chosen in conformity with the inequalities

$$|a_s| < A \quad (s = 1, 2, ..., n),$$

let us consider the functions x_s satisfying equations (1) and taking the values† a_s for $t = t_0$.

On basing our considerations on what precedes, we can affirm that such functions, at least for values of t sufficiently near t_0, always exist and are real whenever the a_s are real (which we assume here), and that moreover we can assign a limit t_1, greater than t_0, such that in the interval t_0 to t_1 inclusively these functions are represented by series ordered in the positive integer powers of the constants a_s.

If the functions defined by these series satisfy for $t = t_1$ the inequalities

$$|x_s| < A \quad (s = 1, 2, ..., n), \tag{38}$$

they admit analytic continuation beyond the limit t_1 and are then represented by series similar to the preceding, ordered in powers of the values of these functions for $t = t_1$.

These new expressions for the functions x_s will only be valid in general for values of t not exceeding a certain limit t_2. But, if for $t = t_2$ inequalities (38) remain satisfied, we shall be able to obtain a new analytic continuation, in the form of series with the same character.

In this manner, starting from the given initial values a_s, we can trace the continued variation of our functions with t, at least as long as inequalities (38) do not cease to be satisfied.

† In giving the constants a_s, the functions x_s are completely defined, at least for values of t sufficiently near to t_0. This results from an easily shown proposition, namely that, except for the obvious solution

$$x_1 = x_2 = ... = x_n = 0,$$

system (1) cannot have another solution where the initial values of all the unknown functions are zero.

It can happen that, for a certain choice of the constants a_s, these inequalities will be satisfied however far we advance in tracing the functions x_s. Then these functions will be defined for all values of t greater than t_0.

In other cases there will exist for t an upper limit t' such that, for $t = t'$, at least one of inequalities (38) will change to equality.

The analytic continuation of our functions beyond such a limit t' certainly calls for a special investigation. But we do not need to concern ourselves with this, inasmuch as for our purpose it will suffice to consider each disturbed motion only as long as the quantities $|x_s|$ remain below given limits as small as we wish.

In all cases, we can choose the constants a_s sufficiently small in absolute value for our analytic expressions for the functions x_s to be valid for all the values of t between t_0 and T, however great the given number T may be, and for the values $\xi_1, \xi_2, ..., \xi_n$ of these functions for $t = T$ to be as small as we wish. Moreover, if we wish to define the functions x_s by their values for $t = T$, we could, however great T may be, choose the ξ_s sufficiently small in absolute value for there to correspond a uniquely determined system of initial values a_s, and for these last to be all as small as we wish.

From this last remark it follows that, for the resolution of questions of stability, it will suffice to consider only values of t greater than a limit T, as large as we wish, on replacing the initial values of the functions x_s by their values corresponding to $t = T$.

In what follows we shall only consider the functions x_s as long as inequalities (38) do not cease to be satisfied, and, in speaking of limits for the quantities $|x_s|$, we shall always suppose these limits to be less than A.

15. [*Some definitions*]

We are going to consider here real functions of the real variables

$$x_1, x_2, ..., x_n, t, \tag{39}$$

subject to conditions of the form

$$t \geqslant T, \quad |x_s| \leqslant H \quad (s = 1, 2, ..., n), \tag{40}$$

where T and H are real constants, of which the first can be supposed as large as we wish, and the second as small as we wish (but not zero).

Furthermore we shall speak only of functions which, under conditions (40), remain continuous and single-valued, and *which vanish for*

$$x_1 = x_2 = ... = x_n = 0.$$

These properties will be common to all the functions which we are going to consider (even when it will not be stated expressly). But our functions will still be able to possess certain more special properties, and when we have to place them in evidence, we shall make use of certain abbreviated expressions, the meanings of which we shall now define.

Suppose that V, the function considered, is such that, under conditions (40), T being sufficiently large and H sufficiently small, it can only take values of one sign.

We shall say then that it is a *function of fixed sign*; and if we have to indicate its sign, we shall say that it is a *positive function* or a *negative function*.

If, further, the function V does not depend on t, and if the constant H can be chosen so small that, under conditions (40), the equality $V = 0$ can only hold if we have

$$x_1 = x_2 = \dots = x_n = 0,$$

we shall call the function V, as is done for a quadratic form, a *definite function*, or indeed, when we wish to call attention to its sign, a *positive-definite function* or a *negative-definite function*.

As far as functions depending on t are concerned, we shall still make use of these terms. But then we shall call the function V *definite* only under the condition that we can find a function W independent of t, which is positive-definite and moreover such that one of the two expressions

$$V - W \quad \text{or} \quad -V - W$$

is a positive function.

In this way, each of the two functions

$$x_1^2 + x_2^2 - 2x_1x_2 \cos t, \quad t(x_1^2 + x_2^2) - 2x_1x_2 \cos t$$

will be of fixed sign. But the first is only of fixed sign, while the second, if $n = 2$, is at the same time definite [recall (40)].

We shall call *limited* every function V for which the constant H can be chosen sufficiently small that, under conditions (40), there exists an upper limit for $|V|$.

By virtue of the properties possessed by all functions which we consider here, this will evidently be so for every function independent of t.

A limited function can be such that, ε being a positive number chosen arbitrarily, we can assign another positive number h sufficiently small that, the variables satisfying the conditions

$$t \geqslant T, \quad |x_s| \leqslant h \quad (s = 1, 2, \dots, n),$$

we have

$$|V| \leqslant \varepsilon.$$

Such will be, for example, every function independent of t. But functions dependent on t, although limited, can fail to satisfy the enunciated condition. This is what occurs, for example, with the function

$$\sin [(x_1 + x_2 + \dots + x_n)t].$$

When for a function V the preceding condition is satisfied, we shall say that it *admits an infinitely small upper limit*.

Such is, for example, the function

$$(x_1 + x_2 + \dots + x_n) \sin t.$$

Let V be a function admitting an infinitely small upper limit. Then, if we know that the variables satisfy the conditions

$$t \geqslant T, \quad |V| \geqslant l,$$

where l is a positive number, we can conclude that there exists another positive number λ, which the greatest of the quantities

$$|x_1|, |x_2|, \dots, |x_n|$$

cannot be below.

At the same time as the function V we shall often have to consider the expression

$$V' = \frac{\partial V}{\partial x_1} X_1 + \frac{\partial V}{\partial x_2} X_2 + \dots + \frac{\partial V}{\partial x_n} X_n + \frac{\partial V}{\partial t},$$

representing its total derivative with respect to t, taken under the hypothesis that $x_1, x_2, \dots, x_n$ are functions of t satisfying the differential equations of the disturbed motion.

In this case we shall always assume the function V such that V', as a function of the variables (39), is continuous and single-valued under conditions (40).

In speaking in what follows of the derivative of a function V, we shall understand that we are referring to the total derivative in question.

16. [*Fundamental propositions*]

We all know the theorem of Lagrange [*Mécanique Analytique*, Paris, 1811, vol. 1, part 1, section 3, arts. 21–27, part 2, section 6, arts. 1–10] on the stability of equilibrium in the case where there exists a force-function [potential energy function multiplied by (-1)], as well as the elegant demonstration which has been proposed for it by Lejeune-Dirichlet [*J. reine angew. Math.* **32** (1846) 85 and *J. Math. pures appl.* **12** (1847) 474]. This last rests on considerations which can serve for the proof of many other analogous theorems.

Guiding ourselves by these considerations, we are going to establish here the following propositions:

THEOREM I. *If the differential equations of the disturbed motion are such that it is possible to find a definite function V, of which the derivative V′ is a function of fixed sign which is opposite to that of V, or reduces identically to zero, the undisturbed motion is stable.*

Let us agree, to fix ideas, that the function found, V, is positive-definite, and that its derivative V' represents a negative function or is identically zero.

Then we shall be able to find constants T and H such that, for all values of the variables $x_1, x_2, \dots, x_n, t$ which satisfy the conditions $t \geqslant T$ and

$$|x_s| \leqslant H \quad (s = 1, 2, \dots, n), \tag{41}$$

we have the following inequalities:

$$V' \leqslant 0, \quad V \geqslant W, \tag{42}$$

where W is a certain positive function of the variables x_s, independent of t, and only becoming zero under conditions (41) for $x_1 = x_2 = \dots = x_n = 0$.

In considering the quantities x_s as functions of t satisfying the differential equations of the disturbed motion, let us suppose that the values ξ_s of these functions for $t = T$ satisfy conditions (41) with inequality signs. Then, by virtue of the continuity of these functions, conditions (41) will be satisfied for all values of t sufficiently near T.

This agreed, let us only consider values of t not less than T.

Then, on designating the value of the function V for $t = T$ by V_0 and taking account of the equality

$$V - V_0 = \int_T^t V'\, dt \tag{43}$$

we shall be able to conclude that, if in the interval from T to t conditions (42) are constantly fulfilled, the functions x_s, in the same interval, will certainly satisfy the condition

$$W \leqslant V_0, \tag{44}$$

of which we can make the right-hand side as small as we wish, on making all the ξ_s sufficiently small in absolute value.

Let us designate by x the greatest of the quantities $|x_1|, |x_2|, \ldots, |x_n|$ and by ε a positive number as small as we wish (and moreover smaller than H), and let us consider all the possible systems of values of the quantities x_s satisfying the condition

$$x = \varepsilon. \tag{45}$$

Let l be the *precise* lower limit of the function W (as a function of the independent variables $x_1, x_2, \ldots, x_n$) under this condition.

The value of l will be necessarily different from zero and positive, for, by the very nature of the function W, this function can become under condition (45) neither negative nor zero, and since l, by virtue of the continuity of this function, is necessarily one of the values that it can take under the said condition.

Consequently we shall always be able to make V_0 less than l, and moreover we shall be able to find a positive number λ such that the inequality $V_0 < l$ is satisfied whenever the ξ_s satisfy the conditions

$$|\xi_s| \leqslant \lambda \quad (s = 1, 2, \ldots, n). \tag{46}$$

This settled, let us agree that the quantities ξ_s are to be actually chosen in accordance with conditions (46).

As the number λ is necessarily less than ε, the functions x_s will then satisfy the inequalities

$$|x_s| < \varepsilon \quad (s = 1, 2, \ldots, n) \tag{47}$$

for all values of t sufficiently near T.

Now these functions, which vary continuously with t, can only cease to satisfy inequalities (47) after having reached values satisfying condition (45). And this, seeing that $V_0 < l$, is incompatible with condition (44).

We must thus conclude that, whatever the ξ_s satisfying conditions (46), the functions x_s will satisfy inequalities (47) for all values of t greater than T.

In this manner, we can regard our theorem as proved.

We see that the theorem of Lagrange is only a particular case. [Take for V the total energy, which is positive-definite, and note that V' is zero by conservation of energy.]

Remark I

If, for the differential equations of the disturbed motion, we know a certain number of integrals $U_1, U_2, \ldots, U_m$ (vanishing, like all the functions considered here, for $x_1 = x_2 = \ldots = x_n = 0$), and if the function found, V, only satisfies conditions (42) (with the previous meaning of the symbol W) for values of the variables subject to the conditions

$$U_1 = 0, \quad U_2 = 0, \quad \ldots, \quad U_m = 0,$$

we shall be able to conclude that the undisturbed motion is stable at least for perturbations which satisfy these last conditions.

The case where the function V itself is one of the integrals, and where the functions $V, U_1, U_2, \ldots, U_m$ do not depend explicitly on t, constitutes a proposition indicated by Routh†. [Recall also the remark on conservative perturbations at the end of Section 1.]

Remark II

If the function V, while satisfying the conditions of the theorem, admits an infinitely small upper limit, and if its derivative represents a definite function, we can show that every disturbed motion, sufficiently near the undisturbed motion, approaches it asymptotically.

For this purpose, let us consider an arbitrary disturbed motion, where the quantities ξ_s are sufficiently small in absolute value for conditions (41) to be constantly satisfied starting from the instant $t = T$.

We easily convince ourselves, on taking account of the properties admitted for the function V (which we assume, as before, positive-definite), that if the constant H is small enough, it is impossible to find a positive number l which is less than all the values which the function V takes in this motion for $t > T$.

In fact, if such a number existed we could find, seeing that the function V admits an infinitely small upper limit, another positive number λ such that we would have $x > \lambda$ (x representing as before the greatest of the quantities $|x_s|$) for all values of t greater than T. And then, for the function $-V'$ there would exist, under the same conditions, a non-zero lower limit l'.

In fact, the function $-V'$, conforming to what we have assumed, is positive-definite. We can thus always suppose the constants T and H to be such that, for $t \geqslant T$ and $x \leqslant H$, we have $-V' \geqslant W'$, where W' is a certain positive function of the variables x_s, independent of t and becoming zero under the condition $x \leqslant H$ only in the case where $x = 0$. Now this last case will be excluded if we subject the x_s to verifying the condition

$$\lambda \leqslant x \leqslant H.$$

Thus, under this condition, the function W' will admit a certain non-zero lower limit l'.

Now, if for $t > T$ we have constantly $-V' > l'$, equation (43) will give

$$V < V_0 - l'(t - T)$$

for all values of t which exceed T. And this is impossible, for the first member of the inequality is a positive function of t and the second becomes negative as soon as t is large enough.

Thus, however small the number l may be, an instant will always arrive when the function V becomes less than l. And as this is a decreasing function of t, it will subsequently remain constantly less than l.

† The advanced part of *A Treatise on the Dynamics of a System of Rigid Bodies*, fourth edition, 1884, pp. 52–53.

Consequently, however small the positive number ε may be, there will always arrive an instant when the function V becomes and subsequently remains less than the exact lower limit of the function W under the condition

$$\varepsilon \leqslant x \leqslant H.$$

And, at least starting from this instant, the functions x_s always remain less in absolute value than ε.

We conclude from this that, the ξ_s being sufficiently small in absolute value, the functions x_s tend towards zero as t approaches infinity.

THEOREM II. *Let V be a function of the variables x_s, t possessing the following properties.*

(1) *It admits an infinitely small upper limit.*
(2) *Its derivative V' is a definite function.*
(3) *For every value of t greater than a certain limit the function V is capable of taking the sign of V', however small in absolute value the x_s may be.*

If such a function V can be formed with the aid of the differential equations of the disturbed motion, the undisturbed motion is unstable.

Let us assume that we have found a function V satisfying these conditions, and that its derivative V' is positive-definite.

We shall then be able to assign constants T and H such that, for all values of the variables satisfying the conditions $t \geqslant T$ and

$$|x_s| \leqslant H, \tag{48}$$

we have

$$V' \geqslant W, \quad |V| < L, \tag{49}$$

where L is a positive constant and W is a function of the variables x_s independent of t, positive and only becoming zero if all the x_s are zero.

This settled, let us suppose that the values ξ_s of the functions x_s for $t = T$ satisfy conditions (48) with the signs of inequality. Then, on designating the value of the function V for the same value of t as V_0 and on resorting to the equation

$$V - V_0 = \int_T^t V'\,dt \tag{50}$$

we deduce from it that

$$V \geqslant V_0 \tag{51}$$

for all values of t exceeding T and such that, in the interval from T to t, conditions (48) do not cease to be fulfilled.

We note now that, in view of the third property of the function V, we can suppose the constant T sufficiently large that, for a suitable choice of the quantities ξ_s subject to the inequalities

$$|\xi_s| < \varepsilon \quad (s = 1, 2, \ldots, n),$$

where ε is a positive number as small as we wish, we can make the constant V_0 positive.

Now, if V_0 is a positive quantity we shall be able to find, because of the first property of the function V, a positive number λ which is less than all the values which x, the greatest of the values $|x_s|$, can take under condition (51), t being greater than T. And then, if we designate by l an arbitrary positive number which is less than all the possible values of the function W under the condition

$$\lambda \leqslant x \leqslant H,$$

we shall have, in view of (50) and (49),

$$V > V_0 + l(t - T), \tag{52}$$

and this inequality will hold for $t > T$, provided that, in the interval from T to t, conditions (48) are constantly fulfilled.

Now, under the same conditions, the function V remains in absolute value below a number L, and this can occur simultaneously with inequality (52) only for values of t less than the number

$$\tau = T + \frac{L - V_0}{l}.$$

We must thus admit that, in the interval from T to τ, there exists a value of t starting from which at least one of conditions (48) ceases to be constantly fulfilled.

In this way we arrive at the conclusion that, however small may be the number ε which has not to exceed the absolute values of the quantities ξ_s, these quantities can always be chosen in such a manner that, during the motion that ensues, at least one of the quantities $|x_s|$ reaches a *fixed* limit H. Thus is manifested the instability of the undisturbed motion.

Example I

Suppose that the given system of differential equations of the disturbed motion has the following form:

$$\frac{dx_1}{dt} = \frac{\partial V}{\partial x_1}, \quad \frac{dx_2}{dt} = \frac{\partial V}{\partial x_2}, \quad \ldots, \quad \frac{dx_n}{dt} = \frac{\partial V}{\partial x_n},$$

where V is a holomorphic function of the variables $x_1, x_2, \ldots, x_n$, not depending explicitly on t and not containing in its development terms below the second degree.

By virtue of these equations we have [see the last equation of Section 15]

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial V}{\partial x_n}\right)^2.$$

We can thus conclude that, if V is a negative-definite function, the undisturbed motion will be stable. On the other hand, this motion will be unstable whenever V is not such a function, at least if we are not dealing with the case where the system of equations

$$\frac{\partial V}{\partial x_1} = 0, \quad \frac{\partial V}{\partial x_2} = 0, \quad \ldots, \quad \frac{\partial V}{\partial x_n} = 0$$

can be satisfied by real values (not simultaneously equal to zero but as small as we wish) of the variables x_s.

This last case will be uncertain and will require a special investigation. Besides, it will only occur if the Hessian of the function V becomes zero when we put

$$x_1 = x_2 = \dots = x_n = 0.$$

Example II

Let us consider the following system of differential equations of order $2k$:

$$\frac{d}{dt}\frac{\partial F}{\partial x'_s} - \frac{\partial F}{\partial x_s} = 0, \quad \frac{dx_s}{dt} = x'_s \quad (s = 1, 2, \dots, k),$$

where

$$F = \frac{1}{2}\sum_{i=1}^{k} x_i'^2 + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} v_{ij}x'_i x'_j + U,$$

$v_{ij} = v_{ji}$ and U being holomorphic functions of the variables $x_1, x_2, \dots, x_k$, independent of t and vanishing when all these variables become zero. [These correspond to Lagrange's equations of motion.] Moreover, the function U is supposed such that its development does not contain terms below the second degree.

This system obviously reduces to the type of system of differential equations of disturbed motion which we consider here.

Let

$$U = U_m + U_{m+1} + \dots,$$

where U_i designates, in a general manner, an entire and homogeneous function of the quantities $x_1, x_2, \dots, x_k$ of degree i. Then, on putting

$$V = x_1\frac{\partial F}{\partial x'_1} + x_2\frac{\partial F}{\partial x'_2} + \dots + x_k\frac{\partial F}{\partial x'_k},$$

we shall have, by virtue of our equations,

$$\begin{aligned}\frac{dV}{dt} &= \sum_{s=1}^{k} x_s\frac{\partial F}{\partial x_s} + \sum_{s=1}^{k} x'_s\frac{\partial F}{\partial x'_s}\\ &= \sum_{i=1}^{k} x_i'^2 + \sum_{i=1}^{k}\sum_{j=1}^{k}\left(v_{ij} + \frac{1}{2}\sum_{s=1}^{k} x_s\frac{\partial v_{ij}}{\partial x_s}\right)x'_i x'_j\\ &\quad + mU_m + (m+1)U_{m+1} + \dots\end{aligned}$$

Suppose that U_m is a positive-definite function of the variables $x_1, x_2, \dots, x_k$ (which implies that m is an even number).

Then this expression dV/dt will be a positive-definite function of the variables $x_1, x_2, \dots, x_k, x'_1, x'_2, \dots, x'_k$ [recall (40)], and all the conditions of Theorem II will be satisfied. We therefore conclude that the undisturbed motion is unstable.

The case considered here can arise, for example, in the question of the stability of equilibrium (in the ordinary sense), when there exists a force-function (the function U).

Whenever, at a position of equilibrium, the force-function becomes a minimum, and this minimum is manifested by the terms of least degree that can be found in the development of the increase of this function in powers of increases of the coordinates, we conclude that the equilibrium is unstable.

THEOREM III. *Let V be a function of the variables x_s, t, possessing the following properties.*

(1) *It is a limited function.*

(2) *Its derivative is of the form*

$$V' = \lambda V + W, \tag{53}$$

λ being a positive constant and W a function of fixed sign (which function may reduce identically to zero).

(3) *For every value of t greater than a certain limit the function V is capable of taking the sign of W (assuming that W is not identically zero), however small the x_s in absolute value.*

If the differential equations of the disturbed motion allow the formation of such a function V, the undisturbed motion is unstable.

Let us suppose that the function found, V, satisfying these conditions, is such that W is a positive function.

Then we shall be able to choose the numbers T and H in such a way that, for values of the variables satisfying the conditions $t \geqslant T$,

$$|x_s| \leqslant H \quad (s = 1, 2, \ldots, n), \tag{54}$$

we have [see property (1) in the theorem]

$$|V| < L, \quad W \geqslant 0,$$

where L is a positive constant. We shall moreover be able to take the number T so large that, for a suitable choice of the values ξ_s which the functions x_s have to take for $t = T$, and on making all the $|\xi_s|$ as small as we wish, we can make positive the corresponding value V_0 of the function V [see property (3)].

Only considering values of t not less than T, and on paying attention to equation (53), we conclude that if we consider V, by virtue of the differential equations of the motion, as a function of t, this function will verify the inequality

$$\frac{dV}{dt} - \lambda V \geqslant 0$$

for all values of t for which conditions (54) remain fulfilled.

Therefore, if from T to t these conditions do not cease to be satisfied, we shall have [on separating the variables in the above inequality and integrating both sides]

$$V \geqslant V_0 e^{\lambda(t-T)}$$

and, by consequence [from the inequality after (54)]

$$L > V_0 e^{\lambda(t-T)}.$$

Now, V_0 being positive, this last inequality can only hold for values of t less than the quantity

$$\tau = T + \frac{1}{\lambda} \log \frac{L}{V_0}.$$

Thus, in the interval from T to τ, conditions (54) cannot be constantly fulfilled.

From this, as in the proof of the preceding theorem, we conclude that the undisturbed motion is unstable.

By varying the conditions which the sought functions must satisfy, we could certainly propose many other theorems similar to the preceding ones. But, for the applications which we have in mind, the theorems which we have given are fully sufficient. That is why we can restrict ourselves to the latter.

Remark

Up to the present, we have assumed that for the variables x_s all sufficiently small real values are possible. But cases can be encountered where, because of the very meaning of these variables, for some among them there will only be allowed values with a predetermined sign (we shall not consider conditions of greater complexity).

In order that this should occur the differential equations (1) must be such that these conditions, which will be of the form

$$x_i \geqslant 0, \quad x_j \leqslant 0, \tag{55}$$

are satisfied throughout the duration of the motion, as soon as we suppose them satisfied at the initial instant.

In such a case, in applying Theorems II and III care must be taken that the function V does not lose the third property, when we take account of the conditions (55). Further, we shall be able to imply these conditions in all the definitions and all the preceding propositions, just as was done as far as conditions (40) were concerned.

CHAPTER II. Study of steady motion

Linear differential equations with constant coefficients

17. [*Determinantal equation. Types of solution corresponding to its simple and multiple roots. Sets of solutions*]

Let us consider the system of linear differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}s_2 + \dots + p_{sn}x_n \quad (s = 1, 2, \dots, n) \tag{1}$$

with constant coefficients $p_{s\sigma}$.

The integration of this system depends on the resolution of the algebraic equation

$$\begin{vmatrix} p_{11}-\chi & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22}-\chi & \cdots & p_{2n} \\ \dots & \dots & \dots & \dots \\ p_{n1} & p_{n2} & \cdots & p_{nn}-\chi \end{vmatrix} = 0$$

of the nth degree relative to the unknown χ.

We shall call this the *determinantal* equation, and the determinant constituting its left-hand side will be called the *fundamental* determinant. In considering the latter as a function of χ, we shall designate it by $D(\chi)$.

To each root of the determinantal equation corresponds a solution of system (1) of the form

$$x_1 = K_1 e^{\chi t}, \quad x_2 = K_2 e^{\chi t}, \quad \dots, \quad x_n = K_n e^{\chi t}, \tag{2}$$

where the K_s are constants, among which at least one is different from zero; and, when the determinantal equation does not have multiple roots, we shall have, on considering all its roots, n solutions of the form (2), which will be independent.

In the case of multiple roots, system (1) will admit, in general, solutions of the following type:

$$x_1 = f_1(t)e^{\chi t}, \quad x_2 = f_2(t)e^{\chi t}, \quad \dots, \quad x_n = f_n(t)e^{\chi t},$$

where the $f_s(t)$ are entire functions [polynomials] of t, of which the degrees do not exceed the number obtained on diminishing by unity the degree of multiplicity of the root χ.

If we consider solutions of type (2) as being included in this last type, to each root χ of multiplicity μ there will correspond μ independent solutions of such form.

Further, if among these solutions there is found one such that the degrees of at least one of the functions $f_s(t)$ attains its upper limit $\mu - 1$, we shall be able, on starting from this solution, to obtain all the μ independent solutions which correspond to root χ. For this purpose we shall have only to replace the functions $f_s(t)$ by their derivatives $f_s^{(r)}(t)$ with respect to t, of various orders. In this manner we shall arrive at the following μ independent solutions:

$$\begin{array}{cccc} f_1(t)e^{\chi t}, & f_2(t)e^{\chi t}, & \dots, & f_n(t)e^{\chi t}, \\ f'_1(t)e^{\chi t}, & f'_2(t)e^{\chi t}, & \dots, & f'_n(t)e^{\chi t} \\ \dots & \dots & \dots & \dots \\ f_1^{(\mu-1)}(t)e^{\chi t}, & f_2^{(\mu-1)}(t)e^{\chi t}, & \dots, & f_n^{(\mu-1)}(t)e^{\chi t}. \end{array}$$

We shall say that, in this case, there corresponds to the root χ a single set of solutions.

This case will arise whenever the root considered, χ, does not make zero at least *one* of the first minors of the fundamental determinant.

It can happen that the root χ, of multiplicity μ, makes zero all the minors of this determinant up to order $k-1$ inclusively, without making zero at least one of the minors of order k.

Then to this root there will correspond k sets of independent solutions, formed similarly to the above set.

The number k has for upper limit the number μ. This limit can be attained, and then all the solutions corresponding to the root will be of type (2).

We can regard all these theorems as so well known by all that it would be superfluous to give proofs, which anyway do not present the least difficulty. [For related expositions see É. Goursat, E. R. Hedrick and O. Dunkel, *Differential Equations*, Boston, 1917, pp. 152–161, and N. G. Chetayev, *The Stability of Motion*, Oxford, 1961, pp. 51–58.]

Let us note that $\chi_1, \chi_2, \dots, \chi_n$ being all the roots of the determinantal equation, the real parts of the numbers

$$-\chi_1, \quad -\chi_2, \quad \dots, \quad -\chi_n$$

will represent for equations (1) what we have called the *characteristic numbers of the system of linear differential equations.*

18. [*Linear transformation of differential equations into the simplest form*]

For system of equations (1) we can find n independent integrals of the form

$$y_1x_1 + y_2x_2 + \dots + y_nx_n,$$

where the y_s are functions of t.

These functions will satisfy the system of equations

$$\frac{dy_s}{dt} + p_{1s}y_1 + p_{2s}y_2 + \dots + p_{ns}y_n = 0 \quad (s = 1, 2, \dots, n), \tag{3}$$

adjoint with respect to (1), and, if

$$\begin{array}{c} y_{11}, y_{21}, \dots, y_{n1}, \\ y_{12}, y_{22}, \dots, y_{n2}, \\ \dots\dots\dots \\ y_{1n}, y_{2n}, \dots, y_{nn} \end{array}$$

is an arbitrary system of n independent solutions of the adjoint system, the n

functions

$$y_{11}x_1 + y_{21}x_2 + \dots + y_{n1}x_n,$$
$$y_{12}x_1 + y_{22}x_2 + \dots + y_{n2}x_n,$$
$$\dots\dots\dots\dots\dots\dots$$
$$y_{1n}x_1 + y_{2n}x_2 + \dots + y_{nn}x_n$$

will be independent integrals of system (1). [For discussion of adjoint equations see Goursat, Hedrick and Dunkel, *loc. cit.*, pp. 156–157.]

The fundamental determinant of the system of equations (3) is obtained on replacing, in the fundamental determinant of system (1), χ by $-\chi$ and on multiplying by $(-1)^n$. It results from this that the determinantal equation of system (3) will only differ in the signs of roots from the determinantal equation of system (1).

Let

$$-\chi_1, -\chi_2, \dots, -\chi_k$$

be all the roots of the determinantal equation of system (3), under the hypothesis that each multiple root is repeated as many times as there correspond to it sets of solutions.

Then, to each of the numbers $-\chi_s$ we shall be able to make correspond a set of solutions, such that all the solutions considered will be independent.

Let n_s be the number of solutions in the set corresponding to the root $-\chi_s$, such that we have

$$n_1 + n_2 + \dots + n_k = n.$$

On taking for the quantities $y_{s\sigma}$ the functions entering into the composition of these sets we shall have, for each root $-\chi_s$, the following n_s integrals of system of equations (1):

$$\left(z_1^{(s)}\frac{t^m}{m!} + z_2^{(s)}\frac{t^{m-1}}{(m-1)!} + \dots + z_m^{(s)}t + z_{m+1}^{(s)}\right)e^{-\chi_s t}$$
$$(m = 0, 1, 2, \dots, n_s - 1), \tag{4}$$

where the $z_j^{(s)}$ are linear forms relative to the quantities $x_1, x_2, \dots, x_n$ with constant coefficients and $m!$ represents, as usual, the product, $1\cdot 2\cdot 3\cdot \dots \cdot m$ when m is non-zero, and 1 when $m = 0$.

The n integrals which we shall obtain here on giving s all the integer values from 1 to k inclusively will be, under our hypothesis, independent.

It results from this that the n forms $z_j^{(s)}$ will be necessarily also independent, and we shall be able, by consequence, to take them as new unknown functions in place of $x_1, x_2, \dots, x_n$.

On doing this, we obtain the following equations:

$$\left.\begin{aligned} &\frac{dz_1^{(s)}}{dt} = \chi_s z_1^{(s)}, \\ &\frac{dz_j^{(s)}}{dt} = \chi_s z_j^{(s)} - z_{j-1}^{(s)} \\ &(j = 2, 3, \dots, n_s;\ s = 1, 2, \dots, k), \end{aligned}\right\} \tag{5}$$

which evidently the quantities $z_j^{(s)}$ must satisfy, expressions (4) representing the integrals of equations (1). [Equations (5) can be obtained by equating to zero the coefficients of $t^m, t^{m-1}, \ldots$ in the derivative of (4). Lyapunov seems here to be using the theory of Weierstrass's elementary divisors. See Chetayev, *loc. cit.*, pp. 58–70, and Goursat, Hedrick and Dunkel, *loc. cit.*, pp. 161–162.]

We conclude from this that, by a linear substitution with constant coefficients, system (1) can be reduced to the form (5).

Let us suppose that all the coefficients $p_{s\sigma}$ in equations (1) are real numbers, and that, in the transformations of these equations, we only wish to consider substitutions with coefficients also real. Then the preceding transformation will only be possible if all the roots of the determinantal equation of system (1) are real numbers. And for the case where there exist complex roots, the most simple form to which these equations can be reduced will be a little different.

To show such a transformation, we note that, under the hypothesis admitted, to each complex root will correspond a conjugate root of the same multiplicity, and that, if we have found all the linear forms $z_j^{(s)}$ which correspond to an arbitrary complex root, we shall have, on replacing in them $\sqrt{-1}$ by $-\sqrt{-1}$, new forms which we may take for the quantities z in the case of the conjugate root.

Let us suppose then that, for the two conjugate roots

$$\chi_1 = \lambda + \mu\sqrt{-1}, \quad \chi_2 = \lambda - \mu\sqrt{-1}$$

we have these values of z:

$$z_j^{(1)} = u_j + v_j\sqrt{-1}, \quad z_j^{(2)} = u_j - v_j\sqrt{-1}$$
$$(j = 1, 2, \ldots, \nu).$$

Then, for new unknown functions, we shall be able to take in place of $z_j^{(1)}$, $z_j^{(2)}$ the quantities u_j, v_j, which will be linear forms in the quantities x_s with coefficients constant and real.

The differential equations which these functions must satisfy reduce easily from (5) [on separating real and imaginary parts] and have the following form:

$$\frac{du_1}{dt} = \lambda u_1 - \mu v_1, \quad \frac{dv_1}{dt} = \lambda v_1 + \mu u_1,$$

$$\frac{du_j}{dt} = \lambda u_j - \mu v_j - u_{j-1}, \quad \frac{dv_j}{dt} = \lambda v_j + \mu u_j - v_{j-1}$$
$$(j = 2, 3, \ldots, \nu).$$

We shall obtain such sets of equations for each pair of conjugate complex roots, and for real roots we shall have sets with the form (5).

Remark

While on the subject of the indicated transformation, let us note that, on using this, we can demonstrate a proposition which is connected with the theory of linear differential equations of which the elements have been expounded in the preceding chapter. In fact (on returning to the hypotheses of Section 10), it is easy to establish that† *for each reducible system of equations, where all coefficients are real functions of t, the transformation into a system with constant coefficients can be effected by means of a substitution* (of the character defined in Section 10) *in which all the coefficients are also real functions of t.*

19. [*Derived determinants and equations obtained on equating them to zero*]

Let us consider the following problem.

Suppose given a partial differential equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = \chi V, \tag{6}$$

† We may prove it in the following manner.

Let us assume that system (1) (in which the coefficients $p_{s\sigma}$ are assumed real functions of t) is reducible. By virtue of what we have shown, we can then assume that, by a substitution satisfying the conditions in Section 10, it is reduced to form (5). We can moreover obviously suppose that all the χ_s are real.

This settled, let

$$z_j^{(s)} = u_j^{(s)} + v_j^{(s)}\sqrt{-1},$$

the $u_j^{(s)}$, $v_j^{(s)}$ being linear forms in the quantities x_σ, of which the coefficients are real functions of t. On considering the following k pairs of functions:

$$u_1^{(1)}, v_1^{(1)};\, u_1^{(2)}, v_1^{(2)}, \ldots;\, u_1^{(k)}, v_1^{(k)}$$

and on taking from each pair only *one* function, let us form all possible combinations, each containing k functions. Since, by the property of the substitutions considered (Section 10), the functional determinant [Jacobian] formed with the partial derivatives of the $z_j^{(s)}$ with respect to the x_σ will not be a vanishing function of t, we shall therefore encounter *at least one* combination such that we cannot form, with the functions which compose it, any linear expression with constant coefficients (not simultaneously equal to zero) which is identically zero, or in which all the coefficients by which the variables x_σ are affected are vanishing functions of t. For definiteness let us suppose that this condition is fulfilled for the following combination:

$$u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(k)}.$$

We now note that, under our hypotheses, every integral (4) of system (1) gives a new integral of the same system, when all the $z_j^{(s)}$ are replaced by the quantities $u_j^{(s)}$. And, under the supposition that we have just admitted, all these integrals will be independent. In fact, if it were not so, we could form with these integrals a linear expression with constant coefficients (not simultaneously equal to zero) which would be identically zero. Now this expression will appear in the form of a sum of products of the quantities $t^\mu e^{-\chi_s t}$ multiplied by linear expressions formed with the $u_j^{(s)}$; and, if we designate by χ the least of the numbers χ_s corresponding to those of the integrals considered which actually appear in the linear expression in question, and by m the greatest of the exponents of the powers of t that are found in those of these last integrals for which $\chi_s = \chi$, we must conclude that, for our expression to be identically zero, it must be that the expression which is there multiplied by $t^\mu e^{-\chi_s t}$ either is also identically zero, or represents a form in the quantities x_σ in which all the coefficients are vanishing functions of t. But neither the one nor the other is possible, for the said expression will be necessarily a linear combination with constant coefficients of the forms $u_1^{(s)}$. Thus our integrals will be independent, and, by consequence, the functional determinant of the quantities $u_j^{(s)}$ with respect to the quantities x_σ will not be identically zero. But then this determinant will be necessarily such that the quantity which is inverse to it will represent a bounded function of t, for the said determinant can only differ by a constant factor from the functional determinant of the quantities $z_j^{(s)}$.

In this way the substitution which replaces the variables x_σ by the variables $u_j^{(s)}$ satisfies all the conditions of the substitutions of Section 10. It moreover possesses real coefficients, and the system of equations (1) is transformed, by this substitution, into a system with constant coefficients.

in which χ designates a constant. We ask to find all the values of χ for which we can satisfy this equation on taking for V an entire and homogeneous function of the variables $x_1, x_2, \ldots, x_s$ of a given degree m.

It is easy to form the algebraic equation which must be satisfied by the sought values of χ.

Let N be the number of coefficients in the function V, so that

$$N = \frac{n(n+1)\ldots(n+m-1)}{1\cdot 2\cdot 3\cdot \ldots \cdot m} = \frac{(m+1)(m+2)\ldots(m+n-1)}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)}$$

[see e.g. C. V. Durell, *Advanced Algebra*, London, 1932, Vol. I, pp. 99–100].

Such will also be the number of equations, linear and homogeneous with respect to these coefficients, that we shall obtain on equating the coefficients of the same products

$$x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

on the two sides of equation (6).

On eliminating between these equations the coefficients of the function V we shall obtain the algebraic equation sought, which will be of the following form:

$$\begin{vmatrix} a_{11}-\chi & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22}-\chi & \ldots & a_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ a_{N1} & a_{N2} & \ldots & a_{NN}-\chi \end{vmatrix} = 0$$

where the a_{ij} represent certain linear forms in the coefficients $p_{s\sigma}$.

This equation will thus be of degree N.

Let us designate by $D_m(\chi)$ the determinant which appears on the left-hand side. On giving to m successively the values $1, 2, 3, \ldots$, we shall have an indefinite sequence of determinants

$$D_1(\chi), D_2(\chi), D_3(\chi), \ldots,$$

where the first term will not differ from the determinant which we have designated as $D(\chi)$, and which we have called *fundamental*. All the other terms will be called *derived determinants*, such that $D_m(\chi)$ will be the $(m-1)$th derived determinant.

Knowing all the roots of the determinantal equation, it is easy to find all the roots of the equation $D_m(\chi) = 0$, for we can demonstrate the following proposition.

THEOREM. *If*

$$\chi_1, \chi_2, \ldots, \chi_n$$

are the roots of the determinantal equation, all the roots of the equation

$$D_m(\chi) = 0$$

will be obtained from the formula

$$\chi = m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n, \tag{7}$$

on giving the numbers $m_1, m_2, \ldots, m_n$ all the non-negative integer values which satisfy the relation

$$m_1 + m_2 + \ldots + m_n = m,$$

and on taking care that one and the same system of values does not occur more than once.

To prove this, let us first suppose that the coefficients $p_{s\sigma}$ are such that there does not exist any relation of the form

$$\mu_1\chi_1 + \mu_2\chi_2 + \dots + \mu_n\chi_n = 0$$

where $\mu_1, \mu_2, \dots, \mu_n$ are integers satisfying the conditions

$$\mu_1 + \mu_2 + \dots + \mu_n = 0,$$
$$|\mu_s| \leqslant m \quad (s = 1, 2, \dots, n),$$

and not being simultaneously all zero.

Then the values of χ defined by formula (7) will all be distinct. We shall assume further that none of them is zero.

This agreed, and on understanding by χ a number different from zero, let us designate by $v_1, v_2, \dots, v_n$ the independent integrals of the system of linear differential equations that may be deduced from (1) on putting

$$t = \frac{1}{\chi}\log V.$$

[Note that (1) and (6) imply $dV/dt = \chi V$, which is satisfied by the above relation.] Then, Φ being an arbitrary function, the equation

$$\Phi(v_1, v_2, \dots, v_n) = 1, \tag{8}$$

since it is soluble with respect to V, will furnish a solution of equation (6). [See Goursat, Hedrick and Dunkel, *loc. cit.*, pp. 214–216.]

Let us suppose that all the integrals v_s are linear with respect to the variables $x_1, x_2, \dots, x_n$.

As, under the agreed hypotheses, all the χ_s will be distinct, these integrals will all be of the form

$$v_s = (\alpha_{s1}x_1 + \alpha_{s2}x_2 + \dots + \alpha_{sn}x_n)V^{-\frac{\chi_s}{\chi}},$$

where the α_{sj} are constants.

Consequently, if we make

$$\Phi(v_1, v_2, \dots, v_n) = v_1^{m_1}v_2^{m_2}\dots v_n^{m_n},$$

on understanding by the m_s non-negative integers, of which the sum is equal to m, and if we put further

$$\chi = m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n$$

equation (8) will lead to the solution

$$V = \prod_{s=1}^{n} (\alpha_{s1}x_1 + \alpha_{s2}x_2 + \dots + \alpha_{sn}x_n)^{m_s},$$

representing an entire and homogeneous function [polynomial] of the quantities x_s of degree m.

It results from this that all the values of χ of the form considered satisfy the equation

$$D_m(\chi) = 0.$$

Now, according to what we have assumed, the number of distinct values of this form is equal to the degree N of the above equation. [When the m_i vary, the number of different expressions

$$m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n$$

is the same as the number of different expressions

$$x_1^{m_1}x_2^{m_2} \dots x_n^{m_n}.]$$

Thus no other value of χ will be able to satisfy this equation.

Next, to prove the theorem in all its generality, it suffices to note that the cases we have omitted can be considered as limiting cases of that which we have just examined. They will thus not present any exceptions; but in these cases, the equation $D_m(\chi) = 0$ will be able to have multiple roots or roots equal to zero.

Remark

Let us draw attention to the following property of the derived determinants.

When the determinantal equation does not have multiple roots and also when, in the case of such roots, each of them makes zero all the minors of the fundamental determinant up to the highest possible order for the multiplicity of the root, each multiple root of the equation $D_m(\chi) = 0$ will possess the same properties with respect to the minors of the determinant $D_m(\chi)$.

This property may be demonstrated on noting that, under the indicated condition, to each root of the equation $D_m(\chi) = 0$, of multiplicity μ, there will correspond μ linearly independent solutions of equation (6), in the form of entire and homogeneous functions of degree m.

20. [*Entire and homogeneous functions satisfying certain linear partial differential equations*]

We can now demonstrate the following propositions.

THEOREM I. *When the roots $\chi_1, \chi_2, \dots, \chi_n$ of the determinantal equation are such that, under the condition $m_1 + m_2 + \dots + m_n = m$, m being a given positive integer, we cannot have any relation of the form*

$$m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n = 0,$$

with non-negative integer values of the m_s, we shall always be able to find, and that in a unique manner, a form V of degree m in the variables $x_1, x_2, \dots, x_n$ satisfying the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = U \tag{9}$$

U being any given form of the same degree m.

In fact, to determine the coefficients of the sought form V, we shall have a system of linear equations of which the number will be equal to that of the coefficients. Moreover, the determinant of this system, which is $D_m(0)$, will not be zero under the conditions of the theorem. [$D_m(0)$ is the product of roots of $D_m(\chi) = 0$; and, because of the theorem of Section 19, none of these roots is zero.]

Remark

The condition of the theorem will be fulfilled for example, and indeed for each value of m, when the real parts of the numbers χ_s are different from zero and have the same signs.

In the two following theorems we shall assume the quantities x_s to be real, whether we consider them as independent variables or as functions of t satisfying equations (1). This is possible by consequence of the assumed reality of the coefficients $p_{s\sigma}$.

THEOREM II. *When the real parts of all the roots χ_s are negative, and when, in equation* (9), *the function U is a definite form of even degree, the form V of the same degree satisfying this equation will also be definite, and its sign will moreover be opposite to that of U.*

To prove this, we note that, the quantities x_s being considered as functions of t satisfying equations (1), we can present equation (9) in the following form:

$$\frac{dV}{dt} = U.$$

From this we conclude that for each solution of system of equations (1) different from $x_1 = x_2 = \dots = x_n = 0$, the form V becomes a function of t, varying constantly, when t increases, in one direction: it increases if U is positive, and decreases if U is negative. Now, under the assumed hypothesis relative to the quantities χ_s, the functions x_s satisfying equations (1) will necessarily be such that, t increasing indefinitely, they will tend towards zero. The same thing will then take place for the function V under consideration. And that, by virtue of what we have noted above [see Theorem II of section 16], is only possible provided that, for every solution different from $x_1 = x_2 = \dots = x_n = 0$, the function V becomes a function of t, such that for any value of t it cannot take the sign of the function U or become zero. Now the latter condition is obviously equivalent to this, that for any choice of the quantities x_s, the function V cannot take a value of the same sign as U or become zero, unless we have $x_1 = x_2 = \dots = x_n = 0$.

THEOREM III. *If among the roots χ_s there are found any of which the real parts are positive and if, m being a given even number, these roots satisfy the condition of Theorem I, then, U being a definite form of degree m, the form V of the same degree satisfying equation* (9) *will certainly not be of fixed sign opposite to that of U.*

In fact, on considering the quantities x_s as functions of t satisfying equations (1), and on presenting equation (9) in the form

$$\frac{dV}{dt} = U,$$

we can conclude that, if by suitable choice of quantities x_s, not simultaneously equal to zero, we can make the function V zero, we shall also be able to give it a value of the same sign as U. Consequently, if the function V could not receive the same sign as U, it would be necessarily definite. And then we would find ourselves in the conditions of Theorem I of Section 16, and we would be able to conclude that, in *every* solution of equations (1), the functions x_s would be bounded (only considering, as before, values of t greater than its initial value).

Now this conclusion would be in disagreement with the hypothesis that among the quantities χ_s there are some with real parts positive, for, under this hypothesis, there will always exist solutions of system (1) in which at least some of the functions x_s will not be bounded.

Thus the function V will necessarily be such that, by a suitable choice of the quantities x_s, we shall always be able to give it the sign of the function U.

Remark

In order that the condition of Theorem I may be fulfilled for any value of m, the determinantal equation must not have zero roots. Moreover, for this condition to be fulfilled for an even value of m, it must be that, among the roots of the determinantal equation, there are no two roots of which the sum is zero.

21. [*Canonical systems of linear differential equations*]

Let us consider a canonical system [Hamiltonian system] of linear differential equations

$$\frac{dx_s}{dt} = -\frac{\partial H}{\partial y_s}, \quad \frac{dy_s}{dt} = \frac{\partial H}{\partial x_s} \quad (s = 1, 2, ..., k), \tag{10}$$

where H is a quadratic form in the variables

$$x_1, x_2, ..., x_k, \quad y_1, y_2, ..., y_k$$

with constant coefficients.

If we put in general

$$\frac{\partial^2 H}{\partial x_i \, \partial x_j} = A_{ij}, \quad \frac{\partial^2 H}{\partial y_i \, \partial y_j} = B_{ij}, \quad \frac{\partial^2 H}{\partial x_i \, \partial y_j} = C_{ij},$$

the fundamental determinant corresponding to this system will only differ by the factor $(-1)^k$ from the determinant

$$\begin{vmatrix} C_{11}+\chi & C_{21} & \dots & C_{k1} & B_{11} & B_{21} & \dots & B_{k1} \\ C_{12} & C_{22}+\chi & \dots & C_{k2} & B_{12} & B_{22} & \dots & B_{k2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ C_{1k} & C_{2k} & \dots & C_{kk}+\chi & B_{1k} & B_{2k} & \dots & B_{kk} \\ A_{11} & A_{21} & \dots & A_{k1} & C_{11}-\chi & C_{12} & \dots & C_{1k} \\ A_{12} & A_{22} & \dots & A_{k2} & C_{21} & C_{22}-\chi & \dots & C_{2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1k} & A_{2k} & \dots & A_{kk} & C_{k1} & C_{k2} & \dots & C_{kk}-\chi \end{vmatrix}$$

Now, by virtue of the relations

$$A_{ij} = A_{ji},$$

$$B_{ij} = B_{ji}$$

this last determinant will not change in value on changing χ into $-\chi$. To see this it suffices, after the indicated replacement, to put rows in place of columns and to use subsequently suitable substitutions [interchanges] for rows as well as columns.

As a consequence, the determinantal equation of system (10) contains only even powers of χ, and therefore to each of its roots χ will correspond the root $-\chi$.

In this way, for the canonical system of equations, we find ourselves in the singular case where the condition of Theorem I (preceding section) is not fulfilled for any even value of m.

We may note that, H being a *definite* form of the variables x_s, y_s, all the roots of the determinantal equation of system (10) will be purely imaginary. Moreover each multiple root with multiplicity μ will make zero all the minors of the fundamental determinant up to order $\mu - 1$ inclusively.

Routh demonstrated this theorem algebraically.† But it is an immediate consequence of the circumstance that H is one of the integrals of system (10). [Lyapunov presumably has in mind that the constancy of H is incompatible with instability of system (10), if H is definite.]

In the case when H is the sum of two quadratic forms, X, of the variables x_s, and Y, of the variables y_s, and when at least one of the forms X or Y is definite, equations (10) have all the properties of the linear differential equations by which are defined, in a first approximation, the small oscillations of a material system in the neighbourhood of a position of equilibrium, when there exists a force-function. [See Routh, *loc. cit.*, Chapter II.] In view of this, and basing ourselves on the known theorems of the theory of small oscillations, we can affirm that in this case the determinantal equation will have only roots of which the squares are real, and that these roots can be all purely imaginary only under the condition that the form H is definite.

When H does not present itself in the form $X + Y$, with the preceding signification of the symbols X and Y, all the roots can be purely imaginary without H being a definite form.

Let us suppose that, from a general point of view, the function H is real and such that the determinantal equation of system (10) has only purely imaginary roots. Let

$$\lambda_1\sqrt{-1},\ \lambda_2\sqrt{-1},\ \dots,\ \lambda_k\sqrt{-1},$$
$$-\lambda_1\sqrt{-1},\ -\lambda_2\sqrt{-1},\ \dots,\ -\lambda_k\sqrt{-1},$$

be these roots, the λ_s designating real numbers different from zero.

Let us assume that all these roots are distinct. Then we shall have for system (10) $2k$ independent integrals of the form

$$(u_s + iv_s)e^{-i\lambda_s t}, \quad (u_s - iv_s)e^{i\lambda_s t} \quad (s = 1, 2, \dots, k), \tag{11}$$

† This proposition was presented by Routh in a different form, since in place of the canonical system of equations he considered the following:

$$\frac{d}{dt}\frac{\partial L}{\partial x'_s} = \frac{\partial L}{\partial x_s}, \qquad \frac{dx_s}{dt} = x'_s \quad (s = 1, 2, \dots, k),$$

where L is a quadratic form in the variables x_s, x'_s. The role of the function H is then played by the function

$$L - \sum \frac{\partial L}{\partial x'_s} x'_s.$$

See the advanced part of *A Treatise on the Dynamics of a System of Rigid Bodies*, fourth edition, 1884, p. 68.

where $i = \sqrt{-1}$, and u_s, v_s are linear forms in the variables x_j, y_j with real constant coefficients.

Let us designate for two arbitrary functions φ and ψ of the variables x_j, y_j (these functions can also contain t) by the symbol (φ, ψ) the quantity

$$\sum_{j=1}^{k} \left(\frac{\partial\varphi}{\partial x_j}\frac{\partial\psi}{\partial y_j} - \frac{\partial\varphi}{\partial y_j}\frac{\partial\psi}{\partial x_j}\right).$$

[This is a Poisson bracket-expression—see E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge, 1937, fourth edition, p. 299; and J. Bertrand, Note VII in J. L. Lagrange, *Mécanique Analytique*, reprinted Paris, 1965, Vol. I, pp. 422–428.] Then, if φ and ψ are integrals of system (10), (φ, ψ) will be, as is known, either an integral of this system, or a determinate constant. [See Whittaker, *loc. cit.*, p. 320.] But, if the functions φ and ψ are taken from the series of integrals (11), the last case is evidently the only possible one.

For this reason, and on noting that, because of the assumption made, none of the numbers $\lambda_s \pm \lambda_\sigma$, s and σ being different, will be zero, we must conclude that all the quantities

$$(u_s + iv_s, u_\sigma + iv_\sigma), \quad (u_s + iv_s, u_\sigma - iv_\sigma), \quad (u_s - iv_s, u_\sigma - iv_\sigma),$$

for which s and σ are different will be zero. As for the quantities

$$(u_s + iv_s, u_s - iv_s) \quad (s = 1, 2, ..., k),$$

they will certainly all be non-zero, for in the contrary case the integrals (11) would not be independent.

We conclude from this that all the brackets (u_s, u_σ), (v_s, v_σ), and also, for s and σ different, all the (u_s, v_σ), will be zero, while all the (u_s, v_s) will be real constants different from zero. We may moreover suppose these constants equal to 1, for we can always assume that each of the functions u_s and v_s includes as a factor the same arbitrary real constant, which we can choose in such a way that (u_s, v_s) becomes equal to 1 in absolute value; and by a suitable choice of the sign of the number λ_s (which up to now has remained undetermined), we shall be able to make the quantity (u_s, v_s) positive.

In this way, on giving a suitable sign to each of the numbers λ_s, we can always suppose the integrals (11) to be such that we have the equations

$$(u_s, u_\sigma) = 0, \quad (v_s, v_\sigma) = 0,$$
$$(u_s, v_s) = 1, \quad (u_s, v_\sigma) = 0 \quad (\sigma \lessgtr s), \quad (s, \sigma = 1, 2, ..., k).$$

Now these are well known equalities, from which we may conclude that, if we form the partial derivatives of functions u_s, v_s with respect to variables x_j, y_j, and if next, considering the latter as functions of the former, we form the partial derivatives of functions x_j, y_j with respect to variables u_s, v_s, we shall have the following relations, due to Jacobi:

$$\frac{\partial u_s}{\partial x_j} = \frac{\partial y_j}{\partial v_s}, \quad \frac{\partial u_s}{\partial y_j} = -\frac{\partial x_j}{\partial v_s}, \quad \frac{\partial v_s}{\partial y_j} = \frac{\partial x_j}{\partial u_s}, \quad \frac{\partial v_s}{\partial x_j} = -\frac{\partial y_j}{\partial u_s} \quad (s, j = 1, 2, ..., k).$$

It results from this that every canonical system of equations

$$\frac{dx_s}{dt} = -\frac{\partial F}{\partial y_s}, \quad \frac{dy_s}{dt} = \frac{\partial F}{\partial x_s} \quad (s = 1, 2, ..., k)$$

(where F is an arbitrary function of the variables x_s, y_s), when we introduce in place of the variables x_s, y_s the variables u_s, v_s, appears again in canonical form:

$$\frac{du_s}{dt} = -\frac{\partial F}{\partial v_s}, \quad \frac{dv_s}{dt} = \frac{\partial F}{\partial u_s} \quad (s = 1, 2, ..., k).$$

Now system (10) reduces in this way to the form

$$\frac{du_s}{dt} = -\lambda_s v_s, \quad \frac{dv_s}{dt} = \lambda_s u_s \quad (s = 1, 2, ..., k).$$

[These equations can be obtained by equating the derivative of the first term in (11) to zero, and then separating real and imaginary parts.] As a consequence, the function H, expressed in terms of the variables u_s, v_s, will be

$$H = \frac{\lambda_1}{2}(u_1^2 + v_1^2) + \frac{\lambda_2}{2}(u_2^2 + v_2^2) + ... + \frac{\lambda_k}{2}(u_k^2 + v_k^2).$$

The preceding analysis, with slight modifications, applies also to the case where the determinantal equation of system (10) has multiple roots, provided that each multiple root makes zero all the minors of the fundamental determinant up to the highest order possible.

To prove this, let us consider two roots, $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$ each of multiplicity m.

Under the indicated condition, there will correspond to these roots $2m$ independent integrals of system (10) with the following form:

$$\left.\begin{matrix} U_1 e^{-i\lambda t}, U_2 e^{-i\lambda t}, ..., U_m e^{-i\lambda t}, \\ V_1 e^{i\lambda t}, V_2 e^{i\lambda t}, ..., V_m e^{i\lambda t}, \end{matrix}\right\} \tag{12}$$

where i, as before, represents $\sqrt{-1}$, all the U_s and V_s being linear forms in the variables x_j, y_j with constant coefficients. We may further suppose that, for each pair of forms U_s, V_s, the coefficients of one are deduced from the coefficients of the other on changing $\sqrt{-1}$ into $-\sqrt{-1}$.

On forming such a system of integrals for each pair of conjugate roots, we shall have a complete system of $2k$ independent integrals of equations (10).

This settled, and on considering all the possible brackets formed with the functions U_s, V_s, we shall evidently obtain

$$(U_s, U_\sigma) = 0, \quad (V_s, V_\sigma) = 0 \quad (s, \sigma = 1, 2, ..., m).$$

We shall also find that all the brackets will be zero which we can form on combining each of the functions U_s or V_s with each of the analogous functions which correspond to other roots.

It results from this that, for each number s taken from the sequence $1, 2, ..., m$, there will be found, in the same sequence, a number σ such that the bracket (U_s, V_σ) represents a non-zero constant. In fact, if all the brackets

$$(U_s, V_1), \quad (U_s, V_2), ..., (U_s, V_m)$$

were zero, it would be the same for all the $2k - 1$ brackets which could be formed on combining the integral $U_s e^{-i\lambda t}$ with all the other linear integrals of the complete system of independent integrals. And this, obviously, is impossible.

It can happen that, among the brackets of the form (U_s, V_s), there are some which are not zero.

Let us assume for example that (U_1, V_1) is not zero. Then we can transform system (12) into an equivalent system of integrals and with the same character,

$$U_1 e^{-i\lambda t},\ V_1 e^{i\lambda t},\ U'_\sigma e^{-i\lambda t},\ V'_\sigma e^{i\lambda t} \quad (\sigma = 2, 3, \ldots, m)$$

for which all the brackets

$$(U'_\sigma, V_1), \quad (U_1, V'_\sigma) \quad (\sigma = 2, 3, \ldots, m)$$

will be zero. For this, we have only to put

$$U'_\sigma = U_\sigma + \alpha_\sigma U_1, \quad V'_\sigma = V_\sigma + \beta_\sigma V_1,$$

on attributing to the constants α_σ, β_σ the following values:

$$\alpha_\sigma = -\frac{(U_\sigma, V_1)}{(U_1, V_1)}, \quad \beta_\sigma = -\frac{(U_1, V_\sigma)}{(U_1, V_1)}$$

and then the function V'_σ will be deduced from the function U'_σ by change of $\sqrt{-1}$ into $-\sqrt{-1}$.

Let us now assume that all the brackets of the form (U_s, V_s) are zero.

As among the brackets (U_1, V_σ) there will certainly be some which are different from zero, let (U_1, V_2) be non-zero. Then on putting

$$U'_1 = U_1 + i(U_1, V_2)U_2, \quad V'_1 = V_1 + i(U_2, V_1)V_2,$$

and taking account of the equalities $(U_1, V_1) = 0$, $(U_2, V_2) = 0$, we shall have

$$(U'_1, V'_1) = 2i(U_i, V_2)(U_2, V_1),$$

and this quantity will certainly not be zero, because (U_2, V_1) is a quantity conjugate with $-(U_1, V_2)$, which is not zero.

By consequence, if we replace the integrals in the first column of array (12) by the integrals

$$U'_1 e^{-i\lambda t}, \quad V'_1 e^{i\lambda t},$$

which leads to a new system of $2m$ independent integrals of the same character (for the function V_1 is deduced from the function U'_1 on changing i into $-i$), we shall find ourselves again with the case we have just examined.

Thus we can suppose that, for system of integrals (12), the bracket (U_1, V_1) represents a constant different from zero, and that all the brackets (U_1, V_σ) and (U_σ, V_1) where $\sigma > 1$ are zero. And then we shall be able to apply the preceding reasoning to the system of $2(m-1)$ integrals which we shall have on deleting the first column of array (12).

We see by this that the system of integrals (12) can always be supposed such that all the brackets (U_s, V_σ), for which s and σ are different, are zero, and that none of the quantities (U_s, V_s) is zero.

After having formed such systems of integrals for each pair of conjugate roots, we can then reason just as in the case of simple roots.

In this way we arrive at the conclusion that, if the determinantal equation of system (10) has only purely imaginary roots, of which the squares are

$$-\lambda_1^2,\ -\lambda_2^2, \ldots,\ -\lambda_k^2,$$

and if, moreover, in the case of multiple roots, each of the latter makes zero all the minors of the fundamental determinant up to the highest order possible, then, with the aid of a linear substitution with real constant coefficients, every canonical system of equations of the form

$$\frac{dx_s}{dt} = -\frac{\partial(H+F)}{\partial y_s}, \quad \frac{dy_s}{dt} = \frac{\partial(H+F)}{\partial x_s} \quad (s = 1, 2, \ldots, k)$$

will be able to be transformed into a canonical system of the following form:

$$\frac{du_s}{dt} = -\lambda_s v_s - \frac{\partial F}{\partial v_s}, \quad \frac{dv_s}{dt} = \lambda_s u_s + \frac{\partial F}{\partial u_s} \quad (s = 1, 2, ..., k),$$

provided that we understand by each λ_s a number with suitable sign.

We may remark that a transformation similar to the preceding one is also possible in the case where the determinantal equation has, besides purely imaginary roots, a zero root, provided that the condition indicated above is fulfilled for each of the multiple roots, among which the zero root will always belong.

Study of the differential equations of the disturbed motion

22. [*Integration by means of series ordered according to powers of the arbitrary constants*]

Let

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n + X_s \quad (s = 1, 2, ..., n) \tag{13}$$

be the proposed differential equations of the disturbed motion.

All the X_s designate here given holomorphic functions of the variables $x_1, x_2, ..., x_n$, for which the expansions

$$X_s = \sum P_s^{(m_1, m_2, ..., m_n)} x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \quad (s = 1, 2, ..., n)$$

do not include terms of degree below the second, and the coefficients $p_{s\sigma}$, $P_s^{(m_1, m_2, ..., m_n)}$ are real constants.

As far as the independent variable t is concerned, inasmuch as there will be no need to attribute complex values to it, we shall take it to be real, as before.

If we omit in equations (13) terms of degree higher than the first, we shall have a system of linear differential equations corresponding to the first approximation. In forming for this system the determinantal equation, we shall say that it is the determinantal equation of system (13).

Let $\chi_1, \chi_2, ..., \chi_n$ be all the roots of this equation.

On integrating system (13) by the procedure expounded in Section 3 we shall obtain, for the functions x_s, the series

$$x_s^{(1)} + x_s^{(2)} + x_s^{(3)} + ... \quad (s = 1, 2, ..., n), \tag{14}$$

of which the mth terms will be of the following form:

$$x_s^{(m)} = \sum T^{(m_1, m_2, ..., m_n)} e^{(m_1\chi_1 + m_2\chi_2 + ... + m_n\chi_n)t}.$$

Here the summation extends over all non-negative integers $m_1, m_2, ..., m_n$ satisfying the condition

$$0 < m_1 + m_2 + ... + m_n \leqslant m,$$

and the coefficients T (entire and homogeneous of degree m with respect to the arbitrary constants) are either constant quantities, or entire functions of t, of which the degrees do not exceed a certain limit† dependent on m.

We shall call the coefficients which are of this second kind *secular*; and we shall make use of the same expression to designate the terms where they occur, when the numbers

$$m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n$$

which correspond to them, are zero or represent purely imaginary quantities.

If we drop the condition introduced in Section 3 that all the functions $x_s^{(m)}$, for which $m > 1$, should be zero for one and the same given value of t, we shall be able to arrange calculations so that expressions (14) become homogeneous of degree m with respect to the exponential functions

$$e^{\chi_1 t}, e^{\chi_2 t}, \dots, e^{\chi_n t}. \tag{15}$$

At the same time, we shall be able to give the coefficients T the form

$$T_s^{(m_1, m_2, \dots, m_n)} = K_s^{(m_1, m_2, \dots, m_n)}\alpha_1^{m_1}\alpha_2^{m_2} \dots \alpha_n^{m_n},$$

$\alpha_1, \alpha_2, \dots, \alpha_n$ being arbitrary constants, on which the coefficients K do not depend, which coefficients represent either constant quantities or entire functions of t.

Then on making some of the constants α_s zero, we shall obtain series where there will enter only some of the function (15).

However, if we wish the series obtained to be convergent, at least within certain limits of the variable t and for sufficiently small values of $|\alpha_s|$, we should, in general, only carry out the calculations in this manner up to a certain limit $m = N$ (but quite arbitrary), and for $m > N$ we should return to the hypothesis of Section 3. From then on, in the expressions of the functions $x_s^{(m)}$ will appear anew all the functions (15) and, relative to the latter, the expressions for $x_s^{(m)}$ will no longer be homogeneous.

The cases where we cannot assign the limit N present a particular interest. Some of them, the most important for our problem, will be indicated in the next section.

It should be noted that the condition for the functions $x_s^{(m)}$ to be homogeneous with respect to quantities (15) does not always suffice to determine completely the constants which are introduced by the integration of the equations on which these functions depend. In such cases there will still remain a certain number of constants which we shall be able to dispose of as we choose.

Let us consider the series where there appear the exponential functions relative to the k roots

$$\chi_1, \chi_2, \dots, \chi_k \tag{16}$$

of the determinantal equation.

It is easy to assure ourselves that, if secular coefficients do not enter into any of the first $m - 1$ approximations

$$x_s^{(1)} + x_s^{(2)} + \dots + x_s^{(\mu)} \quad (\mu = 1, 2, \dots, m-1; s = 1, 2, \dots, n)$$

† It is easy to convince ourselves that these degrees do not exceed the number $(2\mu + 1)m - \mu - 1$, where μ is the highest of these degrees in the case of $m = 1$.

and if, by choice of the non-negative integers $m_1, m_2, \ldots, m_k$, satisfying the condition

$$m_1 + m_2 + \ldots + m_k = m,$$

we cannot satisfy any relation of the form

$$m_1\chi_1 + m_2\chi_2 + \ldots + m_k\chi_k = \chi_s \quad (s = 1, 2, \ldots, n),$$

such coefficients will also not enter into the mth approximation.

For this reason, if the roots (16) have all their real parts with the same sign, the absence or presence of secular coefficients in the series considered will always be able to be revealed with the aid of a limited number of elementary algebraic operations.†

In what follows we shall often consider, in place of equations (13) themselves, different transformations of them obtained by means of linear substitutions with constant coefficients,‡ on guiding our choice of these substitutions by the consideration that, in the transformed equations, the ensembles of terms of first degree should take a special form as simple as possible. Such are the substitutions envisaged in Section 18.

We shall meet with questions where the condition of reality of the coefficients in the differential equations will not play any role. In such a case, by means of the indicated substitutions, we shall be able to reduce equations (13) to the form [compare (5)]

$$\left.\begin{aligned} \frac{dz_1}{dt} &= \chi_1 z_1 + Z_1, \\ \frac{dz_s}{dt} &= \chi_s z_s + \sigma_{s-1} z_{s-1} + Z_s \quad (s = 2, 3, \ldots, n), \end{aligned}\right\} \tag{17}$$

where the Z_j are holomorphic functions of the variables $z_1, z_2, \ldots, z_n$ for which the expansions in powers of the latter begin with terms of degree not less than the second, and have constant coefficients; $\chi_1, \chi_2, \ldots, \chi_n$ are the roots of the determinantal equation corresponding to system (13), and $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are constants, which we shall be able to assume zero if all the χ_j are distinct.

23. [*Theorem on the convergence of these series, obtained from the theorem of Section* 12]

Let us suppose that we have formed series satisfying formally equations (13), ordered according to increasing powers of k arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_k$, and containing the exponential functions relating to the k roots of the determinantal equation

$$\chi_1, \chi_2, \ldots, \chi_k. \tag{18}$$

† Every system of differential equations on which the functions $x_s^{(m)}$ depend, for which $m > 1$, will be integrable with the aid of undetermined coefficients. Then the whole question will reduce each time to the resolution of certain systems of algebraic equations of the first degree.

‡ It is understood that we shall only consider substitutions which allow the new variables to be freely expressed in terms of the old as well as the old in terms of the new.

Suppose that these series are

$$x_s = \sum K_s^{(m_1, m_2, \dots, m_k)} \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_k^{m_k} e^{(m_1 \chi_1 + m_2 \chi_2 + \dots + m_k \chi_k)t}$$

$$(s = 1, 2, \dots, n), \tag{19}$$

the coefficients K not depending on the α_j and representing constant quantities or entire functions of t. As for the summation, it extends over all non-negative integer values $m_1, m_2, \dots, m_k$ with sum not less than 1.

