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HIGH ORDER A-STABLE METHODS FOR THE NUMERICAL SOLUTION OF SYSTEMS OF D.E.'S

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Abstract.

The following note shows how one can obtain one step methods of arbitrarily high order which satisfy Dahlquist's requirements of A-stability. Although most of these methods appear at the moment to be largely of theoretical interest the author is working on several practical applications.

1. Introduction.

In [3] Dahlquist defines A-stability as follows:

DEFINITION. *A k-step method is called A-stable, if all of its solutions tend to zero, as $n \to \infty$, when the method is applied with fixed positive h to any differential equation of the form*

$$\frac{dy}{dx} = qy \tag{1}$$

where q is a complex constant with negative real part.

He then establishes that the trapezoid rule is the most accurate linear multi-step method satisfying his A-stability requirement and also points out that the classical 4th order Runge–Kutta process is not A-stable The same can be said for all other explicit R. K. processes.

2. A-Stable Runge-Kutta Methods.

Butcher [2] has shown that for any positive integer n, there exists an n stage implicit R. K. process of order $2n$, of the form:

$$K_i = f\left(y_0 + h\sum_{j=1}^{n} \beta_{ij}K_j\right) \qquad (i = 1, 2, \ldots, n)$$

$$y_1 = y_0 + h\sum_{i=1}^{n} b_i K_i\,.$$

It is not difficult to show that when Butcher's n stage method is applied to (1) we obtain simply the nth diagonal Padé approximation $P_{nn}(x)$

of $\exp(x)$ for $x=qh$ [7]. This follows from the fact that Butcher's stage method applied to (1) reduces to the quotient of two polynomials of degree n in qh and since this rational function must be an approximation of order $2n$ of the exponential function it follows that it is the diagonal Padé approximation $P_{nn}(qh)$ of $\exp(qh)$ [7]. For example, Butcher's 2 stage method has

$$\beta_{11} = \frac{1}{4} \qquad \beta_{12} = \frac{1}{4}-\frac{\sqrt{3}}{6}$$

$$\beta_{21} = \frac{1}{4}+\frac{\sqrt{3}}{6} \qquad \beta_{22} = \frac{1}{4}$$

$$b_1 = \frac{1}{2} \qquad b_2 = \frac{1}{2}$$

and hence for (1) we would have

$$K_1 = q\left(y_0+\frac{h}{4}K_1+\left(\frac{1}{4}-\frac{\sqrt{3}}{6}\right)hK_2\right),$$

$$K_2 = q\left(y_0+\left(\frac{1}{4}+\frac{\sqrt{3}}{6}\right)hK_1+\frac{h}{4}K_2\right).$$

Hence, solving for K_1 and K_2 using Cramer's rule gives

$$K_1 = qy_0\left(1-\frac{qh\sqrt{3}}{6}\right)\Big/\Delta ,$$

$$K_2 = qy_0\left(1+\frac{qh\sqrt{3}}{6}\right)\Big/\Delta ,$$

where

$$\Delta = 1-\tfrac{1}{2}(qh)+\frac{(qh)^2}{12}.$$

Finally,

$$\begin{aligned} y_1 &= y_0+h(b_1K_1+b_2K_2) \\ &= \left(\frac{1+\frac{1}{2}\,qh\;+\dfrac{(qh)^2}{12}}{1-\frac{1}{2}(qh)+\dfrac{(qh)^2}{12}}\right)y_0 \\ &= P_{22}(qh)y_0 . \end{aligned}$$

Now Birkhoff & Varga [1] have noted that the diagonal Padé approximations $P_{nn}(z)$ have the property that $|P_{nn}(z)| \leqq 1$ for $R(z) \leqq 0$ for

$n=0,1,2,\ldots$ and hence Butcher's n-stage implicit RK processes of order $2n$ are A-stable for $n=1,2,3,\ldots$.

3. Additional A-Stable one step methods.

If one is willing to allow the introduction of second and higher derivatives into the equations being used then a generalized class of linear one step methods first given by Obrechkoff [5] of the form

$$y_{n+1} = y_n + \sum_{i=1}^{j} \alpha_{ij} h^i((-1)^{i+1} y_{n+1}^{(i)} + y_n^{(i)}) \qquad (j=1,2,3,\ldots) \tag{2}$$

also exist which are of order $2j$ when α_{ij} is the ith coefficient in the numerator of the jth diagonal Padé approximation $P_{jj}(x)$ of e^x. Thus for (1), (2) reduces to $y_{n+1}=P_{jj}(qh)y_n$ and (2) is A-stable for all j. Note that for $j=1$ in (2) we have simply Dahlquist's result and for $j=2$ we have Milne's method [4, page 76, equation 35.1] which Ralston [6] observed was unconditionally stable. Milne [4, page 78] has also considered (2) for the case $j=3$ but no discussion of stability appears to have been previously made for the cases $j \geqq 3$.

REFERENCES

1. Birkoff, G., Varga, R. S., *Discretization Errors for Well-set Cauchy Problems*, I, J. Math. and Physics, Vol. 44, (1965), 1–23, MR 31 #4189.
2. Butcher, J. C., *Implicit Runge–Kutta Processes*, Math. Comp., Vol. 18 (1964), 50–64, MR 28 #2641.
3. Dahlquist, G., *A Special Stability Problem for Linear Multistep Methods*, BIT, Vol. 3 (1963), 27–43, MR 30 #715.
4. Milne, W. E., *Numerical Solution of Differential Equations*, John Wiley & Sons, Inc., New York, 1953, MR 16, p. 864.
5. Obrechkoff, N., *Sur les Quadratures Mecaniques* (Bulgarian, French summary), Spisanie Bulgar. Akad. Nauk., Vol. 65 (1942), 191–289, MR 10, p. 70.
6. Ralston, A., *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965, MR 32 #8479.
7. Wall, H. S., *Analytic Theory of Continued Fractions*, van Nostrand, New York, 1948, MR 10, p. 32.

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