This will be a particular case of the series (25) considered in Section 11.

This settled, and on referring to the theorem of Section 12, we can deduce from it the following proposition.

THEOREM. *On going to the case where the real parts*

$$-\lambda_1, -\lambda_2, \dots, -\lambda_k,$$

of the roots (18) *are all non-zero and with the same sign, let us desigate by* τ *a real number, chosen arbitrarily, and let us only consider values of t satisfying the condition*

$$\pm(t - \tau) \geqslant 0, \tag{20}$$

the upper sign corresponding to the case where $\lambda_j (j = 1, 2, \dots, k)$ *are positive numbers, and the lower sign corresponding to where these are negative numbers. Then, if, on understanding by* ε *a real number of the same sign as the* λ_j {*which number, if all the coefficients K in series* (19) *are constants, can be supposed zero*}, *we put*

$$\alpha_j e^{(\chi_j + \varepsilon)t} = q_j \quad (j = 1, 2, \dots, k),$$

and if we then substitute the values of the α_j *which result into series* (19), *the new series*

$$x_s = \sum Q_s^{(m_1, m_2, \dots, m_k)} q_1^{m_1} q_2^{m_2} \dots q_k^{m_k} \quad (s = 1, 2, \dots, n), \tag{21}$$

ordered according to powers of the q_j, *will enjoy the property that, the numbers* τ *and* ε *being fixed, we shall be able to assign a positive number q such that, the moduli of the* q_j *not exceeding q, series* (21) *will be convergent, and indeed uniformly so for all values of t satisfying condition* (20).

Let us assume first that all the λ_j are positive.

Then, if $\tau = 0$, this proposition will be an immediate consequence of the theorem of Section 12, for the quantities q_j which we are now considering only differ from those which we were concerned with in that section by factors of which the moduli are equal to 1.

As for the case where τ is not zero, we may prove the proposition under consideration on applying the theorem of Section 12 to, instead of series (19), those which are deduced from them on replacing

$$t \quad \text{by} \quad t + \tau,$$

$$\alpha_j \quad \text{by} \quad \alpha_j e^{-(\chi_j + \varepsilon)\tau}$$

{and which consequently also formally satisfy equations (13)}. In fact, in the series (21) corresponding to these new series, the coefficients Q will be obtained from the old ones on replacing t by $t + \tau$.

If all the λ_j are negative, we may reduce this case to the preceding one. For this it will suffice, in series (19) and in equations (13), merely to replace t by $-t$.

As far as the possibility of the hypothesis $\varepsilon = 0$ is concerned, when all the coefficients K are constants, this is obvious, without any need for clarification.

From the theorem under consideration, on applying reasoning analogous to that in Section 4 we can conclude that, if the roots (18) taken for forming series (19) have their real parts with the same sign, there is defined by these series a solution of system of equations (13), whether for every value of t greater than a certain limit if the real parts of roots (18) are all negative, or for every value of t less than a certain limit if these real parts are all positive. As for these limits, they depend on the constants α_s in such a way that on making the $|\alpha_s|$ sufficiently small, we shall be able to choose them at will.

These solutions will contain k arbitrary constants, and the number of the latter will not be able to be reduced if, to form series (19), we take k *independent* solutions

$$\begin{array}{c}
K_1^{(1,0,\dots,0)}e^{\chi_1 t},\ K_2^{(1,0,\dots,0)}e^{\chi_1 t},\ \dots,\ K_n^{(1,0,\dots,0)}e^{\chi_1 t},\\
K_1^{(0,1,\dots,0)}e^{\chi_2 t},\ K_2^{(0,1,\dots,0)}e^{\chi_2 t},\ \dots,\ K_n^{(0,1,\dots,0)}e^{\chi_2 t},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
K_1^{(0,0,\dots,1)}e^{\chi_k t},\ K_2^{(0,0,\dots,1)}e^{\chi_k t},\ \dots,\ K_n^{(0,0,\dots,1)}e^{\chi_k t}
\end{array}$$

of the system of differential equations of the first approximation.

In speaking of solutions of this kind, we shall always assume this condition.

In the case where the real parts of *all* the roots of the determinantal equation are different from zero and have the same sign, we can put $k = n$, and then series (19) will define a general integral of system of equations (13).

Remark

If we are dealing with this last case, we can, on forming series (19) under the hypothesis that $k = n$, deduce from them n independent integrals of system (13) of the form

$$e^{-\chi_s t}\sum L_s^{(m_1, m_2, \dots, m_n)} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \quad (s = 1, 2, \dots, n), \tag{22}$$

where the summation extends over all values of the non-negative integers $m_1, m_2, \dots, m_n$ with sum not less than 1, and where the coefficients L are constant quantities or entire functions of t, of which the degrees do not exceed a certain limit dependent on the number $m_1 + m_2 + \dots + m_n$.

The series (22) will be absolutely convergent for every given value of t, as long as the moduli of the quantities x_s do not exceed a certain limit, which will in general depend on t (and will be able to tend to zero when $|t|$ increases indefinitely).

The case where all the coefficients L are constant quantities presents a particular interest. For it to occur, it is necessary and sufficient that all the coefficients K in series (19) should also be constant.

As has already been remarked in the preceding section, it is always easy, under the conditions considered, to recognize whether we have to do with this case.

For this to be possible, every multiple root of the determinantal equation must make zero all the minors of the fundamental determinant up to the highest order possible.

Let us assume that this condition is satisfied and that, moreover, $m_1, m_2, ..., m_n$ being non-negative integers with sum greater than 1, there does not exist any relation of the form

$$m_1\chi_1 + m_2\chi_2 + ... + m_n\chi_n = \chi_j \quad (j = 1, 2, ..., n), \tag{23}$$

Then we can be certain that all the coefficients K and L will be constants.

In general, whenever the coefficients L are constants, we shall have, for the system of differential equations obtained from (13) on eliminating dt, the following system of equations of integrals

$$\left(\frac{\varphi_1}{\alpha_1}\right)^{1/\chi_1} = \left(\frac{\varphi_2}{\alpha_2}\right)^{1/\chi_2} = ... = \left(\frac{\varphi_n}{\alpha_n}\right)^{1/\chi_n}, \tag{24}$$

where $\varphi_1, \varphi_2, ..., \varphi_n$ are holomorphic functions of the variables $x_1, x_2, ..., x_n$, defined by the series which are deduced from (22) on dividing by $e^{-\chi_s t}$.

On associating with them any one of equations of the form

$$\varphi_s = \alpha_s e^{\chi_s t}$$

and on supposing that we only give to t values greater or less than a certain limit (dependent on the constants α_j), according as the real parts of all the χ_j are negative or positive, we shall obtain a complete system of equations of integrals for system (13).

The result we have just indicated represents a theorem given by Mr Poincaré in his Thesis: *Sur les Propriétés des Fonctions Définies par les Équations aux Différences Partielles* (Paris, Gauthier–Villars, 1879, p. 70)†. On making certain hypotheses‡ {including among others, that the roots χ_s do not satisfy any relation of the form (23)}, Mr Poincaré proves the existence of the equations of integrals of the form (24) without going via equations (19). And specifically, he obtains them by considering the partial differential equations

$$\sum_{j=1}^{n} (p_{j1}x_1 + p_{j2}x_2 + ... + p_{jn}x_n + X_j)\frac{\partial\varphi_s}{\partial x_j} = \chi_s\varphi_s \quad (s = 1, 2, ..., n)$$

and by showing that, under certain conditions, these equations admit solutions holomorphic in $x_1, x_2, ..., x_n$.

24. [*Theorems on the conditions for stability and for instability supplied by the first approximation*]

From the theorem of the preceding section, or immediately from the theorems of Section 13, we may obtain the following.

† On the subject of this theorem, Mr Poincaré remarks that it was communicated to him by Mr Darboux.

‡ In place of our hypothesis that the real parts of all the roots χ_s are of the same sign, Mr Poincaré makes a more general one, namely that the points of the complex plane representing these roots are all situated on the same side of a straight line passing through the origin of coordinates. But if we consider, not the quantities χ_s themselves, but only their mutual ratios, {and that is precisely what is required for equations (24)}, the latter hypothesis will not differ essentially from the former.

THEOREM I. *When the determinantal equation corresponding to the system of differential equations of the disturbed motion has only roots with negative real parts, the undisturbed motion is stable, and moreover in such a way that every disturbed motion for which the perturbations are sufficiently small will approach asymptotically the undisturbed motion.*

THEOREM II. *When the determinantal equation admits roots with negative real parts, then, whatever the other roots, there will be a certain conditional stability. And specifically, if the number of these roots is k, the undisturbed motion will be stable provided that the initial values a_s of the functions x_s satisfy a certain set of $n-k$ equations of the form*

$$F_j(a_1, a_2, ..., a_n) = 0 \quad (j = 1, 2, ..., n-k),$$

where the left-hand sides are holomorphic functions of the a_s, becoming zero when all the a_s are zero, and moreover such that all the a_s will be able to be expressed as holomorphic functions of a certain set of k arbitrary parameters.

To these theorems we may now add the following.

THEOREM III. *When among the roots of the determinantal equation there are some for which the real parts are positive, the undisturbed motion is unstable.*

Let us first assume that among the roots there are some which are real and positive.

If there exist several of these, let χ be the greatest of them. Then $m\chi$, for $m > 1$, will certainly not be a root of the determinantal equation. [Hence the equation preceding (17) will not be satisfied.] And, by consequence, if we form series (19) under the supposition $k = 1$, on taking for the first approximation a solution of system (1) of the form

$$x_1^{(1)} = K_1 \alpha e^{\chi t}, \quad x_2^{(1)} = K_2 \alpha e^{\chi t}, \quad ..., \quad x_n^{(1)} = K_n \alpha e^{\chi t}$$

where all the K_s are constant quantities, all the coefficients K in series (19) will also be constants.

We shall assume, as is permissible, that these coefficients are real and independent of the arbitrary constant α.

Let

$$x_1 = f_1(\alpha e^{\chi t}), \quad x_2 = f_2(\alpha e^{\chi t}), \quad ..., \quad x_n = f_n(\alpha e^{\chi t})$$

be the solution of system (13) thus obtained.

All the f_s here are holomorphic functions of the argument $\alpha e^{\chi t}$, becoming zero when the latter becomes zero and only taking real values when this argument remains real.

To this solution, α being real, there will correspond a certain motion, and so this motion will be defined as long as the absolute value of $\alpha e^{\chi t}$ remains small enough for the series representing the functions f_s to be convergent and for the absolute values of their sums not to exceed a certain limit.

Let us assume that this takes place as long as

$$|\alpha e^{\chi t}| \leqslant l,$$

l being a positive number independent of α.

Then our motion will be defined for all values of t which do not exceed the following limit:

$$\tau = \frac{1}{\chi} \log \frac{l}{|\alpha|}.$$

In this manner, if $|\alpha|$ is small enough for this limit to be greater than the initial value of t, we shall obtain a disturbed motion which we shall be able to trace from the initial instant up to the instant when $t = \tau$.

This motion is such that the corresponding initial values of the functions x_s, on making $|\alpha|$ sufficiently small, all become as small in absolute value as we wish, while their values for $t = \tau$

$$x_1 = f_1(\pm l), \quad x_2 = f_2(\pm l), \quad \ldots, \quad x_n = f_n(\pm l),$$

among which there will certainly be some which are different from zero,† do not depend on the absolute value of α.

We must conclude that the undisturbed motion is unstable. [Recall the definition of stability in Section 1.]

Let us now assume that the determinantal equation does not have positive roots, but does have complex roots with positive real parts.

Let us choose from them two conjugate roots

$$\chi_1 = \lambda + \mu\sqrt{-1}, \quad \chi_2 = \lambda - \mu\sqrt{-1}$$

having the greatest possible real part λ.

As expressions of the form

$$m_1\chi_1 + m_2\chi_2 = (m_1 + m_2)\lambda + (m_1 - m_2)\mu\sqrt{-1}$$

m_1 and m_2 satisfying the condition $m_1 + m_2 > 1$, will certainly not be roots of the determinantal equation [so that the equation preceding (17) will not be satisfied], we shall have for system (13) a solution in the form of series (19), with two arbitrary constants α_1 and α_2, where all the coefficients K will be constants.

Let this solution be

$$x_s = f_s(\alpha_1 e^{\chi_1 t}, \alpha_2 e^{\chi_2 t}) \quad (s = 1, 2, \ldots, n),$$

the f_s being holomorphic functions of the arguments $\alpha_1 e^{\chi_1 t}$ and $\alpha_2 e^{\chi_2 t}$, becoming zero when these arguments become zero.

We can suppose the functions f_s to be such that each of the functions

$$f_s(\xi + \eta\sqrt{-1}, \xi - \eta\sqrt{-1}) \quad (s = 1, 2, \ldots, n),$$

ξ and η being real, is real.

---

† If the values of the functions x_s corresponding to a certain value of t are all zero, these functions will be necessarily zero for every value of t (see the footnote to Section 14). And we may safely assume that, among the functions f_s, there exist some which are not identically zero.

Then, to every pair of complex conjugate values of α_1 and α_2, there will correspond a certain motion, which will be defined at least for values of t satisfying inequalities of the form

$$|\alpha_1 e^{\chi_1 t}| \leqslant l, \quad |\alpha_2 e^{\chi_2 t}| \leqslant l,$$

where l designates a positive constant number independent of α_1 and α_2.

Let α be the modulus of α_1 and α_2 such that, i designating $\sqrt{-1}$, we have

$$\alpha_1 = \alpha e^{i\beta}, \quad \alpha_2 = \alpha e^{-i\beta}.$$

Only attributing to α non-zero values, let us put

$$\beta = -\frac{\mu}{\lambda} \log \frac{l}{\alpha}.$$

Then, α being sufficiently small, our solution will define a disturbed motion corresponding to perturbations as small as we wish, and, however, such that at the instant

$$t = \frac{1}{\lambda} \log \frac{l}{\alpha},$$

the functions x_s will take values

$$x_s = f_s(l, l) \quad (s = 1, 2, \ldots, n),$$

independent of α.

Thus, as in the preceding case, we must conclude that the undisturbed motion is unstable.

In this manner we can consider the theorem as proved.†

25. [*Condition for instability of equilibrium in the case where there exists a force-function*]

From what has been proved there follows a complement to the theorem of Lagrange on stability of equilibrium in the case where there is a force-function [i.e. where there is a potential energy function].

This theorem gives, as we know, a sufficient condition for stability which consists in this, that the force-function must attain, at the position of equilibrium, a maximum.

But, in establishing that this condition is sufficient, the theorem in question does not allow any conclusion about the necessity of the same condition.

That is why the question arises: will the position of equilibrium be unstable if the force-function is not maximum?

Posed in its general form, this question has not been resolved up to the present. But, with certain assumptions of a rather general character, we can answer it in a

† This theorem was demonstrated in the same way in my memoir *Sur les mouvements hélicoidaux permanents d'un corps solide dans un liquide* (*Communications de la Société mathématique de Kharkow*, second series, Vol. I, 1888). In this memoir, in pointing out that, under certain conditions, the differential equations of the disturbed motion admit solutions of the form (19), I did not make mention of the work cited above of Mr Poincaré (see Section 23), because this work was not known to me at that time.

precise manner; for the last theorem of the preceding section leads to a proposition which can be considered under certain conditions as the converse of the theorem of Lagrange. These conditions are, moreover, those which we have to deal with most often in applications.

Let

$$q_1, q_2, \ldots, q_k$$

be the independent variables defining the position of the material system under consideration.

We shall suppose them chosen in such a way that, for the position of equilibrium being examined, they all become zero.

The force-function U can depend on all these variables or on only some of them. Let us agree that it depends only on the m following ones:

$$q_1, q_2, \ldots, q_m, \tag{25}$$

and let us assume further that it is a holomorphic function of them.

The *vis viva* [i.e. double the kinetic energy] of our system, which will be a quadratic form of the derivatives

$$q'_1, q'_2, \ldots, q'_k$$

of the variables q_j with respect to t, with coefficients dependent on the q_j, will also be assumed holomorphic with respect to the q_j.

Under our assumption concerning the force-function, the position of equilibrium considered will be one of a series of positions of equilibrium, infinite in number, which will be obtained on giving to the variables

$$q_{m+1}, q_{m+2}, \ldots, q_k$$

arbitrary constant values and on making variables (25) zero.

If U, as a function of the m independent variables (25), becomes a maximum for

$$q_1 = q_2 = \ldots = q_m = 0$$

each of these positions of equilibrium, according to the theorem of Lagrange, will be, with respect to quantities (25), stable.

Let us suppose now that, the m variables in question being all zero, the force-function does not become maximum.

We are going to show that *if this circumstance is manifested by the property that the ensemble of terms of second degree in the expansion of U in powers of the quantities q_j can take positive values, the position of equilibrium under consideration, as well as the other positions of equilibrium indicated above, provided they are sufficiently near, will be unstable*, and that the instability will even hold with respect to quantities (25).

In fact, by the theory of small oscillations, we know that under the stated condition the determinantal equation always has at least one positive root. Consequently, according to previous material, instability will certainly take place with respect to some of the $2k$ following quantities:

$$q_1, q_2, \ldots, q_k, \quad q'_1, q'_2, \ldots, q'_k.$$

Thus it only remains to show that it takes place with respect to the first m of these quantities.

For this, on designating by χ the greatest of the positive roots of the determinantal equation, let us take the corresponding solution

$$q_1 = f_1(\alpha e^{\chi t}), \quad q_2 = f_2(\alpha e^{\chi t}), \quad \ldots, \quad q_k = f_k(\alpha e^{\chi t})$$

of the differential equations of motion. This is a solution of the type considered in the proof of Theorem III.

Our proposition will evidently be demonstrated if we can show that, among the functions

$$f_1, f_2, \ldots, f_m$$

there are some which are not identically zero (we assume, it is understood, that the k functions $f_1, f_2, \ldots, f_k$ are not all identically zero).

Now, this circumstance manifests itself at once, for in every motion (if such motion were possible) where quantities (25) would be identically zero, the equation of *vis viva* [i.e. the equation indicating constancy of total energy] would obviously give for the latter a non-zero constant value, while for the solution of the type considered, the *vis viva* must tend towards zero when $(-t)$ increases indefinitely.

As far as the case is concerned where the absence of a maximum of the force-function is only recognized on examining terms of higher degree than second, Theorem III cannot serve to prove instability.

One case of this sort has been pointed out in Section 16 (Example II), where instability was demonstrated on applying a general theorem of quite another kind.

26. [*New demonstration of the propositions of Section* 24. *General theorem on instability*]

We have arrived at the theorems of Section 24 on considering certain series satisfying the differential equations of the disturbed motion. But Theorems I and III are easily demonstrated without having recourse to these series, and we are now going to show how we can achieve this on starting from the general propositions of Section 16.

Let us assume that the determinantal equation of system (13) has only roots with negative real parts.

We know that under this condition there always exists a quadratic form V of the variables $x_1, x_2, \ldots, x_n$ satisfying the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = x_1^2 + x_2^2 + \ldots + x_n^2, \tag{26}$$

[see Theorem I of Section 20] and, by consequence, such that its total derivative with respect to t,

$$\frac{dV}{dt} = x_1^2 + x_2^2 + \ldots + x_n^2 + \sum_{s=1}^{n} X_s \frac{\partial V}{\partial x_s},$$

formed in accordance with equations (13), represents a positive-definite function [recall that the $|x_s|$ are assumed small—see (40) of Section 15]. We also know that this form [V] will be negative-definite (Section 20, Theorems I and II).

We can thus find a function V satisfying all the conditions of Theorem I (Section 16).

Our form, which represents such a function, will moreover satisfy the conditions of the proposition which is established in the Remark II relating to this theorem.

We must, therefore, conclude that the undisturbed motion is stable and that each disturbed motion for which the perturbations are sufficiently small will tend asymptotically towards the undisturbed motion.

Let us now assume that, among the roots of the determinantal equation, there are some for which the real parts are positive.

If then the determinant $D_2(0)$ (Section 19) is not zero, we shall find, as before, a quadratic form V satisfying equation (26); but this form, under the present hypothesis, will be such that, by means of a suitable choice of real values for the quantities x_s, we shall always be able to make it positive (Section 20, Theorem III). Thus, as it possesses a positive-definite derivative, it will satisfy all the conditions of Theorem II of Section 16.

We must therefore conclude that the undisturbed motion is unstable.

If $D_2(0) = 0$, we take instead of equation (26) the following:

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = \lambda V + x_1^2 + x_2^2 + \dots + x_n^2, \tag{27}$$

on understanding by λ a positive constant.

Assuming that this constant is not a root of the equation $D_2(\chi) = 0$, we shall always find a quadratic form V satisfying equation (27). Now, in satisfying the latter, this form will necessarily satisfy the following:

$$\sum_{s=1}^{n} \left\{p_{s1}x_1 + p_{s2}x_2 + \dots + \left(p_{ss} - \frac{\lambda}{2}\right)x_s + \dots + p_{sn}x_n\right\}\frac{\partial V}{\partial x_s} = x_1^2 + x_2^2 + \dots + x_n^2.$$

And this equation is obtained from (26) on replacing the quantities p_{ss} by the quantities $p_{ss} - \lambda/2$; also all the theorems of Section 20 can be applied to it, provided that, in place of the roots of the determinantal equation of system (13), we consider the roots of the equation

$$D\left(\frac{\lambda}{2} + \chi\right) = 0.$$

Consequently, if we assume that the constant λ is so small that we can satisfy this equation by a value of χ with real part positive, we can be certain that the form V will be capable of taking positive values.

But then the form V, of which the total derivative with respect to t reduces to the form

$$\frac{dV}{dt} = \lambda V + x_1^2 + x_2^2 + \dots + x_n^2 + \sum_{s=1}^{n} X_s \frac{\partial V}{\partial x_s},$$

will satisfy all the conditions of Theorem III of section 16.

We therefore conclude that the undisturbed motion is unstable.

Remark.

The preceding demonstrations evidently apply not only to the case of steady motion, but also to much more general cases, for the assumption that the coefficients $P_s^{(m_1, m_2, \dots, m_n)}$ entering into the expansions of the functions X_s are constant quantities has not played any role in these proofs. These coefficients will be able to be functions of t, and, for the preceding analysis to be applicable, these functions only have to satisfy the general conditions set up at the beginning of Section 11.

For this reason we can obtain from our analysis a general theorem on instability, which will complete in certain respects the propositions of Section 13.

This theorem appears initially as if it were subject to the restriction that the coefficients $p_{s\sigma}$ are constant quantities. But it extends immediately to the condition that the system of differential equations of the first approximation belongs to the class of systems which we have called reducible (see Section 10 and the remark in Section 18). Thus it can be enunciated as follows.

If the system of differential equations of the first approximation is reducible and if, in the set of its characteristic numbers, some are negative, the undisturbed motion is unstable.

On bringing together this result and Theorem I (Section 13), we arrive at the conclusion that, for reducible systems, the question of stability is resolved by the sign of the least of the characteristic numbers. Thus doubt only remains in the case where this number is zero. Then the question cannot be resolved unless, in the differential equations, we have taken account of terms of higher degree than the first.†

27. [*Singular cases where consideration of the first approximation alone is insufficient. Definition of those which will be the subject of subsequent investigations*]

From the preceding analysis it results that, in most cases, the question of stability is resolved by the examination of the first approximation, and this examination fails to answer the question only in the case where the determinantal equation, without having roots with positive real parts, possesses roots of which the real parts are zero.

These singular cases are nevertheless of very great interest, as much as a result of the difficulty of their analysis as because, for many problems, it is only in these cases that absolute stability is possible [Lyapunov has in mind astronomical problems].

Thus, for example, if the system of equations under examination is canonical [Hamiltonian],

$$\frac{dx_s}{dt} = -\frac{\partial H}{\partial y_s}, \quad \frac{dy_s}{dt} = \frac{\partial H}{\partial x_s} \quad (s = 1, 2, ..., k)$$

(H being a holomorphic function of the variables $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k$, not containing terms of degree below the second), absolute stability is only possible if all the roots of the determinantal equation have their real parts zero.

We arrive at this conclusion on taking into account that this equation contains only even powers of the unknown χ (Section 21).

If, among its roots, there were some which had real parts different from zero, there would only exist a certain conditional stability (Section 24, Theorem II) of such a character that, for certain perturbations, the disturbed motions would approach asymptotically the undisturbed motion.

In the case where the real parts of all the roots are zero, it can happen that H is a definite function. Then stability will actually occur [because of Theorem I of Section 16]. But, if H is not such a function, the question becomes in general very difficult and we are unable to indicate methods for resolving it.

† See the note added at the end of this memoir.

It would be natural to have recourse, for this purpose, to the integration of our equations with the aid of series. But the series which give, in the cases which interest us, known methods of integration are not of a nature capable of leading to general conclusions on stability.

Series ordered according to powers of arbitrary constants, which are obtained by the usual method of successive approximations, already present the inconvenience that we encounter, in general, secular terms, which ordinarily enter even in the case when stability actually occurs. And the presence of these terms makes the study of the question very difficult.

It would thus be desirable to have methods of integration which furnish series deprived of secular terms.

We know that, in celestial mechanics, the seeking of such methods constitutes the aim of several modern researches. Among these researches, those of Gyldén and of Lindstedt merit special attention.

The methods proposed by Gyldén rest, as is known, on consideration of elliptic functions.

The simplest method of Lindstedt, in the case where it leads to the desired end, supplies series of sines and cosines of multiples of t, depending not only on the roots of the determinantal equation (which are supposed all purely imaginary), but also on the arbitrary constants introduced by the integration. It is the latter circumstance which allows us to make the secular terms disappear.†

But, although we have thus been able to indicate methods sometimes allowing us to get rid of secular terms, the difficulty is far from being eliminated by this, for it still remains to resolve an essential question, that of the convergence of the series obtained. Now this question, for systems of differential equations of higher order than second, is not easy to resolve, and up to the present nobody has done anything on this subject from which we could profit here.‡

We have in view problems where the said methods are applied to the seeking of the general integral. As for those where we limit ourselves to seeking particular solutions, we could, under certain conditions, obtain for example periodic series similar to those of Lindstedt, for which the convergence would be indubitable.

We shall occupy ourselves with such series at the end of this chapter.

As appears from what we have just said, questions of stability, in the singular cases which interest us, are very difficult. The difficulties moreover become the more serious the greater the number of roots with real parts zero.

Thus, if we wish to arrive at some general methods for these questions, it is necessary to begin with cases where the number of the aforesaid roots is the smallest possible.

We shall limit ourselves here to the examination of the two simplest cases of the following kind: (1) where the determinantal equation has a zero root, all the other

† Lindstedt, Beitrag zur Integration der Differentialgleichungen der Störungstheorie (*Mémoires de l'Académie des Sciences de Saint-Pétersbourg*, seventh series, Vol. XXXI, No. 4, [1883]).

‡ Recently there has appeared a remarkable memoir on the above point by Mr Poincaré: Sur le problème des trois corps et les équations de la dynamique (*Acta Mathematica*, Vol. XIII, [1890]). In this memoir, among other questions, there is considered that of the convergence of the series of Lindstedt for a canonical system of fourth order, and concerning this convergence the author arrives at a negative conclusion.

roots having their real parts negative; and (2) where the equation has two purely imaginary roots, the other roots, being of the same nature as before.†

Our analysis will represent an application of what we have called the second method in Section 5.

First case. Determinantal equation with one root equal to zero

28. [*Reduction of differential equations to a suitable form*]

Let us consider a system of differential equations of order $n+1$ for the disturbed motion and let us suppose that the determinantal equation which corresponds to it has one zero root, the n other roots having their real parts negative.

The system of differential equations of the first approximation will admit, in this case, a linear integral with constant coefficients (Section 18). On taking such an integral (in which the coefficients may be supposed real) for one of the unknown functions, we shall reduce the system considered to the form

$$\left.\begin{aligned} &\frac{dx}{dt} = X, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + p_s x + X_s \\ &\qquad\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{28}$$

where $X, X_1, X_2, \dots, X_n$ are holomorphic functions of the variables $x, x_1, x_2, \dots, x_n$, for which the expansions begin with terms of degree not less than second and possess constant real coefficients, and $p_{s\sigma}$, p_s are real constants. The constants $p_{s\sigma}$ are moreover such that if we designate as before by $D(\chi)$ the fundamental determinant of system (1), the equation

$$D(\chi) = 0$$

will only have roots with negative real parts.

Let us consider in equations (28) the terms not depending on the variables $x_1, x_2, \dots, x_n$. We shall designate the ensembles of these terms in the expansions of the functions $X, X_1, X_2, \dots, X_n$ by $X^{(0)}, X_1^{(0)}, X_2^{(0)}, \dots, X_n^{(0)}$ respectively.

It can happen that all the coefficients $p_1, p_2, \dots, p_n$ are zero. Then if, $X^{(0)}$ not being identically zero, the expansions of the functions $X_1^{(0)}, X_2^{(0)}, \dots, X_n^{(0)}$ in powers of x do not contain terms of degree less than the least power of x entering into the expansion of $X^{(0)}$, or if $X^{(0)}, X_1^{(0)}, X_2^{(0)}, \dots, X_n^{(0)}$ are all identically zero, the question of stability will be resolved, as we shall see, by direct examination of equations (28). In the contrary case, a preliminary transformation will be necessary, and we shall show straightaway that we shall always be able to transform system of equations (28) into a system of the same form for which the conditions which we have just indicated will be fulfilled.

† The case of purely imaginary roots for systems of second order has been considered by Mr Poincaré in his memoir: Sur les courbes définies par les équations différentielles (*Journal de Mathématiques*, fourth series, Vol. I, p. 172, [1885]). The term *stability* is given there a meaning somewhat different from ours.

To this end, let us consider the following system of equations:

$$p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + p_s x + X_s = 0 \quad (s = 1, 2, \dots, n). \tag{29}$$

The left-hand sides of these equations become zero for

$$x_1 = x_2 = \dots = x_n = x = 0.$$

but their functional determinant [Jacobian] with respect to $x_1, x_2, \dots, x_n$ reduces under this hypothesis to $D(0)$, which is a non-zero number [since it equals the product of the non-zero roots]. Therefore, by virtue of a known theorem [see e.g. É. Goursat, *Cours d'Analyse Mathématique*, Paris, 1917, Vol. I, pp. 479–480], these equations are solvable with respect to the quantities $x_1, x_2, \dots, x_n$, and admit a well-defined solution of the form

$$x_1 = u_1, \quad x_2 = u_2, \quad \dots, \quad x_n = u_n,$$

$u_1, u_2, \dots, u_n$ being holomorphic functions of the variable x, becoming zero for $x = 0$.

The coefficients in the expansions of the functions u_s are obtained successively, starting with the smallest power of x; and this will be the least power of x which equations (29) contain in the terms independent of the quantities x_s.

If then all the coefficients p_s are zero and none of the functions X_s includes in its expansion any terms independent of the quantities x_s, all the u_s will be identically zero.

Returning now to the system of differential equations (28), let us transform it by means of the substitution

$$x_1 = u_1 + z_1, \quad x_2 = u_2 + z_2, \quad \dots, \quad x_n = u_n + z_n,$$

where $z_1, z_2, \dots, z_n$ are new variables which replace the old $x_1, x_2, \dots, x_n$.

The transformed system of equations will be of the following form:

$$\frac{dx}{dt} = Z,$$

$$\frac{dz_s}{dt} = p_{s1}z_1 + p_{s2}z_2 + \dots + p_{sn}z_n + Z_s$$

$$(s = 1, 2, \dots, n),$$

where $Z, Z_1, Z_2, \dots, Z_n$ are holomorphic functions of the variables $x, z_1, z_2, \dots, z_n$, for which the expansions begin with terms of degree not less than second; and it is easy to see that, if $Z^{(0)}, Z_1^{(0)}, Z_2^{(0)}, \dots, Z_n^{(0)}$ represent what these functions become when we put $z_1 = z_2 = \dots = z_n = 0$, we shall have

$$Z_1^{(0)} = -\frac{du_1}{dx} Z^{(0)}, \quad Z_2^{(0)} = -\frac{du_2}{dx} Z^{(0)}, \quad \dots, \quad Z_n^{(0)} = -\frac{du_n}{dx} Z^{(0)}.$$

Whence it is clear that in the expansions of the $Z_s^{(0)}$ in powers of x there will not be terms with degrees less than the least power of x in the expansion of $Z^{(0)}$, and that, if $Z^{(0)}$ is identically zero, it will be the same for each of the functions $Z_s^{(0)}$.

Thus the transformed system will have all the required properties.

The function $Z^{(0)}$ is obtained as a result of the substitution

$$x_1 = u_1, \quad x_2 = u_2, \quad \dots, \quad x_n = u_n$$

in the function X.

If the result of this substitution turned out to be identically zero, the system of equations (28) would admit a particular solution with constant values for x, $x_1, x_2, \ldots, x_n$, dependent on an arbitrary constant.

Supposing that

$$u_s = a_s^{(1)}x + a_s^{(2)}x^2 + a_s^{(3)}x^3 + \ldots \quad (s = 1, 2, \ldots, n)$$

are the series defining the functions u_s, we shall be able to represent this solution by the following equations:

$$x = c,$$
$$x_s = a_s^{(1)}c + a_s^{(2)}c^2 + a_s^{(3)}c^3 + \ldots \quad (s = 1, 2, \ldots, n),$$

where c is an arbitrary constant of which the modulus, if the series are to be convergent, must not exceed a certain limit.

To each sufficiently small real value of the constant c, there will correspond in this case a steady motion.† On making this constant vary in a continuous manner, we shall obtain a *continuous series* of such motions, containing the motion under consideration for which the stability is being investigated.

Remark

The substitution by which the preceding transformation was effected is such that the problem of stability with respect to the old variables $x, x_1, x_2, \ldots, x_n$ is entirely equivalent to the problem of stability with respect to the new ones $x, z_1, z_2, \ldots, z_n$; such that, on resolving the one problem in an affirmative or negative sense, we shall resolve the other in the same sense.

Most of the transformations which we shall encounter in the sequel will enjoy this property.

For the rest, it will sometimes happen that we have to do with transformations of a different sort. In such cases, on passing from the original system of equations to the transformed system, we shall be obliged to introduce into the problem certain modifications.

29. [*Study of the general case*]

In view of what has been shown above, we may start our investigation from the supposition that the differential equations of the disturbed motion have the following form:

$$\left.\begin{aligned} &\frac{dx}{dt} = X, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s \\ &\qquad\qquad (s = 1, 2, \ldots, n), \end{aligned}\right\} \tag{30}$$

† If the number $\frac{1}{2}(n + 1)$ is less than that of the degrees of freedom of the material system under consideration, to the given expressions for the quantities $x, x_1, x_2, \ldots, x_n$ as functions of t, there can correspond not one, but an infinity of motions. But we shall agree to consider the entire ensemble of these motions as a single motion, and we shall also do this in analogous cases later.

where the functions X, X_s are such that their expansions satisfy the conditions referred to in the preceding section.

To begin with we shall consider the case where the quantity $X^{(0)}$ is not identically zero, and we shall designate by m the exponent of the lowest degree of x encountered in its expansion.

Then, conforming to our hypothesis, none of the quantities $X_s^{(0)}$ will contain in its expansion terms of degree less than m.

The number m will not be less than 2.

Let us begin with the simplest case, that where $m = 2$.

Let

$$X = gx^2 + Px + Q + R,$$

where g is a non-zero constant, P is a linear form in the variables $x_1, x_2, \ldots, x_n$, Q is a quadratic form in the same variables, and R is a holomorphic function of variables $x, x_1, x_2, \ldots, x_n$, for which the expansion does not contain terms of dimension less than three.

This settled, we note that, for the conditions under consideration, we shall always be able to find forms in the variables x_s, a linear one U and a quadratic one W, which satisfy the equations:

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial U}{\partial x_s} + P = 0,$$

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial W}{\partial x_s} + Q = g(x_1^2 + x_2^2 + \ldots + x_n^2).$$

[See Theorem I of Section 20.] These forms being obtained, let us put

$$V = x + Ux + W.$$

Then, in view of our differential equations (30), we shall have

$$\frac{dV}{dt} = g(x^2 + x_1^2 + x_2^2 + \ldots + x_n^2) + S,$$

where

$$S = x\sum_{s=1}^{n} X_s\frac{\partial U}{\partial x_s} + \sum_{s=1}^{n} X_s\frac{\partial W}{\partial x_s} + UX + R$$

will only contain terms of degree higher than second.

In this manner the derivative of the function V with respect to t will represent a definite function of the variables x, x_s [recall that the latter are assumed small in modulus]. But the function V itself can obviously take positive as well as negative values, however small the absolute values of these variables.

Consequently, on referring to Theorem II of Section 16, we must conclude that the undisturbed motion is unstable.

We shall arrive at the same conclusion in the case where m is any even number [as follows].

By way of generalization, let

$$X = gx^m + P^{(1)}x + P^{(2)}x^2 + \ldots + P^{(m-1)}x^{m-1} + Q + R,$$

where g is a non-zero constant, the $P^{(j)}$ represent linear forms in the quantities x_s,

Q is a quadratic form in the latter, and R is a holomorphic function of the variables x, x_s, for which the expansion does not have terms of degree less than the third and is moreover such that the variable x occurs, in terms linear with respect to the quantities x_s, only in powers not less than the mth, and, in terms independent of the x_s, only in powers not less than the $(m+1)$th.

Further, on designating by k an arbitrary positive integer, let us put

$$X_s = P_s^{(1)}x + P_s^{(2)}x^2 + \dots + P_s^{(k)}x^k + X_s^{(k)} = g_s x^m + X_s'.$$

Here the $P_s^{(j)}$ are linear forms in the quantities x_s; $X_1^{(k)}, X_2^{(k)}, \dots, X_n^{(k)}$ are holomorphic functions of the variables x, x_s, for which the expansions, in terms linear with respect to the quantities x_s, contain x only in powers higher than the kth; the g_s are constants and the X_s' are holomorphic functions for which the expansions, in the terms independent of the x_s, contain x only in powers greater than mth.

Finally let us agree to understand by $U^{(1)}, U^{(2)}, \dots, U^{(m-1)}$ linear forms and by W a quadratic form in the variables x_s, forms which we shall have at our disposal.

This being so, and on assuming that m is an even number, let us put

$$V = x + U^{(1)}x + U^{(2)}x^2 + \dots + U^{(m-1)}x^{m-1} + W.$$

Then by virtue of differential equations (30), we shall get

$$\begin{aligned}\frac{dV}{dt} &= gx^m + P^{(1)}x + P^{(2)}x^2 + \dots + P^{(m-1)}x^{m-1} + Q + R \\ &+ \sum_{k=1}^{m-2} x^k \sum_{s=1}^{n} [p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + P_s^{(1)}x \\ &+ \dots + P_s^{(m-k-1)}x^{m-k-1} + X_s^{(m-k-1)}] \frac{\partial U^{(k)}}{\partial x_s} \\ &+ x^{m-1} \sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}s_2 + \dots + p_{sn}x_n + X_s) \frac{\partial U^{(m-1)}}{\partial x_s} \\ &+ \sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + X_s) \frac{\partial W}{\partial x_s} + X \sum_{k=1}^{m-1} kU^{(k)}x^{k-1}.\end{aligned}$$

On considering here the ensemble of terms linear with respect to quantities x_s, let us settle the choice of the linear forms $U^{(j)}$ in such a way that x does not occur in powers less than the mth. For this we must make

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n) \frac{\partial U^{(1)}}{\partial x_s} + P^{(1)} = 0,$$

$$\begin{aligned}&\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n) \frac{\partial U^{(k)}}{\partial x_s} \\ &\quad + P^{(k)} + \sum_{s=1}^{n} \left(P_s^{(1)} \frac{\partial U^{(k-1)}}{\partial x_s} + \dots + P_s^{(k-1)} \frac{\partial U^{(1)}}{\partial x_s} \right) = 0\end{aligned}$$

$$(k = 2, 3, \dots, m-1).$$

Under the agreed hypotheses, these equations will always be possible [because of Theorem I of Section 20], and we shall extract from them successively $U^{(1)}, U^{(2)}, \dots, U^{(m-1)}$.

If next we choose the quadratic form W to conform with the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial W}{\partial x_s} + Q = g(x_1^2 + x_2^2 + \dots + x_n^2),$$

we shall have

$$\frac{dV}{dt} = g(x^m + x_1^2 + x_2^2 + \dots + x_n^2) + S,$$

where

$$S = \sum_{s=1}^{n}\left\{\sum_{k=1}^{m-2} x^k X_s^{(m-k-1)}\frac{\partial U^{(k)}}{\partial x_s} + x^{m-1}X_s\frac{\partial U^{(m-1)}}{\partial x_s}\right\}$$
$$+\sum_{s=1}^{n} X_s\frac{\partial W}{\partial x_s} + X\sum_{k=1}^{m-1} kU^{(k)}x^{k-1} + R.$$

And this expression for S, account being taken of the meanings of X, R, X_s, $X_s^{(k)}$, can always be presented in the form

$$S = vx^m + \sum_{s=1}^{n}\sum_{\sigma=1}^{n} v_{s\sigma}x_s x_\sigma,$$

where v, $v_{s\sigma}$ are holomorphic functions of the variables x, x_s, becoming zero for

$$x = x_1 = x_2 = \dots = x_n = 0.$$

From this we see that, the forms $U^{(j)}$, W being chosen in the indicated manner, the derivative dV/dt will be a definite function of the variables x, x_s [the latter being small in modulus].

Now, if this is so, the function V will satisfy all the conditions of Theorem II of Section 16. We must thus conclude that the undisturbed motion is unstable.

Let us now consider the case of m odd.

On putting

$$V = W + \tfrac{1}{2}x^2 + U^{(1)}x^2 + U^{(2)}x^3 + \dots + U^{(m-1)}x^m,$$

we shall have, by virtue of equations (30),

$$\frac{dV}{dt} = gx^{m+1} + P^{(1)}x^2 + P^{(2)}x^3 + \dots + P^{(m-1)}x^m + (Q+R)x$$
$$+\sum_{k=2}^{m-1} x^k \sum_{s=1}^{n} [p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n$$
$$+P_s^{(1)}x + \dots + P_s^{(m-k)}x^{m-k} + X_s^{(m-k)}]\frac{\partial U^{(k-1)}}{\partial x_s}$$
$$+x^m\sum_{s=1}^{n}(p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + X_s)\frac{\partial U^{(m-1)}}{\partial x_s}$$
$$+\sum_{s=1}^{n}(p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + g_s x^m + X_s')\frac{\partial W}{\partial x_s} + X\sum_{k=2}^{m} kU^{(k-1)}x^{k-1}.$$

Let us choose the quadratic form W to conform with the equation

$$\sum_{s=1}^{n}(p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial W}{\partial x_s} = g(x_1^2 + x_2^2 + \dots + x_n^2). \tag{31}$$

Next, let us dispose of the linear forms $U^{(j)}$ in such a way that, in this expression for dV/dt, there are no terms linear with respect to quantities x_s in which x occurs in powers less than the $(m+1)$th. For this, let us choose these forms to agree with the equations

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial U^{(1)}}{\partial x_s} + P^{(1)} = 0,$$

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial U^{(k)}}{\partial x_s} + P^{(k)}$$

$$+ \sum_{s=1}^{n}\left(P_s^{(1)}\frac{\partial U^{(k-1)}}{\partial x_s} + \dots + P_s^{(k-1)}\frac{\partial U^{(1)}}{\partial x_s}\right) = 0$$

$$(k = 2, 3, \dots, m-2).$$

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n)\frac{\partial U^{(m-1)}}{\partial x_s} + P^{(m-1)}$$

$$+ \sum_{s=1}^{n}\left(g_s\frac{\partial W}{\partial x_s} + P_s^{(1)}\frac{\partial U^{(m-2)}}{\partial x_s} + \dots + P_s^{(m-2)}\frac{\partial U^{(1)}}{\partial x_s}\right) = 0.$$

Accordingly we shall have

$$\frac{dV}{dt} = g(x^{m+1} + x_1^2 + x_2^2 + \dots + x_n^2) + S,$$

where

$$S = \sum_{s=1}^{n}\left\{\sum_{k=2}^{m-1} x^k X_s^{(m-k)}\frac{\partial U^{(k-1)}}{\partial x_s} + x^m X_s \frac{\partial U^{(m-1)}}{\partial x_s}\right\}$$

$$+ \sum_{s=1}^{n} X'_s\frac{\partial W}{\partial x_s} + X\sum_{k=2}^{m} kU^{(k-1)}x^{k-1} + (Q+R)x.$$

Now this expression for S can be presented in the form

$$S = vx^{m+1} + \sum_{s=1}^{n}\sum_{\sigma=1}^{n} v_{s\sigma}x_s x_\sigma,$$

on understanding, as before, by v, $v_{s\sigma}$ holomorphic functions, becoming zero when all the x, x_s become zero.

The derivative dV/dt will thus be a definite function of the variables x, x_s, and its sign, for small enough values of $|x|$, $|x_s|$, will be the same as that of the constant g.

This settled, and on referring to Theorem II of Section 20, we observe that the form W satisfying equation (31) will be definite, and furthermore of sign opposite to that of g. And from the expression for the form V it is clear that, if W is a positive-definite form, V will be a positive-definite function of the variables x, x_s.

Therefore, for $g < 0$, the function V will be positive-definite, and its derivative negative-definite; thus we find ourselves in the conditions of Theorem I of Section 16, and even in those of the theorem established in Remark II [of that section]. If, on the other hand, $g > 0$, we shall always be able to make the function V a quantity with arbitrary sign, however small the limit which has not to be exceeded by the quantities $|x|$, $|x_s|$; we shall then be in the conditions of Theorem II (Section 16).

Consequently, we arrive at the conclusion that, m being odd, there will be stability or instability according as g is negative or positive, and that, g being negative, every disturbed motion, sufficiently near the undisturbed motion, approaches it asymptotically.

[The results in this section have immediate plausibility, since when $|x_1|, |x_2|, \ldots, |x_n|$ are small (30) yields approximately

$$\frac{dx}{dt} = gx^m.$$

For m even, this equation shows that if $x(t_0)$ has the sign of g then $|x(t)|$ grows continually, indicating instability. For m odd and $g > 0$, again $|x(t)|$ grows, indicating instability. But for m odd and $g < 0$, the equation shows that $|x(t)|$ continually decreases, yielding stability.]

30. [*Auxiliary proposition*]

It still remains for us to consider the case where, in equations (30), none of the functions X, X_s includes, in its expansion, terms independent of the quantities $x_1, x_2, \ldots, x_n$, and where, as a consequence, these equations admit a particular solution of the form

$$x = c, \quad x_1 = x_2 = \ldots = x_n = 0$$

c being an arbitrary constant.

We are going to show that in this case equations (30) will have a *complete* integral,† with equation, depending on an arbitrary constant c, of the following form:

$$x = c + f(x_1, x_2, \ldots, x_n, c),$$

where f is a holomorphic function of the quantities $x_1, x_2, \ldots, x_n, c$ becoming zero for

$$x_1 = x_2 = \ldots = x_n = 0.$$

This proposition can certainly be demonstrated directly; but we prefer to associate it with another one, more general, which may be of use to us in other cases. Here is what we are going to prove.

THEOREM. *Let there be given a system of partial differential equations*

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s)\frac{\partial z_j}{\partial x_s}$$
$$= q_{j1}z_1 + q_{j2}z_2 + \ldots + q_{jk}z_k + Z_j \quad (j = 1, 2, \ldots, k), \qquad (32)$$

where $X_1, X_2, \ldots, X_n$, $Z_1, Z_2, \ldots, Z_k$ *are holomorphic functions of the variables* $x_1, x_2, \ldots, x_n$, $z_1, z_2, \ldots, z_k$, *becoming zero when all these variables become zero. We assume: that the functions* X_s *do not contain in their expansions terms of the first*

† We call *complete* every integral with equation which can be satisfied, on choosing suitably the arbitrary constants which enter into it, by *any* solution of the differential equations.

degree; that the terms of first degree appearing in the functions Z_j do not depend on the quantities $z_1, z_2, \ldots, z_k$; that the $p_{s\sigma}$, q_{jl} are constants, such that, $\chi_1, \chi_2, \ldots, \chi_n$ being the roots of the equation

$$\begin{vmatrix} p_{11}-\chi & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22}-\chi & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}-\chi \end{vmatrix} = 0$$

and $\lambda_1, \lambda_2, \ldots, \lambda_k$ those of the equation

$$\begin{vmatrix} q_{11}-\lambda & q_{12} & \cdots & q_{1k} \\ q_{21} & q_{22}-\lambda & \cdots & q_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ q_{k1} & q_{k2} & \cdots & q_{kk}-\lambda \end{vmatrix} = 0$$

the real parts of all the χ_s are different from zero and have the same sign, and that, moreover, the numbers χ_s and λ_j are not related by any equation of the form

$$m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n = \lambda_j \quad (j = 1, 2, \ldots, k),$$

where all the m_s are non-negative integers satisfying the condition

$$\sum m_s > 0.$$

This agreed, we shall always be able to find a system of holomorphic functions $z_1, z_2, \ldots, z_k$ of the variables $x_1, x_2, \ldots, x_n$, satisfying equations (32) *and becoming zero for*

$$x_1 = x_2 = \ldots = x_n = 0.$$

Moreover there will be only one such system of functions.

To prove this, let us take the following system of ordinary differential equations:

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s \quad (s = 1, 2, \ldots, n),$$

$$\frac{dz_j}{dt} = q_{j1}z_1 + q_{j2}z_2 + \ldots + q_{jk}z_k + Z_j \quad (j = 1, 2, \ldots, k). \tag{33}$$

In view of what was shown in Section 23, we may assert that these equations, under the given assumptions, admit a solution of the following form:

$$x_s = \sum K_s^{(m_1, m_2, \ldots, m_n)} \alpha_1^{m_1}\alpha_2^{m_2} \ldots \alpha_n^{m_n} e^{(m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n)t}$$
$$(s = 1, 2, \ldots, n), \tag{34}$$

$$z_j = \sum L_j^{(m_1, m_2, \ldots, m_n)} \alpha_1^{m_1}\alpha_2^{m_2} \ldots \alpha_n^{m_n} e^{(m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n)t}$$
$$(j = 1, 2, \ldots, k), \tag{35}$$

where all the K and L are constants or entire and rational functions [polynomials] of t independent of the arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, and where the summations are extended over all values of the non-negative integers m_s satisfying the condition $\sum m_s > 0$.

We can moreover assume, and we shall do so, that the ensembles of terms of first degree in the series (34) give a *general* integral of the system of linear differential equations extracted from (33) on dropping the terms of degree higher than the first. From this condition, the functional determinant [Jacobian] of the quantities x_s with respect to the quantities

$$\alpha_1 e^{\chi_1 t}, \alpha_2 e^{\chi_2 t}, \ldots, \alpha_n e^{\chi_n t} \tag{36}$$

will become, when the latter are zero, a constant different from zero.

This settled, we shall be able to resolve equations (34) with respect to quantities (36) and to obtain from them the following:†

$$\alpha_s e^{\chi_s t} = f_s(x_1, x_2, \ldots, x_n, t) \quad (s = 1, 2, \ldots, n),$$

where the right-hand sides are holomorphic functions of the variables $x_1, x_2, \ldots, x_n$, becoming zero for $x_1 = x_2 = \ldots = x_n = 0$, and having for coefficients either constants or entire and rational functions of t.

On substituting these expressions for quantities (36) in equations (35), we shall have

$$z_j = \varphi_j(x_1, x_2, \ldots, x_n, t) \quad (j = 1, 2, \ldots, k), \tag{37}$$

the φ_j being functions of the same character as the f_s; and these functions, because of the way in which they were obtained, will satisfy the following system of partial differential equations:

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s)\frac{\partial z_j}{\partial x_s} + \frac{\partial z_j}{\partial t}$$
$$= q_{j1}z_1 + q_{j2}z_2 + \ldots + q_{jk}z_k + Z_j \quad (j = 1, 2, \ldots, k). \tag{38}$$

[See (33).]

Let us seek to satisfy this system in the most general manner, by supposing that the z_j are holomorphic functions of the variables x_s, becoming zero when the latter are zero, and having, in their expansions, coefficients which are either constants or entire and rational with respect to t.

To simplify the analysis, let us suppose that in equations (38) all the coefficients q_{jl} are zero, with the exception of the following:

$$q_{11} = \lambda_1, q_{22} = \lambda_2, \ldots, q_{kk} = \lambda_k,$$

$$q_{21} = \tau_1, q_{32} = \tau_2, \ldots, q_{k,k-1} = \tau_{k-1}.$$

This supposition is always legitimate for, in other cases, on taking for new unknown functions certain linear forms in the quantities z_j with constant coefficients, we shall be able to transform equations (38) in such a way that, in the new equations, the coefficients q satisfy the above condition {such is the reduction of equations (13) to form (17), indicated in Section 22}. [See also (5) of Section 18.]

Let

$$z_j = z_j^{(1)} + z_j^{(2)} + z_j^{(3)} + \ldots \quad (j = 1, 2, \ldots, k),$$

† [See the reference to Goursat in Section 28.]

where, generally, $z_1^{(m)}, z_2^{(m)}, \ldots, z_k^{(m)}$ designate forms of the mth degree with respect to quantities x_s. Equations (38) will give

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial z_1^{(m)}}{\partial x_s} + \frac{\partial z_1^{(m)}}{\partial t} = \lambda_1 z_1^{(m)} + W_1^{(m)},$$

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial z_j^{(m)}}{\partial x_s} + \frac{\partial z_j^{(m)}}{\partial t}$$
$$= \lambda_j z_j^{(m)} + \tau_{j-1} z_{j-1}^{(m)} + W_j^{(m)} \quad (j = 2, 3, \ldots, k),$$

where $W_1^{(m)}, W_2^{(m)}, \ldots, W_k^{(m)}$ are forms of the mth degree in the variables x_s, deduced in a certain way from the forms $z_j^{(\mu)}$ for which $\mu < m$. If the latter have all their coefficients constant, it will be the same for the coefficients of all the forms $W_j^{(m)}$. For $m = 1$ these coefficients will always be constants, for the forms $W_j^{(1)}$ represent the ensembles of terms of first degree in the expansions of the functions Z_j.

From the equations which we have just written, we shall find successively

$$z_1^{(1)}, z_2^{(1)}, \ldots, z_k^{(1)}, z_1^{(2)}, z_2^{(2)}, \ldots, z_k^{(2)}, \ldots \tag{39}$$

Let v be any one of these forms, and let us assume that all the preceding ones have constant coefficients. Then, in the equation on which the evaluation of v depends, the known term will represent a form also with constant coefficients.

Therefore, if we designate by l the exponent of the highest power of t in the coefficients of the form v, and if, on understanding by $v_0, v_1, \ldots, v_l$ forms with constant coefficients, we make

$$v = v_0 + v_1 t + \ldots + v_l t^l$$

(which represents the most general assumption that we can make concerning v), the form v_l will satisfy the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial v_l}{\partial x_s} = \lambda v_l,$$

where λ is one of the quantities $\lambda_1, \lambda_2, \ldots, \lambda_k$. Now, by assumption, none of these quantities appears in the form $\sum m_s \chi_s$.

Thus (Section 19), whatever the degree of the form v_l, it is impossible to satisfy the equation under consideration other than by putting $v_l = 0$.

The only valid hypothesis will be, by consequence, $l = 0$, and the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial v}{\partial x_s} = \lambda v + w,$$

which has then to be verified by form v, will give for the latter a well-determined expression, whatever the form w, which is supposed known.

Thus, if in sequence (39), for all the forms which precede v the coefficients are constant quantities, it will also be the same for form v. Moreover the coefficients of v will be completely defined by the coefficients of the forms which precede it.

Now the form $z_1^{(1)}$ will necessarily be with constant coefficients, for such is each of the forms $W_j^{(1)}$. Hence all the subsequent forms in series 39 will also possess constant coefficients.

We conclude from this that the functions (37) do not depend on t and that, as a consequence, they satisfy system (32) [this being a special case of system (38)]. We see moreover that it is impossible to obtain further functions of the same character which would satisfy this system.

The theorem is thus proved.†

Let us note that the expansions of the functions z_j will begin with terms of the same degree as the functions to which the Z_j reduce for $z_1 = z_2 = \dots = z_k = 0$. If none of the functions Z_j contains in its expansion terms independent of the quantities z_j, the functions z_j concerned in the theorem will all be identically zero.

Remark

We have assumed that the expansions of the functions X_s begin with terms of degree not less than second. But we could equally well prove the theorem in the case where these expansions contain terms of the first degree, provided that these terms do not depend on the quantities $x_1, x_2, \dots, x_n$, and that the expansions of the functions Z_j begin with terms of degree not less than second. However, it is then necessary to impose on the sought functions z_j the condition that they do not contain terms below the second degree. For the validity of the theorem so modified, it will suffice that the relations of the form $\sum m_s \chi_s = \lambda_j$ do not exist for values of the m_s with sum greater than 1.

31. [*Study of an exceptional case*]

Let us return to equations (30) under the hypothesis that all the functions X, X_s become zero for $x_1 = x_2 = \dots = x_n = 0$.

Let us put

$$x = c + z,$$

on understanding by c an arbitrary constant, with modulus not exceeding a certain limit.

On substituting this value of x in the functions X_s, we shall have

$$X_s = c_{s1}x_1 + c_{s2}x_2 + \dots + c_{sn}x_n + X'_s \quad (s = 1, 2, \dots, n),$$

where the $c_{s\sigma}$ are constants which represent holomorphic functions of the constant c, becoming zero for $c = 0$, and the X'_s are holomorphic functions of the variables $z, x_1, x_2, \dots, x_n$, for which the expansions begin with terms of degree not less than second and possess coefficients holomorphic with respect to c.

An analogous expression will also hold for the function X.

This settled, let us consider the partial differential equation [which is a version of the first of equations (30)]

$$\sum_{s=1}^{n} \{(p_{s1} + c_{s1})x_1 + (p_{s2} + c_{s2})x_2 + \dots + (p_{sn} + c_{sn})x_n + X'_s\} \frac{\partial z}{\partial x_s} = X, \tag{40}$$

on supposing that X is expressed in terms of the variables z, x_s.

By virtue of our assumption that all the roots of the equation $D(\chi) = 0$ have negative real parts, all the conditions of the preceding theorem will be fulfilled for

† This theorem was proved in a special form by Mr Poincaré in his memoir 'Sur les courbes définies par les équations différentielles', *Journal de Mathématiques*, fourth series, Vol. II, p. 155. In the recently published memoir 'Sur le problème des trois corps', *Acta Mathematica*, Vol. XIII, p. 36, Mr Poincaré proved it anew in a generalized form.

equation (40), $|c|$ being sufficiently small. This equation will thus admit, as long as $|c|$ is small enough, a solution of the form

$$z = f(x_1, x_2, ..., x_n, c),$$

where f designates a holomorphic function of variables $x_1, x_2, ..., x_n$, becoming zero for $x_1 = x_2 = ... = x_n = 0$.

The coefficients in the expansion of this function will depend in some way on the constant c, of which they will evidently be holomorphic functions; moreover they will be such that we may take $|c|$ so small that *all* the coefficients will be absolutely convergent. To convince ourselves of this, it suffices to glance at the equations which serve for the calculation of these coefficients.

What we have just said is valid not only for real values of c (which are the only ones suitable for our problem), but also for complex values of this constant.† Because of this we can conclude that if, instead of expanding function f in powers of the x_s, we expand it in powers of the x_s and c, the series obtained will still be absolutely convergent, provided that the moduli of the x_s and c are below certain sufficiently small limits. In other words, we can conclude that the function f will be holomorphic as a function of $n + 1$ arguments $x_1, x_2, ..., x_n, c$.‡

This settled, and on returning to the variable x, we shall have

$$x = c + f(x_1, x_2, ..., x_n, c). \tag{41}$$

This equation will define a solution of the partial differential equation [again expressing the first of equations (30)]

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n + X_s)\frac{\partial x}{\partial x_s} = X.$$

Consequently, as it contains an arbitrary constant c, it [(41)] will represent the equation of a complete integral of system (30). We may therefore replace by equation (41) one of the differential equations of this system.

Let us do this for the first of these equations and then eliminate x from the other equations. The latter will then reduce to the form

$$\frac{dx_s}{dt} = (p_{s1} + c_{s1})x_1 + (p_{s2} + c_{s2})x_2 + ... + (p_{sn} + c_{sn})x_n + X'_s$$

$$(s = 1, 2, ..., n), \tag{42}$$

where the X'_s will be holomorphic functions of the quantities $x_1, x_2, ..., x_n, c$, not containing in their expansions terms of degree less than second with respect to quantities $x_1, x_2, ..., x_n$.

We now note that our problem of stability with respect to the quantities

$$x_1, x_2, ..., x_n, x$$

is entirely equivalent to the problem of stability with respect to the quantities

$$x_1, x_2, ..., x_n, c. \tag{43}$$

† The analysis in the preceding section only assumed that the theorem of Section 23 was applicable. Now this theorem obviously does not depend at all on the supposition that the coefficients in equations (13) are real. We can therefore attribute to c complex values.

‡ These lines replace a rather long passage in the original Russian, where I wanted to consider only real values of c.

In fact, for this to be so, it suffices that, the quantities of one of the two systems having any sufficiently small real values, the quantities of the other should be in the same condition. And that is actually the case, as is seen from equation (41) and from the following,

$$c = x + F(x_1, x_2, \ldots, x_n, x),$$

which is deduced from it on assuming that quantities (43) are sufficiently small in absolute value, and in which F is a holomorphic function of the variables $x_1, x_2, \ldots, x_n, x$, independent of c and becoming zero for

$$x_1 = x_2 = \ldots = x_n = 0.$$

As for the question of stability with respect to the quantities (43), of which the last is a constant, it reduces to the examination of equations (42).

These equations, $|c|$ being sufficiently small, possess all the properties of equations (13), and we can apply the propositions of Section 24. By consequence, since the determinantal equation which corresponds to them only has roots with negative real parts, we can be sure that, c being fixed, we shall be able to find, for every positive number ε, another positive number a such that, the initial values of the x_s satisfying the inequalities

$$|x_1| < a, \quad |x_2| < a, \quad \ldots, \quad |x_n| < a,$$

we shall have throughout the duration of the ensuing motion

$$|x_1| < \varepsilon, \quad |x_2| < \varepsilon, \quad \ldots, \quad |x_n| < \varepsilon,$$

and that the functions x_s, with t increasing indefinitely, tend to zero.

However, we still do not have the right to conclude from this that the undisturbed motion is stable. For such a conclusion to be legitimate, it is necessary that, $|c|$ not exceeding a certain limit, the number a corresponding to a given value of ε can be supposed *independent of c*.

Now, in the case under consideration, this condition will be fulfilled, of which it is easy to assure ourselves with the aid of the method of Section 26.

In fact, the right-hand sides of equations (42) being holomorphic functions not only with respect to quantities x_s, but also with respect to quantities x_s, c, it is easy to find functions V and W, *independent of c* and entire with respect to the x_s, for which the first is negative-definite and the second positive-definite, and which, for all sufficiently small values of the quantities (43), satisfy the inequality

$$\frac{dV}{dt} \geqslant W,$$

the left-hand side representing the total derivative of the function V with respect to t, formed in accordance with equations (42).† And from this the possibility of choosing for the number a a value independent of c becomes evident (see the proof of Theorem I of Section 16).

† Thus, for example, on taking for V a quadratic form satisfying equation (26), we can take for W the function

$$W = \theta(x_1^2 + x_2^2 + \cdots + x_n^2),$$

θ being any fixed positive proper fraction.

In this way we arrive at the conclusion that, in the case where, in equations (30), the functions X, X_s all become zero for $x_1 = x_2 = \ldots = x_n = 0$, the undisturbed motion is stable.

In this case, each disturbed motion, sufficiently near the undisturbed motion, will approach asymptotically a certain steady motion

$$x = c, \quad x_1 = x_2 = \ldots = x_n = 0,$$

which, in general, will be different from the undisturbed motion, but which may be made as close to it as we wish.

It may be noted further that each of these steady motions will be stable, as long as the quantity $|c|$ which corresponds to it is sufficiently small.

32. [*Exposition of the method. Examples*]

The conclusions we have arrived at can be summarized in the following proposition.

THEOREM. *Suppose that the determinantal equation has one root equal to zero, all the other roots possessing negative real parts. After having reduced the system of differential equations of the disturbed motion to form* (28), *let us form equations* (29) *and extract from them* $x_1, x_2, \ldots, x_n$ *as holomorphic functions of the variable* x, *becoming zero for* $x = 0$ (*which is always possible and gives for the* x_s *well-determined values*). *Next, let us substitute the expressions found for the* x_s *in the function* X *and, if the result of this substitution is not identically zero, expand it in increasing powers of* x.

Then, if the least power of x, *in the expansion so obtained, is found to be even, the undisturbed motion will be unstable; if, on the other hand, it is found to be odd, everything will depend on the sign of the corresponding coefficient, and in such a way that the undisturbed motion will be unstable when this coefficient is positive, and stable when it is negative. In the last case, every disturbed motion, the perturbations being sufficiently small, will approach asymptotically the undisturbed motion.*

Finally, if the result of the substitution in question is found to be identically zero, there will exist a continuous series of steady motions, to which the undisturbed motion under consideration will belong, and all the motions of this series sufficiently near the undisturbed motion, including the latter, will be stable. In this case, the perturbations being sufficiently small, every disturbed motion will approach asymptotically one of the steady motions of the series.

Let us apply the rule contained in this theorem to some examples.

Example I

Suppose the following system of differential equations is given:

$$\frac{dx}{dt} = (3m-1)x^2 - (m-1)y^2 - (n-1)z^2 + (3n-1)yz - 2mzx - 2nxy,$$

$$\frac{dy}{dt} = -y + x + (x - y + 2z)(y + z - x),$$

$$\frac{dz}{dt} = -z + x - (x + 2y - z)(y + z - x),$$

where m and n designate constants.

For this system, the roots of the determinantal equation are: $0, -1, -1$.

On designating the right-hand sides of the above equations by X, Y, Z respectively, let us put

$$Y = 0, \quad Z = 0. \tag{44}$$

From this we get [by means of successive approximation]

$$y = x + 2x^2 - 6x^3 - 30x^4 + \ldots,$$

$$z = x - 2x^2 - 6x^3 + 30x^4 + \ldots.$$

and, on substituting these expressions for y and z in function X, we obtain

$$X = 4(5m - 7n)x^4 + 24(m - n)x^5 + \ldots$$

From this expression we see that, if $5m - 7n$ is not zero, the undisturbed motion is unstable. In the contrary case ($5m = 7m$), it is unstable if m and n are positive, and stable if m and n are negative.

If $m = n = 0$, we obtain the following identity:

$$2X = (z - 2y - x)Y + (y - 2z - x)Z,$$

which shows that by virtue of equations (44) we shall then always have $X = 0$.

Thus, in the last case there will exist a continuous series of steady motions, and not only the undisturbed motion under consideration, but also all the motions of this series which are sufficiently neár, will be stable.

Example 11

Let us examine all the possible cases that can be presented by the second-order system

$$\frac{dx}{dt} = ax^2 + bxy + cy^2, \quad \frac{dy}{dt} = -y + kx + lx^2 + mxy + ny^2$$

under different assumptions about the constants a, b, c, k, l, m, n.

From the equation [obtained by equating to zero the right-hand side of the second differential equation]

$$y = kx + lx^2 + mxy + ny^2$$

we extract [by means of successive approximation]

$$y = kx + B_2x^2 + B_3x^3 + \ldots,$$

where

$$B_2 = l + mk + nk^2, \quad B_3 = (m + 2nk)B_2, \ldots$$

and, by virtue of this expression for y we have [for the right-hand side of the first differential equation]

$$ax^2 + bxy + cy^2 = A_2x^2 + A_3x^3 + A_4x^4 + \ldots,$$

where

$$A_2 = a + bk + ck^2, \quad A_3 = (b + 2ck)B_2,$$

$$A_4 = (b + 2ck)B_3 + cB_2^2, \quad \ldots.$$

From this we see that stability will only be possible in the case where

$$a + bk + ck^2 = 0.$$

If then $B_2 = 0$ (which requires all the other B's to be also zero), all the coefficients A will be zero, and by consequence stability will certainly hold.

Let us suppose that B_2 is not zero.

Then, if $b + 2ck$ is not zero, the question will depend on the sign of A_3. If, on the contrary, we have

$$b + 2ck = 0,$$

A_3 will be zero, but A_4 will not be zero as long as c is not zero; stability will thus be possible only in the case $c = 0$. As for this case, for which by virtue of the equations admitted [the previous two displayed equations] we shall also have $a = 0$ and $b = 0$, stability will actually hold [the right-hand side of the first differential equation then being identically zero].

In this way all the possible cases reduce to the six following:

I. $a + bk + ck^2 \gtrless 0$, unstable undisturbed motion;

II. $\begin{cases} a + bk + ck^2 = 0, \\ (l + mk + nk^2)(b + 2ck) > 0, \end{cases}$ unstable undisturbed motion;

III. $\begin{cases} a + bk + ck^2 = 0, \\ (l + mk + nk^2)(b + 2ck) < 0, \end{cases}$ stable undisturbed motion;

IV. $\begin{cases} a = ck^2, \quad b = -2ck, \quad c \gtrless 0, \\ l + mk + nk^2 \gtrless 0, \end{cases}$ unstable undisturbed motion;

V. $\begin{cases} a + bk + ck^2 = 0, \\ l + mk + nk^2 = 0, \end{cases}$ stable undisturbed motion;

VI. $a = b = c = 0$, stable undisturbed motion;

In the last two cases the undisturbed motion belongs to certain continuous series of steady motions.

Second case. Determinantal equation with two purely imaginary roots

33. [*General form to which the differential equations reduce*]

Suppose that the proposed system of differential equations of the disturbed motion is of order $n + 2$, and that the determinantal equation which corresponds to it has two purely imaginary roots and n roots with negative real parts.

Since the coefficients in the differential equations are assumed real, the purely imaginary roots will be necessarily conjugate:

$$\lambda\sqrt{-1}, \quad -\lambda\sqrt{-1},$$

where λ is a non-zero real constant, which we shall suppose, for definiteness, to be positive.

For the system of differential equations of the first approximation, to these roots will correspond two integrals of the form

$$(x + iy)e^{-i\lambda t}, \quad (x - iy)e^{+i\lambda t},$$

where $i = \sqrt{-1}$, and x and y are linear forms, with constant real coefficients, in the variables playing the role of unknown functions in the differential equations (Section 18).

On introducing in place of two of these unknown functions the variables x and y, we shall reduce the proposed system to the following form:

$$\left.\begin{aligned} &\frac{dx}{dt} = -\lambda y + X, \quad \frac{dy}{dt} = \lambda x + Y, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s \\ &\qquad\qquad (s = 1, 2, \dots, n). \end{aligned}\right\} \tag{45}$$

X, Y, X_s are here holomorphic functions of the variables x, y, $x_1, x_2, \dots, x_n$, for which the expansions begin with terms of degree not less than second and possess constant real coefficients, and $p_{s\sigma}$, α_s, β_s are real constants, among which the $p_{s\sigma}$ are such that the equation

$$D(\chi) = 0$$

(with the old notation) only has roots with negative real parts.

We may assume that the functions X and Y become zero when x and y become zero, for, in the opposite case, on replacing variables x and y by certain new variables, we shall always be able to transform system (45) into another of the same kind, but where the functions playing the role of X and Y become zero when the two new variables become simultaneously zero.

In fact, according to the theorem of Section 30 (Remark), we can find holomorphic functions x and y of variables $x_1, x_2, \dots, x_n$ satisfying the equations

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s)\frac{\partial x}{\partial x_s} = -\lambda y + X,$$

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s)\frac{\partial y}{\partial x_s} = \lambda x + Y$$

and only containing in their expansions terms of degree not less than the second.† [See also the paragraph preceding the remark in Section 30. Note that x, y, X, Y correspond to z_1, z_2, Z_1, Z_2 in (32).]

Let

$$x = u, \quad y = v$$

be such solutions of these equations.

Then, on making

$$x = u + \xi, \quad y = v + \eta,$$

and on introducing into equations (45), in place of variables x and y, the variables

† The condition expressed in this theorem, relating to the roots χ_s, λ_j, is obviously satisfied in the case under consideration.

ξ and η, we shall reduce these equations to the form

$$\frac{d\xi}{dt} = -\lambda\eta + \Xi, \quad \frac{d\eta}{dt} = \lambda\xi + \Upsilon,$$

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s\xi + \beta_s\eta + X'_s$$

$$(s = 1, 2, \dots, n),$$

where Ξ, Υ, X'_s represent holomorphic functions of the variables ξ, η, x_s for which the expansions begin with terms of degree not less than the second, and among which the first two, which are defined by the formulae

$$\Xi = X - \lambda v - \sum_{s=1}^{n} \{p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s(u+\xi) + \beta_s(v+\eta) + X_s\} \frac{du}{dx_s},$$

$$\Upsilon = Y + \lambda u - \sum_{s=1}^{n} \{p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s(u+\xi) + \beta_s(v+\eta) + X_s\} \frac{dv}{dx_s}$$

(on supposing that in the functions X, Y, X_s the quantities x and y are replaced by $u + \xi$ and $v + \eta$), become zero for $\xi = \eta = 0$.

The transformation under consideration is moreover such that the new variables can play the same role in our problem as the old ones.

We may suppose that, to form equations (45), we have already effected (if that was necessary) the transformation indicated, and that, by consequence, the functions X and Y become zero for $x = y = 0$.

This being so, let us put

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$

and let us introduce into our equations, in place of variables x and y, the variables r and ϑ.

We shall have [with use of (45)]

$$\frac{dr}{dt} = X\cos\vartheta + Y\sin\vartheta, \quad r\frac{d\vartheta}{dt} = \lambda r + Y\cos\vartheta - X\sin\vartheta.$$

Now, because of what we have assumed, the right-hand sides of these equations, being expressed in terms of r and ϑ, become zero for $r = 0$. Thus the second of these equations reduces to the form

$$\frac{d\vartheta}{dt} = \lambda + \Theta, \tag{46}$$

where Θ is a holomorphic function of the variables $r, x_1, x_2, \dots, x_n$, becoming zero when these variables become simultaneously zero, and having, for the coefficients in its expansion, entire and rational functions of $\cos\vartheta$ and $\sin\vartheta$.

From this equation we see that, as long as the quantities $|r|$, $|x_s|$ do not exceed certain limits, ϑ will be a continuous and increasing function of t [since λ is a positive constant], and that, if the quantities $|r|$, $|x_s|$ remain sufficiently small throughout the duration of the motion, the function ϑ will increase indefinitely with t.

Our problem can be considered as that of stability with respect to the quantities†

$$r, x_1, x_2, \ldots, x_n,$$

and in this problem the variable ϑ will be able to play the same role as t.

Let us take this then for the independent variable instead of t.

We shall then have, for the determination of r, x_s as functions of ϑ, the following equations:

$$\left.\begin{aligned} &\frac{dr}{d\vartheta} = rR, \\ &\frac{dx_s}{d\vartheta} = q_{s1}x_1 + q_{s2}x_2 + \ldots + q_{sn}x_n + (a_s \cos \vartheta + b_s \sin \vartheta)r + Q_s \\ &\qquad\qquad (s = 1, 2, \ldots, n), \end{aligned}\right\} \tag{47}$$

where R, Q_s will represent functions of the same character as Θ, moreover the functions Q_s will not contain in their expansions terms of degree less than the second with respect to quantities r, x_s. As for the coefficients $q_{s\sigma}$, a_s, b_s, they will be given by the formulae

$$q_{s\sigma} = \frac{p_{s\sigma}}{\lambda}, \quad a_s = \frac{\alpha_s}{\lambda}, \quad b_s = \frac{\beta_s}{\lambda},$$

and the $q_{s\sigma}$ will be, by consequence, such that all the roots of the equation

$$\begin{vmatrix} q_{11} - \chi & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} - \chi & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} - \chi \end{vmatrix} = 0 \tag{48}$$

will have negative real parts.

The first of equations (47) shows that, if the initial value of r is zero, r will be zero for each value of ϑ, and that, in the contrary case, r will conserve the sign of its initial value, at least as long as the quantities r, x_s remain sufficiently small in absolute value. In any case, according to the very definition of r, we see that without loss of generality we may limit ourselves to the consideration of values of r of only one sign.

Because of this, we shall suppose that r can take only positive values (or zero).

Remark

The functions Θ, R, Q_s for every value of ϑ are holomorphic with respect to quantities $r, x_1, x_2, \ldots, x_n$. Moreover, because of their very origin, they are such that there will always be positive *constants* $A, A_1, A_2, \ldots, A_n$ satisfying the condition that, for

$$|r| = A, \quad |x_s| = A_s \quad (s = 1, 2, \ldots, n)$$

the expansions of these functions are uniformly convergent for all real values of ϑ.

† [Since $r = (x^2 + y^2)^{1/2}$, stability with respect to x and y holds if and only if stability with respect to r holds. Thus the stability or instability of the ϑ-coordinate is irrelevant.]

As it will often happen in what follows that we shall have to deal with similar functions, we shall use to designate them a special term.

In general, let F be a function of variables $x, y, \ldots$ and of parameters $\alpha, \beta, \ldots$, this function being holomorphic with respect to $x, y, \ldots$ for all values of the parameters $\alpha, \beta, \ldots$ which satisfy certain conditions (A). Then, if it is possible to find non-zero numbers $a, b, \ldots$ *independent of the parameters* in question and such that for

$$x = a, \quad y = b, \quad \ldots$$

the expansion of this function in positive integer powers of $x, y, \ldots$ converges *uniformly* for all the values of $\alpha, \beta, \ldots$ satisfying conditions (A), we shall say that the function F is *uniformly holomorphic* (with respect to the variables $x, y, \ldots$) for all the values under considerations of $\alpha, \beta, \ldots$

Our functions Θ, R, Q_s will thus be, with respect to variables $r, x_1, x_2, \ldots, x_n$, uniformly holomorphic for all real values of θ.†

34. [*Certain characteristic series which satisfy the differential equations formally. General case where these series are not periodic*]

In order to be able to apply to equations (47) the propositions of Section 16, it will be necessary, in general, to submit these equations to certain preliminary transformations.

There will only be no need for such a transformation in the case where all the constants a_s, b_s are zero, and where the functions $R^{(0)}$, $Q_s^{(0)}$ to which R, Q_s reduce for $x_1 = x_2 = \ldots = x_n = 0$ satisfy a certain condition. This condition consists in this, that if the function $R^{(0)}$ is not identically zero, the lowest power in its expansion in powers of r must have a constant coefficient and must moreover be less than the lowest power of r found in the expansions of the $Q_s^{(0)}$, and that, in the case where $R^{(0)}$ is identically zero, all the $Q_s^{(0)}$ must be the same.

The aim of the transformation referred to will actually be to reduce the differential equations to a form such that the said condition is fulfilled.

This transformation turns out to be related to the question of the possibility of a periodic solution for system (47).

Let us seek to satisfy this system by series of the following form,

$$\left.\begin{aligned} r &= c + u^{(2)}c^2 + u^{(3)}c^3 + \ldots, \\ x_s &= u_s^{(1)}c + u_s^{(2)}c^2 + u_s^{(3)}c^3 + \ldots \\ &(s = 1, 2, \ldots, n), \end{aligned}\right\} \tag{49}$$

where c is an arbitrary constant, and $u^{(l)}$, $u_s^{(l)}$ are periodic functions of ϑ, having 2π for their common period, and being independent of c.

† If we wish to consider complex values of ϑ, we shall obviously be able to assert that these functions, with respect to the variables r, x_s, are uniformly holomorphic for all values of ϑ of the form

$$\vartheta = \alpha + \beta\sqrt{-1},$$

where α is an arbitrary real number, and β a real number subject to the condition that its absolute value does not exceed an arbitrarily given limit. [This will ensure that sin ϑ and cos ϑ are limited.]

Such a procedure will not always be possible; but when it is the functions u will be obtained in the form of finite sequences of sines and cosines of integer multiples of ϑ.

However, with respect to the nature of the functions u we can impose a more general condition; namely, we may suppose them to be entire rational functions [polynomials] of ϑ, with coefficients representing finite series of sines and cosines of integer multiples of ϑ. Then the problem of finding these functions so that series (49) satisfies equations (47) at least formally will always become possible to solve.

Let us see how to find such functions.

On making the substitution (49) in equations (47) and then equating coefficients of the same powers of c, we obtain the following systems of equations

$$\frac{du_s^{(1)}}{d\vartheta} = q_{s1}u_1^{(1)} + q_{s2}u_2^{(1)} + \dots + q_{sn}u_n^{(1)} + a_s \cos\vartheta + b_s \sin\vartheta$$

$$(s = 1, 2, \dots, n),$$

$$\left.\begin{aligned} &\frac{du^{(l)}}{d\vartheta} = U^{(l)}, \\ &\frac{du_s^{(l)}}{d\vartheta} = q_{s1}u_1^{(l)} + q_{s2}u_2^{(l)} + \dots + q_{sn}u_n^{(l)} \\ &\qquad + (a_s \cos\vartheta + b_s \sin\vartheta)u^{(l)} + U_s^{(l)} \\ &\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{50}$$

where l is one of the numbers 2, 3, ...

The $U^{(l)}$, $U_s^{(l)}$ here are certain rational and entire functions of the quantities $u^{(i)}$, $u_s^{(i)}$ for $i < l$, with coefficients representing rational and entire functions of $\sin\vartheta$ and $\cos\vartheta$.

When all the $u^{(i)}$, $u_s^{(i)}$ for $i < l$ are already found, the first of equations (50) will give the function $u^{(l)}$, after which the n equations which remain will serve to determine the functions $u_s^{(l)}$.

Under our hypothesis concerning the nature of the functions u, the known terms in these n equations will occur in the form of finite series of sines and cosines of integer multiples of ϑ, where the coefficients will be constants or entire and rational functions of ϑ. On seeking the functions $u_s^{(l)}$ in the form of series of the same kind, we shall obtain, under our supposition with regard to the roots of equation (48), well-determined expressions. These expressions will moreover be periodic, whenever this is so for the known terms in the equations under consideration.

The functions $u_s^{(1)}$ will always be periodic and of the following form,

$$u_s^{(1)} = A_s \cos\vartheta + B_s \sin\vartheta,$$

where A_s, B_s are constants. We can convince ourselves that the functions $u^{(2)}$, $u_s^{(2)}$ are also periodic. But the further functions may contain ϑ outside the signs 'sin' and 'cos'.

Let us assume that all the functions $u^{(l)}$, $u_s^{(l)}$ for which l is less than an integer m are found and represent periodic functions of ϑ. Then we shall be able to present the function $U^{(m)}$ in the form of a finite series of sines and cosines of integer multiples of ϑ, and, if in this series there is no constant term, the function $u^{(m)}$ and, by consequence, all the functions $u_s^{(m)}$ will be periodic. In the contrary case, these functions will contain secular terms, and, among others, the function $u^{(m)}$ will be of the form

$$u^{(m)} = g\vartheta + v, \tag{51}$$

where g is a non-zero constant and v is a finite series of sines and cosines of integer multiples of ϑ.

Let us take up this last case.

On supposing that the calculations are carried out in such a way that all the functions $u^{(l)}$, $u_s^{(l)}$, v are real for real ϑ, let us transform our differential equations (47) by means of the substitution

$$r = z + u^{(2)}z^2 + u^{(3)}z^3 + \dots + u^{(m-1)}z^{m-1} + vz^m,$$

$$x_s = u_s^{(1)}z + u_s^{(2)}z^2 + \dots + u_s^{(m-1)}z^{m-1} + z_s$$

$$(s = 1, 2, \dots, n),$$

where $z, z_1, z_2, \dots, z_n$ are new variables which we introduce in place of the old ones $r, x_1, x_2, \dots, x_n$.

Let

$$\left.\begin{aligned} &\frac{dz}{d\vartheta} = zZ, \\ &\frac{dz_s}{d\vartheta} = q_{s1}z_1 + q_{s2}z_2 + \dots + q_{sn}z_n + Z_s \\ &\qquad (s = 1, 2, \dots, n) \end{aligned}\right\} \tag{52}$$

be the transformed equations.

From (51) and the equations satisfied by functions $u^{(l)}$, $u_s^{(l)}$, we shall have

$$zZ = \frac{rR - U^{(2)}z^2 - U^{(3)}z^3 - \dots - U^{(m-1)}z^{m-1} - (U^{(m)} - g)z^m}{1 + 2u^{(2)}z + 3u^{(3)}z^2 + \dots + (m-1)u^{(m-1)}z^{m-2} + mvz^{m-1}},$$

$$Z_s = Q_s - U_s^{(2)}z^2 - U_s^{(3)}z^3 - \dots - U_s^{(m-1)}z^{m-1} + (a_s \cos\vartheta + b_s \sin\vartheta)vz^m$$
$$-[u_s^{(1)} + 2u_s^{(2)}z + \dots + (m-1)u_s^{(m-1)}z^{m-2}]zZ,$$

where the functions rR, Q_s are supposed expressed in terms of the variables z, z_s.

We see from this that the functions Z, Z_s will be, with respect to variables $z, z_1, z_2, \dots, z_n$, uniformly holomorphic for all real values of ϑ, on which will depend the coefficients in their expansions (these coefficients will present themselves in the form of finite sequences of sines and cosines of integer multiples of ϑ). These functions will become zero when all the z, z_s are zero. Moreover the functions Z_s will not contain, in their expansions, terms of first degree. Finally, if $Z^{(0)}$, $Z_s^{(0)}$ are what Z, Z_s become for

$$z_1 = z_2 = \dots = z_n = 0,$$

the expansion of the function $Z^{(0)}$ in increasing powers of z will begin with the $(m-1)$th power, which will have the constant coefficient g, and the expansions of the functions $Z_s^{(0)}$ will contain z in powers not less than the mth; for, by the very definition of the quantities $U^{(l)}$, $U_s^{(l)}$ the expansions of the functions

$$rR - U^{(2)}z^2 - U^{(3)}z^3 - \dots - U^{(m)}z^m,$$

$$Q_s - U_s^{(2)}z^2 - U_s^{(3)}z^3 - \dots - U_s^{(m)}z^m$$

in the terms independent of quantities z_s, will only be able to contain z in powers exceeding m.

In this manner equations (52) possess all the required properties [stated at the beginning of this section].

Moreover the substitution by means of which they have been obtained is such that, for the resolution of our problem, the new variables $z, z_1, z_2, \dots, z_n$ can play absolutely the same role as the old ones $r, x_1, x_2, \dots, x_n$.

Let us note that, $|z|$ being sufficiently small, the signs of r and z will be the same. Thus, as r has been assumed positive, we must assume that z is also positive.

Remark I

The general expressions for the functions $u^{(l)}$, $u_s^{(l)}$, corresponding to a given value of l, will contain $l-1$ arbitrary constants, which will be introduced by the quadratures with the aid of which we determine the functions $u^{(2)}, u^{(3)}, \dots, u^{(l)}$. But it is easy to see that neither the number m nor the constant g depends on the choice of the values that we may wish to attribute to these arbitrary constants.

In fact, if $h_2, h_3, \dots$ are the values taken by the functions $u^{(2)}, u^{(3)}, \dots$ for $\theta = 0$, and if $v^{(l)}$, $v_s^{(l)}$ represent the functions $u^{(l)}$, $u_s^{(l)}$ obtained under the hypothesis that all the h_j are zero, the general expressions for the functions $u^{(l)}$, $u_s^{(l)}$ will be obtained on seeking the coefficients of c^l in the expansions of the expressions [these are expressions for r and x_s and equate to those in (49)]

$$\gamma + v^{(2)}\gamma^2 + v^{(3)}\gamma^3 + \dots,$$

$$v_s^{(1)}\gamma + v_s^{(2)}\gamma^2 + v_s^{(3)}\gamma^3 + \dots,$$

where [we find on equating the expressions for r at $\theta = 0$]

$$\gamma = c + h_2c^2 + h_3c^3 + \dots$$

As a consequence, if $v^{(m)}$ is the first non-periodic function in the series

$$v^{(2)}, v^{(3)}, \dots, v^{(m)}, \dots,$$

$u^{(m)}$ will be the first non-periodic function in the following one,

$$u^{(2)}, u^{(3)}, \dots, u^{(m)}, \dots,$$

whatever the constants h_j. Moreover the difference $u^{(m)} - v^{(m)}$ will necessarily be a periodic function.

Remark II

On taking account of the way in which the functions R, Q_s have been introduced, we arrive at the conclusion that, if the coefficients in the expansions of these functions in powers of r, x_s are developed in sines and cosines of multiples of θ, the terms with even powers of r will only bring in even multiples of θ, and those with odd powers of r will only bring in odd multiples of θ.

From this, in view of the expressions for the functions $u_s^{(1)}$, it results that, if we expand the function $U^{(2)}$ in sines and cosines of multiples of θ, the series obtained will only contain odd multiples of θ, and that, by consequence, there will not be any constant term. The function $u^{(2)}$ will therefore be always periodic, so that the number m which figures in preceding transformation will never be less than 3.

A deeper examination of equations (50) shows that this number will always be odd.

But actually this property of the number m will be put in evidence by discussion of equations (52) (Section 37, Remark).

35. [*Exceptional case where the series are periodic. Convergence of these periodic series*]

When the functions $u^{(l)}$, $u_s^{(l)}$, starting from a certain value of l, become non-periodic, we can always discover this, having integrated a sufficient number of systems of equations (50). But, when all these functions are periodic, however great the number l, we shall never be able to recognize the fact by making use of this procedure.

Whatever it may be, let us suppose that, in such and such a case, we have succeeded in showing that the functions $u^{(l)}$, $u_s^{(l)}$ are periodic for all values of l.

We are going to show that if the arbitrary constants entering into these functions are determined in a suitable manner, series (49), $|c|$ being sufficiently small, will be absolutely convergent, and uniformly so for all real values of ϑ. These series will then define a periodic solution of differential equations (47), with one arbitrary constant, subject only to the condition that its modulus does not exceed a certain limit.

We shall keep to the assumption that all the functions $u^{(l)}$ become zero for $\vartheta = 0$. This hypothesis will allow the determination of all the arbitrary constants contained by the functions $u^{(l)}$, $u_s^{(l)}$.

We have noticed in Section 22 that by means of a linear substitution with constant coefficients the system of equations (13) can always be reduced to the form (17). Let us make use of a similar substitution to transform equations (47).

Let $\chi_1, \chi_2, \ldots, \chi_n$ be the roots of equation (48). We can then suppose the substitution in question to be such that the coefficients $q'_{s\sigma}$, which play in the transformed equations the role of the coefficients $q_{s\sigma}$, are all zero with the exception of the following:

$$q'_{11} = \chi_1, \quad q'_{22} = \chi_2, \quad \ldots, \quad q'_{nn} = \chi_n,$$
$$q'_{21} = \sigma_1, \quad q'_{32} = \sigma_2, \quad \ldots, \quad q'_{n,n-1} = \sigma_{n-1}.$$

Let us provisionally assume that system (47) already has the transformed form. System (50) will then be of the following form:

$$\frac{du^{(l)}}{d\vartheta} = U^{(l)},$$
$$\frac{du_1^{(l)}}{d\vartheta} = \chi_1 u_1^{(l)} + (a_1 \cos\vartheta + b_1 \sin\vartheta)u^{(l)} + U_1^{(l)},$$
$$\frac{du_s^{(l)}}{d\vartheta} = \chi_s u_s^{(l)} + \sigma_{s-1} u_{s-1}^{(l)} + (a_s \cos\vartheta + b_s \sin\vartheta)u^{(l)} + U_s^{(l)}$$
$$(s = 2, 3, \ldots, n).$$

On supposing that all the functions $u^{(i)}$, $u_s^{(i)}$ for $i<l$ are already found, and taking into account that the real parts of all the χ_s are negative, we extract successively from these equations

$$\left.\begin{aligned}
u^{(l)} &= \int_0^{\vartheta} U^{(l)}\,d\vartheta,\\
u_1^{(l)} &= e^{\chi_1\vartheta}\int_{-\infty}^{\vartheta} e^{-\chi_1\vartheta}\left[(a_1\cos\vartheta+b_1\sin\vartheta)u^{(l)}+U_1^{(l)}\right]d\vartheta,\\
u_s^{(l)} &= e^{\chi_s\vartheta}\int_{-\infty}^{\vartheta} e^{-\chi_s\vartheta}\left[\sigma_{s-1}u_{s-1}^{(l)}+(a_s\cos\vartheta+b_s\sin\vartheta)u^{(l)}+U_s^{(l)}\right]d\vartheta\\
&\qquad (s=2,3,\ldots,n).
\end{aligned}\right\}\tag{53}$$

We now note that $U^{(l)}$, $U_s^{(l)}$ are entire functions of the quantities $u^{(i)}$, $u_s^{(i)}$ already obtained, and that the coefficients in these functions represent linear forms, with positive numerical coefficients, in the coefficients of the expansions of the functions R, Q_s. Consequently, if in general we designate by $v^{(i)}$, $v_s^{(i)}$ upper bounds of the moduli of functions $u^{(i)}$, $u_s^{(i)}$, with ϑ being contained in the interval $(0, 2\pi)$ (i.e. for all real values of ϑ), and by $V^{(l)}$, $V_s^{(l)}$ the results of replacing in functions $U^{(l)}$, $U_s^{(l)}$ quantities $u^{(i)}$, $u_\sigma^{(i)}$ by quantities $v^{(i)}$, $v_\sigma^{(i)}$ and the coefficients in the expansions of R, Q_s by upper bounds for their moduli; if finally we designate by

$$-\lambda_1,\ -\lambda_2,\ \ldots,\ -\lambda_n$$

the real parts of the roots $\chi_1, \chi_2, \ldots, \chi_n$, then by virtue of (53) we shall be able to put

$$\left.\begin{aligned}
v^{(l)} &= 2\pi V^{(l)},\\
\lambda_1 v_1^{(l)} &= \{|a_1|+|b_1|\}v^{(l)}+V_1^{(l)},\\
\lambda_s v_s^{(l)} &= |\sigma_{s-1}|v_{s-1}^{(l)}+\{|a_s|+|b_s|\}v^{(l)}+V_s^{(l)}\\
&\qquad (s=2,3,\ldots,n).
\end{aligned}\right\}\tag{54}$$

On further making

$$\lambda_1 v_1^{(1)}=|a_1|+|b_1|,\quad \lambda_s v_s^{(1)}=|\sigma_{s-1}|v_{s-1}^{(1)}+|a_s|+|b_s|$$
$$(s=2,3,\ldots,n)$$

and on defining by formulae (54) the quantities $v^{(l)}$, $v_s^{(l)}$ for $l>1$, we shall thus obtain for the moduli of functions $u^{(l)}$, $u_s^{(l)}$ upper bounds applicable for all real values of ϑ.

Now, by the nature of the functions R, Q_s, for the moduli of the coefficients in their expansions, θ being real, we can always assign constant upper bounds such that the series to which these expansions reduce, after having replaced in them the coefficients by the upper bounds in question, are convergent, as long as the moduli of the variables r, x_s are small enough. These series will thus then define certain holomorphic functions of the variables r, x_s, which we shall designate respectively by

$$F(r, x_1, x_2, \ldots, x_n),\quad F_s(r, x_1, x_2, \ldots, x_n).$$

These will become zero for $r=x_1=x_2=\ldots=x_n=0$, and moreover the functions F_s will not contain in their expansions terms of first degree.

Now, if we choose in this manner the upper bounds in question, the quantities $v^{(l)}$, $v_s^{(l)}$, defined by the preceding formulae, will represent the coefficients in the expansions

$$\left.\begin{aligned} r &= c + v^{(2)}c^2 + v^{(3)}c^3 + \dots, \\ x_s &= v_s^{(1)}c + v_s^{(2)}c^2 + v_s^{(3)}c^3 + \dots \quad (s = 1, 2, \dots, n) \end{aligned}\right\} \tag{55}$$

in positive integer powers of c for the quantities r, x_s satisfying the equations

$$r = c + 2\pi r F(r, x_1, x_2, \dots, x_n),$$
$$\lambda_1 x_1 = \{|a_1| + |b_1|\}r + F_1(r, x_1, x_2, \dots, x_n),$$
$$\lambda_j x_j = \{|a_j| + |b_j|\}r + |\sigma_{j-1}|x_{j-1} + F_j(r, x_1, x_2, \dots, x_n)$$
$$(j = 2, 3, \dots, n)$$

and becoming zero for $c = 0$.

Then, $|c|$ being sufficiently small, the series (55) will be absolutely convergent, and thus the series

$$|c| + |u^{(2)}c^2| + |u^{(3)}c^3| + \dots,$$
$$|u_s^{(1)}c| + |u_s^{(2)}c^2| + |u_s^{(3)}c^3| + \dots \quad (s = 1, 2, \dots, n)$$

will converge uniformly for all real values of θ.

This settled, let us return to our original equations and to equations (50) which correspond to them.

Since in these equations all the coefficients are real functions of θ, it will be the same for the functions $u^{(l)}$, $u_s^{(l)}$, obtained under the hypothesis that for $\theta = 0$ all the $u^{(l)}$ become zero. Then, c being real, the series (49) will define, under this hypothesis, a real solution of equations (47).

Let us profit from this in transforming our equations.

Let us make

$$r = z + u^{(2)}z^2 + u^{(3)}z^3 + \dots$$
$$x_s = z_s + u_s^{(1)}z + u_s^{(2)}z^2 + \dots \quad (s = 1, 2, \dots, n)$$

and, in place of variables $r, x_1, x_2, \dots, x_n$, let us introduce the variables $z, z_1, z_2, \dots, z_n$.

The transformed equations will be of the form (52), and the functions Z, Z_s which appear in them will be of the same character as in the case considered in the preceding section, with the sole difference that now, for $z_1 = z_2 = \dots = z_n = 0$, all these functions become zero.

It is to be noted that the substitution considered is such that the new variables will be able to play, in our problem, the same role as the old ones.

36. [*Periodic solutions*]

Let us examine more closely the case where all the functions $u^{(l)}$, $u_s^{(l)}$ are periodic.

On supposing that the arbitrary constants in these functions are determined in conformity with the condition considered above, we shall define by series (49), $|c|$ being sufficiently small, a certain periodic solution of system (47).

For system (45), to this solution there will also correspond a periodic solution, which we shall obtain on replacing in the equations

$$\left.\begin{aligned} x = [c + u^{(2)}c^2 + \ldots] \cos \vartheta, \quad y = [c + u^{(2)}c^2 + \ldots] \sin \vartheta, \\ x_s = u_s^{(1)}c + u_s^{(2)}c^2 + \ldots \quad (s = 1, 2, \ldots, n) \end{aligned}\right\} \tag{56}$$

the variable ϑ by its expression as a function of t.

Let us see how this function will be obtained, and what will be the form of the solution in question for system (45).

Let us return to equation (46).

Let us make the substitution (49) in the function Θ and then expand the function

$$\frac{\lambda}{\lambda + \Theta}$$

in increasing powers of c.

As the latter function becomes equal to 1 for $c = 0$, we shall then have

$$\frac{\lambda}{\lambda + \Theta} = 1 + \Theta_1 c + \Theta_2 c^2 + \Theta_3 c^3 + \ldots,$$

where all the Θ_j are periodic functions of ϑ, independent of c, and which we shall be able to present in the form of finite series of sines and cosines of integer multiples of ϑ.

On designating now by t_0 an arbitrary constant, we obtain from equation (46) [on separating the variables and integrating]

$$\vartheta + c\int_0^\vartheta \Theta_1\, d\vartheta + c^2 \int_0^\vartheta \Theta_2\, d\vartheta + \ldots = \lambda(t - t_0).$$

The left-hand side of this equation contains, apart from periodic terms, further terms proportional to ϑ.

If we put in general

$$\frac{1}{2\pi}\int_0^{2\pi} \Theta_m\, d\vartheta = h_m,$$

we may present the ensemble of all these terms in the form†

$$(1 + h_2 c^2 + h_3 c^3 + \ldots)\vartheta.$$

Accordingly we shall be able to give our equation the following form:

$$(1 + h_2 c^2 + h_3 c^3 + \ldots)[\vartheta + c\Phi_1(\vartheta) + c^2\Phi_2(\vartheta) + \ldots] = \lambda(t - t_0),$$

where the $\Phi_j(\vartheta)$ represent finite series of sines and cosines of integer multiples of ϑ, independent of c.

All the preceding operations have been effected under the assumption that ϑ can only take real values and that $|c|$ does not exceed a certain limit.

† We easily assure ourselves that h_1 will always be equal to 0.

Under this assumption the series

$$1 + h_2c^2 + h_3c^3 + \dots, \quad c\Phi_1(\vartheta) + c^2\Phi_2(\vartheta) + \dots$$

are absolutely convergent. Moreover the series

$$|c\Phi_1(\vartheta)| + |c^2\Phi_2(\vartheta)| + |c^3\Phi_3(\vartheta)| + \dots \tag{57}$$

will converge uniformly for all real values of ϑ.

Now, for what is to follow, the consideration of real values of ϑ will no longer be sufficient, and we shall have to attribute to it complex values of the form

$$\vartheta = \alpha + \beta\sqrt{-1},$$

α and β being real numbers, of which the first is arbitrary, while the second is subject to the condition that its absolute value does not exceed a certain limit.

If, in treating the question of convergence of series (49), we had considered such values of ϑ, we would have arrived, as we easily convince ourselves, at the same conclusion as in the case of real values of ϑ.

We can thus be sure that we can always choose $|c|$ small enough for series (57) to be uniformly convergent for all complex values of ϑ of the form indicated above.

After having noted that, let us put

$$\frac{2\pi}{\lambda}(1 + h_2c^2 + h_3c^3 + \dots) = T,$$

$$\frac{2\pi(t - t_0)}{T} = \tau, \quad \vartheta - \tau = \varphi.$$

Our equation will then take the form

$$\varphi + c\Phi_1(\varphi + \tau) + c^2\Phi_2(\varphi + \tau) + \dots = 0. \tag{58}$$

This settled, let us consider τ as a parameter independent of c, and to which may be attributed all values of the form

$$\tau = \rho + \sigma\sqrt{-1},$$

ρ and σ being real numbers, of which the last does not exceed in absolute value a certain given limit.

Then, if we make an analogous assumption with regard to φ, we shall be able to obtain from equation (58) the conclusion that, $|c|$ being sufficiently small, the modulus of the variable φ will become as small as we wish.

Our problem will thus reduce to finding in accordance with equation (58) a function φ, of which the modulus can be made, on making $|c|$ sufficiently small, as small as we wish.

We now note that each of the functions $\Phi_j(\varphi + \tau)$ can be presented in the form of a series, ordered in positive integer powers of φ, and absolutely convergent for all the values of φ and τ. By consequence, the left-hand side of equation (58) will be a holomorphic function of the quantities φ and c (and uniformly so for all values of τ of the above form).

This function, for $\varphi = c = 0$, becomes zero, and its partial derivative with respect to φ then becomes equal to unity.

Therefore, in view of a known theorem,† the sought function φ will be holomorphic with respect to c, and so will present itself, $|c|$ being sufficiently small, in the form of the series

$$\varphi = \varphi_1 c + \varphi_2 c^2 + \varphi_3 c^3 + \dots, \tag{59}$$

the φ_j designating functions of τ independent of c.

The functions φ_j can be calculated successively [from (58)], with the aid of the functions Θ_j and their derivatives $\Theta_j^{(l)}$:

$$\varphi_1 = -\Phi_1(\tau), \quad \varphi_2 = \Phi_1(\tau)\Phi_1'(\tau) - \Phi_2(\tau), \dots$$

We see that all these functions will appear in the form of finite sequences of sines and cosines of integer multiples of τ.

In this way we shall have for ϑ the following expression:

$$\vartheta = \tau + \varphi_1 c + \varphi_2 c^2 + \varphi_3 c^3 + \dots$$

On substituting this expression in equations (56) and on then expanding the right-hand sides in increasing powers of c, we shall present the functions x, y, x_s in the form of series of the same kind as (59).

All these series, for values of c with sufficiently small modulus, will converge uniformly for all the values considered of τ.

On putting in them

$$\tau = \frac{2\pi(t - t_0)}{T} \tag{60}$$

we shall thus obtain the sought solution of system (45).

With respect to t the functions x, y, x_s will be, in this solution, periodic with period

$$T = \int_0^{2\pi} \frac{d\vartheta}{\lambda + \Theta} = \frac{2\pi}{\lambda}(1 + h_2 c^2 + h_3 c^3 + \dots).$$

We can, if we wish, give another form for the solution found. Namely, we may represent the functions x, y, x_s in the form of Fourier series, ordered in sines and cosines of integer multiples of τ. This results from the property that the functions concerned, $|c|$ being sufficiently small, will be synectic‡ for all complex values of τ of the form indicated above.

The new series obtained from this point of view will be of the same character as those considered by Lindstedt (Section 27).

Our periodic solution contains two arbitrary constants c and t_0, and there will correspond to it, for real values of the latter, a periodic motion.

The constant t_0 is not, however, of importance, and the character of this motion depends principally on the constant c.

On making this constant vary in a continuous manner we shall obtain a continuous series of periodic motions, and the undisturbed motion under consideration will play a part in this, being that for $c = 0$.

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† [See a reference to Goursat in Section 28.]

‡ [This term (introduced by Cauchy) is used as a synonym for 'regular' by A. R. Forsyth, *Theory of Differential Equations*, Cambridge, 1902, Vol. IV, p. 4.]

Remark

For the practical calculation of the terms in the series considered, it is not absolutely necessary to resort to the procedure indicated above. For this it will be in general preferable to treat equations (45) directly.

On designating by c an arbitrary constant, and by T the series

$$\frac{2\pi}{\lambda}(1+h_2c^2+h_3c^3+\ldots)$$

with undetermined coefficients h, let us introduce into these equations, in place of t, a new independent variable τ by means of substitution (60). Next let us seek to dispose of the constants h in such a way that the transformed equations are satisfied by the series

$$\left.\begin{aligned} x &= x^{(1)}c + x^{(2)}c^2 + x^{(3)}c^3 + \ldots, \\ y &= y^{(1)}c + y^{(2)}c^2 + y^{(3)}c^3 + \ldots, \\ x_s &= x_s^{(1)}c + x_s^{(2)}c^2 + x_s^{(3)}c^3 + \ldots, \\ &\qquad (s = 1, 2, \ldots, n), \end{aligned}\right\} \tag{61}$$

in which all the $x^{(m)}$, $y^{(m)}$, $x_s^{(m)}$ are periodic functions of τ having 2π for their common period.

For the calculation of these functions (which are supposed independent of c), we obtain systems of differential equations which allow us, when our problem is possible to solve, to obtain successively all the $x^{(m)}$, $y^{(m)}$, $x_s^{(m)}$ in order of increasing m, in the form of finite sequences of sines and cosines of integer multiples of τ, provided that we choose suitably the constants h. We shall then obtain, for each value of m first $x^{(m)}$ and $y^{(m)}$, then the $x_s^{(m)}$. The values that must be attributed to the constants h_m will also be calculated successively in order of increasing m, and in such a way that for every value of m the constant h_{m-1} will be obtained simultaneously with the functions $x^{(m)}$, $y^{(m)}$.

With regard to the equations on which $x^{(1)}$ and $y^{(1)}$ depend, we shall always be able to satisfy them on putting

$$x^{(1)} = \cos\tau, \quad y^{(1)} = \sin\tau.$$

Next we shall be able to carry out the calculations so that the $x^{(m)}$, $y^{(m)}$ for $m > 1$ become zero for $\tau = 0$. Then all the functions sought, as well as the constants h, will become completely determined, and the series (61) will be identical with those considered above.

Keeping with this procedure, let us see how we may obtain the constants h.

Suppose that we have already calculated all the functions $x^{(\mu)}$, $y^{(\mu)}$, $x_s^{(\mu)}$ for $\mu < m$, and all the constants h_j for $j < m-1$. Then, to determine the functions $x^{(m)}$ and $y^{(m)}$, we shall have a system of equations of the form

$$\frac{dx^{(m)}}{d\tau} = -y^{(m)} - h_{m-1}\sin\tau + X^{(m)},$$

$$\frac{dy^{(m)}}{d\tau} = x^{(m)} + h_{m-1}\cos\tau + Y^{(m)},$$

[in these and ensuing equations, h_{m-1} should be replaced by $-h_{m-1}$] where $X^{(m)}$, $Y^{(m)}$ will be known functions, which will be entire and rational with respect to the $x^{(\mu)}$, $y^{(\mu)}$, $x_s^{(\mu)}$ previously found.

The functions $X^{(m)}$, $Y^{(m)}$ will appear in the form of finite series of sines and cosines of integer multiples of τ.

Let us seek the functions $x^{(m)}$, $y^{(m)}$ in the form of series of the same kind.

On seeking the coefficients in these series, we only meet with difficulty for the terms dependent on $\sin\tau$ and $\cos\tau$. Let us therefore restrict attention to these terms.

On designating the other terms by dots, suppose that we have

$$X^{(m)} = A_1 \cos\tau + A_2 \sin\tau + \ldots,$$

$$Y^{(m)} = B_1 \cos\tau + B_2 \sin\tau + \ldots,$$

where A_1, A_2, B_1, B_2 are known constants.

On putting in an analogous manner

$$x^{(m)} = a_1 \cos\tau + a_2 \sin\tau + \ldots,$$

$$y^{(m)} = b_1 \cos\tau + b_2 \sin\tau + \ldots,$$

we shall have, to determine the constants a_1, a_2, b_1, b_2, h_{m-1}, the following equations:

$$a_2 + b_1 = A_1, \quad -a_1 + b_2 + h_{m-1} = A_2,$$

$$-a_2 - b_1 = B_2, \quad -a_1 + b_2 - h_{m-1} = B_1.$$

These equations will only be solvable under the condition [obtained by adding the first equation to the third]

$$A_1 + B_2 = 0, \tag{62}$$

and when this condition is fulfilled they will give

$$h_{m-1} = \frac{A_2 - B_1}{2}, \quad a_2 = A_1 - b_1, \quad b_2 = \frac{A_2 + B_1}{2} + a_1.$$

Since the condition that $x^{(m)}$, $y^{(m)}$ becomes zero for $\tau = 0$ allows the determination of the constants a_1 and b_1, these formulae will give all the constants sought.

The method of calculation which we have just indicated presents only an insignificant modification of that of Lindstedt, such as it would have been in the case which interests us (Section 27).

Let us note that for the application of this method it is not necessary for the functions X and Y to become zero for $x = y = 0$. Thus it does not require the preliminary transformation of equations (45) which we mentioned in Section 33.

If the existence of the periodic solution is not known *a priori*, and if, on applying the preceding method and carrying out the calculation up to a certain value of m, we find that condition (62) is not fulfilled, that will serve to indicate that the solution sought is impossible.

We easily assure ourselves that, in this case (whether or not the functions X and Y become zero for $x = y = 0$), the number m and the constant

$$g = \frac{A_1 + B_2}{2}$$

will be the same as those which we have considered in Section 34. [Equation (51) indicates the case where r has a non-periodic term $g\vartheta c^m$. Correspondingly x and y have terms $g\vartheta c^m \cos\vartheta$ and $g\vartheta c^m \sin\vartheta$ respectively; and the above expressions for $x^{(m)}$ and $y^{(m)}$ generalize to

$$x^{(m)} = a_1 \cos\tau + a_2 \sin\tau + g\tau\cos\tau + \ldots$$
$$y^{(m)} = b_1 \cos\tau + b_2 \sin\tau + g\tau\sin\tau + \ldots$$

Substituting these expressions in the differential equations for $x^{(m)}$ and $y^{(m)}$, and equating coefficients, we find

$$a_2 + b_1 + g = A_1, \quad -a_1 + b_2 - h_{m-1} = A_2$$
$$-a_2 - b_1 + g = B_2, \quad -a_1 + b_2 + h_{m-1} = B_1$$

which yield Lyapunov's above expression for g.

Note also that m and g here must indeed agree with those of Section 34, since otherwise $x^2 + y^2$ and r^2 would have differing expansions in powers of c, contradicting the identity $r^2 = x^2 + y^2$.]

37. [*Study of the general case*]

Let us now return to our problem.

We are going to show how, on starting from equations (52), we may achieve resolution of the question of stability.

Let us first consider the case where, for $z_1 = z_2 = \ldots = z_n = 0$, the function Z does not become identically zero.

Let

$$zZ = gz^m + P^{(1)}z + P^{(2)}z^2 + \ldots + P^{(m-1)}z^{m-1} + R,$$

where g is a non-zero constant, the $P^{(j)}$ are linear forms in the quantities z_s with coefficients† periodic with respect to ϑ, and R represents a holomorphic function of the variables z, z_s, for which the expansion, possessing coefficients of the same kind, does not contain terms of degree less than third. The function R is moreover such that, in the terms which are linear with respect to the quantities z_s, it contains z in powers not less than the mth, and, in the terms independent of these quantities, it contains z in powers not less than the $(m+1)$th.

By the property of equations (52) we must then admit that the expansions of the functions Z_s, in the terms independent of the z_s, do not contain z in powers less than the mth.

Let, k being any positive integer,

$$Z_s = P_s^{(1)}z + P_s^{(2)}z^2 + \ldots + P_s^{(k)}z^k + Z_s^{(k)},$$

where the $P_s^{(j)}$ are linear forms in the quantities z_σ with periodic coefficients, and $Z_1^{(k)}, Z_2^{(k)}, \ldots, Z_n^{(k)}$ are holomorphic functions of the variables z, z_s for which the expansions, in the terms linear with respect to the z_s, can only contain z in powers exceeding the kth.

Proceeding as in Section 29, let us put

$$V = z + W + U^{(1)}z + U^{(2)}z^2 + \ldots + U^{(m-1)}z^{m-1},$$

the $U^{(j)}$ being linear forms and W a quadratic form in the quantities z_s, with undetermined coefficients. But now these coefficients will be supposed constant only for the form W, and, for the forms $U^{(j)}$, we shall suppose them periodic functions of ϑ.

† In general, all the periodic functions with which we are concerned here will be finite series of sines and cosines of integer multiples of ϑ.

After having formed, in accordance with equations (52), the derivative $dV/d\vartheta$, let us seek to assign the coefficients in the forms $U^{(j)}$ in such a way that this derivative, in the terms linear with respect to the quantities z_s, can only contain z in powers not less than the mth. For this we must make

$$\sum_{s=1}^{n} (q_{s1}z_1 + q_{s2}z_2 + \dots + q_{sn}z_n)\frac{\partial U^{(1)}}{\partial z_s} + \frac{\partial U^{(1)}}{\partial \vartheta} + P^{(1)} = 0,$$

$$\sum_{s=1}^{n} (q_{s1}z_1 + q_{s2}z_2 + \dots + q_{sn}z_n)\frac{\partial U^{(k)}}{\partial z_s} + \frac{\partial U^{(k)}}{\partial \vartheta} + P^{(k)} + \sum_{s=1}^{n}\left(P_s^{(1)}\frac{\partial U^{(k-1)}}{\partial z_s}\right.$$
$$\left. + \dots + P_s^{(k-1)}\frac{\partial U^{(1)}}{\partial z_s}\right) = 0 \quad (k = 2, 3, \dots, m-1).$$

[Compare corresponding equations in Section 29.]

From these equations we shall obtain successively

$$U^{(1)}, U^{(2)}, \dots, U^{(m-1)}. \tag{63}$$

Moreover the hypothesis that the coefficients in the forms $U^{(j)}$ are periodic functions of ϑ, namely finite series of sines and cosines of integer multiples of ϑ, will always be realizable and will completely define these coefficients.

In fact, if U is the first of forms (63), or indeed any one of those following, under the hypothesis that all those which precede it are already found in the form indicated, we shall obtain to determine it the equation

$$\sum_{s=1}^{n} (q_{s1}z_1 + q_{s2}z_2 + \dots + q_{sn}z_n)\frac{\partial U}{\partial z_s} + \frac{\partial U}{\partial \vartheta} = A_1 z_1 + A_2 z_2 + \dots + A_n z_n,$$

in which all the A will be finite series of sines and cosines of integer multiples of ϑ. This equation will give for the coefficients a in the form

$$U = a_1 z_1 + a_2 z_2 + \dots + a_n z_n$$

the following system of equations [obtained by equating coefficients of z_1, then those of z_2, and so on]:

$$\frac{da_s}{d\vartheta} + q_{1s}a_1 + q_{2s}a_2 + \dots + q_{ns}a_n = A_s \quad (s = 1, 2, \dots, n).$$

And the latter, the determinantal equation corresponding to it not having purely imaginary roots (all these roots have positive real parts [recall that (48) only has roots with negative real parts]), will always admit a solution, and only one, where all the a are finite series of sines and cosines of integer multiples of ϑ.

After having determined in the way just stated the forms $U^{(j)}$, let us choose the form W in accordance with the equation

$$\sum_{s=1}^{n} (q_{s1}z_1 + q_{s2}z_2 + \dots + q_{sn}z_n)\frac{\partial W}{\partial z_s} = g(z_1^2 + z_2^2 + \dots + z_n^2). \tag{64}$$

Then the expression for the total derivative of the function V with respect to ϑ will take the following form:

$$\frac{dV}{d\vartheta} = g(z^m + z_1^2 + z_2^2 + \dots + z_n^2) + S,$$

if we put

$$S = \sum_{s=1}^{n} \left\{ \sum_{k=1}^{m-2} z^k Z_s^{(m-k-1)} \frac{\partial U^{(k)}}{\partial z_s} + z^{m-1} Z_s \frac{\partial U^{(m-1)}}{\partial z_s} + Z_s \frac{\partial W}{\partial z_s} \right\} + Z \sum_{k=1}^{m-1} k U^{(k)} z^k + R.$$

[Compare similar equations in Section 29.]

Now we can always present this quantity S in the form

$$S = v z^m + \sum_{s=1}^{n} \sum_{\sigma=1}^{n} v_{s\sigma} z_s z_\sigma,$$

where v, $v_{s\sigma}$ are functions of z, z_s, ϑ, becoming zero for

$$z = z_1 = z_2 = \dots = z_n = 0,$$

and being periodic with respect to ϑ and holomorphic with respect to z, z_s, and moreover uniformly so for all real values of ϑ.

It is thus clear that if we introduce the condition

$$z \geqslant 0, \tag{65}$$

the expression found for $dV/d\vartheta$, considered as a function of the variables z, z_s, ϑ, of which the last plays the role of t, will represent a definite function (see remark at the end of Section 16), which, for sufficiently small values of z and the $|z_s|$, will retain the sign of the constant g.

Under the same condition (65), the function V will also be definite and moreover positive, if the form W, as a function of the variables z_s, is positive-definite.

The latter circumstance will effectively hold when $g < 0$, since the form W, which has to satisfy equation (64), will always retain a sign opposite to that of g (Section 20, Theorem II).

On the other hand, if $g > 0$, the function V will be capable of taking any sign, however small the $|z_s|$ and z.

Consequently, if we keep in view condition (65) {and the latter, as has already been mentioned in Section 34, is a consequence of the hypothesis $r \geqslant 0$, which is always possible and does not at all restrict our problem (Section 33)}, we may assert that the function V, for $g > 0$, will satisfy the conditions of Theorem II of Section 16, and, for $g < 0$, the conditions of Theorem I (and even the conditions of the theorem established in Remark II [of that section]).

We must therefore conclude that in the case of positive g the undisturbed motion is unstable, and that in the case of negative g it is stable.

In this last case, the disturbed motions corresponding to sufficiently small perturbations will tend asymptotically to the undisturbed motion.

Remark

We have considered r and z as variables for which negative values are not possible. But we would have been able, with equal right, to have considered them as variables for which positive values are not possible.

In order to examine the question under the latter hypothesis, it would have only required us to modify the preceding analysis a little, on replacing, in equation (64), g by $(-1)^m g$.

Then the new expression for the derivative $dV/d\vartheta$ would represent a function defined subject to the condition $z \leqslant 0$, and its sign would be the same as that of $(-1)^m g$. The function V, under the same condition, would be negative-definite, if $(-1)^m g$ represented a positive number.

We should, by consequence, be led to the conclusion that, under the condition $(-1)^m g > 0$, the undisturbed motion is stable, and, under the condition $(-1)^m g < 0$, unstable.

These new conditions coincide with the preceding ones only in the case where m is an odd number. And as they must necessarily coincide, the result found proves that the number m will always be odd (Section 34, Remark II).

Note that if m were an even number, which could only occur if equations (52), without being the transforms of equations (45), were proposed in themselves, our analysis would lead to the conclusion that, for perturbations subject to one of the two conditions

$$z \geqslant 0 \quad \text{or} \quad z \leqslant 0,$$

the undisturbed motion would be stable, and for perturbations subject to the other condition it would be unstable.

38. [*Study of the exceptional case. Existence of a holomorphic integral independent of t*]

Let us now consider the case where in equations (52) all the functions Z, Z_s become zero for $z_1 = z_2 = \ldots = z_n = 0$, and where, as a consequence, these equations admit the solution

$$z = c, \quad z_1 = z_2 = \ldots = z_n = 0$$

with arbitrary constant c.

We are going to show that in this case we can find for system (52) a complete integral with an arbitrary constant c, with equation presenting itself in the form

$$z = c + f(z_1, z_2, \ldots, z_n, c, \vartheta), \tag{66}$$

where f designates a holomorphic function of the quantities $z_1, z_2, \ldots, z_n, c$, becoming zero for $c = 0$ as well as for $z_1 = z_2 = \ldots = z_n = 0$, and having for coefficients in its expansion in powers of these quantities finite series of sines and cosines of integer multiples of ϑ.

For this purpose we have to show that the partial differential equation [which expresses the first of equations (52)]

$$\sum_{s=1}^{n} (q_{s1} z_1 + q_{s2} z_2 + \ldots + q_{sn} z_n) \frac{\partial z}{\partial z_s} + \frac{\partial z}{\partial \vartheta} = zZ - \sum_{s=1}^{n} Z_s \frac{\partial z}{\partial z_s} \tag{67}$$

has a solution in the form of (66).

Let us put

$$f = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} P_m^{(l)} c^l, \tag{68}$$

on understanding by $P_m^{(l)}$ a form of degree m in the variables z_s and independent of c.

If we replace, in the right-hand side of equation (67), z by its expression (66), on then ordering the result in powers of the quantities z_s, c we shall obtain a series in which, under our hypothesis concerning the functions Z, Z_s, there will not be any terms independent of the quantities z_s.

The result of this substitution will consequently appear in the form

$$-\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} Q_m^{(l)} c^l,$$

where $Q_m^{(l)}$ designates a form of degree m in the quantities z_s, which is deduced in a certain manner from the forms $P_{m'}^{(l')}$ for which

$$m' + l' < m + l$$

(in the case $m + l = 1$ this form will be the ensemble of terms of the first dimension in the function $-Z$).

We shall thus have to satisfy a sequence of equations of the form

$$\sum_{s=1}^{n} (q_{s1} z_1 + q_{s2} z_2 + \dots + q_{sn} z_n) \frac{\partial P_m^{(l)}}{\partial z_s} + \frac{\partial P_m^{(l)}}{\partial \vartheta} = -Q_m^{(l)}, \tag{69}$$

which will serve for the successive calculation of all the $P_m^{(l)}$ in any order for which the number $m + l$ does not decrease.

In these calculations we shall always be able to suppose that the coefficients in the forms $P_m^{(l)}$ are periodic with respect to ϑ (finite series of sines and cosines of integer multiples of ϑ), and such a hypothesis will make our problem completely determinate.

In fact, if all the forms $P_{m'}^{(l')}$ for $m' + l' < m + l$ are already found and possess periodic coefficients, the right-hand side of equation (69) will represent a form in the quantities z_s with coefficients of the same kind. Hence, such will also be the known terms in the system of non-homogeneous linear diffierential equations which this equation will give for the calculation of the coefficients of the form $P_m^{(l)}$. Now the determinantal equation of this system will only have roots with positive real parts, for this equation is obtained by equating to zero the $(m-1)$th derived determinant (Section 19) of the determinant which appears on the left-hand side of equation (48), and by replacing χ by $-\chi$. Thus, the above system will always admit one (and only one) periodic solution.

In this way we see that series (68) will not include anything unknown.†

To examine the convergence of this series, let us consider a certain transformation of it. Specifically, let us consider the series relating to the system which is deduced from that of (52) by means of a linear substitution similar to that which we used in Section 35 for transforming equations (47).

Our question will then reduce to the examination of the convergence of series (68), obtained under hypothesis that, in equation (67), all the coefficients $q_{s\sigma}$ are zero with the exception of the following:

$$q_{11} = \chi_1, \quad q_{22} = \chi_2, \quad \dots, \quad q_{nn} = \chi_n,$$
$$q_{21} = \sigma_1, \quad q_{32} = \sigma_2, \quad \dots, \quad q_{n,n-1} = \sigma_{n-1},$$

among which the first n have negative real parts.

† It goes without saying that, the coefficients in system (52) being real, it will be the same for this series, if it is considered as ordered in powers of the quantities z_s, c (we assume the variable ϑ to be real).

Under this hypothesis, equation (69) will give for the coefficients of form $P_m^{(l)}$ equations such that, being arranged in a suitable order, they will allow the calculation, in a certain order of succession, one after another all the coefficients sought.

Let A be the coefficient of the term containing

$$z_1^{m_1} z_2^{m_2} \dots z_n^{m_n},$$

and suppose that all the coefficients which precede it in the order of succession considered are already found. Then we shall have for the determination of A the equation

$$\frac{dA}{d\vartheta} + (m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n)A = -B,$$

in which B will be a known periodic function.

From this there will appear

$$A = e^{-(m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n)\vartheta} \int_\vartheta^\infty e^{(m_1\chi_1 + m_2\chi_2 + \dots + m_n\chi_n)\vartheta} B \, d\vartheta.$$

We now note that the function B, in its original form, represents an entire and rational function, with positive coefficients, of the previously found coefficients in the form $P_m^{(l)}$ as well as in those which precede it, of the quantities σ_s, and of the coefficients in the expansions of the functions $-Z$, Z_s.

By consequence, it results from the expression obtained for the coefficient A that we shall obtain upper bounds for the moduli of coefficients such as A, if we find the coefficients of the corresponding terms of the series, similar to (68), but independent of ϑ, which is formed under the hypothesis that in equation (67) all the χ_s are replaced by their real parts, all the σ_s by their moduli, and all the coefficients in the expansions of the functions $-Z$, Z_s by constant upper bounds for their moduli, valid for all real values of ϑ. Moreover, by the nature of the functions Z, Z_s, these last upper bounds will always be able to be chosen in such a way that, $|z|$, $|z_s|$ being sufficiently small, the series defining these functions remain convergent after the indicated replacement.

Now the series independent of ϑ which we shall obtain in this manner is a special case of the series considered in Section 31.

We can therefore assert that series (68) defines a function of the quantities z_s, c which is uniformly holomorphic for all real values of ϑ; and, as a consequence, the existence of the integral with equation (66) can be regarded as proved.

Let us go back to our problem.

Supposing the constant c to be real, let us replace the first of equations (52) by equation (66) for the integral, and let us next substitute in the others, in place of z, its expression (66). These equations will then take the form

$$\frac{dz_s}{d\vartheta} = (q_{s1} + c_{s1})z_1 + (q_{s2} + c_{s2})z_2 + \dots + (q_{sn} + c_{sn})z_n + Z'_s$$

$$(s = 1, 2, \dots, n). \tag{70}$$

The $c_{s\sigma}$ here are holomorphic functions of the constant c, becoming zero for $c = 0$, and having for coefficients in their expansions in powers of c real periodic functions of ϑ; and the Z'_s are holomorphic functions of the quantities z_s, c, for which the expansions, possessing coefficients of the same kind, begin with terms of degree not less than second with respect to the variables z_s. All the functions considered are moreover uniformly holomorphic for all real values of ϑ.

Similarly to what we have seen in Section 31, our problem now reduces to the examination of the stability of the motion

$$z_1 = z_2 = \dots = z_n = 0$$

with respect to the variables z_s, satisfying equations (70).

These equations contain the parameter c which is only subject to the condition that its absolute value must not exceed a certain limit, and if we prove that the motion in question is stable independently of the value of this parameter (in the sense defined in Section 31), it will by this also be proved that the steady motion which we had to examine is stable with respect to the variables z, z_s.

Now, having regard to the nature of the functions representing the right-hand sides of equations (70), we prove this easily by the same procedure as that indicated at the end of Section 31.

We can thus be sure that in the case considered the undisturbed motion will always be stable, and that every disturbed motion for which the perturbations are sufficiently small will approach asymptotically one of the periodic motions defined by the equations

$$z = c, \quad z_1 = z_2 = \dots = z_n = 0.$$

Remark I

On resolving equation (66) with respect to the constant c (under the assumption that all the quantities $|z_s|$, $|c|$ are small enough), we shall obtain from it the following:

$$c = z + \varphi(z_1, z_2, \dots, z_n, z, \vartheta), \tag{71}$$

where φ will be a holomorphic function of the quantities z, z_s, becoming zero for $z = 0$ as well as for $z_1 = z_2 = \dots = z_n = 0$, and having for coefficients finite sequences of sines and cosines of integer multiples of ϑ.

The right-hand side of equation (71) will represent one of the integrals of system (52).

On introducing into this integral, in place of the variables z, z_s, the variables r, x_s, we shall obtain an integral for system (47).

Let us consider the square of the latter. It will be of the following form:

$$r^2 + \Phi(x_1, x_2, \dots, x_n, r, \vartheta), \tag{72}$$

where Φ designates a holomorphic function of the quantities r, x_s, for which the expansion begins with terms of degree not less than the third and has for coefficients finite sequences of sines and cosines of integer multiples ϑ.

On introducing into function (72), in place of variables r and ϑ, variables x and y, we shall deduce from it an integral for system (45).

This integral will appear in the form of the following series:

$$x^2 + y^2 + \sum \{U_m^{(m_1, m_2, \dots, m_n)} \\ + \sqrt{x^2 + y^2}\, V_m^{(m_1, m_2, \dots, m_n)}\} x_1^{m_1} x_2^{m_2} \dots x_m^{m_n}. \tag{73}$$

Here the $U_m^{(\ldots)}$, $V_m^{(\ldots)}$ represent rational and homogeneous functions of the variables x and y, of the mth and $(m-1)$th degree respectively. These functions are moreover such that, if they are not entire with respect to x and y, they become so after being multiplied by certain integer powers of the quantity x^2+y^2. As for the summation, it extends over all non-negative values of the integers $m, m_1, m_2, \ldots, m_n$, subject to the conditions

$$m > 1, \quad m + m_1 + m_2 + \ldots + m_n > 2.$$

The mode of convergence of series (73) is indicated by the very extraction of this series from function (72), which is holomorphic with respect to r, x_s, and uniformly so for all real values of ϑ.

The same property, in so far as convergence is concerned, will also belong to the series that may be deduced from (73) on replacing $(x^2+y^2)^{1/2}$ by $-(x^2+y^2)^{1/2}$, for this series will represent the transform into x, y variables of the function obtained by replacing in (72) r by $-r$ and ϑ by $\vartheta+\pi$. Moreover, this new series will evidently also be an integral for system (45).

We conclude from this that the series

$$x^2+y^2+\sum U_m^{(m_1, m_2, \ldots, m_n)} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} = x^2+y^2+F(x_1, x_2, \ldots, x_n, x, y),$$

(which will certainly be convergent, as long as the two preceding ones are) will represent an integral of system (45), for which the transform into variables r and ϑ, similarly to the preceding ones, will be a function of the variables r, x_s which is uniformly holomorphic for all real values of ϑ.

Let us show that this integral will be a holomorphic function of the variables $x, y, x_1, x_2, \ldots, x_n$.

For this, we note at the outset that the coefficients U will necessarily be entire functions of x and y.

We may convince ourselves of this on considering the equation which will be verified by F, namely:

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + \alpha_s x + \beta_s y)\frac{\partial F}{\partial x_s} + \lambda\left(x\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial x}\right)$$
$$= -\sum_{s=1}^{n} X_s \frac{\partial F}{\partial x_s} - X\frac{\partial F}{\partial x} - Y\frac{\partial F}{\partial y} - 2(xX+yY).$$

[This equation is obtained by carrying out the differentiation in

$$\frac{d}{dt}(x^2+y^2+F) = 0$$

and then bringing in equations (45).]

On substituting in it the expression for F in the form of a series we shall deduce from it, for the determination of the functions $U_m^{(m_1, \ldots, m_n)}$, systems of equations such that we shall be able to calculate all the functions which correspond to given values of the numbers

$$m, \quad m_1 + m_2 + \ldots + m_n, \tag{74}$$

after we have calculated all those for which the sum of numbers (74) has a smaller value, as well as all those for which, this sum having the same value, the number

m is smaller.† Now, on examining these systems more closely, we may easily perceive that the functions U cannot be rational without being entire.

Having thus established that all the U will be entire functions of the variables x and y, let us introduce, in place of the latter, the variables ξ and η given by means of the equations

$$\xi = x + y\sqrt{-1}, \quad \eta = x - y\sqrt{-1}.$$

Let

$$U_m^{(m_1, m_2, \ldots, m_n)} = \sum_{k=0}^{m} C_{k,m-k}^{(m_1, m_2, \ldots, m_n)} \xi^k \eta^{m-k}, \tag{75}$$

where the C represent constants.

In accordance with what was noted above, the function F, on putting in it

$$\xi = re^{i\vartheta}, \quad \eta = re^{-i\vartheta} \quad (i = \sqrt{-1})$$

becomes a holomorphic function relative to the quantities r, x_s, and uniformly so for all real values of ϑ.

As a consequence, if, by virtue of the above expressions for ξ and η, we have

$$U_m^{(m_1, m_2, \ldots, m_n)} = r^m \Theta_m^{(m_1, m_2, \ldots, m_n)},$$

we shall always be able to find positive constants A, A_1, A_2, ..., A_n, M, such that for all real values of ϑ we have inequalities of the form [compare (2) of Section 2]

$$|\Theta_m^{(m_1, m_2, \ldots, m_n)}| < \frac{M}{A^m A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}}.$$

Now, because of (75),

$$C_{k,m-k}^{(m_1, m_2, \ldots, m_n)} = \frac{1}{\pi} \int_0^{\pi} \Theta_m^{(m_1, m_2, \ldots, m_n)} e^{i(m-2k)\vartheta}\, d\vartheta.$$

Thus the inequality just written gives

$$|C_{k,m-k}^{(m_1, m_2, \ldots, m_n)}| < \frac{M}{A^m A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}}.$$

From this we see that the function F, being expressed in terms of the variables ξ, η, x_1, x_2, ..., x_n, becomes a holomorphic function. It will thus equally be holomorphic with respect to variables x, y, x_1, x_2, ..., x_n.

Hence, in the case where system (45) admits a periodic solution, it will also admit a holomorphic integral independent of t,

$$x^2 + y^2 + F(x_1, x_2, \ldots, x_n, x, y), \tag{76}$$

where the ensemble of terms of lowest degree will be $x^2 + y^2$.

We moreover easily assure ourselves that if we have found any integral of the form (76), every other holomorphic integral independent of t will be a function of it.

We can also prove that if system (45) admits such an integral, it will also admit a periodic solution, defined by series of form (61).

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† In the case where $m_1 = m_2 = \ldots = m_n = 0$, for m even, we shall encounter an indeterminacy, due to the circumstance that we shall then be able to add to the sought function U the expression $C(x^2 + y^2)^{m/2}$, dependent on an arbitrary constant C.

We may convince ourselves of this on considering the system deduced from (47) by eliminating the variable r with the aid of the equation furnished by this integral. [Alternatively see Section 44.]

We have assumed that, for $x = y = 0$, the functions X and Y become zero. But, for the validity of what we have just said such a hypothesis is not necessary, and in future, in speaking of system (45), we shall no longer retain this assumption.

Remark II†

The conclusion enunciated above on the subject of the stability of the steady motion, in the case where this motion is part of a continuous series of periodic motions defined by the equations

$$z = c, \quad z_1 = z_2 = \ldots = 0,$$

cannot, in general, be extended to the latter motions.

For the steady motion (which corresponds to $c = 0$), the problem of stability with respect to the variables x, y, x_s, which alone interests us here, does not basically differ from the problem of stability with respect to the variables z, z_s. But, for the periodic motions with which we are concerned, these will in general be two different problems.

With respect to the variables z, z_s these motions will still be stable, but, with respect to x, y, x_s they will only enjoy, in general, a certain conditional stability: namely, they will be stable for perturbations which do not change the constant value of the integral (76). With regard to non-conditional stability, it will only hold in the case where the period T (Section 36) does not depend on the constant c, i.e. where the numbers h_j are zero [see the set of equations preceding (58)].

To demonstrate this, let us consider one of the periodic motions, which is defined by the equations

$$z = c; \quad z_1 = z_2 = \ldots = z_n = 0; \tag{I}$$

$$\vartheta = \tau + \varphi_1 c + \varphi_2 c^2 + \ldots, \tag{II}$$

where

$$\tau = \frac{2\pi(t - t_0)}{T},$$

and $\varphi_1, \varphi_2, \ldots$ are certain periodic functions of τ [recall (59)].

On turning to the relations between the variables x, y, x_s and z, ϑ, z_s (Section 35), we readily conclude from them that, if c is not zero, the problem of stability of motion with respect to the first variables is equivalent to the problem of stability with respect to the second. Consequently, for the motion under consideration, which is already stable with respect to z, z_s, to be the same with respect to x, y, x_s, it is necessary and sufficient that it should be stable with respect to ϑ.

† This remark does not appear in the original. It is based on the note 'Contribution à la question de la stabilité', inserted in *Communications de la Société Mathématique de Kharkow* for 1893.

This agreed, let us designate the right-hand side of equation (II) by the letter ψ and, on putting

$$\vartheta = \psi + \zeta,$$

let us form the differential equation which will be satisfied by ζ, under the assumption that, for all the disturbed motions with which is compared the periodic motion under consideration, the constant value of the integral (76) is the same as for the periodic motion.

For all these motions, the constant c in equation (66) will then have the same value as in equations (I) and (II).

Therefore, on eliminating z with the aid of equation (66), we shall have, to determine ζ, an equation of the form

$$\frac{d\zeta}{dt} = Z(z_1, z_2, \ldots, z_n, \zeta, \psi), \tag{III}$$

for which the right-hand side will become zero for $z_1 = z_2 = \ldots = z_n = 0$.

Here Z will be a holomorphic function of the quantities $z_1, z_2, \ldots, z_n, \zeta$, for which the expansion will possess coefficients periodic with respect to ψ, and this function will be uniformly holomorphic for all real values of ψ.

We now note that the constant c can always be supposed small enough in absolute value for the characteristic numbers of the functions z_s {as functions of the variable ϑ satisfying equations (70)} to be all positive, whatever the initial values of these functions.

This accepted, let us designate by χ any positive number less than all of these characteristic numbers.

Next, taking t_0 for the initial value of t, let us designate by z_0 the initial value of the function

$$|z_1| + |z_2| + \ldots + |z_n|.$$

Then, on replacing in the function Z the quantities z_s by their expressions as functions of $\vartheta = \psi + \zeta$, we shall deduce from it a function of τ, ζ and of the initial values of the quantities z_s, such that, M being sufficiently large, we shall have

$$|Z| < M z_0 e^{-\chi\tau}$$

for all values of t greater than t_0, as long as $|\zeta|$ is below a certain limit l and the initial values of all the $|z_s|$ are small enough.

By consequence, on designating by ζ_0 the initial value of the function ζ and on supposing $|\zeta_0|$ and z_0 sufficiently small that the inequality

$$|\zeta_0| + \frac{MT}{2\pi\chi} z_0 < l$$

is fulfilled, we shall be able to deduce from equation (III), t being greater than t_0, the following inequality:

$$|\zeta| < |\zeta_0| + \frac{MT}{2\pi\chi} z_0 (1 - e^{-\chi\tau}),$$

and from this we infer the stability of our motion with respect to ζ, or, what amounts to the same, with respect to ϑ.

This conclusion is obtained on assuming that the perturbations do not change the value of the integral (76).

Let us now consider arbitrary perturbations.

Let c_1 be the constant which will then figure in equation (66) in place of c.

Next let ψ_1 be what ψ will become when we replace in it c by c_1.

In view of what we have just established, we arrive at the following conclusion.

For the motion under consideration, $|c|$ being sufficiently small, to be stable with respect to ϑ, it is necessary and sufficient that, ε being any positive number, we can assign another positive number a such that, c_1 verifying the inequality

$$|c_1 - c| < a,$$

we have

$$|\psi_1 - \psi| < \varepsilon$$

for all values of t greater than t_0.

Now it is obvious that this is possible only in the case where T does not depend on c. [This follows on writing ψ as

$$\psi = \frac{2\pi(t - t_0)}{T} + \varphi_1 c + \varphi_2 c^2 + \ldots]$$

39. [*Particular cases where one can demonstrate existence of a periodic solution or a holomorphic integral*]

We see from the preceding that, in the case which interests us concerning two purely imaginary roots, the question of stability depends in an essential manner on that of the possibility of a periodic solution for system (45), or, if we wish, on the question, very intimately linked with it, of the possibility of a holomorphic integral independent of t for this system.† Unfortunately, all the procedures which we can propose, in general, for the resolution of this last question are such that they only succeed in the case where the answer is negative. However, if it is not possible to indicate any general method which leads to the desired end in all cases, it is appropriate to indicate at least certain special cases where the solution of our question simplifies.

Let us first assume that the functions X and Y do not contain the variables $x_1, x_2, \ldots, x_n$.

The problem is then completely resolved by the examination of the second-order system [see (45)]

$$\left.\begin{aligned} \frac{dx}{dt} &= -\lambda y + X, \\ \frac{dy}{dt} &= \lambda x + Y. \end{aligned}\right\} \tag{77}$$

One of the simplest cases where there exists for this system a holomorphic

† [Refer to the material at the end of Remark I of Section 38, beginning with the sentence containing (76).]

integral independent of t is that where the functions X and Y satisfy the relation

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0,$$

i.e. where system (77) is canonical [Hamiltonian].

In this case the functions x and y which satisfy it will be periodic for all sufficiently small initial values.

Mr Poincaré has indicated a case of another kind where the functions x and y, defined by equations (77), are always periodic. This is the case† where the equations under consideration do not change on replacing simultaneously t by $-t$ and y by $-y$.

Mr Poincaré has shown how the periodicity of the functions x and y can then be established *a priori.*

Now, under the indicated conditions, it is no less easy to establish directly the existence of a holomorphic integral.

In fact, for the case which we have just indicated to hold, the functions X and Y must be of the form

$$X=yf(x,y^2),$$
$$Y=\varphi(x,y^2),$$

where f and φ designate holomorphic functions of their arguments, becoming zero when the latter are simultaneously equal to zero.

Now if this is so we shall have, on eliminating dt,

$$\frac{dy^2}{dx}=-2\frac{\lambda x+\varphi(x,y^2)}{\lambda-f(x,y^2)},$$

and the right-hand side will be a holomorphic function of the quantities x and y^2. Consequently, on considering y^2 as a function of x and on designating by c the value of this function corresponding to $x=0$, we shall have, by virtue of a known theorem [compare Goursat, Hedrick and Dunkel, *loc. cit.*, p. 45],

$$y^2=c+\psi(x,c), \tag{78}$$

where ψ will be a holomorphic function of x and c, becoming zero for $x=0$.

Equation (78) makes apparent that there will indeed be an integral of the required character. This integral will be a holomorphic function of the quantities x and y^2.

In general, for system (77) to admit an integral independent of t, representing a holomorphic function of x and y^2 (or, if we wish, of x and x^2+y^2), it is necessary and sufficient that the functions X and Y can be presented in the form

$$X=yf(x,y^2)+[-\lambda+f(x,y^2)]y^2H(x,y^2),$$
$$Y=\varphi(x,y^2)+[\lambda x+\varphi(x,y^2)]yH(x,y^2),$$

where f, φ and H designate holomorphic functions of x and y^2. [These expressions for X and Y make dy^2/dx equal to a holomorphic function of x and y^2 rather than

† 'Sur les courbes définies par les équations différentielles', *Journal de Mathématiques*, fourth series, Vol. I, p. 193.

of x and y. The resulting differential equation then has a solution for y^2 in terms of x, giving the required integral of (77).]

We may pose the question in a slightly more general manner. Namely, we may seek the conditions under which system (77) admits an integral independent of t, representing a holomorphic function of the quantities

$$ax + by \quad \text{and} \quad x^2 + y^2,$$

where a and b are arbitrary constants. But we shall not stop to treat this case, which reduces to the previous one by a very simple transformation. [If we put

$$u = ax + by, \qquad v = -bx + ay,$$

$$U = aX + bY, \quad V = -bX + aY,$$

system (77) becomes

$$\frac{du}{dt} = -\lambda v + U, \quad \frac{dv}{dt} = \lambda u + V.]$$

There exist cases where, X and Y containing the variables $x_1, x_2, \ldots, x_n$, the question nevertheless reduces to the examination of a system of second order.

Such, for example, is the case where we have

$$-\lambda y + X = (-\lambda y + X')(1 + Z),$$

$$\lambda x + Y = (\lambda x + Y')(1 + Z),$$

X' and Y' being holomorphic functions of the two variables x and y only, and Z being any holomorphic function of $x, y, x_1, x_2, \ldots, x_n$, becoming zero when these variables become simultaneously zero. [In this case dy/dx becomes a function of x and y only.]

Then everything depends on the study of equations of the form

$$\frac{dx}{dt'} = -\lambda y + X',$$

$$\frac{dy}{dt'} = \lambda x + Y'.$$

Such will also be the case where, X and Y being arbitrary, all the functions X_s in system (45) become zero when we put $x_1 = x_2 = \ldots = x_n = 0$, and where all the constants α_s, β_s are zero.

Then, if $X^{(0)}$ and $Y^{(0)}$ are the functions of x and y to which X and Y reduce when the x_s are simultaneously zero, the question will depend on the discussion of the equations

$$\frac{dx}{dt} = -\lambda y + X^{(0)}, \quad \frac{dy}{dt} = \lambda x + Y^{(0)}.$$

The case which we have just indicated is contained in another one, more general, which we can obtain by considering the system of partial differential equations

$$(-\lambda x + X)\frac{\partial x_s}{\partial x} + (\lambda x + Y)\frac{\partial x_s}{\partial y}$$
$$= p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s$$
$$(s = 1, 2, \ldots, n), \tag{79}$$

defining the quantities $x_1, x_2, \ldots, x_n$ as functions of the variables x and y. [These equations interpret dx_s/dt; see (45).]

Whenever we can satisfy this system by holomorphic functions of variables x and y:

$$x_1 = f_1(x, y), \quad x_2 = f_2(x, y), \quad \ldots, \quad x_n = f_n(x, y) \tag{80}$$

becoming zero for $x = y = 0$, the question will reduce to the study of the equations

$$\frac{dx}{dt} = -\lambda y + (X), \quad \frac{dy}{dt} = \lambda x + (Y),$$

in which (X) and (Y) designate the results of the substitution of (80) in the functions X and Y.

On considering more closely equations (79) we may easily convince ourselves that, if we seek the functions x_s in the form of series ordered in positive integer powers of x and y, and not containing constant terms, the equations obtained between coefficients will always be compatible and determinate, while allowing the calculation of these coefficients for the terms of each degree in accordance with those previously found for the terms of lower degree.

In this way we shall always be able to find series in the indicated form which formally satisfy system (79), and these series will be unique.

However, it would be a mistake to believe that they always define a solution of system (79), for cases are possible where these series will not be convergent, no matter how small the moduli of the variables x and y.

Thus, for example, if the equation

$$[-\lambda y - \tfrac{1}{2}x(x^2 + y^2)] \frac{\partial x_1}{\partial x} + [\lambda x - \tfrac{1}{2}y(x^2 + y^2)] \frac{\partial x_1}{\partial y} = -x_1 + x^2 + y^2,$$

is proposed, the series

$$x^2 + y^2 + (x^2 + y^2)^2 + 1 \cdot 2(x^2 + y^2)^3 + 1 \cdot 2 \cdot 3(x^2 + y^2)^4 + \ldots$$

which satisfies it formally will be divergent whenever $x^2 + y^2$ is non-zero, and even, if we consider it as a double series, whenever x and y are not simultaneously zero.

Hence, the indicated reduction will not always be possible, and to recognize whether it is possible, it will in general be necessary to examine the convergence of the series in question.

However, we may encounter cases where these series will be finite, as well as those where we know *a priori* that they must be convergent.

Let us mention the following as one of the cases of the first kind:

$$X = xU, \quad Y = yU, \quad X_s = x_s U \quad (s = 1, 2, \ldots, n),$$

U being any holomorphic function of the variables x, y, x_s, becoming zero when we put $x = y = x_1 = \ldots = x_n = 0$. In this case, we can obviously satisfy system (79) by linear functions† of the variables x and y.

Let us note that, if U is an entire and homogeneous function of odd degree, system (45) will admit in this case a holomorphic integral independent of t.

† [With the above X, Y, X_s, on substituting $x_s = k_{s1}x + k_{s2}y$ in (79) and equating separately to zero the resulting coefficients of x and y, we get $2n$ linear equations for the $2n$ unknowns k_{ij}.]

Let us further mention one of the cases of the second kind.

Let us assume that all the constants α_s, β_s are zero, and that the functions X, Y, X_s satisfy the following relations:

$$(-\lambda y + X)\frac{\partial X_s}{\partial x} + (\lambda x + Y)\frac{\partial X_s}{\partial y} = 0 \quad (s = 1, 2, ..., n), \tag{81}$$

where the partial derivatives are taken on considering the $n+2$ variables $x, y, x_1, x_2, ..., x_n$ as independent.

Then, on considering the equations

$$p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n + X_s = 0 \quad (s = 1, 2, ..., n) \tag{82}$$

and on defining by them the quantities x_s as holomorphic functions of the variables x and y, becoming zero for $x = y = 0$ (which problem, by the nature of the coefficients $p_{s\sigma}$, will always be solvable and completely determinate), we shall find that these functions will satisfy equations (79).

In fact, equations (82), because of (81), yield the following:

$$\sum_{\sigma=1}^{n}\left(p_{s\sigma} + \frac{\partial X_s}{\partial x_\sigma}\right)\left[(-\lambda y + X)\frac{\partial x_\sigma}{\partial x} + (\lambda x + Y)\frac{\partial x_\sigma}{\partial y}\right] = 0 \quad (s = 1, 2, ..., n).$$

[To obtain these equations, multiply the derivative of (82) with respect to x by $(-\lambda y + X)$, multiply that with respect to y by $(\lambda x + Y)$, add the results, and then bring in (81).]

And from the latter, since the determinant

$$\sum \pm \left(p_{11} + \frac{\partial X_1}{\partial x_1}\right)\left(p_{22} + \frac{\partial X_2}{\partial x_2}\right) ... \left(p_{nn} + \frac{\partial X_n}{\partial x_n}\right)$$

cannot be zero for sufficiently small values of $|x|$, $|y|$, $|x_s|$, there results

$$(-\lambda y + X)\frac{\partial x_\sigma}{\partial x} + (\lambda x + Y)\frac{\partial x_\sigma}{\partial y} = 0 \quad (\sigma = 1, 2, ..., n).$$

In this case, provided that the holomorphic functions in question are not all identically zero, system (45) will always admit a holomorphic integral independent of t; and, in the periodic solution which it will possess, all the functions x_s will be constants [since the last equations imply that all the dx_s/dt are zero].

We may note that, if conditions (81) must be satisfied not identically but only by virtue of equations (82), the case that we have just indicated will be the most general one where system (45) admits a periodic solution with constant values for the functions x_s.

In the latter case, the convergence of the series defined by equations (79) coincides with the existence of a periodic solution for system (45).

We easily assure ourselves that, in general, as soon as such a solution is possible for this system, the series we are concerned with will always be convergent, as long as $|x|$ and $|y|$ are sufficiently small.

In fact, under the indicated condition system (47) will admit a periodic solution, defined by equations (49). Now, if we eliminate between these equations the constant c, we shall be able to deduce from them the expressions for the quantities x_s in the form of series ordered in positive integer powers of r, not containing any zero power, and possessing periodic coefficients, which will be finite series of sines and cosines of integer multiples of ϑ. By these series there will be defined functions of the variables r and ϑ, holomorphic with respect to r uniformly for all real values of ϑ, and these functions will satisfy the system of equations representing the transform of system (79) into variables r and ϑ. However, it is easy to convince ourselves that it is not possible to satisfy this system by series in the indicated form if these series do not reduce to ones ordered in positive integer powers of the quantities $r \cos \vartheta$, $r \sin \vartheta$, and having constant coefficients. Thus the series considered must necessarily reduce to these. And if this is so, we shall be able to prove, as in the case considered in the preceding section (Remark I), that they will define holomorphic functions of the quantities $r \cos \vartheta$ and $r \sin \vartheta$. But then, being expressed in terms of variables x and y, these series will represent holomorphic functions of the latter; for if we have to do with the case where the functions X and Y become xero for $x = y = 0$, the variables x and y are respectively equal to $r \cos \vartheta$ and $r \sin \vartheta$, and if we are concerned with the general case, we may pass from one set of variables to the other with the aid of the equations

$$x = r \cos \vartheta + u,$$

$$y = r \sin \vartheta + v,$$

where u and v are the holomorphic functions of the quantities x_s considered in Section 33. As these functions do not include terms of degree less than second, it results from these equations that, if all the x_s are holomorphic functions of the quantities $r \cos \vartheta$ and $r \sin \vartheta$, becoming zero when the latter are zero, the quantities $r \cos \vartheta$ and $r \sin \vartheta$, their moduli being sufficiently small, will be holomorphic functions of x and y, becoming zero for $x = y = 0$.

In this manner, we obtain under our hypothesis holomorphic functions x_s of the variables x and y, becoming zero for $x = y = 0$, and satisfying equation (79).

40. [*Some complements. Exposition of the method*]

In the case where system (45) does not admit a periodic solution of the kind considered, the question of stability is resolved, as we have seen [in Section 37], by the sign of a certain constant g.

For the calculation of this constant there have been proposed in what precedes, two methods, both of which reduce to operations which we would have to carry out in seeking the periodic solution (Section 34 and Section 36, Remark). Let us now show how we may attain the same end on making use of the calculations which arise in seeking a holomorphic integral independent of t.

Let us consider the following expression:

$$U = x^2 + y^2 + f(x_1, x_2, \dots, x_n, x, y),$$

where f represents an entire rational function of the variables x_s, x, y, not containing terms of degree less than third.

If we form in accordance with equations (45) the derivative dU/dt, on expanding it in powers of the quantities x, y, x_s, the series obtained will not have terms of degree less than third, and by a suitable choice of the function f, we shall be able to arrange that it does not contain terms up to a still higher degree.

It can happen that, however great the integer k, we shall be able to dispose of the function f in such a way that, in the expansion of dU/dt there do not appear terms of degree less than kth. In this case we shall obtain a series, ordered in positive integer powers of x, y, x_s, satisfying formally the condition of being an integral of system (45) [since U will satisfy formally $dU/dt = 0$], and, as we shall see later, this system will then actually admit a holomorphic integral independent of t.

But it can also happen (and this will be a general case) that, whatever the function f, we can only make disappear in the expansion of dU/dt all the terms below a definite degree.

Let us suppose that we have to do with this case, and that the function f is chosen so that, in the expansion of the derivative under consideration, there do not appear terms up to the highest degree possible.

Then the ensemble of terms of lowest degree in the expansion of dU/dt will necessarily represent a form of even degree, for if this form, which we shall designate by V, were of odd degree $2N+1$, we should be able to find, in accordance with Theorem I of Section 20, a form v of the same degree, satisfying the equation

$$\lambda\left(x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x}\right) + \sum_{s=1}^{n}(p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s x + \beta_s y)\frac{\partial v}{\partial x_s} = -V;$$

and on adding the latter to the function f, we would form a new function U for which, in the expansion of dU/dt, all the terms would disappear up to the degree $2N+1$ inclusively.

[Note that if V is of even degree $2N$ we can no longer use this last argument, since Theorem I of Section 20 is no longer applicable. The reason is that the determinantal equation now has $n+2$ roots $\beta_1, \beta_2, \dots, \beta_{n+2}$, two of which, say β_1 and β_2, are equal and opposite, being λi and $-\lambda i$. Consequently the disqualifying condition in the theorem, viz.

$$m_1\beta_1 + m_2\beta_2 + \dots + m_{n+2}\beta_{n+2} = 0$$

with

$$m_1 + m_2 + \dots + m_{n+2} = 2N,$$

can be satisfied by non-negative integers m_j on choosing

$$m_1 = m_2 = N, \quad m_3 = m_4 = \dots = m_{n+2} = 0.]$$

Let us agree then that the degree of the form V is equal to an even number $2N$.

We can then suppose the degree of the function f to be not greater than $2N-1$. But, if we wish to introduce into this function also terms of degree $2N$, we shall always be able to reduce the form V to the following:

$$G(x^2+y^2)^N, \tag{83}$$

where G is a constant, which will have a fully determinate value.

In fact, if v designates the ensemble of terms of degree $2N$ in the function f, and V_0 the ensemble of terms of the same degree in the expansion of the expression†

$$\frac{dU}{dt} - \frac{dv}{dt},$$

† [Lyapunov is economizing with symbols here. He has earlier added a first v to a first U to make a second U which satisfies $dU/dt = V_0 + \dots$. He is now adding a second v to the second U to make a third U which will satisfy $dU/dt = G(x^2+y^2)^N + \dots$.]

we should, to effect the said reduction, determine the form v and the constant G to conform with the equation

$$\lambda\left(x\frac{\partial v}{\partial y}-y\frac{\partial v}{\partial x}\right)+\sum_{s=1}^{n}(p_{s1}x_1+\ldots+p_{sn}x_n+\alpha_s x+\beta_s y)\frac{\partial v}{\partial x_s}$$
$$=G(x^2+y^2)^N-V_0. \quad (84)$$

And the latter furnishes for the calculation of the coefficients of form v a system of linear equations in number equal to the number of these coefficients, for which the determinant will be the $(2N-1)$th derived determinant of the fundamental determinant of system (45), under the hypothesis $\chi=0$ (see Section 19). This determinant will be, as a consequence, zero [note that the equation $D_{2N}(\chi)=0$ has one root equal to zero, in view of the theorem in Section 19]; but, among its first minors, there will appear at least one which will be different from zero. Therefore the coefficients of form v will always be able to be eliminated between these equations, and this will give only a single relation between the coefficients in the right-hand side of equation (84). It is this relation, necessary and sufficient for the form v to exist, which will supply, as we shall see straightaway, the sought value of the constant G.

To obtain the relation in question, we can start directly from equation (84). For this, let us replace the variables x_s in it by linear functions of the variables x and y, satisfying the system of equations

$$\lambda\left(x\frac{\partial x_s}{\partial y}-y\frac{\partial x_s}{\partial x}\right)=p_{s1}x_1+p_{s2}x_2+\ldots+p_{sn}x_n+\alpha_s x+\beta_s y$$
$$(s=1,2,\ldots,n)$$

(there will always exist such functions and they will be unique), next let us put $x=r\cos\vartheta$, $y=r\sin\vartheta$ and, after having multiplied both sides of the equation by $r^{-2N}\,d\vartheta$, let us integrate with respect to variable ϑ between limits from zero to π. For the left-hand side we shall evidently obtain zero. [With the previous substitutions v becomes a function of ϑ, say $v(\vartheta)$, and the left-hand side of (84) turns out to reduce to $\lambda(dv/d\vartheta)$. The integration of the latter gives $\lambda\{v(\pi)-v(0)\}$, and this is zero because $v(\vartheta)$ is a form of even degree in $\cos\vartheta$ and $\sin\vartheta$.] Hence, the equality obtained (which will be the sought relation) will give the following value for the constant G:

$$G=\frac{1}{\pi}\int_0^{\pi}r^{-2N}V_0\,d\vartheta.$$

We are thus assured that the function f will always be able to be chosen in such a way that the ensemble of terms of lowest degree in the expansion of dU/dt will be of form (83).

Let us show that the constant G will be related very simply to the constant g.

For this, if we are dealing with the general case of system (45), let us first make use of the transformation in Section 33, in order to get to the case where the functions X and Y become zero for $x=y=0$. On then seeking an entire function f satisfying the preceding condition, we shall evidently be led to the same values of N and G.

On considering such a function f, let us put $x=r\cos\vartheta$, $y=r\sin\vartheta$ in the expression for U, and with the aid of equations (47) let us form the derivative $dU/d\vartheta$.

We shall have

$$\frac{dU}{d\vartheta} = \lambda G r^{2N} + R, \tag{85}$$

where R represents a holomorphic function of the variables x_s, r for which the expansion in powers of x_s, r does not contain terms of degree less than $2N + 1$ and possesses coefficients periodic with respect to ϑ. [We actually have

$$\frac{dU}{d\vartheta} = \frac{dU}{dt} \times \frac{dt}{d\vartheta} = G r^{2N} \times \frac{1}{\lambda} + \ldots$$

with use of (46). Thus it appears that in Lyapunov's (85) and his next equations, λ should be replaced by $1/\lambda$.]

This settled, let us turn to series (49).

If we had wanted to extend this series to infinity, while retaining for the functions u the same form as in Section 34, we would not have been able to arrange for these series, if they were not periodic, to be convergent. But let us only retain this form up to order m, starting from which the functions u cease to be periodic, and let us introduce the condition that, for $\mu > m$, not only the functions $u^{(\mu)}$ but also the $u_s^{(\mu)}$ become zero for $\vartheta = 0$. Then, $|c|$ being small enough, the series in question will be convergent and will represent a solution of system (47), at least for values of ϑ not exceeding a certain limit. We shall moreover be able to take $|c|$ sufficiently small that this limit may be as great as we wish.

We shall suppose that we can make use of these series for all values of ϑ between 0 and 2π.

This agreed, let us substitute series (49) in equation (85). Next, on multiplying both sides by $d\vartheta$, let us integrate them from 0 to 2π and expand the results in increasing powers of c.

Only writing terms of least degree, we shall then evidently have

$$4\pi g c^{m+1} + \ldots = 2\pi\lambda G c^{2N} + \ldots.$$

[Integration of the left-hand side of (85) gives $U(2\pi) - U(0)$, where

$$U(\vartheta) = r^2 + f.$$

Substitution for r from (49) and use of (51) yields

$$U(\vartheta) = c^2 + \ldots + 2g c^{m+1}\vartheta + \ldots$$

in which the first set of dots represents periodic terms. Hence

$$U(2\pi) - U(0) = 4\pi g c^{m+1} + \ldots$$

which is the expression used above by Lyapunov.]

Hence we must conclude that

$$m = 2N - 1, \quad G = \frac{2g}{\lambda}.$$

We thus obtain the desired relation between the constants g and G. At the same time, we achieve a new proof of the proposition according to which the number m will always be odd (Section 34, Remark II).

From the analysis we have just presented it also results that, if we have to do with the case where the function f can be chosen so that in the expansion of dU/dt

all the terms disappear up to any desired degree, system (45) will admit a holomorphic integral independent of t; for it emerges from our analysis that in this case system (47) will certainly have a periodic solution (Section 38, Remark).†

In view of what we have proved, we can now enunciate the following proposition.

THEOREM. *The determinantal equation having two purely imaginary roots and n roots with negative real parts, let us reduce the differential equations of the disturbed motion to the form* (45). *Next, on designating by f an entire and rational function of the variables $x, y, x_1, x_2, ..., x_n$, not containing terms of degree below third, let us consider the expression*

$$2xX + 2yY + (-\lambda y + X)\frac{\partial f}{\partial x} + (\lambda x + Y)\frac{\partial f}{\partial y}$$
$$+ \sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n + \alpha_s x + \beta_s y + X_s)\frac{\partial f}{\partial x_s},$$

which will present itself in the form of a series, ordered in positive integer powers of the quantities x, y, x_s. We shall then find ourselves with one of two cases: either, by the choice of the function f, we shall be able to make disappear, in this expression, all the terms up to a degree as high as we wish, or we shall only be able to do this for terms where the sum of the exponents is less than a certain even number 2N.

In the first case, system (45) *will admit a holomorphic integral independent of t. It will moreover admit a periodic solution containing an arbitrary constant* (*apart from the one which we can add to t*) *and, on making this constant vary, we shall have a continuous series of periodic motions including the undisturbed motion we are concerned with. The latter motion will then be stable, and every disturbed motion sufficiently near the undisturbed motion will tend asymptotically to one of the periodic motions.*

In the second case, there will not exist an integral of the character indicated. But we shall be able to choose the function f in such a way that the ensemble of terms of lowest degree in the above expression reduces to

$$G(x^2 + y^2)^N.$$

Then, if the constant G is found to be positive, the undisturbed motion will be unstable. If on the other hand, it is negative, this motion will be stable, and every disturbed motion sufficiently near the undisturbed motion will approach it asymptotically.‡

Let us show finally that we can get to the evaluation of the constant g on making use of the series referred to at the end of the preceding section, and that it is not even necessary that these series should be convergent.

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† From what has been expounded there also results a theorem already mentioned in Section 38, and to which we shall again return in what follows (see Section 44), a theorem consisting in this, that if system (45) has a holomorphic integral independent of t, it will also have a periodic solution. In fact, we easily prove that if there exists a holomorphic integral independent of t, there will always be found one such in which the ensemble of terms of lowest degree will reduce to the form $x^2 + y^2$.

‡ Let us note that we could propose an entirely analogous theorem for the case examined further back, where the determinantal equation has one zero root.

To this effect, we note that the constant g only depends on a certain number of initial terms in the expansions of the right-hand sides of equations (45); such that, if k represents a sufficiently large integer, this constant will be able to be found by considering any system of equations of the form

$$\left.\begin{aligned} &\frac{dx}{dt} = -\lambda y + X, \quad \frac{dy}{dt} = \lambda x + Y, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s x + \beta_s y + X'_s \\ &\qquad\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{86}$$

where the X'_s are holomorphic functions, only differing from the functions X_s in the terms of degree higher than the kth. [Compare the discussion of (79) in Section 39.] Now we can always choose the latter terms in such a way that we can satisfy the system of partial differential equations

$$\begin{gathered} (-\lambda y + X)\frac{\partial x_s}{\partial x} + (\lambda x + Y)\frac{\partial x_s}{\partial y} \\ = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + \alpha_s x + \beta_s y + X'_s \\ (s = 1, 2, \dots, n) \end{gathered}$$

by entire and rational functions

$$x_1 = \varphi_1(x, y), \quad x_2 = \varphi_2(x, y), \quad \dots, \quad x_n = \varphi_n(x, y), \tag{87}$$

which become zero for $x = y = 0$ and do not contain terms of degree higher than the kth. Then, in view of what we have noted in the preceding section, the question will reduce to the examination of the equations

$$\frac{dx}{dt} = -\lambda y + (X), \quad \frac{dy}{dt} = \lambda x + (Y),$$

which are obtained by replacing in equations (86) the quantities x_s by the functions (87), and these functions will represent ensembles of terms of degree not greater than the kth in the series defined by equations (79).

In this manner, to determine the constant g, we shall be able to treat the equations to which the first two equations of system (45) reduce after we have replaced in them the quantities x_s by the said series, formed up to terms of a sufficiently high degree.

According to this we shall be able, for the question under consideration, to be guided by the following rule.

After having reduced the differential equations of the disturbed motion to form (45), *we consider the system of partial differential equations* (79), *defining the quantities* x_s *as functions of independent variables* x *and* y. *We next introduce new independent variables* r *and* ϑ, *on putting*†

$$\left.\begin{aligned} x &= r\cos\vartheta, \\ y &= r\sin\vartheta, \end{aligned}\right\} \tag{88}$$

† Of course these variables will be, in general, different from those which were represented by the same symbols in the preceding sections. [r and ϑ previously satisfied the last equations of Section 39.]

and we seek to satisfy this system by series, ordered in increasing positive integer powers of r, not containing zero powers, and having for coefficients periodic functions of ϑ *with common period* 2π (such series will always exist and will be unique).

At the same time, on turning to the equation

$$\frac{dr}{d\vartheta} = \frac{r(X\cos\vartheta + Y\sin\vartheta)}{\lambda r + Y\cos\vartheta - X\sin\vartheta},$$

which results because of (88) *from the first two equations of system* (45) [or from the pair of equations preceding (46)], *we represent its right-hand side in the form*

$$\frac{1}{\lambda}(X\cos\vartheta + Y\sin\vartheta)\left\{1 + \frac{X\sin\vartheta - Y\cos\vartheta}{\lambda r} + \left(\frac{X\sin\vartheta - Y\cos\vartheta}{\lambda r}\right)^2 + \dots\right\},$$

on replacing the functions X and Y by their expansions in increasing powers of r, x_s. *We next replace the* x_s *by their expressions as the series referred to above, and proceeding as if these series were absolutely convergent, we present the result in the form of the series*

$$R_2 r^2 + R_3 r^3 + R_4 r^4 + \dots,$$

ordered in increasing powers of r (all the coefficients R will be periodic functions of ϑ with common period 2π).

Finally, on designating by c an arbitrary constant, we form a sequence of functions

$$u_2, u_3, u_4, \dots, \tag{89}$$

independent of c, defined by the condition that, k being any positive integer, the expression

$$\frac{dr}{d\vartheta} - R_2 r^2 - R_3 r^3 - \dots - R_k r^k$$

when we put

$$r = c + u_2 c^2 + u_3 c^3 + \dots + u_k c^k$$

does not include c in powers less than the $(k+1)$*th. We form these functions, one after another, until we arrive at a non-periodic function, at which we stop. Let* u_m *be this function* (the number m always has to be odd for this). *It will always be of the form*

$$u_m = g\vartheta + v,$$

where g represents a constant and v a periodic function of ϑ. *Then, if* λ *is a positive number, the undisturbed motion will be stable or unstable according as g is a negative or positive number.*

Remark

It can happen that in series (89), however far we extend it, all the functions will be periodic. The preceding rule will then no longer lead to a conclusion. But, if it is proved in any manner that we are dealing with such a case, we shall be able to conclude that the undisturbed motion is stable. [The periodic case was treated in Section 38.]

[The theorem in the present section implies that, in the non-periodic case, we can find $U = r^2 + f$ such that

$$\frac{dU}{dt} = Gr^{2N} + \dots$$

i.e. such that

$$\frac{d}{dt}(r^2) = G(r^2)^N + \dots$$

This equation suggests that if G is negative r^2 will continually decrease, indicating stability; and if G is positive r^2 will increase, indicating instability. Thus the stability criterion in the theorem has immediate plausibility. Moreover, this plausible argument will apply whether λ is positive or negative, whereas the derivation given above for the theorem is subject to the assumption (made in Section 33) that $\lambda > 0$.]

41. [*Examples*]

Let us examine a few examples.

Example I

The differential equations of the disturbed motion reduce to an equation of the following form:

$$\frac{d^2x}{dt^2} + x = a\left(\frac{dx}{dt}\right)^{2n+1} + F\left[x, \left(\frac{dx}{dt}\right)^2\right],$$

where a is any constant, n is a positive integer, and F is a holomorphic function of its arguments, not containing terms of degree below the second with respect to the quantities x and dx/dt. It is required to examine the stability of the undisturbed motion ($x = 0$) with respect to these two quantities.

Let us suppose that on putting

$$x = r\sin\vartheta, \quad \frac{dx}{dt} = r\cos\vartheta,$$

we extract from our equation the following:

$$\frac{dr}{d\vartheta} = R_2r^2 + R_3r^3 + \dots, \tag{90}$$

where all the R represent functions of ϑ only.

Then, of the functions

$$R_2, R_3, \dots, R_{2n}, R_{2n+1} - a\cos^{2n+2}\vartheta,$$

evidently none will depend on the constant a (under the hypothesis that F does not depend on it). [The system can be written as

$$\frac{dx}{dt} = y + X, \quad \frac{dy}{dt} = -x + Y$$

where

$$X = 0, \quad Y = ay^{2n+1} + F[x, y^2].$$

Using the expansion in the preceding rule we thus have

$$\frac{dr}{d\vartheta}=\frac{rY\sin\vartheta}{r+Y\cos\vartheta}$$

$$=Y\sin\vartheta\left\{1-\frac{Y}{r}\cos\vartheta+\ldots\right\}$$

$$=(ar^{2n+1}\cos^{2n+1}\vartheta+F)\sin\vartheta\{1-\ldots\}.$$

Since a enters only in the form of a product ar^{2n+1}, the coefficient R_i of r^i in the expansion of the right-hand side will be independent of a for $i<2n+1$. The coefficient of r^{2n+1} will, however, include the term $a\cos^{2n+1}\vartheta\sin\vartheta$. This means that in Lyapunov's above expression $R_{2n+1}-a\cos^{2n+2}\vartheta$, the term $\cos^{2n+2}\vartheta$ should be replaced by $\cos^{2n+1}\vartheta\sin\vartheta$. The same replacement should be made in the integrands appearing in (91) and the subsequent equation.]

Consequently, if for equation (90) we seek a solution in the form of a series

$$r=c+u_2c^2+u_3c^3+\ldots,$$

ordered in increasing powers of arbitrary constant c, all the functions

$$u_2, u_3, \ldots, u_{2n}, u_{2n+1}-a\int_0^{\vartheta}\cos^{2n+2}\vartheta\,d\vartheta \tag{91}$$

will be able to be supposed independent of a. [The u_j are found successively by integration; see the first equations of (53) in Section 35.]

Now, if a were zero the proposed equation would admit a holomorphic integral independent of t (see Section 39).

We can thus assert that the functions (91) will all be periodic, and that by consequence, if a is not zero the constant g will be given by a formula

$$g=\frac{2a}{\pi}\int_0^{\pi/2}\cos^{2n+2}\vartheta\,d\vartheta.$$

[The corrected integrand $\cos^{2n+1}\vartheta\sin\vartheta$ is positive between the limits of integration, and hence

$$\operatorname{sgn} g=\operatorname{sgn} a.]$$

We conclude from this that for $a>0$ the undisturbed motion is unstable, and that for $a\leqslant 0$ it is stable.

Example II

Let the proposed differential equations of the disturbed motion be the following:

$$\frac{dx}{dt}+y=nxz,\quad \frac{dy}{dt}-x=nyz,\quad \frac{dz}{dt}+z=x^2+y^2-2xyz,$$

where n is a constant.

On putting $x = r \cos \vartheta$, $y = r \sin \vartheta$ and on taking ϑ for the independent variable, we deduce from them these equations [which can be obtained by using the equations preceding (46) to evaluate dr/dt and $d\vartheta/dt$, and substituting the results in

$$\frac{dr}{d\vartheta} = \frac{dr}{dt}\bigg/\frac{d\vartheta}{dt} \quad \text{and} \quad \frac{dz}{d\vartheta} = \frac{dz}{dt}\bigg/\frac{d\vartheta}{dt}\Bigg]$$

$$\frac{dr}{d\vartheta} = \frac{nrz \cos 2\vartheta}{1 - nz \sin 2\vartheta} = nrz \cos 2\vartheta + n^2 r z^2 \cos 2\vartheta \sin 2\vartheta + \ldots,$$

$$\frac{dz}{d\vartheta} = \frac{-z + r^2 - r^2 z \sin 2\vartheta}{1 - nz \sin 2\vartheta}$$

$$= -z + r^2 - nz^2 \sin 2\vartheta + (n-1) r^2 z \sin 2\vartheta - n^2 z^3 \sin^2 2\vartheta + \ldots,$$

where, in the expansions, there are written all the terms of degree not higher than the third.

Let us next operate as in Section 34.

Since the above equations do not change when we replace r by $-r$, we do not have to introduce in the series of type (49) which correspond to them, for r the even powers of the constant c, and for z the odd powers.

Let us put, then,

$$r = c + u_3 c^3 + u_5 c^5 + \ldots,$$

$$z = v_2 c^2 + v_4 c^4 + \ldots,$$

where all the u and v are functions of ϑ independent of c.

To calculate them, we shall have the following equations [obtained by substituting these expressions in the preceding differential equations and equating coefficients of powers of c]

$$\frac{dv_2}{d\vartheta} + v_2 = 1, \quad \frac{du_3}{d\vartheta} = nv_2 \cos 2\vartheta,$$

$$\frac{dv_4}{d\vartheta} + v_4 = 2u_3 + (n-1)v_2 \sin 2\vartheta - nv_2^2 \sin 2\vartheta,$$

$$\frac{du_5}{d\vartheta} = n(v_4 + v_2 u_3) \cos 2\vartheta + n^2 v_2^2 \cos 2\vartheta \sin 2\vartheta,$$

....

of which the first three will be satisfied on making

$$v_2 = 1, \quad u_3 = \frac{n}{2} \sin 2\vartheta, \quad v_4 = \frac{n-1}{5}(\sin 2\vartheta - 2\cos 2\vartheta).$$

The fourth equation [which becomes

$$\frac{du_5}{d\vartheta} = \frac{3}{4} n^2 \sin 4\vartheta + \frac{n(n-1)}{5}\{\tfrac{1}{2} \sin 4\vartheta - \cos 4\vartheta - 1\}\Bigg]$$

will then give u_5, and this function, apart from periodic terms, will further contain the following:

$$-\frac{n(n-1)}{5}\vartheta.$$

Thus, if $n(n-1)$ is not zero we shall have

$$g = -\frac{n(n-1)}{5}.$$

As for the case where $n(n-1) = 0$, the proposed differential equations will admit a periodic solution.

In fact, for $n = 0$ this is obvious; and for $n = 1$ we deduce it on noting that the right-hand sides of our equations, which we shall designate by X, Y, Z respectively, then satisfy the relation

$$(-y + X)\frac{\partial Z}{\partial x} + (x + Y)\frac{\partial Z}{\partial y} = 0.$$

so that we then find ourselves with a case indicated in Section 39 [see (81)].

To sum up, we thus arrive at the conclusion that for $n(n-1) \geqslant 0$ the undisturbed motion is stable, and for $n(n-1) < 0$ unstable.

Example III

Suppose given the equations

$$\frac{dx}{dt} + y = \alpha yz, \quad \frac{dy}{dt} - x = \beta xz, \quad \frac{dz}{dt} + kz = \gamma xy,$$

where k designates a positive constant, and α, β, γ are any real constants.

Operating as has been indicated in Section 36 (Remark), let us put

$$t = t_0 + (1 + h_2 c^2 + \ldots)\tau.$$

Next, on noting that the proposed equations do not change when we replace x by $-x$ and y by $-y$, let us seek to satisfy them by putting

$$x = c\cos\tau + x_3 c^3 + x_5 c^5 + \ldots,$$

$$y = c\sin\tau + y_3 c^3 + y_5 c^5 + \ldots,$$

$$z = z_2 c^2 + z_4 c^4 + \ldots,$$

on understanding by x_s, y_s, z_s functions of τ independent of c.

Of these functions, z_2, x_3 and y_3 will be given by the following equations:

$$\frac{dz_2}{d\tau} + kz_2 = \frac{\gamma}{2}\sin 2\tau,$$

$$\frac{dx_3}{d\tau} + y_3 = -h_2\sin\tau + \alpha z_2\sin\tau,$$

$$\frac{dy_3}{d\tau} - x_3 = h_2\cos\tau + \beta z_2\cos\tau.$$

[In these equations h_2 should be replaced by $-h_2$.]

For the first, we find the periodic solution

$$z_2 = \frac{\gamma}{2(k^2+4)}(k \sin 2\tau + 2\cos 2\tau),$$

which we substitute in the right-hand sides of the other two equations. Next, on designating by P and Q the latter sides of the second and third equations, let us form the expression

$$\frac{1}{2\pi}\int_0^{2\pi} (P\cos\tau + Q\sin\tau)\,d\tau = \frac{(\alpha+\beta)\gamma k}{8(k^2+4)}.$$

[Lyapunov is here evaluating $g = \frac{1}{2}(A_1 + B_2)$ where A_1 is the coefficient of $\cos\tau$ in the Fourier expansion of P and B_2 is the coefficient of $\sin\tau$ in that of Q—see the expression for g in the remark at the end of Section 36.]

This expression, if it is not zero, will represent the constant g.

Consequently, on noting that of the two cases $\gamma = 0$ and $\alpha + \beta = 0$ where it does become zero, in the first the proposed system of equations admits the periodic solution

$$x = c\cos(t - t_0), \quad y = c\sin(t - t_0), \quad z = 0,$$

and in the second it admits the integral

$$x^2 + y^2,$$

[again giving a periodic solution], we conclude that for $(\alpha+\beta)\gamma > 0$ the undisturbed motion is unstable, and for $(\alpha+\beta)\gamma \leqslant 0$ stable.

Example IV

Let there be given the equations

$$\frac{dx}{dt} = -\lambda y + (x+y)z, \quad \frac{dy}{dt} = \lambda x + (y-x)z,$$

$$\frac{dz}{dt} = -z + x + y - 2(6x - 3y + z)z,$$

where λ represents any non-zero real constant. [In what follows this system will be treated by the technique described in the first part of Section 40.]

On designating by f a form of third degree in the variables x, y, z, let us determine it by the condition that the derivative with respect to t of the function

$$U = x^2 + y^2 + f,$$

formed in accordance with our differential equations, does not contain terms of third degree.

The equation which has to be verified by form f, namely

$$\lambda\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right) + (z - x - y)\frac{\partial f}{\partial z} = 2(x^2+y^2)z$$

will give

$$f = 2(x^2+y^2)\left(\frac{x-y}{\lambda} + z\right).$$

[This expression can be found by writing f as a third degree form with undetermined coefficients, substituting this form in the above partial differential equation, and choosing the coefficients to make the resulting equation an identity.]

At the same time it will turn out that

$$\frac{dU}{dt} = \frac{4(1-3\lambda)}{\lambda}(x^2+y^2)(2x-y)z. \tag{92}$$

Let us replace z here by the linear solution

$$z = \frac{(1-\lambda)x + (1+\lambda)y}{1+\lambda^2}$$

of the equation [corresponding to the equation after (84)]

$$\lambda\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right) = -z + x + y;$$

next let us put $x = r\cos\vartheta$, $y = r\sin\vartheta$ and, on designating by Θ the function of the variable ϑ to which the result of this substitution reduces after having divided it by r^4, let us consider the expression [compare the equation before (85)]

$$\frac{1}{\pi}\int_0^{\pi}\Theta\, d\vartheta = \frac{2(1-3\lambda)^2}{\lambda(1+\lambda^2)}.$$

If this expression is not zero, it will represent the constant G (Section 40). Now it can only become zero in the case where $\lambda = 1/3$. And then, as we see from (92), the function U will be an integral of the proposed equations.

Therefore the expression obtained allows us to conclude that for λ negative and for $\lambda = 1/3$ the undisturbed motion is stable, and for positive λ different from $1/3$ it is unstable.

[The conclusion regarding negative λ is established only plausibly, in view of the comment inserted at the end of Section 40. To prove it, let us put

$$u = y, \quad v = x, \quad \mu = -\lambda$$

Then the system becomes

$$\frac{du}{dt} = -\mu v + (u-v)z, \quad \frac{dv}{dt} = \mu u + (u+v)z,$$

$$\frac{dz}{dt} = -z + u + v - 2(6v - 3u + z)z.$$

For $\mu > 0$ this system may be investigated by the technique used above for the original system, and we then find that we have stability. Hence the original system indeed yields stability for $\lambda < 0$.]

Example V

The differential equations of the disturbed motion are given in the form

$$\frac{d^2x}{dt^2} + x = az^n, \quad \frac{dz}{dt} + kz = x,$$

where n is an integer greater than 1, k is a positive constant and a is any real constant. It is required to investigate the stability of the undisturbed motion ($x = z = 0$) with respect to the quantities x, dx/dt and z.

[The system can be considered as a feedback loop containing a second-order resonant element followed by a first-order lag followed by a non-linear element az^n.]

Let us proceed according to the rule expounded at the end of the preceding section.

Putting $dx/dt = x'$, let us consider the partial differential equation [corresponding to the equation after (86)]

$$x'\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x'} + kz = x - az^n\frac{\partial z}{\partial x'},$$

which, by means of the substitution $x' = r\cos\vartheta$, $x = r\sin\vartheta$, [and with use of the consequent relations

$$\frac{\partial z}{\partial x} = \sin\vartheta\frac{\partial z}{\partial r} + \frac{\cos\vartheta}{r}\frac{\partial z}{\partial\vartheta}$$

$$\frac{\partial z}{\partial x'} = \cos\vartheta\frac{\partial z}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial z}{\partial\vartheta}\Bigg]$$

reduces to the form

$$\frac{\partial z}{\partial\vartheta} + kz = r\sin\vartheta - az^n\left(\cos\vartheta\frac{\partial z}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial z}{\partial\vartheta}\right).$$

We shall be able to satisfy this equation, at least formally, by substituting for z the series

$$\Theta_0 r + \Theta_1 r^n + \Theta_2 r^{2n-1} + \ldots, \tag{93}$$

ordered in powers of r increasing in steps of $n-1$, with coefficients $\Theta_0, \Theta_1, \ldots$, periodic with respect to ϑ, which may be calculated successively with the aid of differential equations which are easy to form [by equating coefficients of powers of r]. Thus the coefficients Θ_0 and Θ_1 will be obtained from the equations

$$\frac{d\Theta_0}{d\vartheta} + k\Theta_0 = \sin\vartheta,$$

$$\frac{d\Theta_1}{d\vartheta} + k\Theta_1 = -a\Theta_0^n\left(\cos\vartheta\,\Theta_0 - \sin\vartheta\frac{d\Theta_0}{d\vartheta}\right),$$

and will consequently be

$$\Theta_0 = \frac{k\sin\vartheta - \cos\vartheta}{k^2+1}, \quad \Theta_1 = \frac{ae^{-k\vartheta}}{k^2+1}\int_{-\infty}^{\vartheta} e^{k\vartheta}\Theta_0^n\,d\vartheta.$$

Let us now turn to the first of the proposed differential equations. Making use of the substitution employed above [and of the equation after (88)], we deduce from it the following:

$$\frac{dr}{d\vartheta} = \frac{az^n\cos\vartheta}{1 - a\dfrac{z^n}{r}\sin\vartheta}.$$

Let us represent the right-hand side of this equation in the form of the series

$$az^n \cos \vartheta + a^2 \frac{z^{2n}}{r} \sin \vartheta \cos\vartheta + \ldots,$$

and, on replacing z in it by series (93), let us order the result in increasing powers of r.

We shall thus obtain the series

$$R_n r^n + R_{2n-1} r^{2n-1} + \ldots,$$

where the exponents of r increase by $n - 1$, and where the coefficients R are certain periodic functions of ϑ:

$$R_n = a\Theta_0^n \cos \vartheta, \quad R_{2n-1} = a^2\Theta_0^{2n} \sin \theta \cos \vartheta + na\Theta_0^{n-1}\Theta_1 \cos \vartheta, \ldots$$

Considering now the expression

$$\frac{dr}{d\vartheta} - R_n r^n - R_{2n-1} r^{2n-1}$$

and on designating by c an arbitrary constant, and by u_n, u_{2n-1} functions of ϑ independent of c, let us put in it

$$r = c + u_n c^n + u_{2n-1} c^{2n-1}$$

and in the result let us equate to zero the coefficients of the nth and $(2n - 1)$th powers of c.

We obtain from this approach the equations

$$\frac{du_n}{d\vartheta} = R_n, \quad \frac{du_{2n-1}}{d\vartheta} = nu_n R_n + R_{2n-1}, \tag{94}$$

of which the first gives

$$u_n = a \int \Theta_0^n \cos \vartheta \, d\vartheta + \text{const.}$$

Now if we introduce the angle ε defined by the equations

$$\sin \varepsilon = \frac{k}{(k^2 + 1)^{1/2}}, \quad \cos \varepsilon = \frac{1}{(k^2 + 1)^{1/2}},$$

we shall have

$$\Theta_0 = -\frac{\cos (\vartheta + \varepsilon)}{(k^2 + 1)^{1/2}}.$$

[Note also that the further term $\cos \vartheta$ in the above integrand can be written as

$$\cos \vartheta = \cos (\vartheta + \varepsilon) \cos \varepsilon + \sin (\vartheta + \varepsilon) \sin \varepsilon.]$$

Thus the expression for the function u_n will reduce to

$$u_n = \frac{(-1)^n a}{(k^2 + 1)^{(n+1)/2}} \left\{ \int \cos^{n+1} (\vartheta + \varepsilon) \, d\vartheta - \frac{k}{n+1} \cos^{n+1} (\vartheta + \varepsilon) \right\} + \text{const.}$$

We see from this that if n is an odd number the function u_n will contain a secular term [because the integrand $\cos^{n+1} (\vartheta + \varepsilon)$ will then have a non-zero mean value, M say, and the integral of this will give a secular term $M\vartheta$], and the constant g will be given by the formula [found by evaluating the mean M over the interval from $-\varepsilon$ to $\frac{1}{2}\pi - \varepsilon$]

$$g = -\frac{2a}{\pi(k^2+1)^{(n+1)/2}} \int_0^{\pi/2} \cos^{n+1} \vartheta \, d\vartheta.$$

This constant will thus be of sign opposite to that of a.

If on the other hand the number n is even, the function u_n will be periodic. Then, as we see from (94), the constant g will be defined by the equation

$$g = \frac{1}{2\pi} \int_0^{2\pi} R_{2n-1} \, d\vartheta,$$

at least if the integral which enters here is not zero. [Note that, with the use of the first equation of (94), the term nu_nR_n in the second equation of (94) can be expressed as

$$\frac{n}{2} \frac{d}{d\vartheta}(u_n^2)$$

and so has zero mean value; hence it contributes nothing to the secular term.]

Let us show that, a not being zero, this integral will never become zero and that moreover it will be negative.

For this, let us suitably reduce its expression

$$\int_0^{2\pi} R_{2n-1} \, d\vartheta = a^2 \int_0^{2\pi} \Theta_0^{2n} \sin \vartheta \cos \vartheta \, d\vartheta + na \int_0^{2\pi} \Theta_0^{n-1} \Theta_1 \cos \vartheta \, d\vartheta.$$

Making use of the angle ε and noting that in the case of even n

$$\Theta_1 = \frac{ae^{-k(\vartheta+\varepsilon)}}{(k^2+1)^{(n+2)/2}} \int_{-\infty}^{\vartheta+\varepsilon} e^{k\varphi} \cos^n \varphi \, d\varphi,$$

we obtain

$$\int_0^{2\pi} \Theta_0^{n-1} \Theta_1 \cos \vartheta \, d\vartheta = \frac{-a}{(k^2+1)^{n+1}} \int_{-\infty}^{\vartheta} e^{-k\vartheta} \cos^n \vartheta \, d\vartheta \int_{-\infty}^{\vartheta} e^{k\varphi} \cos^n \varphi \, d\varphi$$
$$- \frac{ka}{(k^2+1)^{n+1}} \int_0^{2\pi} e^{-k\vartheta} \cos^{n-1} \vartheta \sin \vartheta \, d\vartheta \int_{-\infty}^{\vartheta} e^{k\varphi} \cos^n \varphi \, d\varphi.$$

[In Lyapunov's double integrals, the second integral is to be interpreted as included in the integrand of the first integral.] Now on putting for brevity

$$\int_0^{2\pi} e^{-k\vartheta} \cos^n \vartheta \, d\vartheta \int_{-\infty}^{\vartheta} e^{k\varphi} \cos^n \varphi \, d\varphi = J,$$

we find, on integrating by parts,

$$n \int_0^{2\pi} e^{-k\vartheta} \cos^{n-1} \vartheta \sin \vartheta \, d\vartheta \int_{-\infty}^{\vartheta} e^{k\varphi} \cos^n \varphi \, d\varphi = \int_0^{2\pi} \cos^{2n} \vartheta \, d\vartheta - kJ.$$

[The integration by parts is here applied to the product uv where

$$u = e^{-k\vartheta}, \quad v = \cos^n \vartheta \int_{-\infty}^{\vartheta} e^{k\varphi} \cos^n \varphi \, d\varphi.]$$

By consequence, on noting that

$$\int_0^{2\pi} \Theta_0^{2n} \sin \vartheta \cos \vartheta \, d\vartheta = \frac{k}{(k^2+1)^{n+1}} \left\{ \int_0^{2\pi} \cos^{2n} \vartheta \, d\vartheta - 2 \int_0^{2\pi} \cos^{2n+2} \vartheta \, d\vartheta \right\},$$

[which may be obtained with the help of the identities

$$\sin \vartheta = \sin (\vartheta + \varepsilon) \cos \varepsilon - \cos (\vartheta + \varepsilon) \sin \varepsilon$$

$$\cos \vartheta = \cos (\vartheta + \varepsilon) \cos \varepsilon + \sin (\vartheta + \varepsilon) \sin \varepsilon]$$

we ultimately arrive at the equation

$$\int_0^{2\pi} R_{2n-1} \, d\vartheta = \frac{4\pi k a^2}{(k^2+1)^{n+1}} \left\{ \frac{(k^2-n)}{4\pi k} J - \frac{1 \cdot 3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \cdot 6 \ldots (2n+2)} \right\}.$$

From this equation follows the validity of what we have said above; for the quantity in the curly brackets on the right-hand side is less than

$$\frac{kJ}{4\pi} - \frac{1 \cdot 3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \cdot 6 \ldots (2n+2)}$$

(since J is positive); and as kJ is obviously less than the integral

$$\int_0^{2\pi} \cos^n \vartheta \, d\vartheta = \frac{1 \cdot 3 \cdot 5 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots n} 2\pi,$$

[as is seen on replacing the second integrand $e^{k\varphi} \cos^n \varphi$ in J by the upper bound $e^{k\varphi}$] this quantity is less than the following:

$$\frac{1 \cdot 3 \cdot 5 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots n} \left\{ \frac{1}{2} - \frac{(n+1)(n+3) \ldots (2n+1)}{(n+2)(n+4) \ldots (2n+2)} \right\},$$

which is certainly negative. [In fact, on changing the order of numerator factors we can write

$$\frac{(n+1)(n+3) \ldots (2n+1)}{(n+2)(n+4) \ldots (2n+2)} = \frac{(n+3)(n+5) \ldots (2n+1)(n+1)}{(n+2)(n+4) \ldots (2n)(2n+2)}$$

$$= \frac{n+3}{n+2} \cdot \frac{n+5}{n+4} \cdot \ldots \cdot \frac{2n+1}{2n} \cdot \frac{1}{2} > \frac{1}{2}.\Bigg]$$

Thus, for n even, the constant g will always be negative.

Consequently we arrive at the conclusion that in the case of odd n the undisturbed motion is stable for positive a and unstable for negative a, and that in the case of even n it is always stable.

Example VI

It is required to investigate the stability in the case of the differential equations

$$\frac{d^2x}{dt^2} + x = az^n, \quad \frac{dz}{dt} + kz = \frac{dx}{dt}.$$

the symbols having the same significance as before. [This system corresponds to the same feedback loop as in the previous example, but with a differentiator inserted as an additional element.]

Operating in entirely the same way as in the preceding example, we shall have: in the case of odd n,

$$g = \frac{2ka}{\pi(k^2+1)^{(n+1)/2}} \int_0^{\pi/2} \sin^{n+1} \vartheta \, d\vartheta,$$

and in the case of even n,

$$g = \frac{2ka^2}{(k^2+1)^{n+1}} \left\{ \frac{1 \cdot 3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \cdot 6 \ldots (2n+2)} - \frac{n+1}{2} \cdot \frac{kJ}{2\pi} \right\}, \tag{95}$$

J representing the same thing as above.

In the first case, the constant g will consequently have the sign of the constant a [recall that k was assumed positive when it was introduced in Example V]. In the second case, its sign will not depend on that of a. Let us show that it will then be negative.

For this purpose let us first of all show that kJ is an increasing function of k.

This may easily be proved with the aid of the reduction formula which relates the values of J for two values of n differing from one another by two. If J as a function of n is designated by J_n, this formula, which is easily obtained by integrating by parts, will be the following:

$$kJ_n = \frac{n^2(n-1)^2}{(k^2+n^2)^2} kJ_{n-2} + \frac{k^2}{k^2+n^2} \left\{ \int_0^{2\pi} \cos^{2n} \vartheta \, d\vartheta + \frac{n(n-1)}{k^2+n^2} \int_0^{2\pi} \cos^{2n-2} \vartheta \, d\vartheta \right\}. \tag{96}$$

[This relation can be obtained by manipulations involving four integrations by parts. If uv represents the integrand in a general integration by parts, for the first and second of the four integrations we can take $u = e^{k\varphi}$, and for the third and fourth, $u = e^{-k\vartheta}$.]

Since it reduces easily to the form [obtained with the help of the relations

$$\int_0^{2\pi} \cos^{2n} \vartheta \, d\vartheta = \frac{2n-1}{2n} \int_0^{2\pi} \cos^{2n-2} \vartheta \, d\vartheta$$

$$\int_0^{2\pi} \cos^{2n-2} \vartheta \, d\vartheta = \frac{2n-3}{2n-2} \int_0^{2\pi} \cos^{2n-4} \vartheta \, d\vartheta \Bigg]$$

$$kJ_n = \int_0^{2\pi} \cos^{2n} \vartheta \, d\vartheta - \frac{n}{2(k^2+n^2)} \int_0^{2\pi} \cos^{2n-2} \vartheta \, d\vartheta$$

$$- \frac{n^2(n-1)^2}{(k^2+n^2)^2} \left\{ \left[\frac{n-2}{2(n-1)^2} + 1 \right] \int_0^{2\pi} \cos^{2n-4} \vartheta \, d\vartheta - kJ_{n-2} \right\}, \tag{97}$$

we conclude from it that if, for $m = n - 2$, we have

$$kJ_m < \int_0^{2\pi} \cos^{2m} \vartheta \, d\vartheta,$$

the same inequality will also hold for $m = n$.

Now we immediately see from (97) that this inequality is valid in the case of $m = 2$. [It is not quite clear that (97) can be applied with $n = 2$. As an alternative, direct evaluation of J_2 shows that

$$kJ_2 = \frac{3k^2+8}{k^2+4}\frac{\pi}{4} < \frac{3\pi}{4} = \int_0^{2\pi} \cos^4 d\vartheta.\Big]$$

Therefore it will be the same for every even value of m greater than 2.

Taking into account the above inequality, we conclude from formula (97) that, if kJ_{n-2} is an increasing function of k, it will be the same for kJ_n. Consequently, on noting that the function

$$kJ_2 = \pi\left(\frac{3}{4} - \frac{1}{k^2+4}\right)$$

increases when k increases, we can conclude that kJ_n enjoys this property for all even values of n.

Having proved that kJ_n, or following our original notation kJ, is an increasing function of k, we shall have a lower limit for it on setting $k = 0$. This limit may easily be obtained from formula (96) [which then simplifies to

$$kJ_n = \frac{(n-1)^2}{n^2} kJ_{n-2}\Big]$$

and will be the following:

$$\left(\frac{1\cdot3\cdot5\ldots(n-1)}{2\cdot4\cdot6\ldots n}\right)^2 2\pi.$$

Returning now to formula (95), let us replace in it kJ by the lower limit which we have just obtained. Then the expression in curly brackets will reduce to the following:

$$\frac{1\cdot3\cdot5\ldots(n-1)}{2\cdot4\cdot6\ldots n}\left\{\frac{(n+1)(n+3)\ldots(2n+1)}{(n+2)(n+4)\ldots(2n+2)} - \frac{n+1}{2}\frac{1\cdot3\cdot5\ldots(n-1)}{2\cdot4\cdot6\ldots n}\right\}.$$

And the latter, which reduces to

$$\frac{1}{2}\frac{1\cdot3\cdot5\ldots(n-1)}{2\cdot4\cdot6\ldots n}\left\{\frac{(n+3)(n+5)\ldots(2n+1)}{(n+2)(n+4)\ldots 2n} - \frac{3\cdot5\cdot7\ldots(n+1)}{2\cdot4\cdot6\ldots n}\right\},$$

is evidently negative. [To see this, consider the two terms in the curly brackets. The first has a factor which is

$$\frac{n+3}{n+2} = \frac{3}{2}\frac{1+(n/3)}{1+(n/2)} < \frac{3}{2}$$

i.e. this factor is less than the corresponding factor in the second term. A similar argument applies to all such factors.]

Then the constant g defined by formula (95) will certainly be negative.

To sum up, we thus arrive at the conclusion that in the case of odd n the undisturbed motion will be stable for negative a and unstable for positive a; as for the case of even n, it will always be stable.

Periodic solutions of the differential equations of disturbed motion

42. [*Demonstration of the convergence of certain periodic series satisfying formally the differential equations*]

We have seen that whenever it is possible to find certain periodic series formally satisfying system (45), the latter actually admits a periodic solution represented by such series. [See Sections 35 and 36.]

The proof of this proposition was based on the assumption that all the roots of the determinantal equation of system (45), with the exception of two, had negative real parts. Now there is nothing essential in this assumption, and the said proposition extends easily to an arbitrary system of differential equations, provided that the determinantal equation which corresponds to it has at least one pair of purely imaginary roots.

Let us show how this can be done, on limiting ourselves however to the case where among the purely imaginary roots there is a pair of *simple* conjugate roots

$$\lambda\sqrt{-1}, \quad -\lambda\sqrt{-1}, \tag{98}$$

the integer multiples of which are not roots, and where the determinantal equation does not have zero roots.

Agreeing that the proposed system of differential equations satisfies these assumptions, and on making use of a suitably chosen linear substitution with constant coefficients, let us reduce it to form (45). Moreover, the roots (98) being simple, we can suppose this substitution to be such that all the α_s, β_s are zero, and we shall do this to simplify our analysis. [For this purpose the substitutions discussed in Section 18 could be used.]

In this way the transformed system will be of the form

$$\left.\begin{aligned} &\frac{dx}{dt} = -\lambda y + X, \quad \frac{dy}{dt} = \lambda x + Y, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s \\ &\qquad (s = 1, 2, \ldots, n) \end{aligned}\right\} \tag{99}$$

in which the functions designated by X, Y, X_s have the same character as before. But the coefficients $p_{s\sigma}$ will now be of a more general character, for we suppose here only that the equation

$$\begin{vmatrix} p_{11} - \chi \cdots p_{1n} \\ \ldots\ldots\ldots\ldots \\ p_{n1} \cdots p_{nn} - \chi \end{vmatrix} = 0$$

does not have roots of the form $m\lambda\sqrt{-1}$, m being a real integer.

Let us put in equations (99)

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \quad x_1 = rz_1, \quad x_2 = rz_2, \quad \ldots, \quad x_n = rz_n$$

and let us extract from them the equations defining $r, z_1, z_2, \ldots, z_n$ as functions of the variable ϑ.

On putting

$$q_{s\sigma} = \frac{p_{s\sigma}}{\lambda} \quad (s, \sigma = 1, 2, \ldots, n)$$

and on designating by $\varphi_1, \varphi_2, \ldots, \varphi_n$ certain quadratic forms in the quantities $\sin\vartheta$ and $\cos\vartheta$, we shall be able to present these equations in the form

$$\frac{dr}{d\vartheta} = R, \quad \frac{dz_s}{d\vartheta} = q_{s1}z_1 + q_{s2}z_2 + \ldots + q_{sn}z_n + \varphi_s r + Z_s$$

$$(s = 1, 2, \ldots, n), \tag{100}$$

where R, Z_s are functions of variables $z_1, z_2, \ldots, z_n, r, \vartheta$, easily deduced from the functions X, Y, X_s. Being expanded in powers of the quantities r, z_s, the functions R, Z_s will not contain terms of degree below the second and will possess coefficients periodic with respect to ϑ. Moreover $|r|, |z_s|$ being below certain fixed numbers, their expansions will be uniformly convergent for all real values of ϑ.

This settled, let us treat equations (100) as we treated equations (47) in Sections 34 and 35.

In seeking for system (100) a solution in the form of the series

$$\left.\begin{aligned} r &= c + u^{(2)}c^2 + u^{(3)}c^3 + \ldots, \\ z_s &= u_s^{(1)}c + u_s^{(2)}c^2 + u_s^{(3)}c^3 + \ldots \\ &(s = 1, 2, \ldots, n), \end{aligned}\right\} \tag{101}$$

ordered in increasing powers of the arbitrary constant c, with coefficients u periodic with respect to ϑ, we shall obtain to determine these coefficients systems of differential equations of the same character as in the case considered in the sections mentioned [see (50)]; and when our problem is solvable, these systems will give the sought coefficients in the form of finite sequences of sines and cosines of integer multiples of ϑ, in the same order of succession as in the case we have just indicated. For each value of l, the determination of the coefficients $u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}$, after having found all those which precede them, will, as before, present no difficulty [compare (53)]; for the determinantal equation

$$\begin{vmatrix} q_{11} - \chi \cdots q_{1n} \\ \cdots\cdots\cdots\cdots \\ q_{n1} \cdots q_{nn} - \chi \end{vmatrix} = 0 \tag{102}$$

of the linear differential equations on which these coefficients depend will have, under the agreed assumptions, neither zero roots nor roots representing integer multiples of $\sqrt{-1}$. The possibility of solution of our problem will then only depend, as before, on the condition that each of the functions $u^{(l)}$, which will be given by quadratures, should be periodic.

Let us assume that this condition is actually satisfied and, on supposing as before that the calculations are carried out so that all the $u^{(l)}$ become zero for $\vartheta = 0$, let us investigate the convergence of the series (101).

For this purpose, let us assume that these series are transformed with the aid of a linear substitution similar to that considered for series (49) in Section 35.

The question will thus reduce to the examination of series (101) formed under the hypothesis that all the coefficients other than the following

$$q_{11} = \chi_1, \quad q_{22} = \chi_2, \quad \ldots, \quad q_{nn} = \chi_n,$$

$$q_{21} = \sigma_1, \quad q_{32} = \sigma_2, \quad \ldots, \quad q_{n,n-1} = \sigma_{n-1},$$

are zero.

In considering these series, let us suppose that all the $u^{(i)}$, $u_s^{(i)}$ for $i < l$ are already known. Then to determine the coefficients $u^{(l)}$, $u_s^{(l)}$ we shall have the equations

$$\frac{du^{(l)}}{d\vartheta} = U^{(l)}, \quad \frac{du_1^{(l)}}{d\vartheta} = \chi_1 u_1^{(l)} + \varphi_1 u^{(l)} + U_1^{(l)},$$

$$\frac{du_j^{(l)}}{d\vartheta} = \chi_j u_j^{(l)} + \sigma_{j-1} u_{j-1}^{(l)} + \varphi_j u^{(l)} + U_j^{(l)}$$

$$(j = 2, 3, \ldots, n),$$

where the known terms $U^{(l)}$, $U_s^{(l)}$ will be the functions of the same character as in the analogous system of equations [those preceding (53)] considered in Section 35.

The first of these equations will give

$$u^{(l)} = \int_0^\vartheta U^{(l)}\, d\vartheta.$$

But, to obtain the periodic solutions of the other equations, it will no longer be permissible to make use of formulae such as (53), for these formulae are only valid in the case where the real parts of all the χ_s are negative.

To obtain formulae which are appropriate for the case under consideration, let us make the general observation that the periodic solution of the equation

$$\frac{du}{d\vartheta} = \chi u + f(\vartheta),$$

where χ is a non-zero constant and $f(\vartheta)$ is a periodic function of ϑ having period ω, is obtained by the formula

$$u = \frac{e^{\chi\vartheta}}{e^{-\chi\omega} - 1} \int_\vartheta^{\vartheta+\omega} e^{-\chi\vartheta} f(\vartheta)\, d\vartheta,$$

provided that $\chi\omega$ does not represent an integer multiple of $2\pi\sqrt{-1}$.

For $\omega = 2\pi$ and $\chi = \chi_s$ this condition is fulfilled, for the quantities χ_s are the roots of equation (102).

Therefore [applying this formula to the differential equations in hand], for the calculation of the coefficients $u_s^{(l)}$ we shall be able to utilize the formulae

$$u_1^{(l)} = \frac{e^{\chi_1\vartheta}}{e^{-2\pi\chi_1} - 1} \int_\vartheta^{\vartheta+2\pi} e^{-\chi_1\vartheta}(\varphi_1 u^{(l)} + U_1^{(l)})\, d\vartheta,$$

$$u_j^{(l)} = \frac{e^{\chi_j\vartheta}}{e^{-2\pi\chi_j} - 1} \int_\vartheta^{\vartheta+2\pi} e^{-\chi_j\vartheta}(\sigma_{j-1} u_{j-1}^{(l)} + \varphi_j u^{(l)} + U_j^{(l)})\, d\vartheta$$

$$(j = 2, 3, \ldots, n).$$

This settled, let

$$\chi_s = \lambda_s + \mu_s\sqrt{-1},$$

$$\rho_s = \frac{\lambda_s}{e^{2\pi\lambda_s} - 1}(1 - 2e^{2\pi\lambda_s}\cos 2\pi\mu_s + e^{4\pi\lambda_s})^{1/2},$$

λ_s, μ_s being real numbers, and the root in the expression for ρ_s being taken as positive. In the case of $\lambda_s = 0$ we shall understand by ρ_s the limit

$$\frac{|\sin \pi\mu_s|}{\pi},$$

towards which its expression tends when λ_s tends to zero.

In the conditions in which we find ourselves, all the ρ_s will be different from zero.

Let us next designate by $v^{(i)}$, $v_s^{(i)}$ upper bounds for the moduli of functions $u^{(i)}$, $u_s^{(i)}$ and by a_s those for the moduli of the functions φ_s, in the domain of real values of ϑ.

Then, if we understand by $V_s^{(l)}$ the constants obtained by treating the functions $U_s^{(l)}$ as was indicated in Section 35, we shall be able to take, in conformity with our formulae,

$$v^{(l)} = 2\pi V^{(l)},$$

$$v_1^{(l)} = \frac{e^{\lambda_1 \vartheta}}{|e^{-2\pi\chi_1} - 1|} \int_\vartheta^{\vartheta + 2\pi} e^{-\lambda_1 \vartheta}(a_1 v^{(l)} + V_1^{(l)})\, d\vartheta,$$

$$v_j^{(l)} = \frac{e^{\lambda_j \vartheta}}{|e^{-2\pi\chi_j} - 1|} \int_\vartheta^{\vartheta + 2\pi} e^{-\lambda_j \vartheta}(|\sigma_{j-1}| v_{j-1}^{(l)} + a_j v^{(l)} + V_j^{(l)})\, d\vartheta \quad (j = 2, 3, \ldots, n).$$

In this way [after exact evaluation of the preceding integrals] we shall obtain the equations

$$v^{(l)} = 2\pi V^{(l)}, \quad \rho_1 v_1^{(l)} = a_1 v^{(l)} + V_1^{(l)},$$

$$\rho_j v_j^{(l)} = |\sigma_{j-1}| v_{j-1}^{(l)} + a_j v^{(l)} + V_j^{(l)} \quad (j = 2, 3, \ldots, n),$$

in which $V^{(l)}$, $V_s^{(l)}$ will only depend on $v^{(i)}$, $v_s^{(i)}$ for $i < l$. We shall be able to make use of these equations for every value of l greater than 1. At the same time we shall be able to take

$$\rho_1 v_1^{(1)} = a_1,$$

$$\rho_j v_j^{(1)} = |\sigma_{j-1}| v_{j-1}^{(1)} + a_j \quad (j = 2, 3, \ldots, n),$$

and then the constants v will be completely defined.

The subsequent line of reasoning will be the same as in Section 35.

In this manner we may demonstrate that, under the agreed hypotheses, $|c|$ being sufficiently small, the series (101) will be absolutely convergent, and that the series of moduli of their terms will converge uniformly for all real values of ϑ.

The preceding analysis, with insignificant modifications, extends easily to complex values of ϑ for which the absolute value of the coefficient of $\sqrt{-1}$ does not exceed a given limit.

In view of this, we shall be able to make use of the reasoning of Section 36 to extract from the periodic solution of system (100), defined by the series we have just considered, a periodic solution for system (99).

The latter will be represented by series of the form (61) and will include two arbitrary constants: c, which we have considered above, and t_0, which will enter together with t in the combination $t - t_0$. The first of these constants will also enter into the expression [compare the expression for T in the equations preceding (58)]

$$T = \frac{2\pi}{\lambda}(1 + h_2 c^2 + h_3 c^3 + \ldots)$$

for the period (corresponding to the variable t), which will be a holomorphic function of c.

For the rest we shall be able to form directly the series expressing this periodic solution by making use of the method expounded in the Remark in Section 36.

Let us pause to consider some circumstances arising in the derivation of this solution.

43. [*Definition of the periodic solutions by the initial values of the unknown functions*]
Let us consider the system of partial differential equations

$$(-\lambda y + X)\frac{\partial x_s}{\partial x} + (\lambda x + Y)\frac{\partial x_s}{\partial y} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + X_s \quad (s = 1, 2, \dots, n), \tag{103}$$

similar to that which we dealt with in Section 39 [see (79)].

We easily assure ourselves [a] that under the present assumptions concerning the $p_{s\sigma}$, we shall always be able, as in the case considered in the section mentioned, to satisfy formally this system by series ordered in positive integer powers of the quantities x and y, without constant terms,† and [b] that such series will be unique.

Now if we consider the hypothesis that there exists for system (99) a periodic solution of the form indicated above, we may demonstrate, just as in [the last part of] Section 39, that these series, $|x|$ and $|y|$ being sufficiently small, will be absolutely convergent, that the holomorphic functions of the variables x and y defined by these series will actually satisfy system (103), and that on equating the x_s to these functions we shall obtain the result of elimination of the constants c and t_0 between the equations expressing the periodic solution.

As a consequence this solution is characterized by the fact that the quantities x_s are well-determined holomorphic functions of the variables x and y.

As far as these variables [x and y] are concerned, they will be functions of t which we shall obtain on seeking the general integral of the equations

$$\frac{dx}{dt} = -\lambda y + (X), \quad \frac{dy}{dt} = \lambda x + (Y), \tag{104}$$

to which the first two equations of system (99) will reduce when the x_s are replaced by the said holomorphic functions.

In view of this, if we introduce the initial values of the functions $x, y, x_1, x_2, \dots, x_n$, designating them respectively by $a, b, a_1, a_2, \dots, a_n$, we may characterize our periodic solution by the condition that all the a_s are holomorphic functions of the quantities a and b, becoming zero for $a = b = 0$, and satisfying the system of partial differential equations obtained on replacing in that of (103) quantities $x, y, x_1, x_2, \dots, x_n$ by quantities $a, b, a_1, a_2, \dots, a_n$.

Let us see to what form the series representing this solution will reduce, if instead of the constants c and t_0 we introduce the constants a and b.

Let us show straightaway that the period T will be a holomorphic function of a and b. For this, we shall make use of the proposition already known to us [refer

† In the case under consideration there will evidently no longer be terms of first degree in these series. [α_s and β_s were taken as zero in Section 42, so the differential equation for x_s in (99) does not have a forcing term component which is linear in x and y.]

to the sentence containing (76)], by virtue of which, under the conditions considered here, system (104) will admit a holomorphic integral independent of t.

As the latter will always be able to be chosen so that the ensemble of terms of lowest degree reduces to $x^2 + y^2$, we shall have, on considering such an integral and on designating by C an arbitrary constant, an equation of the form

$$x^2 + y^2 + F(x, y) = C^2,$$

where F is a holomorphic function of x and y, not containing terms of degree less than the third.

On putting in this equation $x = r \cos \vartheta$, $y = r \sin \vartheta$, we obtain from it the following:

$$r^2 + F(r \cos \vartheta, r \sin \vartheta) = C^2. \tag{105}$$

And the latter, if we consider r as an unknown and ϑ as a given quantity, will have only two solutions satisfying the condition that, for a sufficiently small choice of $|C|$, we can make the modulus of r as small as we wish, and these solutions, holomorphic with respect to C, will be represented by the following common formula:

$$r = \pm C + u^{(2)}C^2 \pm u^{(3)}C^3 \ldots$$

Here the coefficients u will be periodic functions of ϑ such that, ϑ being replaced by $\vartheta + \pi$, all the $u^{(m)}$ corresponding to even m will take again their original values with opposite signs, and none of the $u^{(m)}$ corresponding to odd m will change at all. We convince ourselves of this on noting that equation (105) does not change on replacing r by $-r$ and ϑ by $\vartheta + \pi$.

Consequently, on replacing in each of the two solutions considered C by $-C$ and ϑ by $\vartheta + \pi$, the original value with opposite sign will be obtained for r.

It results from this that if in making use of one of the two values of r we express as a function of ϑ the right-hand side of the equation

$$\frac{dt}{d\vartheta} = \frac{r}{\lambda r + (Y) \cos \vartheta - (X) \sin \vartheta}$$

the result will not change by the substitution indicated. [This equation is similar to that preceding (46). Note that $x(= r \cos \vartheta)$ and $y(= r \sin \vartheta)$ are not changed by the substitution; hence neither are (X) and (Y).]

Thus the integral

$$\int_0^{2\pi} \frac{r\, d\vartheta}{\lambda r + (Y) \cos \vartheta - (X) \sin \vartheta},$$

representing the period T, will not change on replacing C by $-C$, and by consequence the period T, which will be a holomorphic function of C, will be given by the series [compare the equation preceding (103)]

$$T = \frac{2\pi}{\lambda}(1 + h_2 C^2 + h_4 C^4 + \ldots),$$

containing only even powers of C.

Now the square of C, by virtue of the very meaning of this constant [see the equation preceding (105)], represents a holomorphic function of a and b. It will thus be the same for the period T.

This agreed, we note that $T(a, b)$ being the notation for the period T considered as a function of a and b, $T(x, y)$ will necessarily be an integral of system (104). According to this, if on putting

$$dt = T(x, y)\frac{d\tau}{2\pi}$$

we take τ for the independent variable, the functions x and y satisfying system (104) will be periodic with respect to τ and will have period 2π, whatever their initial values, provided that their moduli are small enough.

Moreover these functions will satisfy the equations

$$\frac{dx}{d\tau} = \{-\lambda y + (X)\}\frac{T(x, y)}{2\pi}, \quad \frac{dy}{d\tau} = \{\lambda x + (Y)\}\frac{T(x, y)}{2\pi}, \tag{106}$$

of which the right-hand sides are holomorphic with respect to x and y.

Therefore, if the moduli of their initial values a and b are sufficiently small, these functions will be represented by series ordered in positive integer powers of a and b for all values of τ between limits arbitrarily chosen in advance. And as the series in question, for all values of a and b with sufficiently small moduli, must give periodic functions of τ with period independent of a and b, their coefficients must themselves be necessarily periodic. Now, for this, by the nature of equations (106) these coefficients must be finite series of sines and cosines of integer multiples of τ.

In this way we have succeeded in establishing, quite independently of the original expressions of the functions x and y in terms of the constants c and t_0, that if we express our periodic solution by means of the constants a and b, it will be represented by series ordered in positive integer powers of a and b, where the coefficients will be sums of a limited number of periodic terms representing products of constants by sines and cosines of integer multiples of

$$\tau = \frac{2\pi t}{T(a, b)},$$

and that these series will define, if we consider τ as a parameter independent of a and b, functions of a and b uniformly holomorphic for all real values of τ.

Remark

If we do not limit ourselves to the consideration of real periodic solutions, and if we consider complex values of a and b as valid, we shall be able to extract from our periodic solution two others, each depending on a single constant and remarkable in that they will have for the period a fixed number, which will be equal to $2\pi/\lambda$. We may obtain these solutions by establishing between the constants a and b the relation

$$C^2 = a^2 + b^2 + F(a, b) = 0,$$

which allows each of these constants to be expressed as a holomorphic function of the other, and this in two different ways.

Moreover these solutions can be defined independently of ours. Further, even the existence of the latter is not necessary for them to exist: it suffices for this that equation (99) satisfy the hypotheses enunciated at the beginning of the preceding section.

In speaking in what follows of periodic solutions we shall understand that we have to do with solutions of the type considered above, with two arbitrary constants.

44. [*Case of existence of a holomorphic integral*]

Let us take up again system (99) under the same assumptions concerning the coefficients $p_{s\sigma}$ as before.

Let us assume that we have found for this system a holomorphic integral independent of t and that the ensemble of terms of second degree in it depend on x and y.

We easily assure ourselves that this ensemble will be able to contain x and y only in the form of the combination $x^2 + y^2$.

We must thus admit that our integral, being multiplied by a constant, reduces to the following form:

$$x^2 + y^2 + F(x_1, x_2, ..., x_n, x, y),$$

where F is a holomorphic function of the variables x_s, x, y for which the expansion does not contain terms of degree below the second and, in the terms of second degree, if any, neither x nor y are included.†

We are going to show that under these conditions system (99) will always admit a periodic solution.

For this purpose let us introduce into our integral, instead of the variables x, y, x_s, the variables r, ϑ, z_s, with the aid of the substitution which we have already made use of before. [See equations located between (99) and (100).]

Then, on extracting the square root of our integral, we shall be able to deduce from it the following:

$$r + r\varphi(z_1, z_2, ..., z_n, r, \vartheta), \tag{107}$$

where φ represents a holomorphic function of the quantities z_s, r, becoming zero when these are simultaneously zero, and possessing in its expansion coefficients which are periodic with respect to ϑ.

This agreed, let us provisionally assume that there does not exist a periodic solution, and that, in series (101), this is made apparent for the first time by the terms of mth degree. In other words, let us assume that the coefficients

$$u^{(2)}, u^{(3)}, ..., u^{(m-1)},$$

$$u_s^{(1)}, u_s^{(2)}, ..., u_s^{(m-1)} \quad (s = 1, 2, ..., n)$$

represent periodic functions, while the coefficient $u^{(m)}$ is of the form

$$u^{(m)} = g\vartheta + v,$$

where g is a non-zero constant and v is a periodic function of ϑ.

With this assumption, let us put in expression (107)

$$r = c + u^{(2)}c^2 + ... + u^{(m-1)}c^{m-1} + u^{(m)}c^m,$$

$$z_s = u_s^{(1)}c + u_s^{(2)}c^2 + ... + u_s^{(m-1)}c^{m-1} \quad (s = 1, 2, ..., n),$$

and let us order the result in increasing powers of c.

† Under our assumptions, this integral will not be able to contain terms of first degree.

As this expression is an integral of system (100), the very manner in which the functions u are defined ensures that the terms containing c in powers less than the $(m+1)$th must reduce to constants.

Now this is obviously not possible for the term in c^m, because for the function $r\varphi$ this term will be necessarily periodic [since the coefficient of c^m in $r\varphi$ depends on the $u^{(j)}$ and $u_s^{(j)}$ only for values of j satisfying $j<m$] and will not be able, by consequence, to give a constant sum when added to the term $(g\vartheta+v)c^m$ in the function r.

We must thus conclude that our assumption is impossible, and that consequently, however far we extend series (101), we shall be able to determine their terms so that they are periodic. And with this condition the existence of a periodic solution, as we have seen [in Section 42], is assured.

Remark

We have assumed that the determinantal equation does not have zero roots. But, in the case of such roots, no further difficulty would arise if our system of differential equations would admit a sufficient number of holomorphic integrals independent of t, where there would be terms of first degree of which the ensembles would be independent of one another.

We have in view the case where the number of these integrals attains its upper limit, which is always equal to the multiplicity m of the zero root.

In fact, if we find ourselves with this case we shall have, on equating the said integrals to arbitrary constants $c_1, c_2, \ldots, c_m$, m equations of integrals, which we shall be able to use to lower the order of our system of differential equations by m units. Then, if these calculations have been carried out in a suitable manner, we shall finally obtain a system of differential equations for which the determinantal equation will not have zero roots, as long as the $|c|$ are sufficiently small.

45. [*Periodic solutions of canonical equations*]

The preceding conclusions can find application in many problems of mechanics.

Let us mention for example the question of the motion of a heavy solid body having a fixed point, or supported by its surface touching a smooth horizontal plane. In either case there will exist certain periodic motions in which the components of angular velocity relative to axes fixed in the body, as well as the cosines of the angles which these axes make with the vertical, vary periodically during the course of time.

To indicate applications with a more general character, let us suppose that our system of differential equations has the canonical [Hamiltonian] form

$$\frac{dx_s}{dt}=-\frac{\partial H}{\partial y_s},\qquad \frac{dy_s}{dt}=\frac{\partial H}{\partial x_s}\quad (s=1,2,\ldots,k).$$

We assume here that H is a holomorphic function of the quantities $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ in which the terms of lowest degree reduce to a quadratic form H_2.

Whenever this system satisfies the assumptions made at the beginning of Section 42, it will admit a periodic solution with two arbitrary constants. [See Section 42.]

In fact, this system always admits a holomorphic integral independent of t, for the function H is an integral. Therefore, to establish what we have just said, it suffices (in view of what was remarked in the previous section) to show that the function H_2, transformed with the aid of the linear substitution which reduces our system to the form (99), will contain variables which will play the role of x and y.

Now, under the conditions considered, the determinantal equation not having zero roots, this is already clearly seen by the fact that the Hessian of the function H_2 will not be zero.

If the determinantal equation has only purely imaginary roots

$$\pm\lambda_1\sqrt{-1},\ \pm\lambda_2\sqrt{-1},\ \ldots,\ \pm\lambda_k\sqrt{-1},$$

the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ being such that among their mutual ratios there are no integers, we shall have for our canonical system k periodic solutions, each containing two arbitrary constants.

Taking up this case, let us assume that the ensemble of terms of second degree in the function H is of the form

$$\frac{\lambda_1}{2}(x_1^2+y_1^2)+\frac{\lambda_2}{2}(x_2^2+y_2^2)+\ldots+\frac{\lambda_k}{2}(x_k^2+y_k^2).$$

We know (Section 21) that if we understand by each λ_s a number with an appropriate sign, this ensemble will always be able to be reduced to this [form] by a certain linear transformation of our canonical system.

Under these assumptions, on considering the quantities

$$x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k,$$

$$y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k$$

as functions of the variables x_j and y_j, let us form the following system of partial differential equations [which corresponds to system (103) for the Hamiltonian equations in hand, with x and y taken as x_j and y_j]

$$\frac{\partial H}{\partial x_j}\frac{\partial x_s}{\partial y_j}-\frac{\partial H}{\partial y_j}\frac{\partial x_s}{\partial x_j}=-\frac{\partial H}{\partial y_s},\qquad \frac{\partial H}{\partial x_j}\frac{\partial y_s}{\partial y_j}-\frac{\partial H}{\partial y_j}\frac{\partial y_s}{\partial x_j}=\frac{\partial H}{\partial x_s}$$

$$(s=1,2,\ldots,j-1,j+1,\ldots,k)$$

and let us seek from them a solution where all the x_s, y_s are holomorphic with respect to x_j and y_j and become zero for $x_j=y_j=0$. Such a solution, as we know [from Section 43], will always exist and will be unique.

In considering it, let us form the expression†

$$1+\sum_s\left(\frac{\partial x_s}{\partial x_j}\frac{\partial y_s}{\partial y_j}-\frac{\partial x_s}{\partial y_j}\frac{\partial y_s}{\partial x_j}\right)=[x_j, y_j]$$

(assuming that in the summation we exclude the value $s=j$) and let us designate by $H_j(x_j, y_j)$ the function which H will reduce to when the quantities x_s, y_s are replaced in it by their expressions as found.

† [This is a Lagrange bracket-expression—see Whittaker, *loc. cit.*, p. 298.]

Next let us integrate the equations

$$[x_j, y_j]\frac{dx_j}{dt} = -\frac{\partial H_j}{\partial y_j}, \quad [x_j, y_j]\frac{dy_j}{dt} = \frac{\partial H_j}{\partial x_j}$$

on taking for arbitrary constants the initial values a_j, b_j of the functions x_j and y_j. [These equations, which correspond to (104), follow on evaluating $\partial H_j/\partial y_j$ and $\partial H_j/\partial x_j$ and then substituting for $\partial H/\partial y_s$ and $\partial H/\partial x_s$ from the original Hamiltonian equations.]

Then, if we express all the x, y as functions of t, we shall have one of the periodic solutions of our canonical system, and indeed whatever a_j and b_j may be, provided that their absolute values are sufficiently small.

In this solution, which we can name as that corresponding to the number λ_j, the period T_j relative to the variable t will be defined for sufficiently small values of $|a_j|$ and $|b_j|$ by a formula of the form

$$T_j = \frac{2\pi}{\lambda_j}\{1 + h_j^{(1)}H_j(a_j, b_j) + h_j^{(2)}[H_j(a_j, b_j)]^2 + \dots\},$$

in which the h designate numbers independent of the arbitrary constants. [Compare the expression for T given in Section 43, and note that C^2 is now replaced by the integral H_j.]

Operating as has just been indicated for all the values of j, we shall obtain all the k periodic solutions.

Each of these solutions will be able to be defined by certain conditions relative to the initial values of the unknown functions, and these conditions may be obtained immediately, in view of what precedes.

In this way, in the case of a canonical system satisfying the assumptions which we have just considered, if we cannot completely resolve the question of stability, we can at least indicate for the perturbations a series of conditions under which the undisturbed motion will certainly be stable.

[Lyapunov intends us to solve the above system of partial differential equations for x_s and y_s in terms of x_j and y_j, by substituting a power series for x_s and one for y_s, with undetermined coefficients, then determining the coefficients to make the equations satisfied identically.

The resulting series will be convergent for small enough $|x_j|$ and $|y_j|$, and will define a region of possible values for x_s and y_s ($s = 1, 2, \dots, k$). If the initial perturbations a_s, b_s are in this region, the ensuing motion will be a small periodic oscillation, indicating conditional stability. The period T_j of this oscillation will be close to $2\pi/\lambda_j$.

A similar region of conditional stability will be found corresponding to each of the k values λ_j.

However, no conclusions are drawn regarding unconditional stability, i.e. stability when the perturbations are not restricted to these regions.]

Remark

When for the proposed canonical system the numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ satisfy the condition we have just considered, to obtain all the periodic solutions which correspond to them in conformity with the proposed rule, we must initially effect a certain linear transformation of our system. Namely, we must take as new unknown functions u_s, v_s linear forms in the old ones x_s, y_s, such that the ensemble of terms of second degree in the function H reduces to the form

$$\frac{\lambda_1}{2}(u_1^2+v_1^2)+\frac{\lambda_2}{2}(u_2^2+v_2^2)+\ldots+\frac{\lambda_k}{2}(u_k^2+v_k^2),$$

and such that the differential equations retain the canonical form.

Let us now show how we can avoid this transformation.

For this let us note that in the periodic solution corresponding to the number λ_j, the u_s, v_s for which s is not equal to j are holomorphic functions of u_j and v_j, in which there do not appear terms of degree less than the second [compare the footnote in Section 43]. Thus, if we consider any two linear forms

$$p=\alpha_1 u_1+\beta_1 v_1+\alpha_2 u_2+\beta_2 v_2+\ldots+\alpha_k u_k+\beta_k v_k,$$

$$q=\gamma_1 u_1+\delta_1 v_1+\gamma_2 u_2+\delta_2 v_2+\ldots+\gamma_k u_k+\delta_k v_k$$

in the quantities u_s, v_s, they will reduce for this same periodic solution to holomorphic functions of u_j and v_j in which the ensembles of terms of first degree will be

$$\alpha_j u_j+\beta_j v_j, \qquad \gamma_j u_j+\delta_j v_j.$$

It results from this that if

$$\alpha_j\delta_j-\beta_j\gamma_j \tag{108}$$

is not zero, all the unknown functions in this solution will be holomorphic with respect to p and q.

Let us assume that the condition we have just indicated is fulfilled for all values of j from 1 to k inclusively. Then all the x_s, y_s will be holomorphic functions of p and q for each of the k periodic solutions.

The question will thus reduce to the search for these holomorphic functions and to the integration of the equations

$$\frac{dp}{dt}=P, \qquad \frac{dq}{dt}=Q, \tag{109}$$

which we shall then obtain for the determination of p and q.

In seeking the holomorphic functions in question we must first of all satisfy a certain system of nonlinear algebraic equations, on which will depend the coefficients in their terms of first degree. This system will admit more than k solutions. But, in order to tackle the choice of those of them which correspond to our problem, it will suffice to take account of the condition that, for each of the *periodic* solutions sought, the quantities P and Q, as functions of p and q, must be such that the expression

$$\frac{\partial P}{\partial p}+\frac{\partial Q}{\partial q}$$

becomes zero for $p=q=0$. This condition, which expresses that the sum of the roots of the determinantal equation of system (109) is zero, will only be satisfied for k solutions. On taking up any one of them and then passing on to the determination of the coefficients in the terms of higher degree, for this determination we shall have systems of linear equations, of which the resolution will not offer any more difficulty.

In this way we shall obtain k systems of holomorphic functions, each of which will lead to one of the periodic solutions sought.

It remains to indicate a rule enabling us to choose the linear forms p and q as functions of the variables x_s, y_s without recourse to the formation of the forms u_s, v_s.

For this purpose let us consider at the same time as p and q a further $2k-2$ linear forms

$$p_1, p_2, \ldots, p_{k-1}, q_1, q_2, \ldots, q_{k-1},$$

representing the ensembles of terms of first degree in the expressions for the derivatives

$$\frac{d^2p}{dt^2}, \quad \frac{d^4p}{dt^4}, \quad \ldots, \quad \frac{d^{2k-2}p}{dt^{2k-2}}, \quad \frac{d^2q}{dt^2}, \quad \frac{d^4q}{dt^4}, \quad \ldots, \quad \frac{d^{2k-2}q}{dt^{2k-2}},$$

formed with the aid of our differential equations.

Expressing these forms as functions of the variables u_s, v_s we shall have, by the property of the latter [see the equations at the end of Section 21],

$$(-1)^m p_m = \lambda_1^{2m}(\alpha_1 u_1 + \beta_1 v_1) + \lambda_2^{2m}(\alpha_2 u_2 + \beta_2 v_2) + \ldots + \lambda_k^{2m}(\alpha_k u_k + \beta_k v_k),$$

$$(-1)^m q_m = \lambda_1^{2m}(\gamma_1 u_1 + \delta_1 v_1) + \lambda_2^{2m}(\gamma_2 u_2 + \delta_2 v_2) + \ldots + \lambda_k^{2m}(\gamma_k u_k + \delta_k v_k).$$

We conclude from this, on taking into account that the numbers $\lambda_1^2, \lambda_2^2, \ldots, \lambda_k^2$ in accordance with our assumptions are all distinct, that, if any of quantities (108) is not zero, the forms

$$\left.\begin{matrix} p, p_1, p_2, \ldots, p_{k-1}, \\ q, q_1, q_2, \ldots, q_{k-1} \end{matrix}\right\} \tag{110}$$

will be mutually independent, and conversely, if this last condition is satisfied, some one among the quantities (108) will assuredly not be zero.

Consequently the only condition we have to satisfy in choosing the forms p and q reduces to this, that the $2k$ forms (110) must represent functions of the variables x_s, y_s which are mutually independent.

Example

Let there be proposed the fourth-order system

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = \frac{\partial U}{\partial x}, \qquad \frac{d^2y}{dt^2} + 2\frac{dx}{dt} = \frac{\partial U}{\partial y},$$

in which U represents a given holomorphic function of the variables x and y, not containing terms of degree below the second.

We can, if we wish, transform this system into the canonical one

$$\frac{dx}{dt} = -\frac{\partial H}{\partial \xi}, \qquad \frac{d\xi}{dt} = \frac{\partial H}{\partial x},$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial \eta}, \qquad \frac{\partial \eta}{dt} = \frac{\partial H}{\partial y},$$

on putting

$$\xi = \frac{dx}{dt} - y, \qquad \eta = \frac{dy}{dt} + x,$$

$$H = U - \tfrac{1}{2}(x^2 + y^2) + x\eta - y\xi - \tfrac{1}{2}(\xi^2 + \eta^2).$$

Let us assume that the determinantal equation which corresponds to it has only purely imaginary roots

$$\pm\lambda_1\sqrt{-1}, \qquad \pm\lambda_2\sqrt{-1},$$

which are such that neither of the two ratios

$$\frac{\lambda_1}{\lambda_2}, \quad \frac{\lambda_2}{\lambda_1}$$

represents a whole number.

The proposed system of equations will then admit two periodic solutions, and to determine them according to the method which we have just indicated, we shall be able to take for the forms p and q the variables x and y themselves.

In fact, if the ensemble of terms of second degree in the function U is represented by the expression

$$\tfrac{1}{2}(Ax^2 + 2Bxy + Cy^2),$$

we shall have, on putting $p = x$, $q = y$,

$$p_1 = Ax + By + 2\frac{dy}{dt} = (A-2)x + By + 2\eta,$$

$$q_1 = Bx + Cy - 2\frac{dx}{dt} = Bx + (C-2)y - 2\xi,$$

and the forms p, q, p_1, q_1 will as a consequence be independent, whatever the constants A, B, C. [p, q, p_1, q_1 are linear combinations of x, y, η, ξ, and the determinant formed from their coefficients turns out to evaluate as -4.]

The periodic solutions under consideration will then be obtained on integrating the differential equations

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = \varphi(x, y),$$

of which the right-hand sides are holomorphic functions of the variables x and y, defined by the system of equations [which interpret the two equations at the beginning of this example]

$$f\frac{\partial f}{\partial x} + \varphi\frac{\partial f}{\partial y} - 2\varphi = \frac{\partial U}{\partial x}, \qquad f\frac{\partial \varphi}{\partial x} + \varphi\frac{\partial \varphi}{\partial y} + 2f = \frac{\partial U}{\partial y}$$

with the following two conditions: that these functions become zero for $x = y = 0$, and that in the terms of first degree in their expansions

$$f = ax + by + \cdots, \qquad \varphi = \alpha x + \beta y + \cdots$$

we have for the coefficients a and β the relation $a + \beta = 0$. [This relation corresponds to the requirement that the expression displayed after (109) must equal zero.]

Now, if we proceed to the calculation of the coefficients a, b, α, β [on substituting from the last equations in those which precede them and then evaluating coefficients of linear terms] we shall obtain the following system of equations:

$$a^2 + b\alpha - 2\alpha = A, \qquad ab + b\beta - 2\beta = B,$$

$$b\alpha + \beta^2 + 2b = C, \qquad a\alpha + \alpha\beta + 2a = B,$$

which will have two solutions satisfying the condition $a + \beta = 0$, and we shall get these solutions as the formulae

$$a = \frac{B}{2}, \qquad b = \frac{\lambda^2 + C}{2},$$
$$\alpha = -\frac{\lambda^2 + A}{2}, \quad \beta = -\frac{B}{2},$$

on replacing λ^2 successively by each of the roots of the equation

$$z^2 - (4 - A - C)z + AC - B^2 = 0,$$

which are λ_1^2 and λ_2^2.

To each of these solutions will correspond a pair of functions f and φ and, by consequence, a periodic solution of the proposed system of differential equations.

If we consider the variables x and y as the coordinates of a point moving in a plane, we shall thus have two periodic motions. The trajectory for each of them will be defined by the equation†

$$2U - f^2 - \varphi^2 = \text{const.}$$

[This equation may be verified by differentiating the left-hand side with respect to t, and showing that the result is zero by substituting from the above partial differential equations satisfied by f and φ.]

† The question of periodic solutions of nonlinear differential equations is also considered, although from another point of view, in the latest memoir of Mr Poincaré: 'Sur le problème des trois corps et les équations de la dynamique', *Acta Mathematica*, Vol. XIII.

CHAPTER III. Study of periodic motion

Linear differential equations with periodic coefficients

46. [*Characteristic equation. Types of solution corresponding to its simple and multiple roots. Sets of solutions*]

Let us consider the system of linear differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n \quad (s = 1, 2, \ldots, n) \tag{1}$$

under the assumption that all the coefficients $p_{s\sigma}$ are periodic functions of t with the same real period ω, and that these functions remain determinate and continuous for all real values of t.

Only considering such values of t, let us suppose that we have found for our system n independent solutions

$$\left.\begin{array}{l} x_{11}, x_{21}, \ldots, x_{n1} \\ x_{12}, x_{22}, \ldots, x_{n2} \\ \cdots\cdots\cdots\cdots \\ x_{1n}, x_{2n}, \ldots, x_{nn} \end{array}\right\} \tag{2}$$

the first subscript of x referring, as always, to the unknown function, and the second to the solution.

When we wish to indicate the value attributed to the independent variable we shall write $x_{sj}(t)$ instead of x_{sj}.

This agreed, the set of functions

$$x_{1j}(t + \omega), \quad x_{2j}(t + \omega), \quad \ldots, \quad x_{nj}(t + \omega)$$

corresponding to any value of j taken from the sequence 1, 2, ..., n will represent, by the nature of the system of equations under consideration, again a solution.

Consequently on designating by a_{ij} certain constants we shall have

$$x_{sj}(t + \omega) = a_{1j}x_{s1}(t) + a_{2j}x_{s2}(t) + \ldots + a_{nj}x_{sn}(t)$$
$$(j = 1, 2, \ldots, n) \tag{3}$$

for each value of s.

With the aid of the constants a_{ij} defined in this manner, let us form the following algebraic equation:

$$\begin{vmatrix} a_{11} - \rho & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \rho & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \rho \end{vmatrix} = 0,$$

which will be of degree n with respect to the unknown ρ.

This equation, playing a very important role in the theory of the differential equations under consideration, we shall call the *characteristic* equation corresponding to the period† ω. Similarly the determinant representing the left-hand side will be called the *characteristic determinant.*

If, in the place of (2), we had considered any other system of n independent solutions, we would have obtained in general other values for the constants a_{ij}. But the coefficients A_s in front of the different powers of ρ in the characteristic equation reduced to the form

$$\rho^n + A_1\rho^{n-1} + \dots + A_{n-1}\rho + A_n = 0,$$

would remain the same.

This is one of the fundamental properties of these coefficients, by virtue of which we can call them *invariants.*

For the coefficient A_n this property can already be seen by the expression that we can find for it on making use of the known formula giving the value of the determinant formed from n independent solutions of system (1).

To obtain this expression let us designate the determinant formed from functions (2) by $\Delta(t)$. Then the said formula [see the equation preceding (9) of Section 3] will give the following equality:

$$\Delta(t+\omega) = \Delta(t)e^{\int_0^\omega \Sigma p_{ss}\,dt}.$$

And as, because of relations (3), the determinant $\Delta(t+\omega)$ will be equal to the product of the determinant $\Delta(t)$ by the determinant of the quantities a_{ij}, the equation which we have just written will reduce to the form

$$(-1)^n A_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = e^{\int_0^\omega \Sigma p_{ss}\,dt}. \tag{4}$$

From this equality there follows, among other results, that the characteristic equation cannot have zero roots [since A_n cannot be zero].

Let us note that if all the coefficients $p_{s\sigma}$ in equations (1) are real functions, all the coefficients A_s in the characteristic equation will be necessarily also real. In fact, in this case we can always choose system (2) so that all the functions in it are real; and then all the constants a_{ij} will equally be real.

Formula (4), which defines the product of the roots of the characteristic equation, shows that in this case this product will always be positive and that, as a consequence, if the characteristic equation has negative roots the number of these roots will always be even.

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† We can consider a characteristic equation corresponding to the period $m\omega$, where m is an arbitrary whole number, positive or negative.

In speaking of a characteristic equation corresponding to the period ω, we shall often not make mention of the period, for brevity.

It is known that to each root ρ of the characteristic equation corresponds a solution of system (1) of the form†

$$x_1 = f_1(t)\rho^{t/\omega}, \quad x_2 = f_2(t)\rho^{t/\omega}, \quad \ldots, \quad x_n = f_n(t)\rho^{t/\omega}, \tag{5}$$

where all the f_s are periodic functions of t having for period ω (among which at least one is not identically zero). Therefore, if the characteristic equation does not have multiple roots, on considering all its roots we shall obtain n solutions of this form, and these solutions will be independent.

In the case of multiple roots, system (1) can admit solutions of a more general form. Namely, to a multiple root ρ there can correspond solutions for which the functions f_s in equations (5) will be of the form

$$f_s(t) = \varphi_{s0}(t) + t\varphi_{s1}(t) + t^2\varphi_{s2}(t) + \ldots + t^m\varphi_{sm}(t),$$

where all the φ_{sj} are periodic functions‡ of t.

If we include *zero* in the set of values of m, we shall have for each multiple root for which μ is the multiplicity μ independent solutions of this form.

In any of these solutions the number m will not exceed $\mu - 1$ (we assume that among the functions φ_{sm} at least one is not identically zero), but it will be able to attain this limit and, if the root under consideration makes non-zero at least one of the first minors of the characteristic determinant, there will always correspond to it a solution where we shall have $m = \mu - 1$.

Starting from this solution, we can then obtain all the other solutions which correspond to the same root by a very simple procedure.

In fact, if in any solution of form (5) we replace all the functions $f_s(t)$ by their finite differences of any order, these differences corresponding to an increase ω in the independent variable t, we shall obviously obtain a new solution of system (1). If then, starting from the solution where $m = \mu - 1$, we form for the functions $f_s(t)$ all the differences from the first up to that of order $\mu - 1$ inclusive, we may deduce from them by the indicated approach a further $\mu - 1$ solutions, and these solutions together with the original one will constitute a system of μ solutions which will evidently be independent.

Furthermore, instead of the procedure we have just indicated we can propose another one. Namely, instead of replacing the functions $f_s(t)$ by finite differences, we can for the same purpose replace them by expressions arising from their derivatives with respect to t when the quantities φ_{sj} are considered as constants. In fact, by the known relations between finite differences and derivatives§, we see immediately that we shall obtain in this way further solutions.

On applying this procedure to the case considered above and on taking for the starting point the solution where $m = \mu - 1$, we shall obtain all the μ independent solutions which correspond to the root under consideration.

† We understand by $\rho^{t/\omega}$ the function $e^{(t/\omega)\log\rho}$ corresponding to any fixed determination of the logarithm $\log \rho$.

‡ In speaking of periodic functions without indicating the period, we shall understand that we are concerned with functions with period ω.

§ [See for example E. Whittaker and G. Robinson *The Calculus of Observations*, London, fourth edition, 1944, p. 62. This shows that a derivative of a polynomial can be expressed as a linear combination of finite differences of the polynomial. Lyapunov is applying this result to each of the polynomials t^j ($j = 0, 1, \ldots, m$) which appear in the above expression for $f_s(t)$.]

We shall say that, in this case, to the root ρ corresponds a single set of solutions.

If a multiple root makes zero all the minors of the characteristic determinant up to order $k-1$ inclusive, there will correspond to it k sets of independent solutions, which we shall be able to form by one or other of the two procedures indicated, starting from certain k solutions.

The number k, while never exceeding the multiplicity μ of the root under consideration, can however attain it, and then in each solution corresponding to this root all the functions f_s will be periodic.

The theorems enunciated, which result from the fundamental propositions of the theory of substitutions, can be considered as well known†. Anyway their proofs do not present the slightest difficulty. Thus we may dispense with exposition of these proofs.

Remark

Let

$$\rho_1, \rho_2, \ldots, \rho_n$$

be the roots of the characteristic equation corresponding to the period ω.

Fixing on any determination of the logarithms, let us put

$$\chi_1 = \frac{1}{\omega}\log \rho_1, \quad \chi_2 = \frac{1}{\omega}\log \rho_2, \quad \ldots, \quad \chi_n = \frac{1}{\omega}\log \rho_n.$$

Then the real parts of the quantities

$$-\chi_1, \quad -\chi_2, \quad \ldots, \quad -\chi_n$$

will represent the characteristic numbers of the system of equations (1) [see (5)].

On designating by N a certain real integer, we extract from (4) the equality

$$\sum \chi_s = \frac{1}{\omega}\int_0^\omega \sum p_{ss}\, dt + N\frac{2\pi\sqrt{-1}}{\omega},$$

which shows that the real part of the quantity $\sum \chi_s$ is equal to the characteristic number of the function

$$e^{-\int \sum p_{ss}\, dt}.$$

[In fact, if $T = K\omega$ where K is a large positive integer, we have because of the periodicity of p_{ss}

$$\int_0^T \sum p_{ss}\, dt = K\int_0^\omega \sum p_{ss}\, dt = \frac{1}{\omega}\int_0^\omega \sum p_{ss}\, dt T.\Big]$$

Therefore, in conformity with what has been noted in Section 9, we conclude that system of equations (1) is regular.

† See for example Floquet, 'Sur les équations différentielles linéaires à coefficients périodiques', *Annales Scientifiques de l'École Normale Supérieure*, Vol. XII, 1883.

47. [*Transformation of equations with periodic coefficients into equations with constant coefficients*]

Let us consider the system of equations

$$\frac{dy_s}{dt} + p_{1s}y_1 + p_{2s}y_2 + \ldots + p_{ns}y_n = 0 \quad (s = 1, 2, \ldots, n), \tag{6}$$

adjoint with respect to (1), and let us suppose that the set of functions

$$\begin{array}{c} y_{11}, y_{21}, \ldots, y_{n1}, \\ y_{12}, y_{22}, \ldots, y_{n2}, \\ \ldots\ldots\ldots\ldots \\ y_{1n}, y_{2n}, \ldots, y_{nn} \end{array}$$

is a system of n independent solutions of it. Then the set of functions

$$\begin{array}{c} y_{11}x_1 + y_{21}x_2 + \ldots + y_{n1}x_n, \\ y_{12}x_1 + y_{22}x_2 + \ldots + y_{n2}x_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{1n}x_1 + y_{2n}x_2 + \ldots + y_{nn}x_n \end{array}$$

will represent a system of n independent integrals of system (1). [See Section 18 for discussion of adjoints.]

Let

$$\frac{1}{\rho_1}, \frac{1}{\rho_2}, \ldots, \frac{1}{\rho_k} \tag{7}$$

be all the roots of the characteristic equation of system (6), assuming that each multiple root appears in the series of numbers (7) as many times as there correspond to it sets of solutions. [Recall that no root is zero.] Then to each of the numbers ρ_s we shall be able to make correspond a set of solutions, and in such a way that all these solutions are independent.

Let n_s be the number of solutions in the set corresponding under this assumption to the number ρ_s. The numbers $n_1, n_2, \ldots, n_k$ will certainly satisfy the condition

$$n_1 + n_2 + \ldots + n_k = n.$$

On taking for the $y_{s\sigma}$ the functions constituting these sets, and on choosing in order to form them the second of the two procedures indicated above, for the number ρ_s we shall have n_s independent integrals of system (1) of the form

$$\left(z_1^{(s)} \frac{t^m}{m\,!} + z_2^{(s)} \frac{t^{m-1}}{(m-1)\,!} + \ldots + z_m^{(s)} t + z_{m+1}^{(s)} \right) \rho_s^{-t/\omega}$$
$$(m = 0, 1, \ldots, n_s - 1), \tag{8}$$

where the z_s designate linear forms in the quantities x_σ with periodic coefficients.

On considering the ensemble of all the k sets we shall obtain for system (1) n independent integrals of this form.

The variables x_σ enter into these integrals only via the linear forms $z_j^{(s)}$, the number of which is equal to the number of all the integrals. As a consequence, if the latter are independent the $z_j^{(s)}$ will equally be so. We shall thus be able to take these forms as new unknown functions in place of $x_1, x_2, \ldots, x_n$.

With this approach, on putting

$$\chi_s = \frac{1}{\omega}\log \rho_s,\ \rho_s^{-t/\omega} = e^{-\chi_s t},$$

we arrive at the following system of equations:

$$\left.\begin{aligned} &\frac{dz_1^{(s)}}{dt} = \chi_s z_1^{(s)} \\ &\frac{dz_j^{(s)}}{dt} = \chi_s z_j^{(s)} - z_{j-1}^{(s)} \\ &(j = 2, 3, \dots, n_s;\ s = 1, 2, \dots, k), \end{aligned}\right\} \tag{9}$$

which the quantities $z_j^{(s)}$ must evidently satisfy in view of the very manner in which they enter into integrals (8). [See the comment after (5) of Section 18.]

System (1) is thus found to be transformed into a system with constant coefficients. Moreover the transformation is accomplished by means of a substitution satisfying all the conditions [at the beginning] of Section 10.

In fact, to establish this in the case under consideration, it obviously suffices to show that the quantity inverse to the functional determinant [Jacobian], formed from the partial derivatives of the functions $z_j^{(s)}$ with respect to the variables x_σ, is a bounded function of t. And we assure ourselves of this on noting that this determinant, which is equal to the product of the function

$$e^{(n_1\chi_1 + n_2\chi_2 + \dots + n_k\chi_k)t}$$

and the functional determinant of the integrals (8)†, can only differ from the function

$$e^{\frac{t}{\omega}\int_0^\omega \sum p_{ss}\,dt - \int_0^t \sum p_{ss}\,dt}$$

[in which the exponent is bounded—see the comment at the end of Section 46] by a factor of the form

$$Ce^{\frac{2m\pi}{\omega}it}$$

(where $i = \sqrt{-1}$, m is a real integer and C is a constant). [Use has been made here of the expression for $\sum \chi_s$ appearing towards the end of Section 46.]

We therefore arrive at the conclusion that system (1) is reducible [as defined in Section 10].

At the same time, on considering its transform (9) we conclude that the quantities ρ_s are the roots of the characteristic equation of this system‡. We thus

† [The Jacobian of (8) can be recognized as being the product of two determinants. One of these is the Jacobian of the $z_j^{(s)}$. The other evaluates as the product of elements on a diagonal, which has n_1 elements $\rho_1^{-(t/\omega)}$, n_2 elements $\rho_2^{-(t/\omega)}$, etc. With use of the equation preceding (9) the latter product is thus

$$e^{-(n_1\chi_1 + n_2\chi_2 + \dots + n_k\chi_k)t}.]$$

‡ This conclusion rests on the proposition that if system (1) admits a solution of the form (5), the functions f_s having the character defined above, ρ is a root of the characteristic equation, and that if we have found μ independent solutions of this form, the multiplicity of the root ρ is not less than μ.

obtain the theorem that *the roots of the characteristic equation of the adjoint system are the inverses of the roots of the characteristic equation of the given system.* [Recall (7).]

Let us suppose that the coefficients $p_{\sigma s}$ in system (1) are real functions of t.

We know (Section 18, Remark) that with this condition we shall be able to transform it into a system with constant coefficients, only making use of substitutions with real coefficients. But the question arises of knowing whether we can subject such substitutions to the condition that the coefficients in them are periodic, as occurred in the transformation which we have just indicated.

We easily assure ourselves that if we want the coefficients to have for period the number ω, as occurred in the preceding transformation, this will only be generally possible in the case where the characteristic equation does not have negative roots†. As for the case where there exist such roots, certain conditions have to be satisfied, including one that all the negative roots must be multiple.

On the other hand the substitutions we are concerned with will always be possible if we limit ourselves to assuming that their coefficients have for period the number‡ 2ω.

Some propositions relating to the characteristic equation

48. [*General theorem on the development of invariants in series of powers of certain parameters*]

In each question of stability of periodic motion, the first problem we shall have to concern ourselves with will consist of the finding and examination of the characteristic equation corresponding to the system of linear differential equations which define the first approximation. This is why we believe it necessary to take up here some considerations which we shall be able to make use of in this kind of investigation.

To begin with let us call attention to a general proposition which will serve as a basis for certain methods of calculating the coefficients of the characteristic equation.

This proposition consists of the following.

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† Under our assumption the coefficients of the forms $z_j^{(s)}$ corresponding to positive roots ρ_s can be supposed real functions of t. As for complex roots ρ_s, they will split into conjugate pairs and, starting from the forms $z_j^{(s)}$ corresponding to such a root, we may deduce from them the forms corresponding to the conjugate root by replacing $\sqrt{-1}$ by $-\sqrt{-1}$. Consequently, if the characteristic equation does not have negative roots, we shall obtain, on operating as was shown in Section 18, a real substitution in which the coefficients will be periodic functions of t with period ω.

‡ To convince ourselves of this it suffices to note that, for each negative root ρ_s, we can suppose real the coefficients in the forms

$$z_j^{(s)} e^{i\pi t/\omega} \quad (i = \sqrt{-1}).$$

[Here Lyapunov is perhaps referring obliquely to the following argument. The characteristic equation corresponding to period 2ω has roots β_s which are the squares of the roots of the characteristic equation corresponding to period ω. As a consequence the number of β_s with a given negative value will be even, so that they can be considered as forming conjugate pairs. The technique mentioned in the previous footnote for dealing with conjugate pairs can thus be applied.]

THEOREM. *Suppose that the coefficients $p_{s\sigma}$ in equations* (1) *depend on certain parameters $\varepsilon_1, \varepsilon_2, \ldots$, while conserving their assumed properties as long as the moduli of $\varepsilon_1, \varepsilon_2, \ldots$, are sufficiently small, and suppose that the period ω does not depend on these parameters. Then, if the coefficients $p_{s\sigma}$ can be represented by series ordered in positive integer powers of the parameters $\varepsilon_1, \varepsilon_2, \ldots$, uniformly convergent for all real values of t as long as the moduli of $\varepsilon_1, \varepsilon_2, \ldots$ do not exceed certain non-zero limits $E_1, E_2, \ldots$, the coefficients A_s in the characteristic equation*

$$\rho^n + A_1\rho^{n-1} + \ldots + A_{n-1}\rho + A_n = 0$$

will be holomorphic functions of the parameters under consideration. Moreover, if the constants $E_1, E_2, \ldots$ are chosen so that in the case where

$$|\varepsilon_1| = E_1, \quad |\varepsilon_2| = E_2, \ldots \tag{10}$$

the series obtained by replacing in the expansions of the $p_{s\sigma}$ all terms by their moduli converge uniformly for all real values of t, the series by which the invariants A_s will be represented will certainly converge in the case of equalities (10).

This theorem will be established immediately† if we show that the functions $x_1, x_2, \ldots, x_n$, satisfying equations (1) and for $t = 0$ taking arbitrary given values $a_1, a_2, \ldots, a_n$, independent of the parameters $\varepsilon_1, \varepsilon_2, \ldots$, can be represented by series which are ordered in positive integer powers of these parameters, and are absolutely convergent for every real value of t as long as the moduli of $\varepsilon_1, \varepsilon_2, \ldots$ do not exceed the limits $E_1, E_2, \ldots$ chosen in conformity with the condition indicated.

As for the latter proposition, we may prove it easily by considering, in place of system (1), the following,

$$\frac{dx_s}{dt} = \varepsilon(p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n) \quad (s = 1, 2, \ldots, n),$$

in which ε represents a new parameter, and by seeking the functions x_s in the form of series ordered in the powers of $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ In fact, a glance over the equations arising in such an investigation [see e.g. equations between (13) and (14) below] will suffice for us to conclude that the moduli of the coefficients of the series sought will not exceed the corresponding coefficients in the expansion in powers of these same parameters of the following function‡:

$$ae^{\pm n\varepsilon\int_0^t p\,dt},$$

where a represents the greatest of the quantities $|a_s|$ and p a series, ordered in powers of $\varepsilon_1, \varepsilon_2, \ldots$, in which each coefficient, for any given value of t, is equal to the greatest of the moduli of the corresponding coefficients in the expansions of the $p_{s\sigma}$.

On returning to system (1), let us suppose that the set of functions

$$x_{1s}, x_{2s}, \ldots, x_{ns}$$

represents a solution of this system defined by the condition

$$x_{ss}(0) = 1, \quad x_{js}(0) = 0 \quad (j \lessgtr s).$$

†[By means of an argument which will be given towards the end of this section.]

‡ The upper sign here corresponds to the case of $t > 0$; the lower sign, to that of $t < 0$.

Then, on considering n similar solutions corresponding to $s = 1, 2, \dots, n$, we may extract from equations (3)

$$a_{sj} = x_{sj}(\omega) \quad (s, j = 1, 2, \dots, n).$$

It results from this, by virtue of the proposition which has just been mentioned, that the constants a_{sj} corresponding to our system of particular solutions can be expanded, under conditions (10), in absolutely convergent series in the powers of $\varepsilon_1, \varepsilon_2, \dots$

As a consequence the coefficients A_s in the characteristic equation, which are entire polynomials in the a_{sj}, will be in the same condition.

Let us suppose that we know how to integrate system (1) under the assumption that

$$\varepsilon_1 = \varepsilon_2 = \dots = 0.$$

Then, if we seek the functions x_s in the form of series ordered in powers of the parameters $\varepsilon_1, \varepsilon_2, \dots$, we shall have, for calculating the coefficients in these series, systems of differential equations which we shall be able to integrate successively, equations which will only require quadratures [see e.g. quadratures shown after (13) below]. We shall then obtain the invariants A_s in the form of power series where the coefficients will be expressed by means of certain multiple integrals.

It can happen that the proposed equations do not contain any parameters such that we can expand the constants A_j in their powers. Then we shall be able to replace these equations by others which include such parameters, and which for certain values of the latter reduce to the proposed equations (as has been done for example in the proof of the theorem).

Thus the methods of calculating the invariants A_s based on expansions in powers of parameters can be considered as fully general†.

49. [*Application to a differential equation of second order*]

On making use of the series discussed in the preceding section we can treat several general questions on the subject of the characteristic equation.

Let us show this in connection with the following differential equation:

$$\frac{d^2x}{dt^2} + px = 0, \tag{11}$$

where we understand by p a periodic function of t with real period ω, this function being determinate and continuous for all real values of t.

This equation being replaced by the system

$$\frac{dx}{dt} = x', \quad \frac{dx'}{dt} = -px,$$

† Some applications of these methods have been indicated in my memoir 'Sur la stabilité du mouvement dans un cas particulier du problème des trois corps', *Communications de la Société mathématique de Kharkow*, second series, Vol. II, 1889. [This paper treats the stability of Lagrange's three particles, i.e. three mutually attracting particles with initial velocities such that they always remain at the vertices of an equilateral triangle. The stability of the configuration had been previously investigated by Routh: *Proc. London Math. Soc.*, Vol. VI, 1875, pp. 86–97.]

we immediately conclude, on turning to formula (4) [which gives $A_n = 1$ since $p_{11} = p_{22} = 0$], that the corresponding characteristic equation will be of the form

$$\rho^2 - 2A\rho + 1 = 0. \tag{12}$$

The problem thus reduces to finding the single constant A.

Only considering as before real values of t, we shall assume that the function p always remains real. Then the constant A will also be real.

This settled, two cases can arise: (1) $A^2 \leqslant 1$, when the roots of equation (12) will have their moduli equal to 1, and (2) $A^2 > 1$, when these roots will be real, one being in absolute value greater than 1, the other less. [From (12) the product of the two roots is 1; and in case (1) the roots form a conjugate pair, while in case (2) they are real and unequal.]

The problem of knowing which of these two cases holds is the first which has to be resolved in questions of stability. It is thus appropriate to indicate some criteria which we shall be able to make use of to distinguish one from the other.

We can arrive at certain criteria of this kind by starting from an expression for A in the form of a series.

To form this series let us provisionally consider, instead of equation (11), the following:

$$\frac{d^2x}{dt^2} = \varepsilon px, \tag{13}$$

and let us seek the constant A which corresponds to it in the form of a series ordered in positive integer powers of the parameter ε.

In view of the theorem of the preceding section, this series will be absolutely convergent for every value of t, such that A will be not only a holomorphic function of ε, but also an entire function of this parameter.

Let $f(t)$ and $\varphi(t)$ be particular solutions of equation (13), defined by the conditions

$$f(0) = 1, \quad f'(0) = 0; \quad \varphi(0) = 0, \quad \varphi'(0) = 1.$$

On expanding the functions f and φ in powers of ε, we shall find

$$f(t) = 1 + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \ldots,$$
$$\varphi(t) = t + \varepsilon\varphi_1(t) + \varepsilon^2\varphi_2(t) + \ldots,$$

on designating in general by $f_n(t)$, $\varphi_n(t)$ functions of t which are calculated successively by the formulae

$$f_n(t) = \int_0^t dt \int_0^t pf_{n-1}(t)\, dt, \quad \varphi_n(t) = \int_0^t dt \int_0^t p\varphi_{n-1}(t)\, dt$$

under the hypothesis that

$$f_0(t) = 1, \quad \varphi_0(t) = t.$$

To obtain now the expansion of the constant A, we note that the characteristic equation can be presented in the form [with use of the penultimate equations of Section 48]

$$\begin{vmatrix} f(\omega) - \rho & f'(\omega) \\ \varphi(\omega) & \varphi'(\omega) - \rho \end{vmatrix} = 0$$

and that as a consequence

$$2A = f(\omega) + \varphi'(\omega).$$

The expansion sought will thus be

$$A = 1 + \tfrac{1}{2} \sum_{n=1}^{\infty} [f_n(\omega) + \varphi'_n(\omega)]\varepsilon^n. \tag{14}$$

By making use of the formulae obtained we shall be able to resolve for equation (13) the question posed above, when the parameter ε is small enough in absolute value.

Since

$$f_1(\omega) + \varphi'_1(\omega) = \omega \int_0^{\omega} p\, dt,$$

[which may be verified by applying an integration by parts to $f_1(t)$] this question will depend immediately on the examination of the integral

$$\int_0^{\omega} p\, dt$$

and whenever this is non-zero, the question will be resolved as soon as we know the sign of this integral.

The same formulae lead to the following proposition.

THEOREM I. *If the function p can only take negative or zero values (without being identically zero), the roots of the characteristic equation corresponding to equation* (11) *will always be real, and one of them will be greater than* 1, *and the other less than* 1.

[From the preceding double integrals we can find by induction that $(-1)^n f_n(t) > 0$ and $(-1)^n \varphi_n(t) > 0$ for $t > 0$. Then differentiation of the double integral for $\varphi_n(t)$ shows that $(-1)^n \varphi'_n(t) > 0$ for $t > 0$. Thus (14) with $\varepsilon = -1$ implies that $A > 1$, and the theorem follows.]

Let us now consider the case where the function p can only take positive or zero values, assuming that it is not identically zero.

The functions $f_n(t)$, $\varphi'_n(t)$ will then have the same property. Moreover, we shall be able to prove that for $n > 1$ they will satisfy the inequality

$$(f_{n-1} + \varphi'_{n-1})t \int_0^t p\, dt - 2n(f_n + \varphi'_n) > 0 \tag{15}$$

for every real value of t different from zero.

This may be demonstrated in the following way:

Putting

$$S_n = (f_{n-1} + \varphi'_{n-1})t \int_0^t p\, dt - 2n(f_n + \varphi'_n),$$

we note that we can write

$$S_n = \int_0^t (F_n + p\Phi_n)\, dt,$$

on designating by F_n and Φ_n the following functions:

$$F_n = tf'_{n-1}\int_0^t p\,dt + (f_{n-1} + \varphi'_{n-1})\int_0^t p\,dt - 2nf'_n$$

$$\Phi_n = t\varphi_{n-2}\int_0^t p\,dt + (f_{n-1} + \varphi'_{n-1})t - 2n\varphi_{n-1}.$$

[These equations can be verified by differentiating the first expression for S_n and using the relation $\varphi''_n = p\varphi_{n-1}$, which follows from the above double-integral expression for φ_n.]

Our inequality will thus be proved if we show that for positive values of t we have the inequalities

$$F_n > 0, \quad \Phi_n > 0, \tag{16}$$

and for negative values the inequalities

$$F_n < 0, \quad \Phi_n < 0. \tag{17}$$

Let us note for this purpose that the preceding expressions for the functions F_n and Φ_n easily reduce to the form [as may be checked by similar manipulations to those used above for S_n]

$$F_n = \int_0^t \left(2f'_{n-1}\int_0^t p\,dt + pu_n\right)dt,$$

$$\Phi_n = \int_0^t (2pt\varphi_{n-2} + v_n)\,dt,$$

where u_n and v_n represent the expressions

$$u_n = (\varphi_{n-2} + tf_{n-2})\int_0^t p\,dt + \varphi'_{n-1} + tf'_{n-1} - (2n-1)f_{n-1},$$

$$v_n = (\varphi_{n-2} + t\varphi'_{n-2})\int_0^t p\,dt + f_{n-1} + tf'_{n-1} - (2n-1)\varphi'_{n-1},$$

which we can write thus [as may be verified by the same technique as before]:

$$u_n = \int_0^t (2p(\varphi_{n-2} + tf_{n-2}) + F_{n-1})\,dt,$$

$$v_n = \int_0^t \left(2f'_{n-1} + 2\varphi'_{n-2}\int_0^t p\,dt + p\Phi_{n-1}\right)dt.$$

We conclude from this that if for all positive values of t there hold the inequalities

$$F_{n-1} > 0, \quad \Phi_{n-1} > 0,$$

for the same values of t inequalities (16) will hold, and that if for all negative values of t we have

$$F_{n-1} < 0, \quad \Phi_{n-1} < 0,$$

for the same values of t inequalities (17) will be satisfied. [Note that for $t > 0$ we have $f_n(t) > 0$, $\varphi_n(t) > 0$, $f'_n(t) > 0$, $\varphi'_n(t) > 0$. This follows by induction from the above double-integral expressions for $f_n(t)$ and $\varphi_n(t)$.]

From this it results that the validity of inequalities (16) for $t>0$ and of inequalities (17) for $t<0$ will be assured for every value of n greater than 1, if these inequalities hold for $n=2$.

Now, in this last case we see the result immediately from the following expressions which we extract from our formulae [with use of an integration by parts]

$$F_2 = \int_0^t \left\{ \left(\int_0^t p\,dt \right)^2 + 2p\varphi'_1 \right\} dt, \quad \Phi_2 = 2\int_0^t (pt^2 + 2f_1)\,dt.$$

We may therefore consider inequality (15) as proved.

Let us now go back to our problem.

Formula (14), for equation (11), takes the form

$$A = 1 + \tfrac{1}{2} \sum_{n=1}^{\infty} (-1)^n [f_n(\omega) + \varphi'_n(\omega)].$$

Consequently, on noting that because of (15)

$$f_n(\omega) + \varphi'_n(\omega) < [f_{n-1}(\omega) + \varphi'_{n-1}(\omega)] \frac{\omega}{2n} \int_0^\omega p\,dt,$$

we arrive at these inequalities:

$$A < 1 - \tfrac{1}{2} \sum_{n=1}^{\infty} \left(1 - \frac{\omega}{4n} \int_0^\omega p\,dt\right) [f_{2n-1}(\omega) + \varphi'_{2n-1}(\omega)]$$

$$A > 1 - \frac{\omega}{2} \int_0^\omega p\,dt + \tfrac{1}{2} \sum_{n=1}^{\infty} \left(1 - \frac{\omega}{4n+2} \int_0^\omega p\,dt\right) [f_{2n}(\omega) + \varphi'_{2n}(\omega)].$$

[These inequalities may be obtained by treating separately the positive and negative parts of A, and by using the equation appearing after (14).]

We conclude immediately from this that if

$$\omega \int_0^\omega p\,dt \leqslant 4,$$

we shall necessarily have

$$-1 < A < 1,$$

and in this way we reach the following proposition [on recalling the discussion following (12)]:

THEOREM II. *If the function p can only take positive or zero values (without being identically zero), and if further it satisfies the condition*

$$\omega \int_0^\omega p\,dt \leqslant 4,$$

the roots of the characteristic equation corresponding to equation (11) *will always be complex, their moduli being equal to* 1.

The conditions expressed in this theorem are sufficient, but of course not necessary.

In the particular case where the function p reduces to a constant (we can then take for period ω an arbitrary number), the condition $p > 0$ already suffices by itself for the roots of the characteristic equation, corresponding to any real period, to have their moduli equal to 1.

Thus the question naturally arises whether it will be the same in the general case.

The answer is however negative, for we can cite examples where the function p will always remain positive and the characteristic equation will nevertheless have real roots of which one will be, in absolute value, greater than 1, and the other smaller than 1.

To give an example of this sort, let us consider the Lamé equation:

$$\frac{d^2x}{dt^2} = (h + 2k^2 \operatorname{sn}^2 t)x$$

in one of its most simple cases.

We understand here by h an arbitrary constant and by k a positive fraction representing the modulus of the elliptic function $\operatorname{sn} t$.

Thanks to the investigations of Hermite we know that if in place of h we introduce a new constant λ, by putting

$$h = -1 - k^2 \operatorname{cn}^2 \lambda,$$

one of the particular solutions of the equation under consideration will be given by the expression

$$\frac{H(t+\lambda)}{\Theta(t)} e^{-\frac{\Theta'(\lambda)}{\Theta(\lambda)} t},$$

where H and Θ are the known functions of Jacobi [see Cayley, A., *An Elementary Treatise on Elliptic Functions*, London, 1895, pp. 1–17]. For another independent solution, we may deduce it, in general, on replacing t by $-t$ or λ by $-\lambda$ in this one†.

For the period ω we shall be able to take in the case under consideration the number $2K$, on understanding by K, as usual, the integral

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

and we see from the above expression that the roots of the characteristic equation corresponding to this period are the following‡:

$$-e^{2K \frac{\Theta'(\lambda)}{\Theta(\lambda)}} \quad \text{and} \quad -e^{-2K \frac{\Theta'(\lambda)}{\Theta(\lambda)}}. \tag{18}$$

† See Hermite, *Sur quelques applications des fonctions elliptiques* (Paris, Gauthier-Villars, 1885, p. 14). [Also published in *Oeuvres de Charles Hermite*, Paris, 1912, Vol. 3, pp. 266–418; and in 26 instalments in *Comptes rendus Acad. Sci. Paris*, vols. 85–94, (1877–1882).]

‡ [Compare (5). That the term corresponding to $f_1(t)$ has period $2K$ can be seen from the following properties:

$$H(t+2K) = -H(t), \quad \Theta(t+2K) = \Theta(t), \quad (-1)^{(t+2K)/(2K)} = -(-1)^{t/(2K)}]$$

Suppose that the number λ is real and between zero and $2K$, without however attaining these limits. Suppose moreover that it is small enough for us to have

$$1 - k^2 - k^2 \operatorname{sn}^2 \lambda > 0.$$

Then [with use of the relation $\operatorname{sn}^2 \lambda + \operatorname{cn}^2 \lambda = 1$] the function

$$p = 1 + k^2 \operatorname{cn}^2 \lambda - 2k^2 \operatorname{sn}^2 t$$

will be positive for all real values of t, and yet the numbers (18) will be real, one being greater and the other smaller than 1 in absolute value.

[Thus we have a case of a linear second-order system (11) which becomes unstable when a coefficient p varies periodically over a range of positive values, despite being stable when p is fixed at any positive value.]

50. [*Conclusions on the form of the characteristic equation which follow from certain functional properties of the coefficients of the differential equations*]

Sometimes, by using the functional properties of the coefficients in the differential equations, we can draw some immediate conclusions relating to the characteristic equation.

Thus, for example, if in the system

$$\frac{d^2x_s}{dt^2} = q_{s1}\frac{dx_1}{dt} + q_{s2}\frac{dx_2}{dt} + \dots + q_{sn}\frac{dx_n}{dt} + p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n$$

$$(s = 1, 2, \dots, n)$$

with periodic coefficients $q_{s\sigma}$, $p_{s\sigma}$, all the $q_{s\sigma}$ are odd functions of t and all the $p_{s\sigma}$ are even functions, we may assert that in the characteristic equation which corresponds to it

$$\rho^{2n} + A_1\rho^{2n-1} + \dots + A_{2n-1}\rho + A_{2n} = 0$$

the coefficients A_s will satisfy the relations

$$A_{2n} = 1, \quad A_{2n-s} = A_s \quad (s = 1, 2, \dots, n),$$

so that this equation will belong to the type of equation called *reciprocal.*

We may convince ourselves of this by noting that the system under consideration does not change when we replace t by $-t$. [Thus if

$$x_s = f_s(t)\rho^{t/\omega}$$

represents a solution, so does

$$x_s = f_s(-t)\rho^{-t/\omega}$$

i.e. both ρ and ρ^{-1} are characteristic roots.]

The case just mentioned is contained in a more general case where, in the proposed system of equations of form (1), all those of the coefficients $p_{s\sigma}$ for which the subscripts s and σ do not exceed a certain number k, as well as all those for which the two subscripts are greater than k, , represent odd functions of t, and all the others represent even functions.

Such a system will not change if we replace t by $-t$ and at the same time

$$x_{k+1} \text{ by } -x_{k+1}, \quad x_{k+2} \text{ by } -x_{k+2}, \quad \dots, \quad x_n \text{ by } -x_n.$$

And based on this it is easy to show that there will exist between the coefficients of the corresponding characteristic equation

$$\rho^n + A_1\rho^{n-1} + \dots + A_{n-1}\rho + A_n = 0$$

the following relations:

$$A_n = (-1)^n, \quad A_{n-1} = (-1)^n A_1, \quad A_{n-2} = (-1)^n A_2, \quad \dots \tag{19}$$

[Since p_{ss} is an odd function of t its integral over a period is zero, and then (4) shows that $A_n = (-1)^n$.]

We may consider conditions of a still more general character, namely conditions such that equations (1) do not change following the replacement of t by $-t$, when at the same time the x_s are replaced by certain linear forms in these variables with constant coefficients.

On designating the coefficients $p_{s\sigma}$ by $p_{s\sigma}(t)$ when we have to place in evidence the variable t, let us suppose that they satisfy the following relations:

$$\sum_{j=1}^{n} [\alpha_{sj}p_{j\sigma}(t) + \alpha_{j\sigma}p_{sj}(-t)] = 0 \quad (s, \sigma = 1, 2, \dots, n), \tag{20}$$

where the $\alpha_{s\sigma}$ are constants, for which the determinant

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} \tag{21}$$

will be assumed different from zero.

Then the system of equations

$$\frac{dy_s}{dt} = -p_{s1}(-t)y_1 - p_{s2}(-t)y_2 - \dots - p_{sn}(-t)y_n$$

$$(s = 1, 2, \dots, n) \tag{22}$$

will represent the transform of system (1) by means of the substitution

$$y_s = \alpha_{s1}x_1 + \alpha_{s2}x_2 + \dots + \alpha_{sn}x_n \quad (s = 1, 2, \dots, n). \tag{23}$$

[To verify this result by matrix algebra, let us write (1) and (23) as

$$\frac{dx}{dt} = P(t)x, \quad y = Ax. \tag{I}$$

These yield

$$\frac{dy}{dt} = AP(t)A^{-1}y \tag{II}$$

or, with use of (20) written as

$$AP(t) + P(-t)A = 0, \tag{III}$$

we get

$$\frac{dy}{dt} = -P(-t)y \tag{IV}$$

which is equivalent to (22).]

Based on this it is easy to prove that the invariants A_s will satisfy relations (19).

Thus let ρ be a root of the characteristic equation of system (1), and let us suppose that the equations

$$\left.\begin{aligned} &x_1 = f_1(t)\rho^{t/\omega}, \\ &x_2 = f_2(t)\rho^{t/\omega}, \quad \ldots, \quad x_n = f_n(t)\rho^{t/\omega} \end{aligned}\right\} \tag{24}$$

give for this system one of the solutions corresponding to the root ρ, such that the $f_s(t)$ are periodic functions of t or sums of a finite number of terms representing products of periodic functions by integer powers of t.

Turning to formulae (23) we deduce from them the following solution of system (22):

$$y_1 = \varphi_1(t)\rho^{t/\omega}, \quad y_2 = \varphi_2(t)\rho^{t/\omega}, \quad \ldots, \quad y_n = \varphi_n(t)\rho^{t/\omega},$$

where the functions

$$\varphi_s(t) = \alpha_{s1}f_1(t) + \alpha_{s2}f_2(t) + \ldots + \alpha_{sn}f_n(t)$$

will be of the same character as the functions $f_s(t)$. Moreover, if the functions $f_s(t)$ are not all identically zero (as we shall assume) it will be the same for the functions $\varphi_s(t)$, by virtue of our assumption concerning determinant (21).

Now from each solution of system (22) we deduce, on replacing t by $-t$,' a solution of system (1). We shall thus obtain for the latter the solution

$$\left.\begin{aligned} &x_1 = \varphi_1(-t)\left(\frac{1}{\rho}\right)^{t/\omega}, \\ &x_2 = \varphi_2(-t)\left(\frac{1}{\rho}\right)^{t/\omega}, \quad \ldots, \quad x_n = \varphi_n(-t)\left(\frac{1}{\rho}\right)^{t/\omega}, \end{aligned}\right\} \tag{25}$$

the existence of which shows that $1/\rho$ is one of the roots of the characteristic equation of system (1).

If ρ were a multiple root with multiplicity m we would have for system (1) m independent solutions of form (24), and from this, by the approach we have just indicated, we would deduce m solutions of form (25) which would again be independent, the determinant (21) not being zero. We could thus conclude that $1/\rho$ is a multiple root and that its multiplicity is not less than m. And since the root ρ was taken arbitrarily, it would equally follow that the multiplicity of the root $1/\rho$ cannot be greater than m.

In this way we can affirm that if the characteristic equation of system (1) has a root ρ of multiplicity m, it will also have the root $1/\rho$ of the same multiplicity m, and that therefore the coefficients in this equation must satisfy the relations

$$A_n = \pm 1, \quad A_{n-1} = A_n A_1, \quad A_{n-2} = A_n A_2, \quad \ldots$$

Thus, to prove equations (19) it only remains to demonstrate the first of them†.

For this purpose, on designating by A the determinant (21) and by $A_{s\sigma}$ its minor corresponding to the element $\alpha_{s\sigma}$, we note that equations (20) give the following:

$$\sum_{s=1}^{n}\sum_{\sigma=1}^{n} A_{s\sigma}\sum_{j=1}^{n}[\alpha_{sj}p_{j\sigma}(t) + \alpha_{j\sigma}p_{sj}(-t)] = 0,$$

† If the coefficients p_{ss} are real functions of t, this equality does not need any proof, since by virtue of (4) the quantity $(-1)^n A_n$ is then always positive.

which, being divided by A, reduces to

$$\sum_{s=1}^{n} [p_{ss}(t) + p_{ss}(-t)] = 0$$

[with use of the property that an expansion of a determinant in terms of alien cofactors gives zero (see Aitken, A. C., *Determinants and Matrices*, Edinburgh, 1958, p. 51)] and thus makes apparent that $\sum p_{ss}$ is an odd function of t.

Therefore we obtain

$$\int_0^{\omega} \sum p_{ss}\, dt = 0;$$

and we conclude from this, in view of (4), that $A_n = (-1)^n$.

We may note that in the case of odd n the characteristic equation of system (1), satisfying the condition which we have just considered, will have at least one root equal to 1, and that consequently this system will then admit a periodic solution (other than the obvious one $x_1 = x_2 = \dots = x_n = 0$).

Remark

Let us note that if relations (20) hold with values of the constants $\alpha_{s\sigma}$ such that the equation

$$\begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \lambda & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} - \lambda \end{vmatrix} = 0 \tag{26}$$

has neither multiple roots nor roots which differ only in sign from one another, the integration of system (1) reduces to quadratures.

In fact, we easily assure ourselves that if the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of equation (26) are all different, there will always be a linear substitution with constant coefficients such that system (1) is transformed into a system of the form

$$\frac{dz_s}{dt} = q_{s1}(t)z_1 + q_{s2}(t)z_2 + \dots + q_{sn}(t)z_n$$

$$(s = 1, 2, \dots, n),$$

where the coefficients $q_{s\sigma}$ satisfy the relations

$$\lambda_s q_{s\sigma}(t) + \lambda_\sigma q_{s\sigma}(-t) = 0, \quad (s, \sigma = 1, 2, \dots, n).$$

[To verify this result by matrix algebra, suppose that D is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \dots, \lambda_n$ and that the symbol A now again represents the matrix in (I). Then it is known that A has a diagonalizing matrix B such that

$$BAB^{-1} = D \tag{V}$$

i.e.

$$A = B^{-1}DB. \tag{VI}$$

Substituting (VI) in (III) we get

$$B^{-1}DBP(t) + P(t)B^{-1}DB = 0. \tag{VII}$$

Pre-multiply by B and post-multiply by B^{-1}; this will give

$$DQ(t) + Q(-t)D = 0 \tag{VIII}$$

where

$$Q(t) = BP(t)B^{-1}. \tag{IX}$$

Result (VIII) is equivalent to Lyapunov's last equations.

Also, if we put

$$z = Bx \tag{X}$$

comparison with (I) and (II) shows that

$$\frac{dz}{dt} = Q(t)z \tag{XI}$$

which corresponds to Lyapunov's penultimate equations.

Note further that Lyapunov's last equations, with $-t$ written for t, appear as

$$\lambda_s q_{s\sigma}(-t) + \lambda_\sigma q_{s\sigma}(t) = 0, \quad (s, \sigma = 1, 2, ..., n) \tag{XII}$$

and elimination of $q_{s\sigma}(-t)$ from the two versions leads to

$$(\lambda_s^2 - \lambda_\sigma^2)q_{s\sigma}(t) = 0, \quad (s, \sigma = 1, 2, ..., n). \tag{XIII}$$

Lyapunov has equations (XIII) in mind in his ensuing discussion.]

Now, the squares of all the λ_s being by assumption distinct, these relations will not be possible unless the $q_{s\sigma}$ with differing subscripts s and σ are all zero. And if we have $q_{s\sigma} = 0$ for σ not equal to s, the integration of the transformed system reduces to finding n quadratures

$$\int q_{11}\,dt, \quad \int q_{22}\,dt, \quad ..., \quad \int q_{nn}\,dt.$$

As far as the roots of the characteristic equation are concerned, subject to the assumptions under consideration, let us note that determinant (21) not being zero all these roots will be equal to 1 [from Lyapunov's last equations the above integrals will *all* have odd periodic integrands, and will consequently be periodic also]; and if this determinant is zero, one root will be able to be arbitrary, while all the others will be equal to 1 [if the determinant is zero one of the λ's will be zero, say λ_j, and then $q_{jj}(t)$ will no longer necessarily be an odd function of t].

51. [*On the characteristic equation of the canonical system*]

Sometimes the relations between invariants mentioned in the preceding section [see (19)] can result from the very form of the differential equations, whatever may be the functional properties of their coefficients.

Let us indicate one of the most important cases of this kind.

Suppose that the proposed system is canonical [Hamiltonian]:

$$\frac{dx_s}{dt} = -\frac{\partial H}{\partial y_s}, \quad \frac{dy_s}{dt} = \frac{\partial H}{\partial x_s} \quad (s = 1, 2, ..., k), \tag{27}$$

H representing a quadratic form in the variables $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k$ with periodic and continuous coefficients.

Let

$$\left.\begin{matrix} x_{11}, x_{21}, \ldots, x_{k1}, y_{11}, y_{21}, \ldots, y_{k1}, \\ x_{12}, x_{22}, \ldots, x_{k2}, y_{12}, y_{22}, \ldots, y_{k2} \end{matrix}\right\} \tag{28}$$

be any two solutions of this system.

Designating by H_1 and H_2 what H becomes on replacing in it the x_s, y_s by the x_{s1}, y_{s1} and x_{s2}, y_{s2}, respectively, we find

$$\frac{d}{dt}\sum_{j=1}^{k}(x_{j1}y_{j2}-x_{j2}y_{j1}) = \sum_{j=1}^{k}\left(x_{j1}\frac{\partial H_2}{\partial x_{j2}} - x_{j2}\frac{\partial H_1}{\partial x_{j1}} + y_{j1}\frac{\partial H_2}{\partial y_{j2}} - y_{j2}\frac{\partial H_1}{\partial y_{j1}}\right).$$

Now the right-hand side of this equality is identically zero, for its partial derivatives with respect to quantities (28), considered as independent variables, are identically zero. [Note that the right-hand side is quadratic in the variables and hence cannot reduce to a non-zero constant.] Thus, for example, its partial derivative with respect to x_{s1} is equal to

$$\frac{\partial H_2}{\partial x_{s2}} - \sum_{j=1}^{k}\left(x_{j2}\frac{\partial^2 H}{\partial x_j\,\partial x_s} + y_{2j}\frac{\partial^2 H}{\partial y_j\,\partial x_s}\right) = 0.$$

Our equality therefore leads to the following relation:

$$\sum_{j=1}^{k}(x_{j1}y_{j2}-x_{j2}y_{j1}) = \text{const.},$$

by which will thus be related any two solutions of system (27).

This settled, let us consider $2k$ independent solutions of this system

$$x_{1s}, x_{2s}, \ldots, x_{ks}, y_{1s}, y_{2s}, \ldots, y_{ks} \quad (s = 1, 2, \ldots, 2k).$$

By virtue of what we have just shown, there will exist between them $k(2k-1)$ relations of the form

$$\sum_{j=1}^{k}(x_{js}y_{j\sigma} - x_{j\sigma}y_{js}) = C_{\sigma s}, \tag{29}$$

in which the constants $C_{\sigma s} = -C_{s\sigma}$ $(s, \sigma = 1, 2, \ldots, 2k)$ [these relations follow from an interchange of subscripts in (29)], because of the independence of the solutions under consideration, will be such that among the constants

$$C_{s1}, C_{s2}, \ldots, C_{s,2k},$$

whatever the given number s, there will exist at least one of them which will not be zero.

On now designating by $\rho_1, \rho_2, \ldots, \rho_{2k}$ the roots of the characteristic equation of system (27) and on starting with the case where these roots are all distinct, let us suppose that our solutions have been chosen so that the functions x_{js}, y_{js} are of the form

$$x_{js} = f_{js}(t)\rho_s^{t/\omega}, \quad y_{js} = \varphi_{js}(t)\rho_s^{t/\omega},$$

f_{js}, φ_{js} designating periodic functions of t.

Then from equation (29), which will take the form

$$(\rho_s\rho_\sigma)^{t/\omega}\sum_{j=1}^{k}[f_{js}(t)\varphi_{j\sigma}(t) - f_{j\sigma}(t)\varphi_{js}(t)] = C_{s\sigma},$$

we may conclude that if $C_{s\sigma}$ is not zero we shall necessarily have

$$\rho_s \rho_\sigma = 1.$$

Now, in view of what was noted above, for every given number s there will correspond a number σ such that the constant $C_{s\sigma}$ is not zero, and this number σ will obviously be different from s. Therefore, for each root ρ_s of the characteristic equation there will correspond a root equal to $1/\rho_s$.

According to this we can conclude that if the characteristic equation

$$\rho^{2k} + A_1 \rho^{2k-1} + \ldots + A_{2k-1}\rho + A_{2k} = 0$$

of system (27) does not have multiple roots, its coefficients will satisfy the relations

$$A_{2k} = 1, \quad A_{2k-s} = A_s \quad (s = 1, 2, \ldots, k-1). \tag{30}$$

Now, these relations holding in the case of simple roots, they will necessarily be fulfilled in all cases.

To prove this we may reason in the following way.

In the function H, which will have an expression of the form

$$H = \sum_{s=1}^{k} \sum_{\sigma=1}^{k} (p_{s\sigma} x_s x_\sigma + q_{s\sigma} y_s y_\sigma + r_{s\sigma} x_s y_\sigma),$$

let us replace the coefficients $p_{s\sigma}$, $q_{s\sigma}$, r_{ss} and $r_{s\sigma}$ (for s and σ different) by

$$\varepsilon p_{s\sigma}, \quad \varepsilon q_{s\sigma}, \quad \chi_s + \varepsilon(r_{ss} - \chi_s), \quad \varepsilon r_{s\sigma},$$

respectively, on understanding by ε an arbitrary parameter and by $\chi_1, \chi_2, \ldots, \chi_k$ any constants such that the numbers

$$e^{\chi_1 \omega}, e^{\chi_2 \omega}, \ldots, e^{\chi_k \omega}, e^{-\chi_1 \omega}, e^{-\chi_2 \omega}, \ldots, e^{-\chi_k \omega} \tag{31}$$

are all different, and let us consider the canonical system corresponding to the function H so modified.

This system, for $\varepsilon = 0$, will reduce to a system with constant coefficients, for which the numbers

$$\chi_1, \chi_2, \ldots, \chi_k, -\chi_1, -\chi_2, \ldots, -\chi_k$$

will be the roots of the determinantal equation and, as a consequence, the numbers (31) will be the roots of the characteristic equation corresponding to the period ω.

Therefore, on noting that for our new canonical system the invariants A_s will be continuous with respect to ε, for because of the theorem of Section 48 they will be certain entire (transcendental) functions of ε, and on taking into account that by assumption the numbers (31) are all different, we can assert that the characteristic equation will not have multiple roots, either for $\varepsilon = 0$ or for non-zero values of ε for which the moduli are sufficiently small. Thus, for such values of ε, relations (30) will be satisfied. But then these relations, holding as they do between entire functions of ε, will necessarily be satisfied for all values of ε. They will therefore be satisfied in particular for $\varepsilon = 1$, when our new canonical system reduces to the original one.

In this way we obtain the following theorem.

THEOREM. *If the proposed system of linear differential equations with periodic coefficients has the canonical form, the characteristic equation which corresponds to it is always reciprocal*†.

Our proof rests on the existence of a certain relation between two arbitrary solutions of system (27).

Now we can indicate other systems admitting such relations, from which we shall be able to draw similar conclusions about the characteristic equation.

Such is, for example, the case where the coefficients $p_{s\sigma}$ in system (1) are related to one another by the equalities

$$\sum_{j=1}^{n} (\alpha_{js}p_{j\sigma} - \alpha_{j\sigma}p_{js}) = 0 \quad (s, \sigma = 1, 2, ..., n),$$

in which $\alpha_{s\sigma}$ are constants such that we have $\alpha_{s\sigma} + \alpha_{\sigma s} = 0$ for all values of s and σ taken from the sequence $1, 2, ..., n$.

It can happen that the proposed system, without being canonical, reduces to this with the aid of a linear substitution with constant or periodic coefficients. Whenever this is so and the substitution satisfies the conditions [at the beginning] of Section 10, we shall be able to assert that the characteristic equation for this system is reciprocal.

Thus, for example, let there be proposed the system

$$\frac{d^2x_s}{dt^2} = \sum_{\sigma=1}^{k} \left[\alpha_{s\sigma} + \int_0^t (p_{s\sigma} - p_{\sigma s})\, dt\right] \frac{dx_\sigma}{dt} + \sum_{\sigma=1}^{k} p_{s\sigma}x_\sigma$$
$$(s = 1, 2, ..., k),$$

in which the coefficients $p_{s\sigma}$ satisfy the conditions

$$\int_0^\omega (p_{s\sigma} - p_{\sigma s})\, dt = 0 \quad (s, \sigma = 1, 2, ..., k),$$

and $\alpha_{s\sigma}$ are any constants verifying the relations

$$\alpha_{s\sigma} + \alpha_{\sigma s} = 0 \quad (s, \sigma = 1, 2, ..., k).$$

On making

$$y_s = \frac{dx_s}{dt} - \tfrac{1}{2} \sum_{\sigma=1}^{k} \left[\alpha_{s\sigma} + \int_0^t (p_{s\sigma} - p_{\sigma s})\, dt\right] x_\sigma$$
$$(s = 1, 2, ..., k)$$

and putting

$$H = \tfrac{1}{4} \sum_{s=1}^{k} \sum_{\sigma=1}^{k} \left[p_{s\sigma} + p_{\sigma s} - \tfrac{1}{2} \sum_{i=1}^{k} q_{si}q_{\sigma i}\right] x_s x_\sigma + \tfrac{1}{2} \sum_{s=1}^{k} \sum_{\sigma=1}^{k} q_{s\sigma}x_s y_\sigma - \tfrac{1}{2} \sum_{s=1}^{k} y_s^2,$$

† This theorem is also indicated by Mr Poincaré in his memoir 'Sur le problème des trois corps et les équations de la dynamique' (*Acta Mathematica*, Vol. 13 [1890], pp. 99–100), where the author also bases it on the relations of the form (29). But I knew it before the publication of his memoir, and in February 1900 I communicated it in the above form to the Mathematical Society of Kharkov, with other propositions relating to the characteristic equation (*Communications de la Société Mathématique de Kharkow*, second series, Vol. II; extract from the verbal proceedings of the sessions).

where

$$q_{s\sigma} = \alpha_{s\sigma} + \int_0^t (p_{s\sigma} - p_{\sigma s})\, dt,$$

we shall reduce this system to the form (27). We can therefore state that the corresponding characteristic equation will be reciprocal.

52. [*Some particular procedures for the study of the characteristic equation*]

If in equations (1) the coefficients $p_{s\sigma}$ are real functions of t (as we shall assume here), we shall be able to arrive at some conclusions on the characteristic equation by making use of similar procedures to those which we have proposed for the study of stability, under the name of the *second method.*

These procedures always allow us to obtain for the moduli of the roots of the characteristic equation bounds, upper and lower, and more or less precise. To get to them we can, for example, operate as was done in Section 7 for proving Theorem I.

But the same method can sometimes also serve to put in evidence some other properties of the characteristic equation.

Let us take for example the following system:

$$\frac{d^2x_s}{dt^2} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n$$

$$(s = 1, 2, \dots, n), \tag{32}$$

where the coefficients $p_{s\sigma}$, representing real periodic functions of t, are assumed such that the equation

$$\begin{vmatrix} 2(p_{11}-k) & p_{12}+p_{21} & \dots & p_{1n}+p_{n1} \\ p_{21}+p_{12} & 2(p_{22}-k) & \dots & p_{2n}+p_{n2} \\ \dots & \dots & \dots & \dots \\ p_{n1}+p_{1n} & p_{n2}+p_{2n} & \dots & 2(p_{nn}-k) \end{vmatrix} = 0$$

in the unknown k does not have negative roots for any value of t (we shall only consider, as previously, real values of t).

Let p be smallest of its roots (which are, as is known, all real [a result established by Cauchy in 1829]).

The coefficients in our differential equations being continuous for all the values of t under consideration, it will be the same for the function p. Moreover this function will be periodic and its period ω will be the same as that of the coefficients $p_{s\sigma}$.

We shall assume that the function p is not identically zero (however, it can possibly become zero for certain values of t).

Then we shall be able to show that *the characteristic equation of system* (32) *has n roots with moduli greater than* 1 *and n roots with moduli less than* 1.

For this purpose, on putting

$$x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} + \dots + x_n\frac{dx_n}{dt} = X,$$

we note that our equations give

$$\frac{dX}{dt} = \sum_{s=1}^{n} \sum_{\sigma=1}^{n} p_{s\sigma} x_s x_\sigma + \sum_{s=1}^{n} \left(\frac{dx_s}{dt}\right)^2.$$

From this, in view of a known property of quadratic forms, the x_s being supposed real, we deduce

$$\frac{dX}{dt} \geqslant p \sum_{s=1}^{n} x_s^2 + \sum_{s=1}^{n} \left(\frac{dx_s}{dt}\right)^2. \tag{33}$$

[To verify by matrix algebra the first term on the right-hand side of (33), suppose that P is the matrix of coefficients in (32), P' being its transpose, and that Q is defined by

$$Q = \tfrac{1}{2}(P + P') = Q'. \tag{XIV}$$

It is known that there is an orthogonal matrix which diagonalizes the symmetric matrix Q, i.e. there is a matrix R such that

$$RR' = I, \quad RQR' = K, \quad Q = R'KR \tag{XV}$$

where K is diagonal. Hence

$$x'Qx = x'R'KRx = y'Ky \tag{XVI}$$

where

$$y = Rx. \tag{XVII}$$

If the diagonal elements of K are $k_1, k_2, \ldots, k_n$ with p the smallest of these, we have from (XIV) and (XVI)

$$x'Px = x'Qx = \sum_{s=1}^{n} k_s y_s^2 \tag{XVIII}$$

$$\geqslant p \sum_{s=1}^{n} y_s^2 = py'y = px'R'Rx \tag{XIX}$$

or with use of (XV)

$$x'Px \geqslant px'x = p \sum_{s=1}^{n} x_s^2. \tag{XX)]}$$

Now the right-hand side of this inequality [(33)] is greater than

$$2\sqrt{p}\left(x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} + \ldots + x_n \frac{dx_n}{dt}\right),$$

[with use of inequalities having the general form

$$a^2 + b^2 = (a - b)^2 + 2ab \geqslant 2ab].$$

We thus have

$$\frac{dX}{dt} \geqslant 2\sqrt{p}X.$$

This settled, let us designate by X_0 the value of the function X for $t = 0$, and let us only consider positive values of t. Then the above inequality will give [on separating the variables and integrating both sides]

$$X \geqslant X_0 e^{2\int_0^t \sqrt{p}\, dt}.$$

On putting

$$\frac{1}{\omega}\int_0^\omega \sqrt{p}\, dt = \lambda,$$

[from which there follows

$$\int_0^t \sqrt{p}\, dt > \lambda(t-\omega)\Big]$$

we conclude from this that if X_0 is a positive quantity we shall be able to make the function

$$Xe^{-2(\lambda-\varepsilon)t}$$

as large as we wish, however small the positive number ε, by choosing t sufficiently large.

Let us suppose that we have found for system (32) $2n$ independent real solutions, and let

$$X_1, X_2, \ldots, X_{2n} \tag{34}$$

be the functions to which the expression X reduces for these solutions.

Whatever our solutions, none of these functions will be identically zero, since the equality $X = 0$, because of (33), would only be possible for the solution

$$x_1 = x_2 = \ldots = x_n = 0,$$

which does not enter into those which we consider.

We may further always assume that our solutions are chosen so that we have

$$X_1 = r_1^{2t/\omega}F_1(t), \quad X_2 = r_2^{2t/\omega}F_2(t), \quad \ldots, \quad X_{2n} = r_{2n}^{2t/\omega}F_{2n}(t),$$

where $r_1, r_2, \ldots, r_{2n}$ are the moduli of the roots of the characteristic equation corresponding to the period ω, and the $F_s(t)$ designate certain real functions of t such that each of the functions

$$F_1(t), F_2,(t), \ldots, F_{2n}(t),$$
$$F_1(-t), F_2(-t), \ldots, F_{2n}(-t)$$

has zero for its characteristic number.

We now note that there will always be values of t for which none of the functions (34) will be zero [otherwise each of these functions would be identically zero, and from (33) this would require all the x_s to be identically zero].

For definiteness, let us assume that the value $t = 0$ satisfies this condition. Let us further assume that for $t = 0$ the functions

$$X_1, X_2, \ldots, X_m$$

become positive and all the others become negative.

By virtue of what has been proved we can then state that, however small the positive number ε, the functions

$$X_1 e^{-2(\lambda-\varepsilon)t}, \quad X_2 e^{-2(\lambda-\varepsilon)t}, \quad \ldots, \quad X_m e^{-2(\lambda-\varepsilon)t}$$

t being sufficiently large, will all become as large as we wish. And that is only possible under our hypothesis concerning the form of the functions X_s if

$$r_1 \geqslant e^{\lambda\omega}, \quad r_2 \geqslant e^{\lambda\omega}, \quad \ldots, \quad r_m \geqslant e^{\lambda\omega}$$

(we suppose the number ω to be positive).

Let us now consider, in place of (32), the system deduced from it on replacing t by $-t$.

This new system will obviously satisfy all the hypotheses made relative to the original one. We can therefore apply to it the preceding reasoning on replacing the functions (34) by the following:

$$X'_1 = -r_1^{-2t/\omega} F_1(-t), \quad X'_2 = -r_2^{-2t/\omega} F_2(-t), \quad \ldots, \quad X'_{2n} = -r_{2n}^{-2t/\omega} F_{2n}(-t).$$

Now among the latter the functions

$$X'_{m+1}, \quad X'_{m+2}, \quad \ldots, \quad X'_{2n}$$

in conformity with what was assumed above, become positive for $t = 0$. We can thus conclude, similarly as in the preceding argument, that we shall have

$$\frac{1}{r_{m+1}} \geqslant e^{\lambda\omega}, \quad \frac{1}{r_{m+2}} \geqslant e^{\lambda\omega}, \quad \ldots, \quad \frac{1}{r_{2n}} \geqslant e^{\lambda\omega}.$$

So we arrive at the conclusion (on taking into account that λ is a positive number) that, under our assumptions, the characteristic equation of system (32) will have m roots with moduli

$$r_1, r_2, \ldots, r_m,$$

greater than 1, and $2n - m$ roots with moduli

$$r_{m+1}, r_{m+2}, \ldots, r_{2n},$$

less than 1.

Let us make apparent that we shall necessarily have $m = n$.

For this we first note that if the coefficients $p_{s\sigma}$ in our differential equations satisfy the condition $p_{s\sigma} = p_{\sigma s}$ for all s and σ taken from the sequence $1, 2, \ldots, n$, the equality $m = n$ will be a consequence of what was shown in the previous section. In fact, according to what we have seen there the characteristic equation of system (32) will in this case be reciprocal. [Equations (32) become a special case of the second set of equations after the theorem in Section 51.]

This agreed, and on returning to the general case, let us replace in system (32) the coefficients $p_{\sigma s}$ by the following expressions:

$$q_{s\sigma} = \tfrac{1}{2}(p_{s\sigma} + p_{\sigma s}) + \frac{\varepsilon}{2}(p_{s\sigma} - p_{\sigma s}) \quad (s, \sigma = 1, 2, \ldots, n),$$

ε being an arbitrary real parameter.

Whatever the number ε we shall have

$$q_{s\sigma} + q_{\sigma s} = p_{s\sigma} + p_{\sigma s}$$

whence we see that our new system, for every value of ε, will satisfy the assumptions made relative to system (32).

Therefore, on applying to it what we have just established, we can assert that the characteristic equation of this new system will not have roots with moduli equal to 1 for any value of ε.

But then, on taking into account that the coefficients A_s in this equation,

$$\rho^{2n} + A_1\rho^{2n-1} + \dots + A_{2n-1}\rho + A_{2n} = 0$$

will be continuous functions of ε for all values of the latter (Section 48), we must conclude that the number of roots of this equation for which the moduli are greater than 1 (or less than 1) will always be the same, whatever ε may be. To determine this number it suffices, by consequence, to consider the hypothesis that $\varepsilon = 0$. Now under this hypothesis we have $q_{s\sigma} = q_{\sigma s}$ for all values of s and σ. Therefore, by virtue of what was noted above, the number sought must be equal to n.

We may thus consider our theorem as proved, for on putting $\varepsilon = 1$ we arrive at system (32).

Having proved that the characteristic equation of this system has n roots with moduli greater than 1, and the same number of roots with moduli smaller than 1, we have at the same time found a lower bound $e^{\lambda\omega}$ for the moduli of the roots of the first set, and an upper bound $e^{-\lambda\omega}$ for the moduli of the roots of the second set.

It may be noted that Theorem I of Section 49 is only a special case of what we have just proved.

To give a further example, let us assume that for the system (1) we have managed to find an integral representing a quadratic form in the variables x_s with constant or periodic coefficients. Let us further assume that this integral is a definite function (Section 15) such that, $t, x_1, x_2, \dots, x_n$ being real, it cannot become less in absolute value than the function

$$N(x_1^2 + x_2^2 + \dots + x_n^2),$$

where N represents a positive constant.

Once such an integral exists, we shall be able to conclude that in each real solution of system (1) all the functions x_s will always remain less in absolute value than a certain limit, whatever t may be, positive or negative. And this is only possible under the condition that all the roots of the characteristic equation possess moduli equal to 1, and that further, in solutions of type (5) corresponding to multiple roots, all the functions $f_s(t)$ are periodic.

We find ourselves with such a case, for example, when the coefficients $p_{s\sigma}$ in system (1) satisfy the condition

$$p_{s\sigma} + p_{\sigma s} = 0$$

for all s and σ taken from the sequence $1, 2, \dots, n$. This system will then admit the integral

$$x_1^2 + x_2^2 + \dots + x_n^2.$$

[This result can be confirmed by matrix algebra as follows:

$$\frac{d}{dt}(x'x) = \frac{dx'}{dt}x + x'\frac{dx}{dt} \tag{XXI}$$

$$= x'P'x + x'Px = x'(P' + P)x. \tag{XXII}$$

Now Lyapunov's penultimate relation is equivalent to

$$P' + P = 0 \tag{XXIII}$$

and we obtain from (XXII) and (XXIII)

$$\frac{d}{dt}(x_1^2 + x_2^2 + \dots + x_n^2) = 0. \qquad \text{(XXIV)]}$$

53. [*Application of the theory of functions of a complex variable. A case where the logarithms of the roots of the characteristic equation are obtained algebraically with the aid of certain definite integrals*]

Up to now we have only considered real values of the variable t. But if we also consider complex values (representing them as usual by points on a plane), and if we make suitable assumptions concerning the coefficients $p_{s\sigma}$, we shall be able to profit from the general principles of the theory of the singular points of linear differential equations, in obtaining the solution to various questions concerning the system (1) and in particular to the problem of the determination of the invariants A_s.

Suppose we draw in the plane of the complex variable t two straight lines parallel to the real axis, on either side of this axis and at distances equal to h, and assume that the coefficients $p_{s\sigma}$ (supposed as before to be periodic with real period ω) are given for the region of the plane between these straight lines as functions of the complex variable t, not having any singular points there.†

Under this condition, if we trace a circle of radius h with centre the point $t = 0$, for all points situated in the interior of or on the circle itself we shall be able to represent the coefficients $p_{s\sigma}$ as well as the functions x_s satisfying equations (1) by series ordered in positive integer powers of t.

As a consequence, if $\omega \leqslant h$ (we assume ω positive) we shall be able, by making use of these series, to determine the values of the functions x_s for $t = \omega$ in terms of the values that we give them for $t = 0$. And the series by which these values will be expressed will at the same time furnish series for the calculation of the invariants A_s. [See (3) and the two equations following it.]

When $\omega > h$ the use of these series, to be sure, will not always be legitimate. But we shall then be able to obtain for the calculation of the invariants series of another kind, by making use for example of the procedures indicated by Hamburger and Poincaré.‡

We shall not stop to examine these series, nor all the other series which we could propose for the calculation of the invariants, and we shall limit ourselves here to indicating a case where the invariants can be calculated without making use of series.

† We shall not consider points reaching infinity.

‡ HAMBURGER, 'Über ein Princip zur Darstellung des Verhaltens mehrdeutiger Functionen, etc.' *J. für Mathematik*, Vol. LXXXIII.

POINCARÉ, 'Sur les groupes des équations différentielles linéaires'. *Acta Mathematica*, Vol. IV.

See also the recent memoir of Mittag-Leffler, 'Sur la représentation analytique des intégrals et des invariants d'une équation différentielle linéaire et homogène', *Acta Mathematica*, Vol. XV.

This case is deduced from the known theorem of Fuchs on regular solutions of linear differential equations in the neighbourhood of their singular points [see Goursat, Hedrick and Dunkel, *loc. cit.*, p. 134].

Let us put

$$e^{i(2\pi t/\omega)} = z \quad (i = \sqrt{-1}).$$

On taking z in place of t as independent variable, we shall transform the system (1) into the following:

$$\frac{2\pi i}{\omega} z \frac{dx_s}{dz} = = p_{s1}(z)x_1 + p_{s2}(z)x_2 + \ldots + p_{sn}(z)x_n$$

$$(s = 1, 2, \ldots, n). \tag{35}$$

In agreement with the assumptions made, the coefficients $p_{s\sigma}(z)$ here will be functions of the complex variable z, not having singular points in the region of the plane situated between two concentric circles with radii $e^{2\pi h/\omega}$ and $e^{-2\pi h/\omega}$ and with common centre at the point $z = 0$; these functions will moreover be single-valued. [Lyapunov is using $p_{sj}(z)$ to represent $p_{sj}(t(z))$. If, for clarity, we write the first of these as $p_{sj}\{z\}$, we have

$$p_{sj}\{z\} = p_{sj}\left(\frac{\omega}{2\pi i}\log z\right)$$

and the left-hand side will not change value if we add $2\pi Ni$ (N integer) to $\log z$ in the right-hand side, because of the periodicity of $p_{sj}(t)$. Thus $p_{sj}\{z\}$ is indeed single-valued.]

We shall now assume that these coefficients do not have singular points throughout the interior of the circle with radius $e^{2\pi h/\omega}$.

This being so, system (35) will satisfy the conditions of the theorem of Fuchs for the point $z = 0$ [and therefore (35) will have an analytic solution in the neighbourhood of this singular point].

We can therefore state that, $\chi_1, \chi_2, \ldots, \chi_n$ being the roots of the equation

$$\begin{vmatrix} p_{11}(0) - \chi & p_{12}(0) & \ldots & p_{1n}(0) \\ p_{21}(0) & p_{22}(0) - \chi & \ldots & p_{2n}(0) \\ \dots & \dots & \dots & \dots \\ p_{n1}(0) & p_{n2}(0) & \ldots & p_{nn}(0) - \chi \end{vmatrix} = 0,$$

the numbers

$$e^{\chi_1\omega}, e^{\chi_2\omega}, \ldots, e^{\chi_n\omega} \tag{36}$$

will be the roots of the characteristic equation of system (35), corresponding to a circuit round the point $z = 0$ along a circle with sufficiently small radius with this point as centre. [Lyapunov seems to have in mind the following argument, which for brevity will be set out in terms of matrices. For small $|z|$ the coefficients $p_{sj}\{z\}$ in (35) are approximately the constants $p_{sj}\{0\}$. Thus we replace (35) by

$$\frac{2\pi i}{\omega} z \frac{dx}{dz} = P\{0\}x$$

Reverting to the original independent variable t, we may write this equation as

$$\frac{dx}{dt} = P\{0\}x$$

which, being constant-coefficient, will give as solutions for $x_1, x_2, ..., x_n$ linear combinations of

$$e^{\chi_1 t}, e^{\chi_2 t}, ..., e^{\chi_n t}$$

(assuming the χ_s are all different). When the real part of t increases by ω these terms become multiplied by the factors (36), indicating that the latter are characteristic roots for system (35).] And as, in accordance with our assumptions, we can make the radius of this circle equal to 1, the numbers (36) will equally represent the roots in which we are interested of the characteristic equation of the system (1) corresponding to a change in t, supposed real, by period ω.

It is anyway easy to prove this without having recourse to the theorem of Fuchs.

For this, on designating by ε an arbitrary parameter, let us consider in place of (35) the system

$$\frac{2\pi i}{\omega} z \frac{dx_s}{dz} = p_{s1}(\varepsilon z)x_1 + p_{s2}(\varepsilon z)x_2 + ... + p_{sn}(\varepsilon z)x_n$$
$$(s = 1, 2, ..., n),$$

which is deduced from it on changing z into εz.

It is clear that the invariants of this new system, corresponding to a circuit round the point $z = 0$ along the circle with radius 1 and with centre at this point, will be the same for all values of ε for which the moduli do not exceed the number $e^{2\pi h/\omega}$, greater than 1. And from the theorem of Section 48 we conclude that by making the modulus of ε sufficiently small we shall be able to make these invariants differ as little as we wish from the corresponding invariants of the system

$$\frac{2\pi i}{\omega} z \frac{dx_s}{dz} = p_{s1}(0)x_1 + p_{s2}(0)x_2 + ... + p_{sn}(0)x_n$$
$$(s = 1, 2, ..., n). \tag{37}$$

By consequence, the invariants of system (35) relating to the said circuit will necessarily be identical with the corresponding invariants of system (37).

Now this last system may be integrated in a well-known manner [see the previous inserted comment], and the roots of its characteristic equation are obtained precisely as has been indicated above.†

For the principal object of our study, the only case of interest is where the coefficients in the differential equations are real for all real values of t; and the systems of equations we have just considered do not obviously belong to this case, at least when their coefficients do not reduce to constants. There are however systems with real coefficients which can be reduced to them by means of certain transformations.

Let us consider for example the following system:

$$\left.\begin{aligned} \frac{dx_s}{dt} &= p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n \\ &\qquad - q_{s1}y_1 - q_{s2}y_2 - ... - q_{sn}y_n, \\ \frac{dy_s}{dt} &= q_{s1}x_1 + q_{s2}x_2 + ... + q_{sn}x_n + p_{s1}y_1 + p_{s2}y_2 \\ &\qquad + ... + p_{sn}y_n \quad (s = 1, 2, ..., n), \end{aligned}\right\} \tag{38}$$

† The procedure we have just made use of applies easily to a proof of the actual theorem of Fuchs.

supposing that the coefficients $p_{s\sigma}$, $q_{s\sigma}$ are periodic functions of t with real period ω, having singular points neither on the real axis nor at distances from this axis equal to or less than a certain limit h. We shall assume further that these coefficients satisfy the relations

$$\int_0^\omega p_{s\sigma} \cos \frac{2\pi mt}{\omega}\, dt = \int_0^\omega q_{s\sigma} \sin \frac{2\pi mt}{\omega}\, dt,$$
$$\int_0^\omega p_{s\sigma} \sin \frac{2\pi mt}{\omega}\, dt = -\int_0^\omega q_{s\sigma} \cos \frac{2\pi mt}{\omega}\, dt$$

for every positive integer value of the number m.

These relations express the property that, if the expansions of the functions $p_{s\sigma}$ in series of sines and cosines of integer multiples of $2\pi t/\omega$ are the following:

$$p_{s\sigma} = a_{s\sigma}^{(0)} + \sum_{m=1}^{\infty} \left(a_{s\sigma}^{(m)} \cos \frac{2\pi mt}{\omega} - b_{s\sigma}^{(m)} \sin \frac{2\pi mt}{\omega} \right),$$

the expansions of the functions $q_{s\sigma}$ will be of the form

$$q_{s\sigma} = b_{s\sigma}^{(0)} + \sum_{m=1}^{\infty} \left(a_{s\sigma}^{(m)} \sin \frac{2\pi mt}{\omega} + b_{s\sigma}^{(m)} \cos \frac{2\pi mt}{\omega} \right).$$

It is known that, under the assumptions considered here, we can represent the coefficients $p_{s\sigma}$, $q_{s\sigma}$ by such series for all values of t for which the representative points are displaced from the real axis by distances less than h.

Returning now to our system of equations, we note that if we take for the unknown functions the quantities

$$u_s = x_s + iy_s, \quad v_s = x_s - iy_s \quad (s = 1, 2, \ldots, n),$$

i designating $\sqrt{-1}$, this system will decompose into two systems

$$\frac{du_s}{dt} = (p_{s1} + iq_{s1})u_1 + (p_{s2} + iq_{s2})u_2 + \ldots + (p_{sn} + iq_{sn})u_n,$$

$$\frac{dv_s}{dt} = (p_{s1} - iq_{s1})v_1 + (p_{s2} - iq_{s2})v_2 + \ldots + (p_{sn} - iq_{sn})v_n,$$

$$(s = 1, 2, \ldots, n),$$

which may be integrated separately.

Now each of these systems will satisfy the conditions of the system considered above.

In fact, if we put

$$e^{i(2\pi t/\omega)} = z,$$

the coefficients of the first system will be represented by the series

$$p_{s\sigma} + iq_{s\sigma} = \sum_{m=0}^{\infty} (a_{s\sigma}^{(m)} + ib_{s\sigma}^{(m)})z^m,$$

not containing negative powers of z and defining, as a consequence, functions of the complex variable z not having singular points in the interior of the circle with radius $e^{2\pi h/\omega}$ and with centre at the point $z = 0$. Similarly, if we put

$$e^{-i(2\pi t/\omega)} = \zeta,$$

the coefficients of the second system will be represented by the series

$$p_{s\sigma} - iq_{s\sigma} = \sum_{m=0}^{\infty} (a_{s\sigma}^{(m)} - ib_{s\sigma}^{(m)})\zeta^m,$$

not including negative powers of ζ and, by consequence, defining functions of the complex variable ζ not having singular points in the interior of the circle of radius $e^{2\pi h/\omega}$ and with centre at the point $\zeta = 0$.

Consequently we can state that the roots of the characteristic equation of system (38) are obtained as follows:

Putting

$$\frac{1}{\omega}\int_0^{\omega} p_{s\sigma}\,dt = a_{s\sigma}, \quad \frac{1}{\omega}\int_0^{\omega} q_{s\sigma}\,dt = b_{s\sigma},$$

[$a_{s\sigma}$ and $b_{s\sigma}$ were previously represented by $a_{s\sigma}^{(0)}$ and $b_{s\sigma}^{(0)}$, and are the values of the functions $p_{s\sigma}\{z\}$ and $q_{s\sigma}\{z\}$ at $z = 0$] we replace the coefficients $p_{s\sigma}$, $q_{s\sigma}$ in this system by the quantities $a_{s\sigma}$, $b_{s\sigma}$ and we form the determinantal equation for the system with constant coefficients thus obtained. Let $\chi_1, \chi_2, \ldots, \chi_{2n}$ be the roots of this equation. Then the numbers [corresponding to (36)]

$$e^{\chi_1\omega}, e^{\chi_2\omega}, \ldots, e^{\chi_{2n}\omega}$$

will be the sought roots of the characteristic equation corresponding to the period ω.

Let us add that the relevant determinantal equation will be of the form $\Delta\Delta' = 0$, where

$$\Delta = \begin{vmatrix} a_{11} + ib_{11} - \chi & a_{12} + ib_{12} & \ldots & a_{1n} + ib_{1n} \\ a_{21} + ib_{21} & a_{22} + ib_{22} - \chi & \ldots & a_{2n} + ib_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} + ib_{n1} & a_{n2} + ib_{n2} & \ldots & a_{nn} + ib_{nn} - \chi \end{vmatrix},$$

and Δ' is deduced from it by replacing i by $-i$. [Recall that the original system has been decomposed into two independent subsystems.]

Study of the differential equations of the disturbed motion

54. [*Integration with the aid of series ordered according to powers of the arbitrary constants*]

Let there be given the differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s$$

$$(s = 1, 2, \ldots, n), \tag{39}$$

where the X_s designate as usual functions of $x_1, x_2, \ldots, x_n, t$, developable in series

$$X_s = \sum P_s^{(m_1, m_2, \ldots, m_n)} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

in positive integer powers of the variables $x_1, x_2, \ldots, x_n$ and not containing terms of degree less than the second.

We are going to consider here these equations under the assumption that all the coefficients $p_{s\sigma}$, $P_s^{(m_1, \ldots, m_n)}$ are periodic functions of t with one and the same real period ω.

Moreover, in considering exclusively real values of t, we shall suppose that these coefficients always remain determinate, continuous and real, and that the series by which the X_s are expressed represent uniformly holomorphic functions of the variables $x_1, x_2, \ldots, x_n$ for all real values of t (Section 33, Remark).

The coefficients in these series being periodic, the latter hypothesis is only another expression of the hypothesis [at the beginning] of Section 4 or of Section 11.

In place of system (39) we shall often consider various transformations of it, including among others transformations by means of linear substitutions with periodic coefficients.

These latter transformations will always be such that the coefficients in the transformed system will enjoy all the properties set out above.

We shall moreover be able to choose the said substitutions in such a way that, for the transformed system, the coefficients in the terms of first degree become constants, and such a transformation will be possible with substitutions with real coefficients, provided that the period ω is chosen so that the number $\omega/2$ is again a period for the coefficients of the system (39) (Section 47).

We shall often speak of the characteristic equation of the system of differential equations of the disturbed motion, understanding by this the characteristic equation of the system of linear differential equations relating to the first approximation. Further, we shall always assume that we have to do with the characteristic equation corresponding to the period ω, which, for definiteness, will be supposed positive.

Let us consider the series obtained by integrating the system (39) by the method indicated in Section 3.

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the roots of the characteristic equation of this system.

Fixing on any determination of the logarithms, let us put

$$\frac{1}{\omega} \log \rho_1 = \chi_1, \quad \frac{1}{\omega} \log \rho_2 = \chi_2, \quad \ldots, \quad \frac{1}{\omega} \log \rho_n = \chi_n.$$

Then if

$$x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}$$

are the ensembles of terms, in the series in question, of the mth dimension with respect to the arbitrary constants, we shall have for the quantities $x_s^{(m)}$ expressions of the form

$$x_s^m = \sum T_s^{(m_1, \ldots, m_n)} e^{(m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n)t}, \tag{40}$$

where the summation extends over all values of the non-negative integers $m_1, m_2, \ldots, m_n$ subject to the condition

$$0 < m_1 + m_2 + \ldots + m_n \leqslant m,$$

and where the $T_s^{(m_1, \ldots, m_n)}$ represent either periodic functions of t, or sums of a limited number of terms representing products of periodic functions and non-negative integer powers of t.†

We assure ourselves of this by considering in detail the expressions for the $x_s^{(m)}$ given in Section 3, and by taking into account formulae which we are going to write straightaway.

Let $f(t)$ be a periodic function of t with period ω, let m be a positive or zero integer and let χ be a constant such that the number $\chi\omega$ does not appear in the form $2\pi N\sqrt{-1}$, N being a real integer. We shall then have

$$\int e^{\chi t} t^m f(t)\, dt = e^{\chi t}[t^m f_0(t) + t^{m-1} f_1(t) + \dots + f_m(t)] + \text{const.},$$

$$\int t^m f(t)\, dt = \frac{h}{m+1} t^{m+1} + t^m \varphi_0(t) + t^{m-1}\varphi_1(t) + \dots + \varphi_m(t),$$

where all the $f_s(t)$, $\varphi_s(t)$ represent periodic functions of t with period ω, and h represents the following constant:

$$h = \frac{1}{\omega}\int_0^\infty f(t)\, dt.$$

[These relations can be verified by induction, with use of integration by parts.]

If, to form the series under consideration, the calculations are carried out so that the $x_s^{(m)}$ for $m > 1$ become zero for $t = 0$, these series, when the moduli of the arbitrary constants are sufficiently small, will actually represent functions satisfying our equations, at least within certain limits of variation of t.

But, on discarding the condition indicated, we can conduct the calculations in such a way that in expressions (40) all the terms for which

$$m_1 + m_2 + \dots + m_n < m,$$

disappear, and the expressions for the $T_s^{(m_1, m_2, \dots, m_n)}$ take the form

$$T_s^{(m_1, m_2, \dots, m_n)} = K_s^{(m_1, m_2, \dots, m_n)} \alpha_1^{m_1}\alpha_2^{m_2} \dots \alpha_n^{m_n},$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are arbitrary constants and the $K_s^{(m_1, \dots, m_n)}$ are functions of t independent of these constants.

If we consider the series thus obtained as ordered in powers of the quantities

$$\alpha_1 e^{\chi_1 t}, \alpha_2 e^{\chi_2 t}, \dots, \alpha_n e^{\chi_n t},$$

the coefficients in them will be sums of a finite number of periodic and secular terms.‡

With regard to the convergence of these series, we shall not in general be able to draw any conclusion. But in the case where among the numbers χ_s there are found

$$\chi_1, \chi_2, \dots, \chi_k, \tag{41}$$

† The periodic functions we are concerned with here possess the period ω and remain determinate and continuous for all real values of t. In general, all the periodic functions of t which we shall encounter in the sequel will enjoy the same properties. But, for brevity, we shall not always mention this expressly.

‡ We shall call *secular* all terms of the form $t^m f(t)$, where m is a positive integer and $f(t)$ is a periodic function.

for which the real parts are different from zero and all of the same sign, and when these series are formed under the assumption that

$$\alpha_{k+1} = \alpha_{k+2} = \ldots = \alpha_n = 0,$$

we shall have for them a theorem entirely similar to that which was enunciated in Section 23.

In this case, for an arbitrary choice of the constants $\alpha_1, \alpha_2, \ldots, \alpha_k$ the series under consideration will define a solution of system (39), either for every value of t greater than a certain limit (depending on the choice of the constants α_s) when the real parts of the numbers (41) are all negative, or for every value of t less than a certain limit when the real parts of these numbers are all positive.

55. [*Theorems on the conditions for stability and for instability supplied by a first approximation. Singular cases. Definition of those which will be the subject of subsequent investigations*]

From what precedes we next extract some propositions relating to conditions for stability in the case which now interests us.

Thus, from Theorem II of Section 13 we deduce the following.

THEOREM I. *Whenever the characteristic equation has roots with moduli less than* 1, *the undisturbed motion will possess a certain conditional stability, and among the perturbations there will be some for which the disturbed motion will approach asymptotically the undisturbed motion. If the number of the said roots is k, these perturbations will depend on k arbitrary constants.*

With respect to absolute stability, Theorem I of the cited section, account being taken of what was noted in Section 26 (Remark), leads to the following proposition.

THEOREM II. *When the characteristic equation only has roots of which the moduli are less than* 1, *the undisturbed motion will be stable, and in such a manner that every disturbed motion which is sufficiently near will approach it asymptotically. But if among the roots of this equation there are any for which the moduli are greater than* 1, *this motion will be unstable.*

It results from this theorem that doubt concerning stability only remains in the case where the characteristic equation, without having roots with moduli greater than 1, has roots with moduli equal to 1.

However, for many problems, such cases, which we may call *singular*, are the only ones where absolute stability is possible.

Such are, for example, problems in which the system of differential equations of the disturbed motion has the canonical [Hamiltonian] form.

We know (Section 51) that, for such a system, to each root ρ of the characteristic equation there will correspond a root equal to $1/\rho$. Consequently, absolute stability will only be possible if all the roots have moduli equal to 1.

Problems of stability in singular cases, even for steady motion, are very difficult. And for periodic motion the difficulties assuredly become still greater. However, in certain cases of this sort (under the condition that we have managed to integrate the system of linear differential equations corresponding to the first approximation), we can propose general methods of which we can make use in this kind of investigation; this is what we are going to do now.

In a similar way to that of the preceding chapter, we are going to consider here the following two cases in succession:

(1) the characteristic equation has one root equal to 1, the other roots having moduli less than 1; and

(2) this equation has two conjugate complex roots with moduli equal to 1, all the other roots, as in the first case, having moduli less than 1.

We have not indicated the case where the characteristic equation has one root equal to -1, the other roots having moduli less than 1, since this case reduces to the first of the two previous ones, on taking for the period a number twice as great as the original period. [Recall that the roots of the characteristic equation corresponding to period 2ω are the squares of the roots of the characteristic equation corresponding to period ω.]

First case. Characteristic equation with one root equal to unity

56. [*Reduction of the differential equations to a suitable form*]

Let us assume that the characteristic equation of the system under consideration (which will be taken to be of order $n + 1$) has one root equal to 1 and n roots with moduli less than 1.

By virtue of what has been expounded in Section 47, we can suppose that by means of a linear substitution with periodic coefficients our system is reduced to the following form:

$$\left.\begin{aligned} &\frac{dx}{dt} = X, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + p_s x + X_s \\ &\qquad\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{42}$$

where X, X_s, which represent holomorphic functions of the variables $x, x_1, x_2, \dots, x_n$, do not contain in their expansions terms of degree less than the second.

The coefficients in these expansions, like the coefficients p_s, are periodic functions of t. As for the coefficients $p_{s\sigma}$, we shall suppose them to be constants and, in agreement with what we have assumed [namely that the characteristic equation has n roots with moduli less than 1], such that the equation

$$\begin{vmatrix} p_{11} - \chi & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} - \chi & \dots & p_{2n} \\ \dots & \dots & \dots & \dots \\ p_{n1} & p_{n2} & \dots & p_{nn} - \chi \end{vmatrix} = 0 \tag{43}$$

only has roots with negative real parts.

We shall assume moreover that all the coefficients in system (42) are real.

In two cases, as we shall see, the question of stability will be resolved immediately by the form of equations (42).

If $X^{(0)}$, $X_s^{(0)}$ are what X, X_s become when we put

$$x_1 = x_2 = \dots = x_n = 0,$$

one of these cases will be that where all the p_s are zero, and where the expansion of the function $X^{(0)}$ in powers of x begins with a term with a constant coefficient and with degree not greater than the least power of x entering into the expansions of the functions $X_s^{(0)}$. The other case will be that where $X^{(0)}$, all the $X_s^{(0)}$ and all the p_s are identically zero.

As far as all other possible cases are concerned, they will reduce, as we shall show immediately, to the two cases which we have just mentioned.

Let us seek to satisfy equations (42) by the series [similar to (49) of Section 34]

$$\left.\begin{aligned} x &= c + u^{(2)}c^2 + u^{(3)}c^3 + \dots \\ x_s &= u_s^{(1)}c + u_s^{(2)}c^2 + u_s^{(3)}c^3 + \dots \\ &(s = 1, 2, \dots, n), \end{aligned}\right\} \tag{44}$$

ordered in positive integer powers of the arbitrary constant c, with this condition, that the coefficients $u^{(l)}$, $u_s^{(l)}$ represent either periodic functions of t or sums of a finite number of periodic and secular terms.

The calculation of these coefficients will depend on differential equations which will be basically of the same character as the equations we dealt with in Section 34, and, just as there, we shall arrive at the conclusion that, if among the functions $u^{(l)}$, $u_s^{(l)}$ there exist any non-periodic ones, we shall find them already in the series of functions

$$u^{(2)}, u^{(3)}, u^{(4)}, \dots,$$

and that if $u^{(m)}$ is the first non-periodic function in this series, the functions

$$u_s^{(1)}, u_s^{(2)}, \dots, u_s^{(m-1)} \quad (s = 1, 2, \dots, n)$$

will all be periodic, and this $u^{(m)}$ will be of the form [compare (51) of Section 34]

$$u^{(m)} = gt + v,$$

where g is a non-zero constant and v is a periodic function of t.

Taking it that we are concerned with this case and that the calculation is carried out in such a way that all the $u^{(l)}$, $u_s^{(l)}$ become real, let us transform system (42) by means of the substitution

$$x = z + u^{(2)}z^2 + \dots + u^{(m-1)}z^{m-1} + vz^m,$$

$$x_s = u_s^{(1)}z + u_s^{(2)}z^2 + \dots + u_s^{(m-1)}z^{m-1} + z_s$$

$$(s = 1, 2, \dots, n).$$

We shall then arrive at a system of the original form

$$\left.\begin{aligned} &\frac{dz}{dt} = Z, \\ &\frac{dz_s}{dt} = p_{s1}z_1 + p_{s2}z_2 + \dots + p_{sn}z_n + Z_s \\ &(s = 1, 2, \dots, n), \end{aligned}\right\} \tag{45}$$

but satisfying the conditions of the first of the two cases indicated above. In fact, we easily convince ourselves that if $Z^{(0)}$, $Z_s^{(0)}$ are what Z, Z_s become for $z_1 = z_2 = \dots = z_n = 0$, the expansion of the function $Z^{(0)}$ in increasing powers of z will begin with the mth power, which will have a constant coefficient g, and at the same time the expansions of the functions $Z_s^{(0)}$ will not contain z with powers less than the mth. [Compare the treatment after (52) of Section 34.]

Let us now assume that we are dealing with the case where $u^{(l)}$, $u_s^{(l)}$ are all found to be periodic, however great the number l.

Then, just as in Section 35, we may show that if the calculation is carried out according to the rule that all the $u^{(l)}$ become zero for one and the same value of t, for example $t = 0$, the series (44), $|c|$ being sufficiently small, will converge uniformly for all real values of t.

These series will then define a periodic solution of system (42), and for every sufficiently small real value of c there will correspond to this solution a periodic motion. We shall thus find ourselves with the case where there exists a continuous series of periodic motions including the undisturbed motion under consideration.

In this case, on transforming system (42) by means of the substitution [compare the last equations of Section 35]

$$x = z + u^{(2)}z^2 + u^{(3)}z^3 + \dots,$$
$$x_s = z_s + u_s^{(1)}z + u_s^{(2)}z^2 + \dots$$
$$(s = 1, 2, \dots, n),$$

we shall obtain a system of the form (45), where Z and all the Z_s will become zero for $z_1 = z_2 = \dots = z_n = 0$. We shall find ourselves, as a consequence, with the second of the two cases indicated above.

In the two cases our transformations are such that the problem of stability with respect to the old variables x, x_s will be entirely equivalent to the problem of stability with respect to the new ones z, z_s.

Let us note further that if the functions X, X_s are, with respect to the variables x, x_σ, *uniformly* holomorphic for all real values of t (which will hold by virtue of what was assumed in [the initial part of] Section 54), it will be the same for the functions Z, Z_s with respect to the variables z, z_σ.

57. [*Study of the general case*]

Let us consider the system (45) under the assumption that it satisfies the conditions of the first case. [This case was specified in the previous section.]

On designating by g a non-zero constant, by $P^{(1)}, P^{(2)}, \dots, P^{(m-1)}$ linear forms in the quantities z_s and by Q a quadratic form in the same quantities, all these forms having coefficients independent of z and periodic with respect to t, let us suppose that we have

$$Z = gz^m + P^{(1)}z + P^{(2)}z^2 + \dots + P^{(m-1)}z^{m-1} + Q + \dots,$$

such that those of the subsequent terms which are linear with respect to the z_s (including those which do not depend on them at all) are at least of degree $m + 1$, and the others are at least of third degree with respect to z, z_s.

As we are dealing by assumption with the first case, the functions Z_s, in the terms independent of the quantities z_σ, will not contain z in powers less than the mth.

As a consequence, only considering besides these terms those which are linear with respect to the quantities z_σ, and ordering the ones and the others in increasing powers of z, we may take it that

$$Z_s = g_s z^m + \dots + P_s^{(1)} z + P_s^{(2)} z^2 + P_s^{(3)} z^3 + \dots .$$

Here the g_s are periodic functions of t and the $P_s^{(j)}$ are linear forms in the variables z_σ with periodic coefficients.

This settled, let us designate by $U^{(1)}, U^{(2)}, \dots, U^{(m-1)}$ linear forms and by W a quadratic form in the variables z_s with undetermined coefficients which will be supposed periodic functions of t.

Considering first the case of m even, let us put

$$V = z + U^{(1)} z + U^{(2)} z^2 + \dots + U^{(m-1)} z^{m-1} + W$$

and let us seek to dispose of the linear forms $U^{(k)}$ so that in the expression for the derivative dV/dt, formed in accordance with our differential equations, all the terms disappear which are linear with respect to the quantities z_s and contain z in powers less than the mth.

For this we must choose these forms in such a way that they satisfy the equations [compare similar equations in Sections 29 and 37]

$$\sum_{s=1}^{n} (p_{s1} z_1 + \dots + p_{sn} z_n) \frac{\partial U^{(k)}}{\partial z_s} + \frac{\partial U^{(k)}}{\partial t} + P^{(k)}$$
$$+ \sum_{s=1}^{n} \left(P_s^{(1)} \frac{\partial U^{(k-1)}}{\partial z_s} + \dots + P_s^{(k-1)} \frac{\partial U^{(1)}}{\partial z_s} \right) = 0$$
$$(k = 1, 2, \dots, m-1)$$

[the left-hand side here represents the contribution to the coefficient of z^k in the expression for dV/dt, such that this contribution is linear in the z_s] (where the second sum, for $k = 1$, must be replaced by zero); and this is always possible, the roots of equation (43) having their real parts negative. Moreover, the condition that the coefficients of the forms $U^{(k)}$ are periodic makes this problem completely determinate. [See the discussion following (63) of Section 37.]

The forms $U^{(k)}$ being so chosen, we shall choose the form W in accordance with the equation

$$\sum_{s=1}^{n} (p_{s1} z_1 + p_{s2} z_2 + \dots + p_{sn} z_n) \frac{\partial W}{\partial z_s} + \frac{\partial W}{\partial t} + Q = g(z_1^2 + z_2^2 + \dots + z_n^2),$$

which equally is always possible.

Then we shall have [similar manipulations are described in more detail in Section 29]

$$\frac{dV}{dt} = g(z^m + z_1^2 + z_2^2 + \dots + z_n^2) + S,$$

on understanding by S an expression of the form

$$S = vz^m + \sum_{s=1}^{n} \sum_{\sigma=1}^{n} v_{s\sigma} z_s z_\sigma,$$

where v, $v_{s\sigma}$ are functions of the variables $t, z, z_1, z_2, \ldots, z_n$, becoming zero for

$$z = z_1 = z_2 = \ldots = z_n = 0,$$

and being periodic with respect to t, and uniformly holomorphic with respect to z, z_s for all real values of t.

Our function V will satisfy, by consequence, all the conditions of Theorem II of Section 16. We must thus conclude that the undisturbed motion is unstable.

Let us now consider the case of m odd.

On putting

$$V = W + \tfrac{1}{2}z^2 + U^{(1)}z^2 + U^{(2)}z^3 + \ldots + U^{(m-1)}z^m,$$

let us choose the quadratic form W with constant coefficients in conformity with the equation

$$\sum_{s=1}^{n} (p_{s1}z_1 + p_{s2}z_2 + \ldots + p_{sn}z_n)\frac{\partial W}{\partial z_s} = g(z_1^2 + z_2^2 + \ldots + z_n^2). \tag{46}$$

Next let us dispose of the linear forms $U^{(j)}$ so that in the expression for the derivative dV/dt, formed in accordance with our differential equations, all the terms disappear which are linear with respect to the quantities z_s and contain z in powers less than the $(m+1)$th; which requires these forms to be determined by the equations [similar to the equations following (31) in Section 29]

$$\sum_{s=1}^{n} (p_{s1}z_1 + \ldots + p_{sn}z_n)\frac{\partial U^{(k)}}{\partial z_s} + \frac{\partial U^{(k)}}{\partial t} + P^{(k)}$$
$$+ \sum_{s=1}^{n}\left(P_s^{(1)}\frac{\partial U^{(k-1)}}{\partial z_s} + \ldots + P_s^{(k-1)}\frac{\partial U^{(1)}}{\partial z_s}\right) = 0$$
$$(k = 1, 2, \ldots, m-2),$$

$$\sum_{s=1}^{n} (p_{s1}z_1 + p_{s2}z_2 + \ldots + p_{sn}z_n)\frac{\partial U^{(m-1)}}{\partial z_s} + \frac{\partial U^{(m-1)}}{\partial t} + P^{(m-1)}$$
$$+ \sum_{s=1}^{n}\left(g_s\frac{\partial W}{\partial z_s} + P_s^{(1)}\frac{\partial U^{(m-2)}}{\partial z_s} + \ldots + P_s^{(m-2)}\frac{\partial U^{(1)}}{\partial z_s}\right) = 0.$$

We shall then have

$$\frac{dV}{dt} = g(z^{m+1} + z_1^2 + z_2^2 + \ldots + z_n^2) + S,$$

S being an expression of the form

$$S = vz^{m+1} + \sum_{s=1}^{n}\sum_{\sigma=1}^{n} v_{s\sigma}z_s z_\sigma,$$

with the same significance for the symbols v, $v_{s\sigma}$ as in the preceding case.

In this way, the forms W, $U^{(j)}$ being chosen as has just been shown, the derivative of the function V will be a definite function of the variables z, z_s, t, [$|z|$, $|z_s|$ being small], and its sign when $|z|$, $|z_s|$ are sufficiently small will be that of g.

Therefore, on noting that because of equation (46) the form W will be definite and of sign opposite to that of g (Section 20, Theorem II), we conclude, as in Section 29, that in the case of $g < 0$ the undisturbed motion will be stable, and in that of $g > 0$ it will be unstable.

We can further state that, for $g < 0$, the disturbed motions corresponding to any sufficiently small perturbations will approach asymptotically the undisturbed motion.

58. [*Study of an exceptional case*]

Let us now consider system (45) under the assumption that it satisfies the conditions of the second case, i.e. under the assumption that the Z, Z_s all become zero for $z_1 = z_2 = \dots = z_n = 0$.

As in Section 38, we shall demonstrate that in this case there will exist a complete integral with equation of the form

$$z = c + f(z_1, z_2, \dots, z_n, c, t),$$

where c is an arbitrary constant and f is a function of the quantities z_s, c, t which is holomorphic with respect to z_s, c and uniformly so for all real values of t. We suppose that this function contains in its expansion neither terms of first degree with respect to z_s, c, nor terms independent of the quantities z_s, and that the coefficients in it are real periodic functions of t.†

On making use of this equation to eliminate the variable z and on considering the system that we deduce in this way from (45), we may easily prove (Section 38) that, under our assumption, the undisturbed motion will always be stable, and that every disturbed motion sufficiently near this motion will approach asymptotically one of the periodic motions

$$z = c, \quad z_1 = z_2 = \dots = z_n = 0,$$

which will again be stable, as long as $|c|$ is small enough.

Remark

From what we have just said it results that, if we can satisfy system (42) by the periodic series (44), this system will admit a holomorphic integral of the form

$$x + F(x, x_1, x_2, \dots, x_n, t), \tag{47}$$

where F represents a holomorphic function of the variables $x, x_1, \dots, x_n$, for which the expansion does not contain terms of degree less than the second and possesses periodic coefficients with respect to t.

Thus every holomorphic integral, periodic with respect to t, will be a holomorphic function of an integral with the form (47).

It is also easy to establish the converse proposition: if system (42) admits an integral, periodic with respect to t and holomorphic with respect to x, x_s, it will equally admit a periodic solution of the form (44) (Section 38, Remark, and Section 44).

† System (45) is a little more general than that with which we were concerned in Section 38 [since the coefficients in the expansions of Z, Z_s can now have an infinite number of terms in their Fourier expansions]. But this circumstance, from which arises the more general form for the integral considered here, does not entail essential modifications to the proof. This will be based as before on the following three assumptions: (1) that the roots of equation (43) have all their real parts non-zero and with one and the same sign; (2) that the functions Z, Z_s all become zero for $z_1 = z_2 = \dots = z_n = 0$; (3) that these functions, with respect to z, z_σ are uniformly holomorphic for all real values of t.

59. [*Exposition of the method. Example*]

The conclusions we have arrived at can be summarized in the following manner.

The differential equations of the disturbed motion being reduced to form (42), *we seek a solution of them depending on an arbitrary constant c, in the form of the series* (44) *ordered in positive integer powers of this constant, in accordance with the following condition which is always achievable: that the coefficients* $u_s^{(1)}$ *in these series are periodic functions of t, that all the* $u_s^{(2)}$ *are equally so if the coefficient* $u^{(2)}$ *is a periodic function and, in general, that all the* $u_s^{(l)}$ *are periodic if all the* $u^{(j)}$ *for* $j \leqslant l$ *are periodic. Under this hypothesis let us suppose that* $u^{(m)}$ *is the first non-periodic function in the series*

$$u^{(2)}, u^{(3)}, u^{(4)}, \dots \tag{48}$$

Then, if m is an even number, we must conclude that the undisturbed motion is unstable. If on the other hand m is an odd number, to resolve the question we have to consider the expression for the function $u^{(m)}$, *which will always be of the form*

$$u^{(m)} = gt + v,$$

where g is a non-zero constant and v is a periodic function of t. The question will then be resolved in accordance with the sign of the constant g: in the case of $g > 0$ *the undisturbed motion will be unstable, and in that of* $g < 0$ *it will be stable.*

It can happen that in series (48), *however far it is extended, all the functions* $u^{(l)}$ *are periodic. In this case there will exist a continuous series of periodic motions, including the undisturbed motion under consideration, and all the motions of this series sufficiently near the undisturbed motion, the latter included, will be stable.*

Remark 1

To form functions (48) we can if we wish also make use of a procedure similar to that which was indicated at the end of Section 40, when an analogous problem was treated.

For this we shall consider the following system of partial differential equations:

$$X\frac{\partial x_s}{\partial x} + \frac{\partial x_s}{\partial t} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + p_s x + X_s$$

$$(s = 1, 2, \dots, n),$$

defining the quantities x_s as functions of the independent variables x and t.

We easily convince ourselves that, under the conditions considered here, it will always be possible to satisfy this system (at least formally) by series ordered in positive integer powers of the variable x not containing terms independent of x, and possessing coefficients periodic with respect to t. Further, we see without difficulty that this problem will be completely determinate.

On introducing these series into the expression for the function X and on representing the result in the form of the series

$$X^{(2)}x^2 + X^{(3)}x^3 + X^{(4)}x^4 + \dots,$$

ordered in increasing powers of x, we shall next be able to calculate the functions (48) successively in accordance with the condition that, for every integer value of k

greater than 1, all the terms containing the constant c in powers less than the $(k+1)$th disappear in the expression

$$\frac{dx}{dt} - X^{(2)}x^2 - X^{(3)}x^3 - \ldots - X^{(k)}x^k$$

after we have put in it

$$x = c + u^{(2)}c^2 + u^{(3)}c^3 + \ldots + u^{(k)}c^k$$

It may be noted that in the case where the system (42) admits a periodic solution, the series under consideration, $|x|$ being sufficiently small, will certainly be convergent. When on the other hand there does not exist such a solution, we shall not be able to say anything, in general, on the subject of their convergence. But this circumstance is not of any importance for our problem.

Remark II

We have supposed that the coefficients $p_{s\sigma}$ in equations (42) are constant quantities. But this assumption was only made to simplify the proofs, and it is not at all necessary for the applicability of the procedure which has just been indicated.

Example

Let there be proposed the equations

$$\frac{dx}{dt} = ay^k, \quad \frac{dy}{dt} + py = bx^n,$$

where p designates a real periodic function of t, having for period ω and such that the integral

$$\int_0^\omega p\, dt$$

has a positive value, and a, b, k, n are real constants, among which k and n represent positive whole numbers, with k not less than 2.

[The linear system of the first approximation is (if $n > 1$)

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} + py = 0,$$

with general solution

$$x = A, \quad y = Be^{-\int_0^t p\, dt}$$

where A and B are arbitrary constants. It follows that the roots of the characteristic equation are

$$1 \quad \text{and} \quad e^{-\int_0^\omega p\, dt}$$

and because of the above stipulation the second of these has modulus less than 1.]

We shall assume that neither of the constants a and b is zero, for in the contrary case the question of stability would be resolved at once in the affirmative sense.

Operating in accordance with what has been expounded, let us seek to satisfy our equations by the series [corresponding to (44)]

$$x = c + u_2 c^2 + u_3 c^3 + \dots,$$

$$y = v_1 c + v_2 c^2 + v_3 c^3 + \dots,$$

on requiring the coefficients u_l, v_l to be periodic functions of t, as far as is possible.

[To begin with, on substituting these series in the second differential equation above and equating coefficients of c, c^2, c^3, ..., we find (for $n > 1$)

$$\frac{dv_j}{dt} + pv_j = 0 \quad (j = 1, 2, \dots, n-1).$$

Separation of the variables and integration of both sides yields

$$v_j = K_j e^{-\int_0^t p\,dt} \qquad (j = 1, 2, \dots, n-1)$$

where the K's are constants. Because of the initial assumption, $p(t)$ has a non-zero mean value, so its integral is non-periodic. Hence v_j here is periodic only if K_j is zero.]

We shall then find that all the v_l for $l < n$ will be zero; that v_n, as the periodic solution of the equation

$$\frac{dv_n}{dt} + pv_n = b,$$

will be given by the formula

$$v_n = be^{-\int_0^t p\,dt} \int_{-\infty}^t e^{\int_0^t p\,dt}\,dt,$$

[the periodicity of this formula can be shown on writing

$$e^{\int_0^t p\,dt} = e^{Mt} F(t)$$

where M represents the mean value of $p(t)$, and the factor $F(t)$, being periodic, is expressed as a Fourier expansion] and that the first non-periodic function in the series u_2, u_3, ... will be u_{kn}, which will be obtained from the equation

$$\frac{du_{kn}}{dt} = av_n^k.$$

From this we conclude that

$$g = \frac{ab^k}{\omega} \int_0^\omega e^{-k\int_0^t p\,dt} \left(\int_{-\infty}^t e^{\int_0^t p\,dt}\,dt \right)^k dt$$

[this being the mean value of av_n^k]. Moreover we obtain [as the subscript of the first of the u's to be non-periodic]

$$m = kn.$$

Consequently [with application of the rule stated at the beginning of this section], if at least one of the numbers k and n is even, the undisturbed motion will be unstable [m being even]. If on the other hand both these two numbers are odd, this motion will be stable or unstable according as the signs of the constants a and b are different or the same [m is now odd, and g has the sign of ab^k since the main integrand in the above expression for g is positive].

Second case. Characteristic equation with two complex roots with moduli equal to unity

60. [*General form to which the differential equations reduce*]

Let us now consider the case where the characteristic equation of the proposed system has two complex conjugate roots with moduli equal to 1, assuming that all the other roots of this equation (if it is of higher degree than the second) have moduli less than 1.

Let

$$e^{\lambda\omega\sqrt{-1}} \quad \text{and} \quad e^{-\lambda\omega\sqrt{-1}}$$

be the two roots having moduli equal to unity.

We understand here by λ a real number which is for the moment not subject to any restriction. But in investigations which are going to follow we shall assume that $\lambda\omega/\pi$ is an incommensurable [i.e. irrational] number.

On this subject it is to be noted that if the number $\lambda\omega/\pi$ were commensurable, the case under consideration would reduce to that where the characteristic equation has two roots equal to 1. It would suffice for this to take for the period a certain integer multiple of the original period ω. [If

$$\frac{\lambda\omega}{\pi} = \frac{M}{N}$$

where M and N are integers, we have as roots of the characteristic equation corresponding to period $2N\omega$

$$(e^{\pm\lambda\omega\sqrt{-1}})^{2N} = e^{\pm M2\pi\sqrt{-1}} = 1.]$$

Now such a case [two roots equal to 1] requires a special investigation which we have no intention of taking up here.

We can assume that our system of differential equations (which we shall designate as having order $n+2$) is reduced, by means of a linear substitution with periodic coefficients, to the form [similar to (45) of Section 33]:

$$\left.\begin{aligned} &\frac{dx}{dt} = -\lambda y + X, \quad \frac{dy}{dt} = \lambda x + Y, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + p_s x + q_s y + X_s \\ &\qquad\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{49}$$

where X, Y, X_s are holomorphic functions of the variables $x, y, x_1, x_2, \dots, x_n$ for which the expansions, possessing real and peiodic coefficients with respect to t, do not contain terms of degree less than the second. The coefficients p_s, q_s are real periodic functions of t, and the coefficients $p_{s\sigma}$ are real constants such that the equation of form (43) only has roots with negative real parts.

We may further suppose that the functions X and Y become zero for $x = y = 0$, for every other case reduces to this with the aid of a transformation similar to that which we considered in Section 33.

This transformation arises from the proposition that, under the conditions considered, we can always satisfy the system of partial differential equations [which interpret the first two of equations (49)]

$$\left.\begin{aligned}\sum_{s=1}^{n}(p_{s1}x_1+p_{s2}x_2+\dots+p_{sn}x_n+p_sx+q_sy+X_s)\frac{\partial x}{\partial x_s}+\frac{\partial x}{\partial t}&=-\lambda y+X,\\ \sum_{s=1}^{n}(p_{s1}x_1+p_{s2}x_2+\dots+p_{sn}x_n+p_sx+q_sy+X_s)\frac{\partial y}{\partial x_s}+\frac{\partial y}{\partial t}&=\lambda x+Y\end{aligned}\right\}\quad(50)$$

by holomorphic functions of the variables $x_1, x_2, \dots, x_n$, not containing in their expansions terms below the second dimension, and possessing periodic coefficients with respect to t.

As for this proposition, it is easily proved with the aid of the same reasoning as we made use of in the proof of the theorem of Section 30.

For this, we note that if $\chi_1, \chi_2, \dots, \chi_n$ are the roots of equation (43), these roots having all their real parts negative, system (49) will admit a solution containing n arbitrary constants $\alpha_1, \alpha_2, \dots, \alpha_n$, in which the functions x, y, x_s will be given by series proceeding in positive integer powers of the quantities

$$\alpha_1 e^{\chi_1 t}, \alpha_2 e^{\chi_2 t}, \dots, \alpha_n e^{\chi_n t}, \tag{51}$$

with coefficients representing either periodic functions of t or sums of a finite number of periodic and secular terms (Section 54). Moreover the series by which the functions x and y will be expressed will not contain terms of degree less than the second with respect to quantities (51). On the other hand the series representing the functions x_s will also contain terms of the first degree, and the coefficients in these terms will be such that their determinant will be a non-zero constant.

Consequently, on eliminating the quantities (51), we shall be able to deduce from this solution expressions for the functions x and y in the form of series ordered in positive integer powers of the quantities x_s, with coefficients of the same character as before, and these series will not contain terms of degree less than the second with respect to the variables x_s.

These series will define, for every real value of t, holomorphic functions of the variables x_s satisfying system (50).

It only remains for us, as a consequence, to prove that the coefficients in the series thus obtained will be necessarily periodic functions of t. Now we may prove this without difficulty by considering more closely the equations which we shall have to satisfy in order that our series define a solution of system (50), and by taking into account that the χ_s have their real parts negative.

We shall then see also that system (50) can only admit one solution of the character considered.

Let u and v be the expressions for the functions x and y in this solution.

Then, to bring system (49) to the required form, we shall only have to introduce instead of x and y the variables ξ and η, by means of the substitution

$$x = u + \xi, \quad y = v + \eta.$$

With this transformation system (49) will not lose any of its properties; and moreover the problem of stability with respect to the variables x, y, x_s will be entirely equivalent to that of stability with respect to the variables ξ, η, x_s.

This settled, we can consider system (49) under the assumption that the functions X and Y become zero when we put $x = y = 0$.

With this assumption, on introducing in place of the variables x and y variables r and ϑ by means of the substitution

$$x = r\cos\vartheta, \quad y = r\sin\vartheta,$$

we shall arrive at the equations [similar to (46) and (47) of Section 33]:

$$\left.\begin{aligned} &\frac{dr}{dt} = rR, \quad \frac{d\vartheta}{dt} = \lambda + \Theta, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n \\ &\qquad\qquad + (p_s\cos\vartheta + q_s\sin\vartheta)r + X_s \\ &\qquad\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{52}$$

where, in the functions X_s, the quantities x and y are taken as replaced by their expressions in r and ϑ.

We have here designated by R and Θ holomorphic functions of the quantities r, x_s becoming zero for $r = x_1 = \dots = x_n = 0$, for which the coefficients can be presented in the form of finite sequences of sines and cosines of integer multiples of ϑ with coefficients periodic with respect to t; it is to be noted that the coefficients in the expansions of the functions X_s in powers of the quantities $r, x_1, x_2, \dots, x_n$ will be of the same character.

Our problem is thus reduced to that of stability with respect to the quantities r, x_s, and in treating it we shall be able to impose the condition $r \geqslant 0$, as we did in the previous chapter when studying an analogous case (Section 33).

61. [*Certain characteristic series depending on two arguments. General case where these series are not periodic*]

On considering the quantities r, x_s as functions of the independent variables ϑ and t, let us form the following system of partial differential equations [which interpret (52)]:

$$\begin{aligned} &\frac{\partial r}{\partial t} + (\lambda + \Theta)\frac{\partial r}{\partial\vartheta} = rR, \\ &\frac{\partial x_s}{\partial t} + (\lambda + \Theta)\frac{\partial x_s}{\partial\vartheta} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n \\ &\qquad\qquad\qquad\qquad + (p_s\cos\vartheta + q_s\sin\vartheta)r + X_s \\ &\qquad\qquad\qquad\qquad (s = 1, 2, \dots, n), \end{aligned}$$

and let us seek to satisfy them by the series

$$\left.\begin{aligned} &r = c + u^{(2)}c^2 + u^{(3)}c^3 + \dots, \\ &x_s = u_s^{(1)}c + u_s^{(2)}c^2 + u_s^{(3)}c^3 + \dots \\ &\qquad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{53}$$

ordered in powers of the arbitrary constant c, under the condition that the coefficients $u^{(l)}$, $u_s^{(l)}$ appear in the form of finite sequences of sines and cosines of integer multiples of ϑ, where the coefficients are periodic functions of t or sums of a finite number of periodic and secular terms.

Let us suppose in this investigation that $\lambda\omega/\pi$ is an incommensurable number.

To determine the functions $u^{(l)}$, $u_s^{(l)}$ we shall have systems of equations of the following form [compare (50) of Section 34]:

$$\frac{\partial u^{(l)}}{\partial t} + \lambda \frac{\partial u^{(l)}}{\partial \vartheta} = U^{(l)},$$

$$\frac{\partial u_s^{(l)}}{\partial t} + \lambda \frac{\partial u_s^{(l)}}{\partial \vartheta} = p_{s1}u_1^{(l)} + p_{s2}u_2^{(l)} + \dots + p_{sn}u_n^{(l)}$$

$$+ (p_s \cos \vartheta + q_s \sin \vartheta)u^{(l)} + U_s^{(l)}$$

$$(s = 1, 2, \dots, n),$$

where $U^{(l)}$, $U_s^{(l)}$, if $l = 1$, are identically zero, and if $l > 1$, represent entire and rational [polynomial] functions of the $u^{(i)}, u_s^{(i)}, \partial u^{(i)}/\partial\vartheta, \partial u_s^{(i)}/\partial\vartheta$ for $i < l$, with coefficients of the same character as in the expansions of the functions R, Θ, X_s.

Assuming that all the $u^{(i)}$, $u_s^{(i)}$ for $i < l$ are already found, let us present the functions $U^{(l)}$, $U_s^{(l)}$ in the form of finite sequences of sines and cosines of integer multiples of ϑ, next transforming them into those of the form

$$\sum F_k e^{k\vartheta\sqrt{-1}}. \tag{54}$$

[Use is made here of the relations (where $j = \sqrt{-1}$)

$$\sin k\vartheta = \tfrac{1}{2}j(e^{-jk\vartheta} - e^{jk\vartheta}), \quad \cos k\vartheta = \tfrac{1}{2}(e^{-jk\vartheta} + e^{jk\vartheta}).]$$

The summation here extends over all integer values, positive and negative, which are contained between certain limits $-N$ and $+N$, and the coefficients F_k represent functions of t alone.

If all the $u^{(i)}$, $u_s^{(i)}$ are periodic with respect to t, it will be the same for all the F_k for each of the functions $U^{(l)}$, $U_s^{(l)}$.

Under this hypothesis, let us seek the functions $u^{(l)}$, $u_s^{(l)}$ in the form of sequences like (54).

We must then begin with the function $u^{(l)}$, and if we put

$$u^{(l)} = \sum f_k e^{k\vartheta\sqrt{-1}},$$

on understanding by the f_k functions of t alone, to determine these functions we shall obtain [from the first equation after (53)] equations of the form

$$\frac{df_k}{dt} + k\lambda\sqrt{-1}f_k = F_k. \tag{55}$$

[The general solution of this equation is

$$f_k = e^{-k\lambda\sqrt{-1}t}\left(\int_0^t e^{k\lambda\sqrt{-1}t}F_k(t)\,dt + K\right) \tag{XXV}$$

where K is an arbitrary constant; and this solution will be periodic if the integrand has mean value zero and $K = 0$.]

Now, under our assumption about λ, such an equation, when k is not zero, will supply for the function f_k a completely determined expression of the required character; and this expression, in accordance with what was assumed for the F_k, will represent a periodic function of t. Therefore the function $u^{(l)}$ will only be able to be non-periodic with respect to t in the case where the function f_0 is found to be non-periodic.

But let us assume that the latter is periodic, which supposes that we have

$$\int_0^\omega F_0\,dt = 0$$

[i.e. $F_0(t)$ has zero mean value].

Then, passing to the functions $u_s^{(l)}$, and taking into account the property assumed for the roots of equation (43), we may easily demonstrate that these functions will equally be periodic.

It results from this that if among the functions $u^{(l)}$, $u_s^{(l)}$, starting from a certain value of l, there appear some non-periodic ones with respect to t (and that will occur in most cases), there will already be found some in the series

$$u^{(2)}, u^{(3)}, u^{(4)}, \ldots, \tag{56}$$

and that if the first non-periodic function in this series is $u^{(m)}$, the functions

$$u_s^{(1)}, u_s^{(2)}, \ldots, u_s^{(m-1)} \quad (s = 1, 2, \ldots, n)$$

will all be periodic, while the function $u^{(m)}$ will be of the form

$$u^{(m)} = gt + v,$$

where g is a non-zero constant and v is a finite sequence of sines and cosines of integer multiples of ϑ with periodic coefficients with respect to t.

Assuming that it is this case we are dealing with and that the calculations are carried out so that all the $u^{(l)}$, $u_s^{(l)}$ are real for all real values of ϑ and t, let us put [compare the equations before (52) of Section 34]

$$r = z + u^{(2)}z^2 + u^{(3)}z^3 + \ldots + u^{(m-1)}z^{m-1} + vz^m,$$

$$x_s = u_s^{(1)}z + u_s^{(2)}z^2 + u_s^{(3)}z^3 + \ldots + u_s^{(m-1)}z^{m-1} + z_s$$

$$(s = 1, 2, \ldots, n)$$

and let us introduce the variables z, z_s into system (52) in place of variables r, x_s.

The transformed system will be of the form [similar to (52) of Section 34]

$$\left.\begin{aligned}
&\frac{dz}{dt} = zZ, \quad \frac{d\vartheta}{dt} = \lambda + \Theta, \\
&\frac{dz_s}{dt} = p_{s1}z_1 + p_{s2}z_2 + \ldots + p_{sn}z_n + Z_s \\
&\qquad (s = 1, 2, \ldots, n)
\end{aligned}\right\} \tag{57}$$

and the functions Z, Θ, Z_s (with respect to the variables z, z_σ, ϑ, t) will be of the same character as the functions R, Θ, X_s (with respect to the variables r, x_σ, ϑ, t) in system (52). But, by the nature of our transformation, the functions Z, Z_s will be such that if $Z^{(0)}$, $Z_s^{(0)}$ are what these become for $z_1 = \ldots = z_n = 0$, the expansion of $Z^{(0)}$ in increasing powers of z will not contain terms below the degree $m - 1$, and the term of this degree will contain a constant coefficient g; as for the expansions of the $Z_s^{(0)}$, they will not have powers of z less than the mth. [See the discussion after (52) of Section 34.]

Our problem will thus be reduced to that of stability with respect to the quantities z, z_s, of which the first is subject to the condition $z \geqslant 0$. [$r \geqslant 0$ was assumed at the end of Section 60.]

Remark I

Each of the functions $u^{(l)}$, $u_s^{(l)}$ can contain a certain number of arbitrary constants. But all these constants will reduce to those which can enter into the functions $u^{(l)}$ in the form of the constant terms, and, whatever they may be, we shall always get the same values of m and g. [Compare Remark I of Section 34.]

We may note that the number m will always be odd (next section, Remark).

Remark II

If the number $\lambda\omega/\pi$ were commensurable, equation (55) would be able not to have a periodic solution, even for k non-zero [since the integrand in (XXV) would no longer necessarily have zero mean value]; this is why the first non-periodic function in series (56) would then not be of the type indicated above. But whenever, $\lambda\omega/\pi$ being commensurable (including here the case of $\lambda = 0$), we can arrange the calculations so that this function does become of this type, the preceding transformation [refer to the equations before (57)] will be possible, and we shall be able to draw the conclusions which we arrive at in the next section.

Let us note that in the case where the number $\lambda\omega/\pi$ is commensurable, the problem of finding the functions $u^{(l)}$, $u_s^{(l)}$ allows a much greater indeterminateness than in the case considered above; for, if $\lambda\omega/\pi = \alpha/\beta$ where α and β are whole numbers, we shall be able to add to each of the functions $u^{(l)}$ a series of sines and cosines of even multiples (and, for α even, also of odd multiples) of $\beta(\vartheta - \lambda t)$ with arbitrary constant coefficients. [The reason is that in this case the arbitrary constant K in (XXV) no longer has to be zero, for $f_k(t)$ to be periodic with period ω.]

62. [*Study of the case where the series are not periodic*]

On understanding by $P^{(1)}, P^{(2)}, \ldots, P^{(m-1)}$ linear forms in the variables z_s with coefficients periodic† with respect to ϑ and t, let us suppose that we have

$$zZ = gz^m + P^{(1)}z + P^{(2)}z^2 + \ldots + P^{(m-1)}z^{m-1} + \ldots$$

the subsequent terms, as long as they are below degree $m+1$ with respect to the z, z_s, being at least of second dimension with respect to the z_s.

Next, only considering in the functions Z_s linear terms with respect to the quantities z_σ, and ordering them in increasing powers of z, let us assume that we have

$$Z_s = P_s^{(1)}z + P_s^{(2)}z^2 + P_s^{(3)}z^3 + \ldots,$$

where the $P_s^{(j)}$ are linear forms in the quantities z_σ with coefficients of the same character as in the forms $P^{(j)}$.

† By functions periodic with respect to ϑ and t we understand finite series of sines and cosines of integer multiples of ϑ with coefficients periodic with respect to t.

Finally, only considering in the function Θ terms independent of the quantities z_s, and ordering them in increasing powers of z, let us assume that

$$\Theta = \Theta^{(1)}z + \Theta^{(2)}z^2 + \Theta^{(3)}z^3 + \dots,$$

where the $\Theta^{(j)}$ are functions of only two variables, ϑ and t, and are periodic with respect to the one and the other.

Designating now by $U^{(1)}, U^{(2)}, \dots, U^{(m-1)}$ linear forms in the quantities z_s with coefficients periodic with respect to ϑ and t, and by W a quadratic form in the same quantities with constant coefficients, let us put

$$V = z + W + U^{(1)}z + U^{(2)}z^2 + \dots + U^{(m-1)}z^{m-1},$$

and after having formed with the aid of equations (57) the total derivative of this function V with respect to t, let us seek to dispose of the linear forms $U^{(j)}$ so that, in the expression for this derivative, all terms disappear which are linear with respect to the z_s and which at the same time contain z in powers less than the mth. Such a problem will always be solvable and completely determinate, for, to resolve it, we shall only have to satisfy the following system of equations [analogous to those preceding (63) of Section 37]:

$$\begin{aligned}\sum_{s=1}^{n} (p_{s1}z_1 + p_{s2}z_2 + \dots + p_{sn}z_n)\frac{\partial U^{(k)}}{\partial z_s} + \frac{\partial U^{(k)}}{\partial t} + \lambda\frac{\partial U^{(k)}}{\partial \vartheta} + P^{(k)} \\ + \sum_{s=1}^{n}\left(P_s^{(1)}\frac{\partial U^{(k-1)}}{\partial z_s} + \dots + P_s^{(k-1)}\frac{\partial U^{(1)}}{\partial z_s}\right) + \Theta^{(1)}\frac{\partial U^{(k-1)}}{\partial \vartheta} + \dots + \Theta^{(k-1)}\frac{\partial U^{(1)}}{\partial \vartheta} = 0\end{aligned}$$

$$(k = 1, 2, \dots, m-1)$$

(where the expression which figures in the second line must be replaced by 0 when $k = 1$).

These equations will furnish successively $U^{(1)}, U^{(2)}, \dots, U^{(m-1)}$. [See the discussion after (63) of Section 37.]

Having thus determined the forms $U^{(j)}$, let us choose the form W to conform with the equation

$$\sum_{s=1}^{n} (p_{s1}z_1 + p_{s2}z_2 + \dots + p_{sn}z_n)\frac{\partial W}{\partial z_s} = g(z_1^2 + z_2^2 + \dots + z_n^2)$$

according to which we shall have

$$\frac{dV}{dt} = g(z^m + z_1^2 + z_2^2 + \dots + z_n^2) + S$$

S being an expression which we shall be able to present in the form

$$S = vz^m + \sum_{s=1}^{n}\sum_{\sigma=1}^{n} v_{s\sigma}z_s z_\sigma$$

where v, $v_{s\sigma}$ are holomorphic functions of the quantities $z, z_1, \dots, z_n$, becoming zero for

$$z = z_1 = z_2 = \dots = z_n = 0$$

and possessing in their expansions coefficients periodic with respect to ϑ and t. [Compare the treatment after (64) of Section 37.] Moreover, as the functions Z, Θ, Z_s will be (with respect to the quantities z, z_σ) uniformly holomorphic for all real values of ϑ and t, we shall be able to choose the same for the functions v, $v_{s\sigma}$.

We see from this that, whatever real function of t the variable ϑ is expressed as, the derivative dV/dt, under the condition $z \geqslant 0$, will be a definite function and its sign for small enough values of z, $|z_s|$ will be the same as that of the constant g.

As a consequence, on noting that, by the nature of the form W (Section 20, Theorem II) [this theorem implies that W is a definite quadratic form with sign opposite to that of g], the function V, under the same condition $z \geqslant 0$, will be positive-definite if $g < 0$ and will be able to change sign if $g > 0$, we conclude that the undisturbed motion will be unstable in the case of $g > 0$, and stable in the case of $g < 0$.

In this last case, every disturbed motion for which the perturbations are sufficiently small will tend asymptotically towards the undisturbed motion.

Remark

If instead of the condition $z \geqslant 0$ we had assumed that $z \leqslant 0$, we would have obtained, as in Section 37 (Remark), a result from which comparison with the preceding one would have shown that m is an odd number.

63. [*Exposition of the method. Example*]

In what has just been expounded there is already contained a method for resolving the question in which we are interested. We are now going to give a variant of it in the form of a rule to follow.

Let us take the following system of partial differential equations [similar to (79) of Section 39]:

$$(-\lambda y + X)\frac{\partial x_s}{\partial x} + (\lambda x + Y)\frac{\partial x_s}{\partial y} + \frac{\partial x_s}{\partial t} = p_{s1}x_1 + \dots + p_{sn}x_n + p_s x + q_s y + X_s$$

$$(s = 1, 2, \dots, n). \tag{58}$$

We ascertain without difficulty that, under the conditions considered, we shall always be able to satisfy it formally (and in only one way) by series proceeding in positive integer powers of the quantities x and y and becoming zero for $x = y = 0$, where the coefficients are periodic with respect to t.

Although we could not say anything about the convergence of these series, this circumstance is of no importance here, inasmuch as we shall only be concerned with sums of terms for which the degrees do not exceed a certain limit.

Supposing that m is the number referred to in preceding sections, let us assume that

$$x_1 = f_1(x, y, t), \quad x_2 = f_2(x, y, t), \quad \dots, \quad x_n = f_n(x, y, t) \tag{59}$$

are the ensembles of terms of less than mth degree in the series under consideration. [Compare (87) of Section 40.]

Next let (X) and (Y) be what X and Y become after the quantities x_s have been replaced in them by expressions (59). Then, if we treat as before the system of equations

$$\frac{dx}{dt} = -\lambda y + (X), \quad \frac{dy}{dt} = \lambda x + (Y), \tag{60}$$

transformed by means of the substitution

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \tag{61}$$

we shall encounter in series (56), discontinued at the term $u^{(m)}$, the same functions as before, and in this way we shall arrive at the previous value for the constant g.

We have assumed that the functions X and Y in system (49) become identically zero for $x = y = 0$. We have also indicated a transformation by means of which every other case reduces to this one (Section 60). We shall now point out that, if we do not find ourselves with this case, we shall be able, instead of transforming system (49), to submit to a corresponding transformation system (60), formed as has just been indicated†. If, further, we wish to pass directly to the variables r and ϑ, this will reduce to transforming system (60) with the aid of a substitution of the form

$$\left.\begin{aligned} x &= r\cos\vartheta + F(r\cos\vartheta, \quad r\sin\vartheta, \quad t), \\ y &= r\sin\vartheta + \Phi(r\cos\vartheta, \quad r\sin\vartheta, \quad t), \end{aligned}\right\} \tag{62}$$

where F and Φ represent certain holomorphic functions of the quantities $r\cos\vartheta$ and $r\sin\vartheta$, *not containing in their expansions terms of degree less than the second*, and possessing periodic coefficients with respect to t. [Compare the discussion at the end of Section 39.]

Now we easily convince ourselves that if instead of substitution (62) we use (61) as before, on next operating as was indicated in Section 61, although we then obtain another sequence of functions (56), the first non-periodic function that we shall find will be as before $u^{(m)}$, and its examination will lead to the previous value for the constant g.

Therefore, whatever the case under consideration (i.e. whatever the functions X and Y), we can guide ourselves in our problem by the following rule [which is analogous to the rule given towards the end of Section 40].

The differential equations of the disturbed motion being reduced to form (49), *we form the system of partial differential equations* (58) *and on introducing into them by means of the substitution* (61) *the variables r and ϑ instead of x and y, we seek to satisfy this system* (at least formally) *by series ordered in positive integer powers of r, not containing the zero power, and possessing coefficients periodic with respect to ϑ and t* (see the footnote in Section 62). *Such series will always exist and will be completely determined; and if we substitute them for the quantities x_s in the expansions in the powers of these quantities for the expressions*

$$\frac{d\vartheta}{dt} - \lambda = \frac{Y\cos\vartheta - X\sin\vartheta}{r} \quad \textit{and} \quad \frac{dr}{dt} = X\cos\vartheta + Y\sin\vartheta,$$

these last will present themselves in the form of the series

$$\Theta_1 r + \Theta_2 r^2 + \Theta_3 r^3 + \ldots, \quad R_2 r^2 + R_3 r^3 + R_4 r^4 + \ldots,$$

ordered in positive integer powers of r with coefficients Θ and R periodic with respect to ϑ and t.

On forming these coefficients we at the same time form the functions

$$u_2, u_3, u_4, \ldots \tag{63}$$

† We have in mind problems for which terms of degree higher than the mth in the differential equations do not have any importance.

of the variables ϑ and t, defined by the condition that, k being any integer greater than 2, the expression

$$\frac{\partial r}{\partial t}+(\lambda+\Theta_1 r+\Theta_2 r^2+\ldots+\Theta_{k-2}r^{k-2})\frac{\partial r}{\partial\vartheta}-R_2r^2-R_3r^3-\ldots-R_kr^k$$

after we put in it

$$r=c+u_2c^2+u_3c^3+\ldots+u_kc^k$$

does not include the arbitrary constant c in powers less than the $(k+1)$th, and that moreover each of the functions (63) *appears in the form of a finite sequence of sines and cosines of integer multiples of ϑ, where the coefficients are periodic functions of t or sums of a finite number of periodic and secular terms.*

Suppose that we find ourselves with the case of incommensurable $\lambda\omega/\pi$. Then, in forming the functions (63) *until we encounter a function non-periodic with respect to t, we shall have for this function, say u_m* (the number m will be odd), *an expression of the form*

$$u_m=gt+v,$$

where g designates a non-zero constant and v a periodic function of ϑ and t. This being so, the question of stability will be resolved immediately, in the case of $g>0$ in the negative sense, and in the case of $g<0$ in the affirmative sense.

Remark I

We have assumed that the coefficients $p_{s\sigma}$ in system (49) are constants. This is permissible since the case of $p_{s\sigma}$ periodic reduces to this one with the aid of a linear transformation. But, for the indicated rule to be applicable, it is not necessary to effect such a transformation.

Remark II

We have assumed that $\lambda\omega/\pi$ is an incommensurable number. But if, this number being commensurable, we have found for the first non-periodic function in series (63) an expression of the type indicated above, we would have the right to draw the same conclusions on stability as before.

Example

Let the following system of equations be proposed:

$$\frac{dx}{dt}+\lambda y=z^2\cos t,\quad \frac{dy}{dt}-\lambda x=-z^2\sin t,\quad \frac{dz}{dt}+z=xy,$$

for which we can take $\omega=2\pi$.

On putting

$$x=r\cos\vartheta,\quad y=r\sin\vartheta,$$

we may transform it into the following [with use of the first two equations of the preceding rule]:

$$\left.\begin{aligned}\frac{d\vartheta}{dt}-\lambda=-\frac{z^2}{r}\sin(\vartheta+t),\quad \frac{dr}{dt}=z^2\cos(\vartheta+t),\\ \frac{dz}{dt}+z=r^2\sin\vartheta\cos\vartheta.\end{aligned}\right\}\tag{64}$$

This established, let us form the partial differential equation [equivalent to (58) expressed in terms of r and ϑ instead of x and y]

$$\frac{\partial z}{\partial t}+\left[\lambda-\frac{z^2}{r}\sin(\vartheta+t)\right]\frac{\partial z}{\partial \vartheta}+z^2\cos(\vartheta+t)\frac{\partial z}{\partial r}+z=r^2\sin\vartheta\cos\vartheta,$$

which we shall seek to satisfy by a series ordered in increasing powers, integer and positive, of r (in which the least power is the second), and possessing coefficients periodic with respect to ϑ and t.

This series will evidently contain neither the third nor the fourth power of r. [The differential equations for z_3 and z_4, the coefficients of r^3 and r^4, turn out to be like those written below for z_2 and z_5, but with zero right-hand sides. Thus they are satisfied by $z_3=z_4=0$.] Consequently, on writing only the first two terms, we can take it to be

$$z_2r^2+z_5r^5+\dots \tag{65}$$

The coefficients z_2 and z_5 will be calculated with the aid of the equations

$$\frac{\partial z_2}{\partial t}+\lambda\frac{\partial z_2}{\partial \vartheta}+z_2=\tfrac{1}{2}\sin 2\vartheta,$$

$$\frac{\partial z_5}{\partial t}+\lambda\frac{\partial z_5}{\partial \vartheta}+z_5=z_2^2\left[\sin(\vartheta+t)\frac{\partial z_2}{\partial \vartheta}-2z_2\cos(\vartheta+t)\right],$$

of which the first gives for z_2 an expression independent of t, namely:

$$z_2=\frac{\sin 2\vartheta-2\lambda\cos 2\vartheta}{2(1+4\lambda^2)}.$$

On introducing the angle ε defined by the equalities

$$\cos\varepsilon=\frac{1}{\sqrt{1+4\lambda^2}},\quad \sin\varepsilon=\frac{2\lambda}{\sqrt{1+4\lambda^2}},$$

and putting

$$2\vartheta-\varepsilon=\varphi,$$

we may present this expression in the form

$$z_2=\frac{\sin\varphi}{2\sqrt{1+4\lambda^2}}.$$

If next we take for independent variables φ and $\tau=\vartheta+t$ instead of ϑ and t, and if we make use of the expression found for the function z_2, the second of the equations written above [the second after (65)] will reduce to the form

$$(1+\lambda)\frac{\partial z_5}{\partial \tau}+2\lambda\frac{\partial z_5}{\partial \varphi}+z_5=\frac{\sin^2\varphi(\cos\varphi\sin\tau-\sin\varphi\cos\tau)}{4(1+4\lambda^2)^{3/2}}.$$

From this we get

$$z_5=P\cos\tau+Q\sin\tau,$$

on understanding by P and Q periodic functions of φ, defined by the equations

$$2\lambda\frac{dP}{d\varphi}+P+(1+\lambda)Q=-\frac{\sin^3\varphi}{4(1+4\lambda^2)^{3/2}},$$
$$2\lambda\frac{dQ}{d\varphi}+Q-(1+\lambda)P=\frac{\sin^2\varphi\cos\varphi}{4(1+4\lambda^2)^{3/2}}.$$

Putting $\sqrt{-1}=i$ we next find

$$P+Qi=ae^{i\varphi}+be^{-i\varphi}+ce^{3i\varphi}, \tag{66}$$

where a, b, c are constants, of which the first two are given by the formulae

$$\begin{aligned} a&=\frac{\lambda-1+i}{8[1+(\lambda-1)^2](1+4\lambda^2)^{3/2}},\\ b&=\frac{3\lambda+1-i}{16[1+(3\lambda+1)^2](1+4\lambda^2)^{3/2}}. \end{aligned} \tag{67}$$

[If we define

$$S=P+Qi,\quad K=\frac{i}{16(1+4\lambda^2)^{3/2}}$$

it follows from the two differential equations before (66) that S satisfies

$$2\lambda\frac{dS}{d\varphi}+\{1-(1+\lambda)i\}S=K(2e^{i\varphi}-e^{-i\varphi}-e^{3i\varphi}).$$

This will have as a periodic solution a linear combination of $e^{i\varphi}$, $e^{-i\varphi}$ and $e^{3i\varphi}$, leading to (66) and (67).]

Let us now [continuing to follow the rule] introduce series (65) in place of z in the right-hand sides of [the first two of] equations (64) and let us order the results in increasing powers of r.

In the series obtained in this manner

$$\Theta_3r^3+\Theta_6r^6+\ldots,\quad R_4r^4+R_7r^7+\ldots$$

the coefficients Θ_3, R_4 and R_7 will have the following expressions:

$$\Theta_3=-z_2^2\sin\tau,\quad R_4=z_2^2\cos\tau,\quad R_7=2z_2z_5\cos\tau.$$

Operating next according to the rule, let us form the expression [on taking $k=7$]

$$\frac{\partial r}{\partial t}+(\lambda-z_2^2\sin\tau\, r^3)\frac{\partial r}{\partial\vartheta}-r^4(z_2^2+2z_2z_5r^3)\cos\tau,$$

and, on making

$$r=c+u_4c^4+u_7c^7,$$

[if we start instead from

$$r=c+u_2c^2+u_3c^3+\ldots+u_7c^7$$

due investigation shows that we arrive at

$$u_2 = u_3 = u_5 = u_6 = 0]$$

let us seek to dispose of the functions u_4 and u_7 so that, in this expression, all terms disappear which contain the constant c in powers less than the eighth.

For this we must subject these functions to verifying the equations

$$\frac{\partial u_4}{\partial t} + \lambda \frac{\partial u_4}{\partial \vartheta} = z_2^2 \cos \tau,$$

$$\frac{\partial u_7}{\partial t} + \lambda \frac{\partial u_7}{\partial \vartheta} = 4u_4 z_2^2 \cos \tau + 2z_2 z_5 \cos \tau + z_2^2 \sin \tau \frac{\partial u_4}{\partial \vartheta}.$$

[Expressing these equations in terms of partial derivatives with respect to τ and φ, we have

$$(1+\lambda)\frac{\partial u_4}{\partial \tau} + 2\lambda \frac{\partial u_4}{\partial \varphi} = z_2^2 \cos \tau, \tag{XXVI}$$

$$(1+\lambda)\frac{\partial u_7}{\partial \tau} + 2\lambda \frac{\partial u_7}{\partial \varphi} = 4u_4 z_2^2 \cos \tau + 2z_2 z_5 \cos \tau + z_2^2 \sin \tau \left[\frac{\partial u_4}{\partial \tau} + 2\frac{\partial u_4}{\partial \varphi}\right]. \tag{XXVII}$$]

Assuming that λ is an incommensurable number, we may always satisfy the first of these equations by a periodic function of ϑ and t, and this function, being expressed in φ and τ, will appear in the form

$$u_4 = M \cos \tau + N \sin \tau + \text{const.},$$

where [on substituting in (XXVI) this expression for u_4 and a previous expression for z_2]

$$M = -\frac{\lambda \sin 2\varphi}{2(1+4\lambda^2)(5\lambda+1)(3\lambda-1)},$$

$$N = \frac{(\lambda+1)\cos 2\varphi}{8(1+4\lambda^2)(5\lambda+1)(3\lambda-1)} + \frac{1}{8(1+4\lambda^2)(\lambda+1)}.$$

Under the same assumption we may satisfy the second equation by an expression of the form

$$u_7 = gt + 2u_4^2 + v,$$

where g is a constant and v is a periodic function of ϑ and t.

Moreover, on considering the equation [found by substituting this expression for u_7 in (XXVII) and then making use of (XXVI)]

$$g + (1+\lambda)\frac{\partial v}{\partial \tau} + 2\lambda \frac{\partial v}{\partial \varphi} = 2z_2 z_5 \cos \tau + z_2^2 \sin \tau \left(\frac{\partial u_4}{\partial \tau} + 2\frac{\partial u_4}{\partial \varphi}\right),$$

which v must satisfy as a function of the variables φ and τ, we obtain [after integrating with respect to φ and τ from 0 to 2π, to get rid of zero-mean periodic terms]

$$g = \frac{1}{4\pi^2}\int_0^{2\pi} d\varphi \int_0^{2\pi} \left\{2z_2 z_5 \cos \tau + z_2^2 \sin \tau \left(\frac{\partial u_4}{\partial \tau} + 2\frac{\partial u_4}{\partial \varphi}\right)\right\} d\tau$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{P z_2 + z_2^2 \left(\frac{dN}{d\varphi} - \frac{1}{2}M\right)\right\} d\varphi,$$

[the last line being found by substituting for x_5 and u_4 and integrating with respect to τ; note further that the term

$$z_2^2\left(\frac{dN}{d\varphi}-\frac{1}{2}M\right)$$

is periodic and zero-mean with respect to φ and thus contributes nothing to the above integral] and in this way we arrive at the following expression for the constant g:

$$g=\frac{1}{2\pi}\int_0^{2\pi}Pz_2\,d\varphi=\frac{A}{4\sqrt{1+4\lambda^2}},$$

where A designates the coefficient of $\sin\varphi$ in the expansion of the function P in sines and cosines of integer multiples of φ.

This expression, after we have substituted in it the value for A which is easily obtained from formulae (66) and (67) [and is $(\operatorname{Im} b-\operatorname{Im} a)$], reduces to

$$g=-\frac{1}{64(1+4\lambda^2)^2}\left\{\frac{1}{1+(3\lambda+1)^2}+\frac{2}{1+(\lambda-1)^2}\right\}$$

and gives, as a consequence, for g an always negative value.

We therefore conclude [on referring to the rule] that the undisturbed motion will always be stable.

This conclusion has been obtained under the assumption that λ is an incommensurable number. But it will be equally true for all commensurable values of λ for which we can take the functions u_4 and v to be periodic.

On considering the expression for the function u_4, we see directly that there are only three singular values of λ which must be excluded, namely -1, $1/3$ and $-1/5$ [these being the real poles in the above expressions for M and N]. And if we go to the function v, we must further associate with them these: 0, 1 and $-1/3$; and these will be the only singular values of λ. [v will be of the form

$$v=C\sin 2\tau+D\cos 2\tau+E$$

and evaluation of the periodic functions $C(\varphi)$, $D(\varphi)$ and $E(\varphi)$ will yield the additional singular values.]

Our conclusion will thus be certainly valid for all real values of λ, with the exception of the following six:

$$0,\ 1,\ -1,\ \tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{5},$$

which will require special discussion.

64. [*Exceptional case. Difficulties which it presents. Case of a canonical system of second order*]

Under the assumption that $\lambda\omega/\pi$ is an incommensurable number, we have examined completely one of the two possible cases, that where among functions (63) are found some non-periodic ones. Now we have to consider the other case, that where all these functions are periodic.

In this case all coefficients $u^{(l)}$, $u_s^{(l)}$ in the series (53) will equally be periodic.

In analogous cases in the preceding treatment, whenever we have been able to demonstrate that we are dealing with such cases, the question of stability was resolved in the affirmative sense [Section 38]. Here it will not be the same, and in general the case under consideration will remain doubtful.

Such difference arises from the circumstance that the periodic series which we have previously encountered in similar cases could always be made convergent; while the series (53) do not enjoy this property, and in general the discussion of their convergence presents great difficulties.

These difficulties do not disappear even in the case of $n = 0$, i.e. when the proposed system is of second order.

Such systems have been studied by Mr Poincaré, and he has shown that, $\lambda\omega/\pi$ being incommensurable, the case under consideration occurs for every canonical system† [of second order].

This circumstance can be seen immediately from the equations which define functions (63).

In fact, let the following system be proposed:

$$\frac{dx}{dt} = -\lambda y - \frac{\partial F}{\partial y}, \quad \frac{dy}{dt} = \lambda x + \frac{\partial F}{\partial x},$$

in which F designates a holomorphic function of x and y, not containing in its expansion terms of degree less than the third and processing coefficients periodic with respect to t with period ω, and λ is a constant such that $\lambda\omega/\pi$ is an incommensurable number.

Putting

$$x = r \cos \vartheta, \quad y = r \sin \vartheta,$$

let us form the partial differential equation [both sides of which express dr/dt]

$$\frac{\partial r}{\partial t} + \left(\lambda + \frac{1}{r}\frac{\partial F}{\partial r}\right)\frac{\partial r}{\partial \vartheta} = -\frac{1}{r}\frac{\partial F}{\partial \vartheta}, \tag{68}$$

which r must satisfy as a function of the variables ϑ and t, obtained by resolution with respect to r of the equation of any complete integral containing an arbitrary constant.

We have to show that in the series of the known type

$$c + u_2 c^2 + u_3 c^3 + \ldots, \tag{69}$$

satisfying formally this equation, all the coefficients u_l, which we assume periodic with respect to ϑ, are equally periodic with respect to t.

Suppose that substitution of this series for r in the functions F and r^2 leads to the following expansions

$$F_3 c^3 + F_4 c^4 + F_5 c^5 + \ldots \quad \text{and} \quad c^2 + v_3 c^3 + v_4 c^4 + \ldots$$

The coefficients F_m, v_m will be deduced in a certain way from the coefficients u_l. Moreover

$$F_m \quad \text{and} \quad v_m - 2u_{m-1}$$

---

† Poincaré, 'Sur les courbes définies par les équations différentielles'. *Journal de Mathématiques* [*pures et appliquées*], 4th series, Vol. II, 1886, pp. 199 and 200. Here we prove a slightly more general proposition.

will evidently only depend on the u_l for $l < m - 1$. [This can be seen on expressing F in the form

$$F = r^3 P(r, \vartheta)$$

and on writing r^2 as

$$r^2 = c^2\{1 + 2(u_2 c + u_3 c^2 + \ldots) + (u_2 c + u_3 c^2 + \ldots)^2\}.]$$

By consequence, on noting that equation (68), which can be written thus:

$$\frac{\partial r^2}{\partial t} + \lambda \frac{\partial r^2}{\partial \vartheta} = -2\left(\frac{\partial F}{\partial \vartheta} + \frac{\partial F}{\partial r}\frac{\partial r}{\partial \vartheta}\right),$$

furnishes for every value of m (in the sequence $3, 4, \ldots$) an equation of the form

$$\frac{\partial v_m}{\partial t} + \lambda \frac{\partial v_m}{\partial \vartheta} = -2\frac{\partial F_m}{\partial \vartheta}, \tag{70}$$

we shall calculate v_m from the latter, and we shall thus have u_{m-1} after having found all the u_l for $l < m - 1$. [The right-hand side of (70) can be justified as follows. If we define the above series for F as $G(c, \vartheta)$:

$$G(c, \vartheta) = F_3 c^3 + F_4 c^4 + \ldots$$

we have

$$G(c, \vartheta) = F(r(c, \vartheta), \vartheta)$$

so that

$$\frac{\partial G}{\partial \vartheta} = \frac{\partial F}{\partial \vartheta} + \frac{\partial F}{\partial r}\frac{\partial r}{\partial \vartheta}.$$

Therefore the coefficient of c^m in the expression in the right-hand side here will be the same as in the left-hand side, viz. $\partial F_m/\partial \vartheta$.]

Now, if we present the right-hand side of equation (70) in the form of a series of sines and cosines of integer multiples of ϑ, this series will obviously not include any term independent of ϑ. [If F_m is a non-trivial function of $\sin \vartheta$ and $\cos \vartheta$, so will be $\partial F_m/\partial \vartheta$.] Therefore, if all the $u^{(l)}$ for $l < m - 1$ are periodic with respect to t it will be the same for the function v_m (Section 61) and, as a consequence, also for the function u_{m-1}.

Thus, for every canonical system, series (69) will always be periodic, provided that $\lambda\omega/\pi$ is incommensurable.

In the questions of stability, in this case we shall have to begin with study of the convergence of this series, and if we succeed in showing that, $|c|$ being sufficiently small, this series converges uniformly for all real values of ϑ and t, the question will be resolved in the affirmative sense.

It will be the same, as we are going to show, in the general case.

Returning to system (52), let us assume that in this or that case we have succeeded in finding the periodic series (53) and in showing that, $|c|$ being sufficiently small, they converge uniformly for all real values of ϑ and t.

Supposing that all the $u^{(l)}$ and $u_s^{(l)}$ are real functions, let us make

$$r = z + u^{(2)}z^2 + u^{(3)}z^3 + \ldots,$$

$$x_s = z_s + u_s^{(1)}z + u_s^{(2)}z^2 + \ldots \quad (s = 1, 2, \ldots, n)$$

and in place of the variables r, x_s let us introduce into our system the variables z, z_s.

We shall arrive then at a system of the form (57), in which the functions Z, Z_s will become zero for $z_1 = ... = z_n = 0$, while fully conserving the other properties referred to in Section 61.

In this way our problem will be reduced to that of stability with respect to the quantities z, z_s.

On discarding in system (57) the equation containing the derivative $d\vartheta/dt$, let us consider, in the others, ϑ as a *given* function of t, continuous and real but otherwise arbitrary.

Then, if it is proved that for every given positive value of ε we can assign, *independently of the choice of the function* ϑ, a positive number a such that, the conditions

$$|z| < a, \quad |z_1| < a, \quad |z_2| < a, \quad ..., \quad |z_n| < a$$

being fulfilled at the initial instant, the inequalities

$$|z| < \varepsilon, \quad |z_1| < \varepsilon, \quad |z_2| < \varepsilon, \quad ..., \quad |z_n| < \varepsilon,$$

will be satisfied throughout subsequent time, our problem will be resolved, and in the positive sense.

We shall prove in the following section a proposition from which the postulate which we have just enunciated will actually follow, seeing that the functions Z, Z_s in our equations will be *uniformly* holomorphic (with respect to z, z_σ) for all real values of ϑ and t.

A generalization

65. [*General form to which the differential equations reduce in the singular cases considered previously. Existence of holomorphic integrals with bounded coefficients. Conclusions on stability*]

Let us pose the problem in a somewhat more general way.

Suppose that the proposed system is the following:

$$\left.\begin{aligned} &\frac{dz_1}{dt} = Z_1, \quad \frac{dz_2}{dt} = Z_2, \quad ..., \quad \frac{dz_k}{dt} = Z_k, \\ &\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n + X_s \\ &\qquad\qquad (s = 1, 2, ..., n), \end{aligned}\right\} \tag{71}$$

where X_s, Z_j are holomorphic functions of the variables $x_1, x_2, ..., x_n, z_1, z_2, ..., z_k$, becoming zero for

$$x_1 = x_2 = ... = x_n = 0$$

and not containing in their expansions terms of degree less than the second.

We shall assume that the coefficients in these expansions are any continuous, real and bounded functions of t, and such that all the X_s, Z_j are uniformly holomorphic functions for all real values† of t. As for the coefficients $p_{s\sigma}$, we shall assume that these are real constants such that all the roots of equation (43) possess negative real parts. [Note that the differential equations for the z_i in the linear system of the first approximation are

$$\frac{dz_i}{dt} = 0 \quad (i = 1, 2, ..., k)$$

with solution

$$z_i = \text{const.} \quad (i = 1, 2, ..., k).$$

Thus, although the determinantal equation now has a root of multiplicity k at the origin, the linear system does not have a solution with secular terms, i.e. terms such as at^j $(j > 0)$.]

[Note further that the following treatment is analogous to that in Section 38, where a more detailed exposition is given.]

If such are the differential equations of the disturbed motion, we can demonstrate that the undisturbed motion is stable.

For this purpose, let us show first of all that the system (71) will always admit an integral of the form

$$L + F(x_1, x_2, ..., x_n, z_1, z_2, ..., z_k, t),$$

where L represents a linear form in the quantities $z_1, z_2, ..., z_k$ with arbitrary constant coefficients, and F represents a holomorphic function of the x_s, z_j, not containing in its expansion terms of degree less than the second, becoming zero for $x_1 = x_2 = ... = x_n = 0$, and having for coefficients bounded functions of t.

For this let us consider the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + ... + p_{sn}x_n)\frac{\partial F}{\partial x_s} + \frac{\partial F}{\partial t} = -\sum_{s=1}^{n} X_s \frac{\partial F}{\partial x_s} - \sum_{j=1}^{k} Z_j \left(\frac{\partial F}{\partial z_j} + \frac{\partial L}{\partial z_j}\right),$$

which has to be verified by F [and which interprets $d(L + F)/dt = 0$].

Let

$$F = \sum P_m^{(l_1, l_2, ..., l_k)} z_1^{l_1} z_2^{l_2} ... z_k^{l_k}, \tag{72}$$

where the $P_m^{(...)}$ are forms of the mth degree in the quantities x_s, and where the summation extends over all values of the non-negative integers $m, l_1, l_2, ..., l_k$ satisfying the conditions

$$m > 0, \quad m + l_1 + l_2 + ... + l_k > 1.$$

On substituting this expression for the function F into our equation we shall represent the right-hand side of the latter in the form

$$-\sum Q_m^{(l_1, l_2, ..., l_k)} z_1^{l_1} z_2^{l_2} ... z_k^{l_k},$$

where the summation extends over the values indicated above for the numbers m,

† If we wish we need not consider all real values of t, but only those which are greater than a certain limit t_0.

l_j, and where $Q_m^{(l_1, l_2, \ldots, l_k)}$ represents a form of the mth degree in the quantities x_s, which is deduced in certain manner from those of the forms $P_{m'}^{(l'_1, \ldots, l'_k)}$ for which

$$m' + l'_1 + l'_2 + \ldots + l'_k < m + l_1 + l_2 + \ldots + l_k. \tag{73}$$

This established, the equations

$$\sum_{s=1}^{n} (p_{s1}x_1 + \ldots + p_{sn}x_n)\frac{\partial P_m^{(l_1, \ldots, l_k)}}{\partial x_s} + \frac{\partial P_m^{(l_1, \ldots, l_k)}}{\partial t} = -Q_m^{(l_1, \ldots, l_k)}, \tag{74}$$

which we shall have to satisfy will allow the calculation of all the $P_m^{(l_1, \ldots, l_k)}$ in any order such that the number $m + l_1 + l_2 + \ldots + l_k$ does not decrease. [Compare (69) of Section 38.]

Suppose that all the $P_{m'}^{(l'_1, \ldots, l'_k)}$ for which inequality (73) is satisfied are found, and that they possess bounded coefficients. Then it will be the same for the coefficients of the right-hand side of equation (74), and this equation, under our assumption concerning the $p_{s\sigma}$, will always admit as a solution a form $P_m^{(l_1, \ldots, l_k)}$ with bounded coefficients; such a solution will moreover be unique. This is what may be easily seen on considering a certain transformation of equation (74).

In this way, for every fixed choice of the form L series (72) will be completely determined. Moreover, if the form L possesses real coefficients it will be the same for the coefficients of the series under consideration.

Let us come to the question of the convergence.

Let $\chi_1, \chi_2, \ldots, \chi_n$ be the roots of equation (43).

No longer keeping to the assumption that the coefficients in the equations under consideration are real, we shall be able by effecting a certain linear transformation to reduce the general case to the one where all the $p_{s\sigma}$ not contained in the series

$$p_{11} = \chi_1, \quad p_{22} = \chi_2, \quad \ldots, \quad p_{nn} = \chi_n,$$
$$p_{21} = \sigma_1, \quad p_{32} = \sigma_2, \quad \ldots, \quad p_{n,n-1} = \sigma_{n-1},$$

are zero.

Let us fix then on this hypothesis concerning the $p_{s\sigma}$ and seek the form P satisfying equation (74). The coefficients in this form will be calculated in a certain succession easy to establish, and the coefficient A of

$$x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

will be obtained from an equation of the form

$$\frac{dA}{dt} + (m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n)A = -B,$$

whence we get

$$A = e^{-(m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n)t} \int_t^{\infty} e^{(m_1\chi_1 + m_2\chi_2 + \ldots + m_n\chi_n)t} B \, dt.$$

[Compare the last few equations with the corresponding equations between (69) and (70) of Section 38.]

The function B which appears here will depend in a certain way on the coefficients A found previously: namely, by its very origin it will necessarily be an entire and rational function of these coefficients. Moreover, the coefficients of this function will be sums of products: of the quantities σ_i, of the coefficients in the form

L, of the coefficients in the expansions of the functions X_s, Z_j, and of certain positive whole numbers.

Therefore, on replacing in the function B the quantities we have just listed by constant upper bounds for their moduli, suitable for all real values of t, and on designating the result by $\boldsymbol{B}$, [the above integral becomes easy to evaluate, and] we shall have for the modulus of the coefficient A under consideration the following upper bound:

$$\frac{\boldsymbol{B}}{m_1\lambda_1 + m_2\lambda_2 + \dots + m_n\lambda_n}, \tag{75}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the real parts of the numbers $-\chi_1, -\chi_2, \dots, -\chi_n$.

We can further replace here each of the numbers λ_s by a smaller positive number, and these new numbers may be chosen to be distinct.

In this way our problem will be reduced to a similar problem where all the χ_s will be different, and in this case we can always assume the preliminary linear transformation to be such that all the σ_i are zero [see the equation before (74)].

Now, on considering the problem under this assumption, we can next, in formulae of the form (75), replace all the λ_s by the smallest among them.

We thus see from this that it will suffice to examine the convergence of our series under the hypothesis that all the $p_{s\sigma}$ for s and σ different are zero, that

$$p_{11} = p_{22} = \dots = p_{nn} = -\lambda,$$

λ being a positive number, and that the coefficients in the expansions of the functions X_s, Z_j are constants.

We can further suppose these coefficients to be such that all the X_s, Z_j become functions of only the two arguments

$$x_1 + x_2 + \dots + x_n \quad \text{and} \quad z_1 + z_2 + \dots + z_k,$$

and that we moreover have the equalities

$$X_1 = X_2 = \dots = X_n, \quad Z_1 = Z_2 = \dots = Z_k,$$

for every other case reduces to this on replacing the coefficients by upper bounds for their moduli, chosen in a suitable manner.

Finally, for the form L we shall be able to take the following:

$$L = z_1 + z_2 + \dots + z_k.$$

On making these hypotheses, and taking into account that the X_s and the Z_j must become zero for $x_1 = x_2 = \dots = x_n = 0$, let us put

$$X_s = (x_1 + x_2 + \dots + x_n)X, \quad Z_j = (x_1 + x_2 + \dots + x_n)Z.$$

Then the equation [before (72)] defining the functions F, which we must now take to be independent of t, will reduce to the form

$$\lambda \sum_{s=1}^{n} x_s \frac{\partial F}{\partial x_s} = (x_1 + x_2 + \dots + x_n)\left\{X \sum_{s=1}^{n} \frac{\partial F}{\partial x_s} + Z \sum_{j=1}^{k} \frac{\partial F}{\partial z_j} + kZ\right\}.$$

Now we shall always be able to satisfy this equation on assuming that the function F only depends on the two arguments

$$x_1 + x_2 + \dots + x_n = x \quad \text{and} \quad z_1 + z_2 + \dots + z_k = z,$$

in which case our equation becomes

$$(\lambda - nX)\frac{\partial F}{\partial x} = kZ\left(1 + \frac{\partial F}{\partial z}\right)$$

and therefore gives for $\partial F/\partial x$ a holomorphic expression with respect to x, z, $\partial F/\partial z$. Thus, in view of a known theorem of Cauchy, it always admits one and only one solution, such that the function F, becoming zero for $x = 0$, appears in the form of a series proceeding in positive integer powers of x and z, as long as the moduli of these variables are small enough. [See Goursat, Hedrick and Dunkel, *loc. cit.*, pp. 53–57.]

This series, if we replace x and z in it by their expressions, on then considering it as ordered in powers of the x_s, z_j, will be precisely the one for which the convergence has to be examined.

Therefore, the $|x_s|$ and $|z_j|$ being small enough, the convergence of series (72), under the conditions which we have just considered, is established. So, in view of what has been expounded above, it is equally established under the most general conditions. Moreover, on going back to these, we can state that series (72) represents a *uniformly* holomorphic function of the variables x_s, z_j for all real values of t.

In this way, on fixing on any choice of the form L, we shall find for system (71) a completely determined integral of the required character.

By taking for L successively $z_1, z_2, \ldots, z_k$ we shall obtain k integrals of this kind. These integrals, which will evidently be independent, may be called *elementary*, since every holomorphic integral with bounded coefficients will necessarily be a holomorphic function of them.

Let us now return to our problem.

Let us consider the integral which is equal to the sum of the squares of the elementary integrals. It will be of the following form:

$$z_1^2 + z_2^2 + \ldots + z_k^2 + R,$$

where R only includes terms of degree greater than the second with respect to the variables x_s, z_j.

Next let us consider the quadratic form W in the quantities $x_1, x_2, \ldots, x_n$ defined by the equation

$$\sum_{s=1}^{n} (p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n)\frac{\partial W}{\partial x_s} = -(x_1^2 + x_2^2 + \ldots + x_n^2).$$

This form, as we know, will be positive-definite (Section 20, Theorem II).

It will as a consequence be the same for the function

$$V = z_1^2 + z_2^2 + \ldots + z_k^2 + W + R$$

[the $|x_s|$, $|z_j|$ being small].

Let us form the total derivative of this function with respect to t, with the aid of equations (71). This derivative will be

$$\frac{dV}{dt} = -(x_1^2 + x_2^2 + \ldots + x_n^2) + \sum_{s=1}^{n} X_s\frac{\partial W}{\partial x_s}.$$

Now, the functions X_s all becoming zero for $x_1 = x_2 = \dots = x_n = 0$, we can write

$$\sum_{s=1}^{n} X_s \frac{\partial W}{\partial x_s} = \sum_{s=1}^{n} \sum_{\sigma=1}^{n} v_{s\sigma} x_s x_\sigma,$$

understanding by the $v_{s\sigma}$ holomorphic functions of the quantities x_s, z_j with bounded coefficients, becoming zero when all these quantities are simultaneously zero, and moreover uniformly holomorphic for all real values of t.

Therefore the derivative under consideration will represent a negative function, and as a consequence our function V will satisfy all the conditions of Theorem I of Section 16.

In this manner the stability of the undisturbed motion, in the case under consideration, is found to be proved.

We easily see that every disturbed motion for which the perturbations are sufficiently small will approach asymptotically one of the motions defined by the equations

$$z_1 = c_1, \quad z_2 = c_2, \quad \dots, \quad z_k = c_k, \quad x_1 = x_2 = \dots = x_n = 0, \tag{76}$$

where $c_1, c_2, \dots, c_k$ are arbitrary constants.

We may assure ourselves of this on considering those of the equations of system (71) which contain the derivatives dx_s/dt, and on regarding the quantities z_j in them as given real functions of t, for which the absolute values never exceed sufficiently small limits for values of t greater than its initial value.

We may also easily prove that the motions of series (76), for which the $|c_j|$ are sufficiently small, will be stable.

Remark

We can reduce to systems of the form (71) certain more general systems:

$$\left.\begin{aligned} \frac{dz_j}{dt} &= q_{j1} z_1 + q_{j2} z_2 + \dots + q_{jk} z_k + Z_j \quad (j = 1, 2, \dots, k), \\ \frac{dx_s}{dt} &= p_{s1} x_1 + p_{s2} x_2 + \dots + p_{sn} x_n + X_s \quad (s = 1, 2, \dots, n), \end{aligned}\right\} \tag{77}$$

in which the coefficients q_{ji}, $p_{s\sigma}$, instead of being constants, are bounded functions of t.

Let us consider the case where all the coefficients in system (77) are periodic functions of t. [The transformation which makes the periodic linear system of the first approximation become constant-coefficient is discussed in Section 47.]

Supposing as before that all the X_s, Z_j become zero for $x_1 = x_2 = \dots = x_n = 0$, let us next assume that the characteristic equation of the system

$$\frac{dz_j}{dt} = q_{j1} z_1 + q_{j2} z_2 + \dots + q_{jk} z_k \quad (j = 1, 2, \dots, k), \tag{78}$$

only has roots with moduli equal to 1, and that of the system

$$\frac{dx_s}{dt} = p_{s1} x_1 + p_{s2} x_2 + \dots + p_{sn} x_n \quad (s = 1, 2, \dots, n)$$

only has roots with moduli less than 1.

Then, if to each root ρ of the characteristic equation of system (78) there correspond only solutions

$$z_1 = f_1(t)\rho^{t/\omega}, \quad z_2 = f_2(t)\rho^{t/\omega}, \quad \ldots, \quad z_k = f_k(t)\rho^{t/\omega},$$

where all the $f_s(t)$ are periodic functions of t, we shall be able, in view of what has just been expounded, to state that the undisturbed motion will be stable, and that every disturbed motion, for which the perturbations are sufficiently small, will approach asymptotically one of the motions for which $x_1, x_2, \ldots, x_n$ are all zero and $z_1, z_2, \ldots, z_k$ satisfy system (78).

Such will be, for example, the case where the characteristic equation of this system does not have multiple roots.

When, on the other hand, there exist such roots and we encounter secular terms in the functions $f_s(t)$ which correspond to them, the undisturbed motion will be unstable.

[Actually the conclusion stated above, that when all the $f_s(t)$ are periodic the undisturbed motion will be stable, does *not* follow from the previous assumption that the roots of the characteristic equation of system (78) have moduli equal to unity. In view of equations (9) and those preceding them, the determinantal equation of the constant-coefficient system into which (78) is transformed then has roots on the imaginary axis, and not necessarily at the origin. Consequently system (77) cannot then, in general, be reduced to system (71). To amend the treatment, we need to replace the assumption that the roots of the characteristic equation of system (78) all have moduli equal to unity by one to the effect that these roots are all equal to unity.]

NOTE. Complement to the general theorems on stability

In the preceding (Section 26), on supposing that in the differential equations of the disturbed motion, reduced to the normal form, the right-hand sides are series proceeding in positive integer powers of the unknown functions, and on making certain further general hypotheses, I have indicated a condition under which the question of stability does not depend on the terms of degree greater than the first, in these series; but I have demonstrated only that this condition is sufficient. Now I propose to show that it is also necessary.

Let $x_1, x_2, \ldots, x_n$ be the quantities with respect to which stability is under investigation, and which must, in the differential equations of the disturbed motion, play the role of unknown functions of time t.

These quantities are certain given functions of the coordinates and velocities of the material system under consideration, for which the expressions can moreover depend explicitly on the time t.

I assume that these functions are chosen in such a way that, for the motion of which the stability is being studied, and which I call the undisturbed motion, they all become zero, and that for the disturbed motion they satisfy differential equations of the form

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \ldots + p_{sn}x_n + X_s \quad (s = 1, 2, \ldots, n), \tag{1}$$

where $p_{s\sigma}(s, \sigma = 1, 2, \ldots, n)$ are real constants and $X_1, X_2, \ldots, X_n$ are known functions of the quantities $x_1, x_2, \ldots, x_n$ and t, represented for sufficiently small values of

Then, if to each root ρ of the characteristic equation of system (78) there correspond only solutions

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where all the $f_s(t)$ are periodic functions of t, we shall be able, in view of what has just been expounded, to state that the undisturbed motion will be stable, and that every disturbed motion, for which the perturbations are sufficiently small, will approach asymptotically one of the motions for which $x_1, x_2, \dots, x_n$ are all zero and $z_1, z_2, \dots, z_k$ satisfy system (78).

Such will be, for example, the case where the characteristic equation of this system does not have multiple roots.

When, on the other hand, there exist such roots and we encounter secular terms in the functions $f_s(t)$ which correspond to them, the undisturbed motion will be unstable.

[Actually the conclusion stated above, that when all the $f_s(t)$ are periodic the undisturbed motion will be stable, does *not* follow from the previous assumption that the roots of the characteristic equation of system (78) have moduli equal to unity. In view of equations (9) and those preceding them, the determinantal equation of the constant-coefficient system into which (78) is transformed then has roots on the imaginary axis, and not necessarily at the origin. Consequently system (77) cannot then, in general, be reduced to system (71). To amend the treatment, we need to replace the assumption that the roots of the characteristic equation of system (78) all have moduli equal to unity by one to the effect that these roots are all equal to unity.]

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Let $x_1, x_2, \dots, x_n$ be the quantities with respect to which stability is under investigation, and which must, in the differential equations of the disturbed motion, play the role of unknown functions of time t.

These quantities are certain given functions of the coordinates and velocities of the material system under consideration, for which the expressions can moreover depend explicitly on the time t.

I assume that these functions are chosen in such a way that, for the motion of which the stability is being studied, and which I call the undisturbed motion, they all become zero, and that for the disturbed motion they satisfy differential equations of the form

$$\frac{dx_s}{dt} = p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n + X_s \quad (s = 1, 2, \dots, n), \tag{1}$$

where $p_{s\sigma}(s, \sigma = 1, 2, \dots, n)$ are real constants and $X_1, X_2, \dots, X_n$ are known functions of the quantities $x_1, x_2, \dots, x_n$ and t, represented for sufficiently small values of the $|x_s|$ by the series

$$X_s = \sum P_s^{(m_1, m_2, \dots, m_n)} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \quad (m_1 + m_2 + \dots + m_n > 1),$$

proceeding in positive integer powers of the quantities x_s and not containing terms of degree less than the second. I further assume that in these series the coefficients $P_s^{(\dots)}$, which represent either real constants or continuous real functions of time, are such that we can find positive constants M and A for which are satisfied inequalities of the form

$$\left|P_s^{(m_1, m_2, \dots, m_n)}\right| < \frac{M}{A^{m_1 + m_2 + \dots + m_n}}$$

for all values of t greater than that which is taken for its initial value.

The problem of stability with respect to the quantities x_s reduces to recognizing whether we can assign, for any given positive number l, another positive number ε such that, the functions x_s having at the initial instant any real values satisfying the conditions

$$|x_1| \leqslant \varepsilon, \quad |x_2| \leqslant \varepsilon, \quad \dots, \quad |x_n| \leqslant \varepsilon,$$

the inequalities

$$|x_1| < l, \quad |x_2| < l, \quad \dots, \quad |x_n| < l$$

are satisfied throughout the duration of the ensuing motion.

When this question is resolved in the affirmative sense, the undisturbed motion with respect to the quantities x_s is stable; in the opposite case it is unstable.

In what precedes there has been indicated a condition which the constants $p_{s\sigma}$ must satisfy for this question not to depend on special hypotheses concerning the functions X_s.

This condition is imposed on the roots of the equation

$$\begin{vmatrix} p_{11} - \chi & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} - \chi & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} - \chi \end{vmatrix} = 0,$$

and if

$$\lambda_1, \lambda_2, \dots, \lambda_n \tag{2}$$

are the real parts of these roots taken with a minus sign, it is enunciated as: *the smallest of the numbers* (2) *must not be zero.*

That this condition is sufficient is demonstrated by showing that in the case where the smallest of the numbers (2) is positive the undisturbed motion is stable, while in the case where this number is negative it is unstable, and this is independent of every special assumption concerning the functions X_s.

To establish that the same condition is necessary, I now need to prove the following.

Whatever the constants $p_{s\sigma}$, provided they are such that the smallest of the numbers (2) *is zero, we can always choose the functions X_s so that stability or instability holds at will.*

That we can always choose, under this assumption, the said functions so that instability holds already follows from some results obtained previously, and it is anyway very easy to prove directly.

[By means of a suitable linear transformation we can arrange for the linear terms in (1) to be those on the right-hand sides of equations (5) of Section 18. Thus if say $\chi_1 = 0$, one resulting equation will be of the form

$$\frac{dz_1}{dt} = Z_1.$$

If we choose $Z_1 = z_1^3$, the solution will be

$$z_1(t) = \{1 - 2z_1^2(0)t\}^{-\frac{1}{2}} z_1(0),$$

indicating instability.

If, on the other hand, χ_1 is imaginary, the equations after (5) of Section 18 will apply and we shall have

$$\frac{du_1}{dt} = -\mu v_1 + U_1, \quad \frac{dv_1}{dt} = \mu u_1 + V_1.$$

Choosing $U_1 = u_1(u_1^2 + v_1^2)$, $V_1 = v_1(u_1^2 + v_1^2)$, and defining $r^2 = u_1^2 + v_1^2$, we find that r^2 satisfies the equation

$$\frac{dr^2}{dt} = (r^2)^2,$$

with a solution again exhibiting instability:

$$r^2(t) = \{1 - r^2(0)t\}^{-1} r^2(0).]$$

It thus only remains to prove that if the smallest of the numbers (2) is zero, we can always choose the functions X_s so that the undisturbed motion is stable.

I am now going to consider two special cases where the numbers (2) will all be zero.

Let us suppose that system (1) has the following form:

$$\left.\begin{aligned} &\frac{dx_1}{dt} = X_1, \\ &\frac{dx_i}{dt} = x_{i-1} + X_i \quad (i = 2, 3, \ldots, n). \end{aligned}\right\} \tag{3}$$

On understanding by $\varphi_1, \varphi_2, \ldots, \varphi_n$ the functions calculated successively (for $s = n, n-1, \ldots, 2, 1$) from equations of the form

$$\varphi_s = x_s^2 + \varphi_{s+1}^2,$$

with the condition

$$\varphi_{n+1} = 0,$$

it is easy to convince ourselves that if

$$X_s = -2x_{s+1}\varphi_{s+1} \quad (s = 1, 2, \ldots, n),$$

the function φ_1 will be an integral of system (3).

[It can be proved by induction that

$$\frac{d\varphi_1}{dt} = 2^{i-1}\varphi_2\varphi_3 \dots \varphi_i \left\{\frac{d\varphi_i}{dt} - 2x_{i-1}x_i\right\} \quad (i = 2, 3, \dots, n)$$

and we also find

$$\frac{d\varphi_n}{dt} = 2x_{n-1}x_n.$$

With $i = n$ these two equations yield

$$\frac{d\varphi_1}{dt} = 0,$$

confirming that φ_1 is an integral of system (3).]

Now this function (representing an entire polynomial) is such that, for real values of the x_s, it can only become zero for

$$x_1 = x_2 = \dots = x_n = 0.$$

As a consequence, with the indicated choice of the functions X_s, the undisturbed motion will certainly be stable.

[In fact φ_1 is of the form

$$\varphi_1 = x_1^2 + (x_2^2 + (x_3^2 + \dots + (x_n^2)^2)^2 \dots)^2$$

and is thus a positive-definite function. Moreover its rate of change has just been shown to be zero. Hence, in view of Theorem I of Section 16, stability holds.]

I shall now assume that the system (1) is of even order $n = 2m$ and has the following form:

$$\left.\begin{aligned} &\frac{dx_1}{dt} = -\mu y_1 + X_1, && \frac{dy_1}{dt} = \mu x_1 + Y_1, \\ &\frac{dx_i}{dt} = -\mu y_i + x_{i-1} + X_i, && \frac{dy_i}{dt} = \mu x_i + y_{i-1} + Y_i \quad (i = 2, 3, \dots, m), \end{aligned}\right\} \tag{4}$$

where y_s, Y_s are new notations for the quantities x_{m+s}, X_{m+s}.

Let $\varphi_1, \varphi_2, \dots, \varphi_m$ be the functions calculated successively from equations of the form

$$\varphi_s = x_s^2 + y_s^2 + \varphi_{s+1}^2,$$

with the condition

$$\varphi_{m+1} = 0.$$

Then, if

$$X_s = -2x_{s+1}\varphi_{s+1}, \quad Y_s = -2y_{s+1}\varphi_{s+1} \quad (s = 1, 2, \dots, m),$$

the function φ_1 will be, as is easy to convince ourselves, an integral of system (4); and since this function, for real values of x_s, y_s, can only become zero if

$$x_1 = x_2 = \dots = x_m = y_1 = y_2 = \dots = y_m = 0,$$

we must conclude, as before, that with the indicated choice of the functions X_s, Y_s the undisturbed motion will be stable.

[We find by induction that

$$\frac{d\varphi_1}{dt} = 2^{i-1}\varphi_2\varphi_3 \dots \varphi_i \left\{\frac{d\varphi_i}{dt} - 2(x_{i-1}x_i + y_{i-1}y_i)\right\} \quad (i = 2, 3, \dots, m)$$

and we also have

$$\frac{d\varphi_m}{dt} = 2(x_{m-1}x_m + y_{m-1}y_m),$$

so that again

$$\frac{d\varphi_1}{dt} = 0.$$

Further, φ_1 is now of the form

$$\varphi_1 = z_1^2 + (z_2^2 + (z_3^2 + \dots (z_m^2)^2)^2 \dots)^2$$

where

$$z_i^2 = x_i^2 + y_i^2 \quad (i = 1, 2, \dots, m).$$

Hence φ_1 is positive-definite and with zero rate of change; and we can again apply Theorem I of Section 16 and deduce stability.]

Passing now to the general case, I note that, whatever the constants $p_{s\sigma}$, there will always be a linear substitution with constant real coefficients which will transform the system (1) into a system decomposing into sets of equations belonging to one of the following two types:

$$\left.\begin{aligned} \frac{dy_1}{dt} &= -\lambda y_1 + Y_1, \\ \frac{dy_i}{dt} &= -\lambda y_i + y_{i-1} + Y_i \quad (i = 2, 3, \dots, k) \end{aligned}\right\} \tag{5}$$

or

$$\left.\begin{aligned} \frac{dy_1}{dt} &= -\lambda y_1 - \mu z_1 + Y_1, & \frac{dz_1}{dt} &= \mu y_1 - \lambda z_1 + Z_1, \\ \frac{dy_i}{dt} &= -\lambda y_i - \mu z_i + y_{i-1} + Y_i, & \frac{dz_i}{dt} &= \mu y_i - \lambda z_i + z_{i-1} + Z_i \quad (i = 2, 3, \dots, k), \end{aligned}\right\} \tag{6}$$

where Y_s, Z_s designate the ensembles of terms of degree greater than the first with respect to the unknown functions. [Compare related equations in Section 18.]

I do not exclude here the case of $k = 1$, where the set of the form (5) reduces to a single equation, the first one, and where the set of the form (6) reduces to the two equations in the first line.

In these equations λ represents one of the numbers (2).

Therefore, if among these numbers there do not appear any negative ones, we shall arrive at the case of stability if, in the sets of equations for which $\lambda > 0$ as well as in those for which $k = 1$, we put $Y_s = Z_s = 0$, and if, in the sets where we have simultaneously $\lambda = 0$, $k > 1$, we choose the terms of degree greater than the first as has been indicated in the two special cases considered above.

We can thus regard as proved the necessity of our condition.

It is however to be noted that this condition will only be necessary when we consider general systems of the form (1); and if we wish to consider only systems of a specific particular type, our condition, while remaining completely sufficient, need not be necessary.

Thus it is that if we consider exclusively systems of canonical equations with constant coefficients, it will certainly not be necessary. [For discussion of canonical systems and their transformations, see Section 21. Suppose that in such a system

$$\frac{dx_s}{dt} = -\frac{\partial H}{\partial y_s}, \quad \frac{dy_s}{dt} = \frac{\partial H}{\partial x_s} \quad (s = 1, 2, ..., k)$$

the Hamiltonian H is of the form

$$H = Q + F$$

where Q is a fixed positive-definite quadratic form with constant coefficients and F is a choosable expansion beginning with terms of the third degree. Then $dH/dt = 0$, and when the $|x_r|$, $|y_s|$ are small H is positive-definite. Thus H satisfies the conditions of Theorem I of Section 16, and therefore any choice of F results in stability.]

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INT. J. CONTROL, 1992, VOL. 55, NO. 3, 775–784

Biography of A. M. Lyapunov

V. I. SMIRNOV

Translated by *J. F. Barrett*† from the Russian article in *A. M. Lyapunov: Izbrannie Trudi* (A. M. Lyapunov: Selected Works) (Leningrad: Izdat. Akad. Nauk SSSR, 1948) edited by V. I. Smirnov, pp. 325–340.

Aleksandr Mikhailovich Lyapunov was born on 25th May 1857 (old style date [1]) in Yaroslavl' where his father was then director of the Demidovsk Lycée, the higher general educational establishment.

We shall give brief details about the ancestors and relatives of Lyapunov. His grandfather, Vasili Mikhailovich Lyapunov, in 1826 took an administrative post at the University of Kazan. The eldest son of V. M. Lyapunov, Victor, was grandfather of Academician A. N. Krylov [2] (through his mother, Sophia Victorovich) and the younger daughter, Ekaterina, was married to R. M. Sechenov, brother of the physiologist I. M. Sechenov [3]. From this marriage was born the daughter Natalia Rafailovna, the cousin of Lyapunov. In 1886 she became his wife.

In the numerous family of V. M. Lyapunov was the son Mikhail Vasilievich, the father of A. M. Lyapunov. M. V. Lyapunov finished at Kazan University in 1839, he became astronomical observer at Kazan University in 1840, and in 1850 he founded an observatory. He worked at the University of Kazan until 1855 [4]. In 1856, M. V. Lyapunov became Director of the Demidovsk Lycée in Yaroslavl' about which we have already spoken. In 1852, M. V. Lyapunov was married to Sophia Aleksandrovna Shipilova. They had seven children of which four died in infancy. Of the remaining three sons, the eldest was Aleksandr Mikhailovich. The middle son, Sergei Mikhailovich (1859–1924), was the well-known composer [5] (student of M. A. Balakirev) and the youngest, Boris Mikhailovich (1864–1942), was an active member of the Academy of Science of the U.S.S.R. with specialization in Slavonic philology.

In 1863 the father of Aleksandr Mikhailovich went into retirement and resided initially at the estate of his parents and later on the estate of his wife at Bolobonov in the upper Simbirsk province [6] where he died in 1868.

The first instruction of Lyapunov was obtained from his father. His further education continued, after the death of his father, in the family of his uncle R. M. Sechenov, about whom we have already spoken. Here, together with his cousin (and future wife) Natalia Rafailovna, he prepared for entry into the gymnasium. All circumstances, both at home as well as with his closest relatives with whom Lyapunov associated, awakened in him an interest in science. In 1870, Lyapunov's mother, together with her three sons, settled in Nizhny-Novgorod (later known as Gorki) and Lyapunov entered the 3rd class of the gymnasium which he completed with a gold medal in 1876. In the same year he entered the Physico-Mathematical

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0020-7179/92 $3.00

Faculty of St. Petersburg University but in a month transferred to the Mathematics Department.

This was the time of flowering of the famous St. Petersburg mathematics school created by the great P. L. Chebyshev [7]. Among the university professors in mathematics then were such outstanding scholars as P. L. Chebyshev himself and his well-known students A. N. Korkin and E. I. Zolotarev. Among the teachers of Aleksandr Mikhailovich were also such brilliant professors as K. A. Posse and D. K. Bobylev. In 1878, two years before Lyapunov, there had finished at St. Petersburg University A. A. Markov [8], with whom A. M. Lyapunov all his life maintained a close scientific relationship and with whom he was connected in work in the Academy of Science after 1902. These circumstances at St. Petersburg University provided favourable soil for the development of the exceptional mathematical talents of Lyapunov. Later, in the description of the scientific work of Lyapunov, we shall dwell more closely on his connection with the St. Petersburg school.

At the university Lyapunov devoted his greatest attention to the lectures of P. L. Chebyshev who, as Lyapunov himself acknowledged:

> 'through his lectures and subsequent guidance imparted a creative influence on the character of his subsequent research activities'

(*Biographical Dictionary. D.* Members of the Acad. Sci. I, p. 430). An extraordinarily vivid characterization of Chebyshev as professor and scholar was given by Lyapunov in an article dedicated to the memory of Chebyshev (Kharkov 1895).

The first independent scientific steps of Lyapunov were carried out under the direction of the professor of mechanics at St. Petersburg University, D. K. Bobylev. In 1880 Lyapunov received a gold medal for an essay on a theme proposed by the faculty on hydrostatics. This essay was founded on two previously published works: *On the Equilibrium of a Heavy Body in a Heavy Fluid Contained in a Vessel of a Fixed Form* and *On the Potential of Hydrostatic Pressure*. In Lyapunov's autobiography we read:

> 'In 1881 on the advice of Bobylev, directing Lyapunov's activity and always encouraging him in attempts at independent work, there were published in the Journal of the Physico-Chemical Society his first two works relating to hydrostatics.'

On finishing the university course (1880) Lyapunov was received on Bobylev's recommendation into the faculty of mechanics. The relationship between Lyapunov and Bobylev continued until the death of the latter (20th February 1917). In a speech dedicated to the memory of Bobylev, Lyapunov said:

> 'Almost 40 years I knew the deceased who was my teacher and supervisor of my activities in the first years after finishing my university course. Remembering these years in which I knew Dimitri Konstantinovich especially closely, I have to express to him my deepest thanks for the readiness with which he, often very busy, gave me his time in looking over my first youthful work, often of a fairly naïve character, or explaining what seemed to me obscure passages of authors studied. Assuredly, other students of Dimitri Konstantinovich, knowing him as closely as myself, regarded with the same gratitude the memory of his shining personality.'

In 1882 Lyapunov finished the taking of the master's examination and it became necessary for him to embark on work on the master's dissertation. In a supplement to the presently collected lectures *On the Form of Celestial Bodies* we read:

> 'In 1882 wishing to seek out a suitable theme for a master's dissertation, I discussed several times with Chebyshev according to the occasion various mathematical questions

> and Chebyshev always stated to me that the consideration of easy, even though new, questions which could be solved by generally known methods was not worthwhile and that every young scholar, if he has already acquired some practice in the solution of mathematical questions, should test his strength on some serious theoretical question presenting known difficulties. In connexion with this he proposed to me the following question: "It is known that for certain values of the angular velocity, ellipsoidal forms cease to serve as forms of equilibrium of rotating fluids. Do they not go over in this case to some new forms of equilibrium which, for a small increase of angular velocity, would differ little from ellipsoids?" He added to this "If you solve this question, your work would immediately receive attention." [9]'

Further Lyapunov continues:

> 'Subsequently I realised that Chebyshev had proposed this same question to other mathematicians such as Zolotarev, then a young scholar whose brilliant lectures I heard at the University, and Sonia Kovalevskaya. I do not know if Zolotarev or Kovalevskaya had attempted to solve this question. I became strongly interested in this question, the more so since Chebyshev had not given any observations for its solution, and I immediately set to work.'

Further Lyapunov writes:

> 'After several unsuccessful attempts I had to postpone the solution of the question for an indefinite time. But the question led me on to another, namely that of the stability of ellipsoidal forms of equilibrium, which I took as the subject of my master's dissertation.'

This dissertation under the title *On the Stability of Ellipsoidal Forms of Equilibrium of Rotating Fluids* was defended in January [1884] at St. Petersburg University. The opponents were D. K. Bobylev and a Professor of the Artillery Academy who was working temporarily at St. Petersburg University, N. S. Budaev.

This dissertation made the name of Lyapunov known in Europe. Immediately after publication, a short summary of it appeared in *Bulletin Astronomique* [10]. In 1904 it was, on the initiative of E. Cosserat, translated into French and published in *Annales de l'Université de Toulouse*. In the spring of 1895 Lyapunov was granted the title privat-dozent and subsequently was called to accept the chair of mechanics in Kharkov University, where he went in autumn of the same year.

With this finishes the first period of the life of Lyapunov in St. Petersburg, to which he returned in 1902 after his election as Acting Member of the Academy of Science.

We cite a short excerpt from the reminiscences of B. M. Lyapunov about A. M. Lyapunov concerning the period 1881 to 1885, when the brothers lived together in St. Petersburg.

> 'We lived in one room in the apartment of the widow Mikhailovska, the sister of the Professor of Physiology, I. M. Sechenov, and I was a witness to the pressure of work on my brother at the time of preparation for taking his master's examination and the defence of the first dissertation. At that time he liked to work at night. Once a week at our landlady's apartment there gathered relatives, among them the physiologist I. M. Sechenov who loved to relax in the circle of studious youth coming together on Sundays at his sister's. I remember also that at that time my brother A. M. gave lessons to I. M. Sechenov in those parts of mathematics which he considered especially important for the physiologist Ivan Mikhailovich who was taking the warmest pleasure in all scientific successes of A. M. Lyapunov.'

The close relation between A. M. Lyapunov and I. M. Sechenov was sustained until the end of the life of I. M. Sechenov.

Of the beginning of the Kharkov period, Lyapunov, in his autobiography writes:

> 'Here at first, the research activity of Lyapunov was cut short. It was necessary to work out courses and put together notes for students, which took up much time'.

These lithographed courses of mechanics were in many aspects original. Their analysis was presented in a speech of Academician A. N. Krylov delivered on 3rd May 1919 and published in the same year in *Izvestia Akademia Nauk*.

At the same meeting, Academician V. A. Steklov, the first student of Lyapunov at Kharkov University, delivered a speech in memory of his teacher in which he described the first appearance of Lyapunov at Kharkov University. We follow his words:

> 'In 1884, as is known, the statutes of 1863 were abolished and the Delyanov reaction started [11]. In 1885 I was a student of course III and as a former student under the 1863 statutes stood with the majority of my colleagues in extreme opposition to the new order. When we students learnt that there was coming to us from St. Petersburg a new professor of mechanics, we immediately decided that this must be some pitiful mediocrity from Delyanov's creatures. But into the auditorium together with the elderly Dean Professor Levakovski, respected by all the students, there entered a handsome young man almost of the age group of some of our colleagues and who, on the departure of the Dean, began to read in a voice quivering with agitation, instead of the course on dynamical systems, a course on the dynamics of material points which we had already followed with Professor Delarue – the course of mechanics already known to me. But, from the beginning of the lecture, I heard something which I had not heard or encountered from a single instructor or from any of my known textbooks. And all antipathy to the course was immediately blown to dust; by the strength of his talent, by the charm of which in most cases youth unconsciously yields, Aleksandr Mikhailovich, even without knowing it, overcame in one hour the prejudiced attitude of the auditorium. From that day A. M. took a completely special place in the eyes of the students and they came to regard him with exceptionally respectful esteem. The majority, who were no strangers to an interest in science, strained with all strength to approach, however little, those heights to which A. M. took his audience.'

At the time of the winter break in activity of the 1885/86 academic year Lyapunov arrived in St. Petersburg and on the 17th January 1886 married his first cousin Natalia Rafailovna Sechenova.

Until 1890 Lyapunov conducted alone all instruction in mechanics at Kharkov University. In the first two years of stay in Kharkov, apart from working on the construction of courses, he published in *Communications of the Kharkov Mathematical Society* two articles on potential theory. Already in 1885 he had proposed delivering a special course on potential theory at St. Petersburg University. This course did not take place in view of Lyapunov's departure to Kharkov.

In 1888 began the appearance in print of the works of Lyapunov devoted to questions of the stability of motion of mechanical systems with a finite number of degrees of freedom. In 1892 he produced his remarkable work *The General Problem of the Stability of Motion* in which the question of the stability or instability of motion (or equilibrium) of mechanical systems with a finite number of degrees of freedom was, for the first time, considered with exceptional depth and accuracy on the one hand and with generality on the other hand. This work served as a doctoral dissertation. The defence took place in September 1892 in Moscow University and the opponents were N. E. Zhukovski [12] and V. B. Mlodzeevski. This dissertation, as with the master's, was translated into French and published in *Annales de l'Université de Toulouse*. After the defence of the doctoral dissertation, Lyapunov was in 1893 promoted to ordinary professor. In a series of subsequent works Lyapunov introduced substantial additions to the mentioned dissertation. The publication of this cycle of works on stability finished in 1902 [13].

To the Kharkov period belong also two other directions of work of Lyapunov: in the theory of potential and in the theory of probability.

Under the influence of Lyapunov there had developed among the Kharkov mathematicians a great interest in questions of mathematical physics and primarily in the fundamental boundary value problem for the equation of Laplace. The investigation of this problem is closely connected to the theory of potential. Lyapunov revealed a series of faults and imperfections in this apparently classical branch of mathematical physics. He, as we mentioned above, had also earlier been interested in the theory of potential. His works on the theory of potential, especially the memoir *Sur Certaines Questions qui se Rattachent au Problème de Dirichlet* (1897), for the first time rigorously clarified a series of fundamental features of the theory of potential and served as directing features for further work, in particular, for the work of V. A. Steklov.

The works of Lyapunov in the theory of probability, presented in 1900 and 1901 to A. A. Markov of the Academy of Science, are devoted to the proof of the applicability under highly general assumptions of the Central Limit Theorem of Laplace for a sum of independent random quantities [14]. Their special meaning in the theory of probability stems not only from the results obtained but also in the application in these works of new methods (characteristic functions) which then took on widespread application in the theory of probability. During 1879–1880 Lyapunov listened to the lectures of P. L. Chebyshev on the theory of probability. These lectures in the notes of A. M. were published by Academician A. N. Krylov. In these lectures Chebyshev sketches the proof of the limiting theorems for a sum of independent random quantities and at the end of this exposition we read:

> 'The lack of rigour of the derivation consists in the fact that we made various assumptions not proving the conclusion resulting from this progression. This conclusion however cannot be given by any satisfactory form of mathematical analysis in its present state.'

At the end of the Kharkov period, Lyapunov delivered lectures in the theory of probability at the University. The method outlined by Chebyshev in his lectures was then developed by him in one of his works and carried through completely rigorously in the work of A. A. Markov. Naturally all this directed the attention of Lyapunov to the limit theorem of the theory of probability.

Apart from scientific and educational works Lyapunov also took an active part in general university matters. We quote the characterization of Lyapunov given by a professor of Kharkov University, Academician V. P. Buzeskul:

> 'A. M. Lyapunov belonged to those professors who made up the true spirit of the university by which it lived and flourished and which carried in itself the ideal of the professor and scholar. All baseness was alien to him. He constantly soared in the sphere of science. Frequently in the professors' room in the intervals between lectures you would see him in the circle of his professionally closest colleagues always conversing about scientific questions. In the course of time to these themes there were associated also other closely related painful university issues.'

The activity of Lyapunov in the Kharkov Mathematical Society had great significance. From 1899 to 1902 he was president of this society and the editor of its *Communications*. He reported all his works of the Kharkov period to sessions of the Society. Here also were published the works of his students V. A. Steklov and N. N. Saltykov.

In 1900 Lyapunov was elected Corresponding Member of the Academy of Sciences and on 6th November 1901 Ordinary Professor in the Faculty of Applied Mathematics. This post became vacant after the death of P. L. Chebyshev (1894). In this way Lyapunov became the successor of his renowned teacher in the Academy. In the autumn of 1902 Lyapunov moved to St. Petersburg and with this finished the Kharkov period of his life. In the published speech of V. A. Steklov we read:

> 'Subsequently he remembered this period of his life (from 1885 to 1902) with special affection and in conversations with me often called it his happiest.'

In St. Petersburg Lyapunov undertook no teaching work and devoted all his time exclusively to science. He returned to the problem of Chebyshev with which he started his scientific activity and in a substantial way widening its formulation he, in a series of works vast in volume and exceptional for the strength of analysis, carried the whole question to a conclusion. In his speech V. A. Steklov said:

> 'With that achievement on a subject with which he had attempted to start his research activity he brilliantly finished, as we saw, his distinguished life, so prematurely cut short. The perfect work by Aleksandr Mikhailovich must necessarily be called something more than an achievement.'

The works of Lyapunov of the second period of his life in St. Petersburg relate to the fundamentals of the theory of celestial bodies, i.e. to the question of the forms of equilibrium of uniformly rotating fluids the particles of which mutually attract according to Newton's law. In these works Lyapunov for the first time proved the existence of figures of equilibrium close to ellipsoidal in form but different from ellipsoids and also investigated the stability of these new figures of equilibrium. The problem was solved by Lyapunov both in the case of homogeneous as also in the case of non-homogeneous fluids. In the first two works of this cycle (1903–1904) a slowly rotating non-homogeneous fluid is considered, the form of the surface of which is close to a sphere. These works have an immediate connexion with those of Clairaut and Laplace and in them the formulated problem is, for the first time, rigorously solved to completion.

On moving to St. Petersburg Lyapunov began initially to be engaged on the question of figures of equilibrium of homogeneous fluids close to ellipsoids. At that time he received from V. A. Steklov news that Poincaré had published a book on figures of equilibrium (*Figures d'équilibre d'une masse fluide*, lectures at the Sorbonne in 1900). Lyapunov interrupted his work and became occupied with the above mentioned question on the forms of equilibrium of a slowly rotating inhomogeneous fluid. We quote excerpts from the letter of this time from Lyapunov to Steklov. On 15th February 1903 Lyapunov wrote:

> 'Thank you for your communication which spared me a pointless loss of time. How annoying this would have been with the work now fit for throwing away; for, according to what you write, Poincaré did what would have been the subject for my investigations, and there is no doubt that he set out from the same premises which served as the exact point of departure in my researches and thanks to which I also gave significance to my work: otherwise he could not have made a step forward in the question considered.'

Receiving the book of Poincaré and becoming familiarized with it, Lyapunov wrote to Steklov on 21 February 1903:

> 'To my greatest surprise I did not find anything significant in this book. The greatest part of the book is devoted to the exposition (which is, it is necessary to note, highly disorganized) of results already known. As regards questions of interest to me, Poincaré only repeats, and in a very abridged form, that which he said in his old memoir of 1886 [15]. In no way is there a sign of proof of the existence of forms of equilibrium close to the ellipsoids of MacLaurin and Jacobi and evidently Poincaré stands at the same point as he did 17 years ago on this question. Thus my work has not suffered and I apply myself to it afresh. Because of this the week's interruption of this work seemed very useful, for in this interval I made a start on another work related to the question of the stability of equilibrium in a non-homogeneous rotating fluid. The theory of Clairaut and Laplace requires highly substantial additions since the existence of the forms of equilibrium investigated there is not proved. In this question I have for a long time also wished to become engaged. But it seems to me much more complicated than the question of the form of equilibrium of a homogeneous fluid close to ellipsoidal. I propose therefore to solve from the beginning the last problem and afterwards to apply the same principles to the solution of the second.
>
> Now, becoming involved with this question, I am convinced that it is much easier than the first. The calculations have the same character but are incomparably simpler (the form of equilibrium being little different from a sphere). Completing this calculation I noted also in the first question possibilities of significant simplification of the calculation and that I had gone about it in a too involved way. Thus the break in the work proved to be useful in two respects: I applied myself to a new work and clarified the possibility of significant simplification in a former. Now I propose to continue the second work (relating to the theory of Laplace), as it may more quickly be taken to a conclusion, and then to set about the first.'

Finally, in a letter of 7th April 1903 Lyapunov writes:

> 'Only in the last week I succeeded in removing all difficulties in the proof of the convergence of the series which expresses the solution of the problem of Laplace (on the form of a non-homogeneous rotating fluid for small angular velocity). Now it is necessary to take up the simplification of this proof which at present is extremely complicated. And then, publishing a shortened note on this question I shall go over to that on which I started to become engaged in January (on the forms of equilibrium of homogeneous rotating fluids close to ellipsoidal). Detailed memoirs on these questions I shall hardly be able to edit earlier than two years hence, since the corrections frequently demand very much time.'

These quotations give a clear idea of the beginning of that huge work which was carried out by Lyapunov in the second St. Petersburg period of his life [16]. In the introductory lecture to his course *On the form of celestial bodies* which he started to read in 1918 in Odessa, Lyapunov indicated that the problem of Laplace is solvable for much more general assumptions than previously. The corresponding materials have not been discovered up to the present time.

In St. Petersburg Lyapunov led a closed form of life. His activities were limited to the Academy of Science. His circle of acquaintances was limited to close relatives and his teachers and colleagues in science: D. K. Bobylev, A. H. Korkin, A. A. Markov, K. A. Posse, A. N. Krylov and V. A. Steklov who in 1906 moved from Kharkov to St. Petersburg. In the summer Lyapunov departed for his village of birth (Bolobonovo, previously the Simbirsk province) but even there he did not interrupt his scientific work. B. M. Lyapunov (the brother of A. M.) in his sketch of the life and activity of Lyapunov writes:

> '...in the hours of rest A. M. loved to give himself up to the delight of the beauty of nature, loved and knew how to plant and cultivate indoor and garden trees. Both the Kharkov and later the Petersburg apartments were beautified by his own cultivated ficus and palms.'

In 1908 Lyapunov took part in the Fourth International Mathematical Congress in Rome. Even before this time he corresponded on scientific questions with a number of foreign mathematicians. Among them were: Poincaré, Picard, Korn and Cosserat. At the congress he became personally acquainted with many mathematicians. In 1909 Lyapunov took part in the publication of the collected works of Euler. In particular, he was one of the editors of two mathematical volumes (18th and 19th) which appeared after his death. This publication, carried out by the Swiss Society for Natural Science, is far from being completed even at the present time.

The academic achievements of Lyapunov received wide recognition. He was Honorary Member of the Petersburg, Kharkov and Kazan Universities, Foreign Member of the Paris Academy, Honorary Member of the Kharkov Mathematical Society and member of a number of other scientific societies.

At the end of June 1917, Lyapunov left with his wife for Odessa [17] where at that time his brother Boris Mikhailovich lived. In the spring of 1918 N. R. Lyapunova suffered worsening tuberculosis of the lungs, from which she had suffered earlier, and towards the end of the year the illness took on an ominous character. In September 1918 Lyapunov started in the New-Russian University the reading of a special course *On the form of celestial bodies* devoted to the exposition of his latest work. The introductory lectures of the course are included in the present publication [i.e. in the *Selected Works*].

On 31 October 1918 N. R. Lyapunova died and, after a further three days on 3 November, A. M. Lyapunov died [18]. In a letter which he left he asked to be buried in the same grave as his wife.

After the death of Lyapunov, a large manuscript of finished work was found in which was given the proof of figures of equilibrium close to ellipsoidal in the case of a non-homogeneous fluid. This manuscript was published in the Bicentenary Jubilee of the Academy of Science (*Sur certaines séries de figures d'équilibre d'une liquide hetérogène en rotation* 1925–1927).

Thus an outstanding scholar and a man rare in his personal qualities prematurely departed from life. His closest student V. A. Steklov, personally associated with Lyapunov throughout thirty years, gives in his speech a clear characterization of him as a man. We quote it in conclusion of this short sketch.

> 'Brought up from the beginning by his father, a colleague of N. I. Lobachevski at Kazan University, then in the circle of personalities close to our physiologist I. M. Sechenov, living out his youth amid a small enlightened part of our society at that time whose outlook continued to be influenced by N. A. Dobrolyubov and N. G. Chernyshevski, A. M. Lyapunov personified the best idealistic type for 60 years—at the present time perhaps one not intelligible to everybody.
>
> All the strength which he derived from that series of predecessors he devoted to the selfless service of science, for which he lived, and in which alone he saw sense in life and frequently said that without scientific creativity even life itself for him had nothing.
>
> At the very beginning of his academic activity he worked day by day up to four or five o'clock in the morning and frequently appeared at lectures (at Kharkov University) not having slept the whole night.
>
> He did not allow himself any entertainment and if he appeared occasionally (once or twice a year) at the theatre or in a concert then it was for a very special reason, as, for example, the rare concerts of his brother, the well-known composer S. M. Lyapunov.
>
> The circle of acquaintances of A. M. was extremely restricted and consisted of the closest of his relatives and a small number of scholars, mainly mathematicians, as well as a sparse group of colleagues, among whom A. M. Lyapunov became pre-eminent,

especially in the Kharkov period of his life, to the highest degree in instructive conversation on current scientific matters.

This is partly why he sometimes produced on people who knew him little the impression of a taciturn gloomy closed man who was frequently so much preoccupied with his scientific reflections that he looked but did not see and listened but did not hear. In reality, however, the external dryness and severity of A. M. Lyapunov concealed a man of great temperament with a sensitive and, it could be said, childlike purity of spirit, a highly developed sense of honour and of inner dignity, which struck everyone, even on casual acquaintance, as something impressive and unique.'

Such was the life of the great Russian scholar and of one of the clearest and best representatives of Russian educated society at the end of the 19th and beginning of the 20th centuries, Aleksandr Mikhailovich Lyapunov.

ACADEMICIAN V. I. SMIRNOV

Translator's notes on biography

[1] 7th June 1857 by modern calendar.

[2] Aleksei Nikolaevich Krylov (1863–1945). Outstanding marine engineer engaged both in practical ship design and in theoretical studies. His work laid foundations for the theory of ship motion. Not to be confused with N. N. Krylov of the Krylov–Bogolyubov method for non-linear oscillations.

[3] Ivan Mikhailovich Sechenov (1829–1905), the 'Father of Russian physiology'. His book: *Reflexes of the Brain* influenced Pavlov.

[4] Lyapunov's father and grandfather were colleagues at Kazan University of N. I. Lobachevski (1793–1856) famous for his work on non-euclidean geometry who was rector of the University from 1827 to 1846. He worked on non-euclidean geometry from 1826 to 1855.

[5] Sergei Lyapunov is known especially for his piano studies and for his collection of Russian folk music. He died in exile in Paris.

[6] Simbirsk is present-day Ulyanovsk situated on the Volga to the South of Kazan. Bolobonovo is now renamed Pilna Raion, Gorki province.

[7] Pafnuti L'vovich Chebyshev (1821–1894). Known for his work in the theory of numbers, probability theory, functional approximation, and the theory of mechanisms.

[8] Andrei Andreevich Markov (1856–1922). Known for his work in the theory of numbers and in probability theory. His collaboration with Lyapunov on the Central Limit Theorem led on in 1909 to his theory of events in a chain, i.e. Markov chains.

[9] The problem of the stability of the ellipsoidal forms had been discussed three years before in 1879 by Thomson and Tait and this attracted the attention of Poincaré. Chebyshev's acquaintance with the stability problem had, however, apparently originated many years previously from investigations by Liouville and others. (See S. Chandrasekhar: *Ellipsoidal Figures of Equilibrium*, Dover, 1969, p. 8.)

[10] Actually, the review in *Bulletin Astronomique* appeared in print the following year, 1885. This was also the year of publication in *Acta Mathematica* of Poincaré's major memoir on rotating fluid masses where he proved stability of the ellipsoidal forms of MacLaurin and Jacobi. Lyapunov had proved the same result in his thesis one year previously although his proof remained inaccessible in Western Europe until its later publication in French in 1904. In his memoir Poincaré also claimed to have shown the existence of 'pear-shaped' forms of equilibrium which the ellipsoidal forms go over into at higher angular speeds (cf. Chebyshev's problem). On this subject, though, he was mistaken as Lyapunov was later to show.

[11] In 1884 Count Delyanov, State Secretary and Minister of Public Education, passed a statute depriving universities of autonomy.

[12] Nikolai Egorovich Zhukovski (1847–1921), more familiar as 'Joukowsky' in the French transliteration of his name. He wrote a doctoral thesis on stability and worked in theoretical mechanics and machine control writing an influential textbook *Theory of Regulation of the Speed of Machines* in 1909. He is now mainly known for his later

pioneering work in aeronautics which earned for him the title of 'Father of Russian aviation' and left his name associated with the Kutta–Joukowsky lift formula.

[13] Lyapunov made no further contribution to his stability theory after this date, his later work on stability of rotating fluids being discussed by an energy criterion and the calculus of variations. His stability theory first became applied to control through work in the 1930s by Chetaev and Malkin at the Kazan Aviation Institute.

[14] For a modern statement of Lyapunov's result see e.g. B. V. Gnedenko: *Theory of Probability*, Chelsea, 1962, Chapter 8.

[15] Lyapunov seems here to be referring to Poincaré's *Acta Mathematica* paper. Poincaré had still not proved to Lyapunov's satisfaction the existence of the pear-shaped figures of equilibrium.

[16] Although Lyapunov was primarily interested in its mathematical aspects, his work on rotating fluids during his second St Petersburg period had important implications for astronomy. At that time, G. H. Darwin in Cambridge was attempting to show how the formation of satellites from rotating fluids could be explained by evolution from Poincaré's pear-shaped forms, this theory having particular application to the Earth–Moon system. Lyapunov however showed that the pear-shaped forms were unstable and so could have no physical existence. In his Adams Prize Essay of 1917, Jeans showed how their instability follows from Darwin's own calculations (see R. E. Lyttleton: *The Stability of Rotating Fluid Masses*, Cambridge University Press, 1953).

[17] When they left, St Petersburg (then renamed Petrograd) would have been in turmoil. Following the formation of the Provisional Government and the Tsar's abdication, Lenin had returned and was soon after in July to make a first attempt to seize power.

[18] Lyapunov shot himself. Apart from suffering his wife's death he was going blind from cataract.

Translator's acknowledgment

Thanks are due to Mr R. Beerman of Glasgow University Department of Slavonic and East European Studies who kindly advised on the translation and to Dr A. T. Fuller for his friendly criticism and help.

INT. J. CONTROL, 1992, VOL. 55, NO. 3, 785–790

Bibliography of A. M. Lyapunov's work

J. F. BARRETT†

The following bibliography has been compiled from the extensive bibliography in Russian given in Lyapunov's *Selected Works*‡ which lists, apart from Lyapunov's own publications and reviews of them in journals, also publications of other authors, either on Lyapunov's work, or on work related to Lyapunov's interests. Since Lyapunov's major interest was the theory of rotating fluid masses, most of the quoted references have also to do with this subject and are consequently of specialized interest. For this reason, and for the sake of brevity, the present list is confined to publications of Lyapunov himself. As such it is, to the translator's knowledge, essentially complete (certain minor notes and comments having been omitted). References to recent reprintings of Lyapunov's work have been added, as have cross references to his *Collected Works*§ and *Selected Works*. In each of Sections A, B, C and D below, the references are arranged in chronological order.

The abbreviations adopted for Russian journals are transliterations of those used by Lukomskaya, and are as follows.

Izv. Akad. Nauk (Izvestiya Imperatorskoi Akademii Nauk: Bulletin of the Imperial Academy of Science)

Izv. Akad. Nauk po fiz.-matem. otd. (Izvestiya Akademii Nauk S.S.S.R., otdelenie fiziko-matematicheskikh nauk: Bulletin of the Academy of Science of U.S.S.R., physico-mathematical section)

Matem. sb. (Matematicheskii sbornik: Mathematical Collection, issued by Moscow Mathematical Society)

Protok. zased. fiz.-matem. otd. Akad. Nauk (Protokoly zasedanii fiziko-matematicheskogo otdeleniya Akademii Nauk: Minutes of sessions of the physico-mathematical section of the Academy of Science)

Protok. zased. Obshch. sobr. Akad. Nauk (Protokoly zasedanii Obshchego sobranniya Akademii Nauk: Minutes of sessions of general meetings of the Academy of Science)

Soobshch. i protok. zased. Matem. obshch. pri Khar'k. univ. (Soobshcheniya i protokoly zasedanii Matematicheskogo obshchestva pri Khar'kovskom universitete: Communications and minutes of the Mathematical Society at Kharkov University)

Soobshch. Khar'k. matem. obshch. (Soobshcheniya Khar'kovskogo matematicheskogo obshchestva: Communications of the Kharkov Mathematical Society)

Tr. Otd. fiz. nauk Obshch. lyubit. estestvozn. (Trudy Otdeleniya fizicheskikh nauk Obshchestva lyubitelei estestvoznaniya, antropologii i etnografii: Works of the physics section of the Society of Amateurs of Natural Science, Anthropology and Ethnography, Moscow)

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† Southampton, U.K.

‡ *A. M. Lyapunov*: *Izbrannye Trudy* (Leningrad, 1948) edited by V. I. Smirnov (see Reference 6 of Section D below). The bibliography was originally compiled by Lukomskaya, who subsequently published a further version: Lukomskaya, A. M. *Aleksandr Mikhailovich Lyapunov*: *Bibliografia* (Moscow: Izdat. Akad. Nauk S.S.S.R., 1953, 268 pp. In Russian).

§ *Akademik A. M. Lyapunov*: *Sobranie Sochinenii* (Moscow, Leningrad: Izdat. Akad. Nauk S.S.S.R., 1954–1965).

Zap. Akad. Nauk po fiz.-matem. otd. (Zapiski Imperatorskoi Akademii Nauk, po fiziko-matematicheskomy otdeleniyu: Memoirs of the Imperial Academy of Science, physico-mathematical section)

Zap. Khar'k. univ. (Zapiski Imperatorskogo Khar'kovskogo universiteta: Memoirs of the Imperial University of Kharkov)

Zhurn. Russk. fiz.-khim. obshch. (Zhurnal Russkogo fiziko-khimicheskogo obshchestva pri Imperatorskogo St-Petersburgskogo universitete: Journal of the Russian Physico-Chemical Society at the Imperial St-Petersburg University)

In addition, Lyapunov's Collected Works and Selected Works will be abbreviated as follows:

Coll. Works (Akademik A. M. Lyapunov: Sobranie Sochinenii)

Sel. Works (A. M. Lyapunov: Izbrannye Trudy)

A. Scientific works

[1] On the equilibrium of a heavy body in a heavy fluid contained in a vessel of a certain form. Candidate's dissertation, University of St Petersburg, 1881. *Zhurn. Russk. fiz.-khim. obshch.*, **13** (1881) issue 5, fizich. otd., 197–238, issue 6, 273–307 (in Russian); *Coll. Works*, **I**, 191 (in Russian).

[2] On the potential of hydrostatic pressure. *Zhurn. Russk. fiz.-khim. obshch.*, **13** (1881) issue 8, fizich. otd., 249–376 (in Russian).

[3] *On the stability of ellipsoidal forms of equilibrium of rotating fluids.* Master's dissertation, University of St Petersburg, 1884 (in Russian). Republished in French in 1904.

[4] Some generalizations of the formula of Lejeune-Dirichlet for the potential function of an ellipsoid at an internal point. *Soobshch. i protok. zased. Matem. obshch. pri Khar'k. univ.*, **2** (1885) 120–130 (in Russian); *Coll. Works*, **I**, 19 (in Russian).

[5] On the body of greatest potential. *Soobshch. i protok. zased. Matem. obshch. pri Khar'k. univ.*, **2** (1886) 63–73 (in Russian); *Coll. Works*, **I**, 26 (in Russian).

[6] On the spiral motion of a rigid body in a fluid. *Soobshch. Khar'k. matem. obshch.*, **1** (1888) 7–60 (in Russian); *Coll. Works*, **I**, 276 (in Russian).

[7] On the stability of motion in a special case of the problem of three bodies. *Soobshch. Khar'k. matem. obshch.*, **2** (1889) Nos 1 & 2, 1–94 (in Russian); *Coll. Works*, **I**, 327 (in Russian).

[8] *The general problem of the stability of motion.* Doctoral dissertation, University of Kharkov, 1892. Published by Kharkov Mathematical Society, 250 pp. (in Russian); *Coll. Works*, **II**, 7 (in Russian); chap. 1 is in *Sel. Works*, 7. Republished in French in 1908 and in Russian in 1935.

[9] On a question of the stability of motion. *Zap. Khar'k. univ.* (1893) No. 1, 99–104 (in Russian); *Coll. Works*, **II**, 267 (in Russian). (Addition to 1892 doctoral dissertation.) Republished with corrections in *Soobshch. Khar'k. matem. obshch.*, **3** (1893) No. 6, 265–272. Republished in the 1935 Russian edition of the 1892 dissertation.

[10] Investigation of one of the singular cases of the problem of stability of motion. *Matem. sb.*, **17** (1893) issue 2, 253–333 (in Russian). Republished in the 1935 Russian edition of the 1892 dissertation, and also in the 1963 reprint and its 1966 English translation.

[11] New case of integrability of the differential equations of motion of a rigid body in a fluid. *Soobshch. Khar'k. matem. obshch.*, **4** (1893) Nos 1 & 2, 81–85 (in Russian); *Coll. Works*, **I**, 320 (in Russian).

[12] On a property of the differential equations of the problem of the motion of a heavy rigid body having a fixed point. *Soobshch. Khar'k. matem. obshch.*, **4** (1894) No. 3, 123–140 (in Russian); *Coll. Works*, **I**, 402 (in Russian).

[13] Some remarks concerning the article of G. G. Applerot: 'On a paragraph of the first memoir of S. V. Kovalevskaya "On the problem of rotation of a body about a fixed point"' *Soobshch. Khar'k. matem. obshch.*, **4** (1895) Nos 5 & 6, 292–297 (in Russian).

[14] On the series proposed by Hill for the representation of the motion of the moon. *Tr. Otd. fiz. nauk Obshch. lyubit. estestvozn.*, **8** (1896) issue 1, 1–23 (in Russian); *Coll. Works*, **I**, 418 (in Russian).

[15] On a question concerning linear differential equations of the second order with periodic

coefficients. *Soobshch. Khar'k. matem. obshch.*, **5** (1896) Nos 3–6, 190–254 (in Russian); *Coll. Works*, **II**, 332 (in Russian).

[16] On a series relating to the theory of linear differential equations with periodic coefficients. *Comptes rendus Acad. Sci.* Paris, **123** (1896) 1248–1252 (in French); *Coll. Works*, **II**, 387 (in Russian).

[17] On the instability of equilibrium in certain cases where the function of forces is not a maximum. *J. de Math. pures appl.*, **3** (1897) 81–94 (in French); republished in Russian (1935); *Coll. Works*, **II**, 391 (in Russian).

[18] On the potential of the double layer. *Comptes rendus Acad. Sci.* Paris, **125** (1897) 694–696 (in French); also in *l'Éclairage électrique* **13** (1897) 423–424 (in French); *Coll. Works*, **I**, 33 (in Russian).

[19] On the potential of the double layer. *Soobshch. Khar'k. matem. obshch.*, **6** (1897) Nos 2 & 3, 129–138 (in Russian); *Coll. Works*, **I**, 36 (in Russian).

[20] On certain questions relating to the problem of Dirichlet. *Comptes rendus Acad. Sci.* Paris, **125** (1897) 803–810 (in French); *Coll. Works*, **I**, 45 (in Russian).

[21] On certain questions relating to the problem of Dirichlet. *J. de Math. pures appl.* **4** (1898) 241–311 (in French); *Coll. Works*, **I**, 48 (in Russian); *Sel. Works*, 97 (in Russian).

[22] On a linear differential equation of the second order. *Comptes rendus Acad. Sci.* Paris, **128** (1899) 910–913 (in French); *Coll. Works*, **II**, 401 (in Russian).

[23] On a transcendental equation and linear differential equations of the second order with periodic coefficients. *Comptes rendus Acad. Sci.* Paris, **128** (1899) 1085–1088 (in French); *Coll. Works*, **II**, 404 (in Russian).

[24] On a proposition of the theory of probabilities. *Izv. Akad. Nauk*, **8** (1900) No. 4, 359–386 (in French); *Coll. Works*, **I**, 125 (in Russian); *Sel. Works*, **179** (in Russian).

[25] On a series relating to the theory of a linear differential equation of the second order. *Comptes rendus Acad. Sci.* Paris, **131** (1900) 1185–1188 (in French); *Coll. Works*, **II**, 407 (in Russian).

[26] Reply to P. A. Nekrasov. *Zap. Khar'k. univ.* (1901) No. 3, 51–63 (in Russian).

[27] On a theorem of the calculus of probabilities. *Comptes rendus Acad. Sci.* Paris, **132** (1901) 126–128 (in French); *Coll. Works*, **I**, 152 (in Russian).

[28] A general proposition of the calculus of probabilities. *Comptes rendus Acad. Sci.* Paris, **132** (1901) 814–815 (in French); *Coll. Works*, **I**, 155 (in Russian).

[29] New form of the theorem on the limit of probability. *Zap. Akad. Nauk po fiz.-matem. otd.*, **12** (1901) No. 5, 1–24 (in French); *Coll. Works*, **I**, 157 (in Russian); *Sel. Works*, 219 (in Russian).

[30] On a series encountered in the theory of linear differential equations of the second order with periodic coefficients. *Zap. Akad. Nauk po fiz.-matem. otd.*, **13** (1902) No. 2, 1–70 (in French); *Coll. Works*, **II**, 410 (in Russian).

[31] On the fundamental principle of the method of Neumann in the problem of Dirichlet. *Soobshch. Khar'k. matem. obshch.*, **7** (1902) Nos 4 & 5, 229–252 (in French); *Coll. Works*, **I**, 101 (in Russian).

[32] Researches in the theory of the form of celestial bodies. *Zap. Akad. Nauk po fiz.-matem. otd.*, **14** (1903) No. 7, 1–37 (in French); *Coll. Works*, **III**, 114 (in Russian); *Sel. Works*, 251 (in Russian).

[33] On the equation of Clairaut and the more general equations of the theory of the form of the planets (in French). *Zap. Akad. Nauk po fiz.-matem. otd.*, **15** (1904) No. 10, 1–66; *Coll. Works*, **III**, 147 (in Russian).

[34] On the stability of ellipsoidal forms of equilibrium of rotating fluids (in French). *Ann. de la Fac. des Sci. de l'Univ. de Toulouse*, **6** (1904) 5–116 (French translation of 1884 master's dissertation.)

[35] On a problem of Chebyshev (in French). *Zap. Akad. Nauk po fiz.-matem. otd.*, **17** (1905) No. 3, 1–32; *Coll. Works*, **III**, 207 (in Russian).

[36] *On the Figures of Equilibrium Slightly Different from Ellipsoids for a Homogeneous Liquid Mass Given a Motion of Rotation. Part I: General Study of the Problem.* (St Petersburg: Academy of Science, 1906), 225 pp. (in French); *Coll. Works*, **IV**, 9 (in Russian).

[37] Problem of minimum in a question of stability of equilibrium of a fluid mass in rotation. *Zap. Akad. Nauk po fiz.-matem. otd.*, **22** (1908) No. 5, 1–140 (in French); *Coll. Works*, **III**, 237 (in Russian).

[38] The general problem of the stability of motion. *Ann. de la Fac. des Sci. de l'Univ. de Toulouse*, **9** (1908) 203–474 (French translation of the 1892 doctoral dissertation, with corrections and the addition of the material of Reference 9 above. Reprinted 1966).
[39] *On the Figures of Equilibrium Slightly Different from Ellipsoids for a Homogeneous Liquid Mass Given a Motion of Rotation. Part II: Figures of Equilibrium Derived from the Ellipsoids of Maclaurin.* (St Petersburg: Academy of Science, 1909), 203 pp (in French); *Coll. Works*, **IV**, 211 (in Russian).
[40] On a class of figures of equilibrium of a liquid in rotation. *Ann. Scientif. de l'Éc. norm. supér.*, **26** (1909) 473–483 (in French); *Coll. Works*, **V**, 387 (in Russian).
[41] *On the Figures of Equilibrium Slightly Different from Ellipsoids for a Homogeneous Liquid Mass Given a Motion of Rotation. Part III: Figures of Equilibrium Derived from the Ellipsoids of Jacobi.* (St Petersburg: Academy of Science, 1912), 228 pp (in French); *Coll. Works*, **IV**, 383 (in Russian).
[42] *On the Figures of Equilibrium Slightly Different from Ellipsoids for a Homogeneous Liquid Mass Given a Motion of Rotation. Part IV: New Formulae for the Investigation of Figures of Equilibrium.* (St Petersburg: Academy of Science, 1914), 112 pp. (in French); *Coll. Works*, **IV**, 557 (in Russian).
[43] On series of polynomials. *Izv. Akad. Nauk*, **9** (1915) No. 17, 1857–1868 (in French); *Coll. Works*, **I**, 179 (in Russian).
[44] On the equations which pertain to the surfaces of figures of equilibrium derived from ellipsoids for a homogeneous liquid in rotation. *Izv. Akad. Nauk*, **10** (1916) No. 3, 139–168 (in French); *Coll. Works*, **V**, 395 (in Russian).
[45] New considerations relating to the theory of figures of equilibrium derived from ellipsoids in the case of a homogeneous liquid. Parts 1 & 2. *Izv. Akad. Nauk* **10** (1916) No. 7, 471–502, No. 8, 589–620 (in French); *Coll. Works*, **V**, 419 (in Russian).
[46] On a formula of analysis. *Izv. Akad. Nauk*, **11** (1917) No. 2, 87–118 (in French); *Coll. Works*, **V**, 469 (in Russian).

B. Courses of lectures

Extant lithographic notes of courses given by Lyapunov at Kharkov. All are in Russian. Items 2, 3, 13, 14 and 15 are without title pages. Item 11 is mentioned in the obituary 'Aleksandr Mikhailovich Lyapunov' by A. N. Krylov.

[1] Analytical mechanics (1885), 155 pp.
[2] Dynamics of material points (1886), 156 pp.
[3] Statics (1887), 75 pp.; 2nd version 124 pp.
[4] Theoretical mechanics (1890) (Technological Institute), 344 pp.; 2nd version (1893), 468 pp.
[5] Theory of gravitation (1897), 131 pp.; 2nd version, 75 pp.
[6] Kinematics (1900), 127 pp.
[7] Dynamics of material points (1900), 328 pp.
[8] Theory of gravitation (1900), 56 pp.
[9] Fundamentals of the theory of deformable bodies and hydrostatics (1900), 140 pp.
[10] Dynamics of a system of points (1900), 526 pp.
[11] Dynamics of a system of material points. 415 pp.
[12] Dynamics of invariable systems of points. 104 pp.
[13] Kinematics. 8 pp.
[14] Mechanics of a system of points. 191 pp.
[15] Fundamentals of the theory of deformable bodies and hydrostatics. 120 pp.

C. Reviews, translations, obituaries, etc.

All are in Russian unless otherwise stated.

[1] Notes on the lectures of P. L. Chebyshev delivered in 1879–1880 on the theory of probability. (Published 1936.)

[2] Commendation on the work of privat-dozent Steklov 'On the motion of a rigid body in a fluid' presented in the Physico-Mathematical Faculty as dissertation for master of applied mathematics. *Zap. Khar'k. univ.* (1894) No. 4, 94–100.

[3] Obituary: Pafnuti Lvovich Chebyshev. *Soobshch. Khar'k. matem. obshch.*, **4** (1895) Nos 5 & 6, 263–273; Bibliography of Chebyshev. *Ibid.*, 274–280. Reprinted in *P. L. Chebyshev: Selected Mathematical Works* (Moscow, Leningrad, 1946).

[4] Translation from French into Russian of P. L. Chebyshev's 'On the integration of the differential $(x + A)/(x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta)^{1/2}\,dx$. *See* Chebyshev's *Oeuvres*, Vol. I (St Petersburg, 1899), pp. 517–530.

[5] Translation from French into Russian of P. L. Chebyshev's 'On a modification of the articulated parallelogram of Watt'. *See* Chebyshev's *Oeuvres*, Vol. I (St Petersburg, 1899), pp. 533–538.

[6] Translation from French into Russian of P. L. Chebyshev's 'On the integration of irrational differentials'. *See* Chebyshev's *Oeuvres*, Vol. I (St Petersburg, 1899), pp. 511–514.

[7] Speech delivered on 12th September 1901 at a solemn public gathering in memory of Ostrogradski: 'On the contributions of M. V. Ostrogradski in the area of mechanics'. In *Mikhail Vasilievich Ostrogradski—a centennial celebration by the Poltavski circle of Amateurs of Physico-Mathematical Science*, edited by P. Tripolski (Poltava, 1902), pp. 115–118.

[8] Note in connection with the work of B. B. Galitsin 'On seismometric observations.' *Protok. zased. fiz.-matem. otd. Akad. Nauk*, **III**, pril. k protok. zased., 5th March 1903. Reprinted in the book *Debate Between Academicians F. A. Bredikhin, O. A. Backlund, B. B. Galitsin, A. M. Lyapunov and A. A. Markov* (St Petersburg, 1903), pp. 22–28.

[9] Remark on the objection of B. B. Galitsin in the preface to the proceedings of the session of 19th March 1903. *Protok. zased. fiz.-matem. otd. Akad. Nauk*, **III**, pril. k protok. zased., 16th April 1903. Reprinted in the book mentioned in the previous reference, pp. 48–50.

[10] Commendation of the work of Professor V. A. Steklov 'General methods of solution of fundamental problems of mathematical physics.' *Zap. Khar'k. univ.* (1903) No. 1, 25–34.

[11] Recommendation of A. M. Lyapunov on the work of A. N. Krylov 'On an integrator of ordinary differential equations.' *Protok. zased. fiz.-matem. otd. Adad. Nauk* 14th Jan. 1904 (in French); *Izv. Akad. Nauk*, **20** (1904) Nos 1 & 2, 8–9.

[12] (Coauthored with O. Backlund and O. Sonin) Note on the academic achievements of A. A. Belopolskogo. *Protok. zased. fiz.-matem. otd. Akad. Nauk*, 11th Jan. 1906.

[13] Translation from French into Russian of P. L. Chebyshev's 'On the limiting values of integrals'. *See* Chebyshev's *Oeuvres*, Vol. II (St Petersburg, 1907), pp. 183–185.

[14] Translation from French into Russian of P. L. Chebyshev's 'On the simplest parallelograms which provide a rectilinear motion to within terms of the fourth order'. *See* Chebyshev's *Oeuvres*, Vol. II (St Petersburg, 1907), pp. 359–374.

[15] Report on the Fourth International Mathematical Congress, Rome 1908. *Izv. Akad. Nauk*, **2** (1908) No. 9, 709–710.

[16] (Coauthored with N. Sonin) Report on the scientific achievements of Vito Volterra. *Izv. Akad. Nauk* (1909) No. 4, **III**, pril. k protok. zased. fiz.-matem. otd., 10th Dec. 1908, 213–215.

[17] (Coauthored with A. A. Markov and N. Sonin) Note on the academic works of Ordinary Professor of St Petersburg University Vladimir Andreevich Steklov. *Protok. zased. Obshch. sobr. Akad. Nauk*, 2nd Oct. 1910; pril. k protok. fiz.-matem. otd., 15th Sept. 1910.

[18] (Coauthored with N. Sonin and O. Backlund) Note on the academic works of V. A. Steklov. *Protok. zased. fiz.-matem. otd. Akad. Nauk*, pril. k protok., 7th Dec. 1911.

[19] (Coauthored with A. A. Markov, V. A. Steklov, N. Ya. Tsinger, D. K. Bobylev and A. N. Krylov) Report of the commission for the discussion of certain questions relating to the teaching of mathematics in secondary schools. *Protok. zased. fiz.-matem. otd. Akad. Nauk*, pril. k protok., 18th Nov. 1915, 250–267; *Izv. Akad. Nauk* (1916) No. 2, 66–80.

[20] Dimitri Konstantinovich Bobylev—obituary (with list of publications.) *Izv. Akad. Nauk*, **11** (1917) No. 5, 301–306.

[21] Gaston Darboux (1842–1917)—obituary. *Izv. Akad. Nauk*, **11** (1917) No. 6, 351–352.

D. Posthumous and reprinted publications

[1] *On Certain Series of Figures of Equilibrium of a Heterogeneous Fluid in Rotation*. Part 1. (Leningrad: Academy of Science, 1925), in French.

[2] *On Certain Series of Figures of Equilibrium of a Heterogeneous Fluid in Rotation*. Part 2. (Leningrad: Academy of Science, 1927), in French. Parts 1 and 2 were published, with prefaces by Steklov and Smirnov, on the occasion of the bicentenary of the Academy of Science of the U.S.S.R. Also in *Coll. Works*, V, 7 (in Russian).

[3] On the form of heavenly bodies. *Izv. Akad. Nauk po fiz.-matem. otd.* (1930) No. 1, 25–41 (in Russian); *Coll. Works*, **III**, 361; *Sel. Works*, 301. Lectures delivered in 1918 at the New Russian University, Odessa. Published under the editorship of A. N. Krylov.

[4] *The General Problem of the Stability of Motion*, second edition. (Moscow, Leningrad: Academy of Science, 1935), 386 pp., with a portrait (in Russian). Includes additions of the 1908 French version, the 1897 *J. de Math. pures appl.* paper in Russian, and an obituary by Steklov.

[5] *Theory of probability*, edited by A. N. Krylov (Moscow: Academy of Science, 1936), 253 pp. (Lyapunov's notes on lectures by Chebyshev delivered in 1879–1880).

[6] *A. M. Lyapunov: Selected Works*, edited by V. I. Smirnov with contributions by V. I. Smirnov, N. G. Chetaev, L. N. Sretenskii, S. N. Bernstein, A. N. Krylov and A. M. Lukomskaya (Leningrad: Academy of Science, 1948), in Russian.

[7] *The General Problem of the Stability of Motion* (Princeton University Press: Annals of Mathematics Studies No. 17, 1949), in French, reproduction of the 1908 French version of Lyapunov's 1892 doctoral dissertation.

[8] *The General Problem of the Stability of Motion*, third edition, (Moscow: Academy of Science, 1950), in Russian.

[9] *Academician A. M. Lyapunov: Collected Works*, Vol. I (Moscow: Academy of Science, 1954), in Russian (works on potential, probability, series, and mechanics).

[10] *Academician A. M. Lyapunov: Collected Works*, Vol. II (Moscow: Academy of Science, 1956), in Russian (works on stability and differential equations).

[11] *Academician A. M. Lyapunov: Collected Works*, Vols III & IV (Moscow: Academy of Science, 1959), in Russian (works on rotating fluids).

[12] *Investigation of one of the singular cases of the problem of the stability of motion*, edited by V. P. Basov (Leningrad University Press, 1963), republication of the 1893 paper, with an introduction by V. P. Basov.

[13] *Academician A. M. Lyapunov: Collected Works*, Vol. V (Moscow: Nauka, 1965), in Russian (works on rotating fluids and analysis).

[14] *Stability of Motion* (New York: Academic Press, 1966) in English, translation by F. Abramovici and M. Shimshoni of the 1963 reprinting, and including translation of a related paper by V. A. Pliss.

[15] The general problem of the stability of motion. *International Journal of Control* (1992), in English, this issue, translation into English of the French version of 1908.

Acknowledgment

The author thanks Dr A. T. Fuller for his very substantial assistance in the preparation of this paper.