Numerical Solution of Bifurcation and Nonlinear Eigenvalue Problems

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1. Introduction.

We show in this paper how a large class of bifurcation and nonlinear eigenvalue problems can be solved and the numerical methods justified. Only equilibrium problems are treated here, say in the general form

(1.1) $$G(u, \lambda) = 0$$

where $G : \mathbb{B} \times \mathbb{R} \to \mathbb{B}$ for some Banach space, $\mathbb{B}$. We find it most instructive to refer occasionally to the cannonical example of the matrix eigenvalue problem:

(1.2) $$Au - \lambda u = 0 \ ,$$

where A is an $n \times n$ real matrix with real eigenvalues.

By a smooth branch or arc of solutions

(1.3) $$\Gamma_{ab} : [u(s), \lambda(s)] \ , \quad s_a \leq s \leq s_b \ .$$

we mean a one parameter family of solutions of (1.1), $u(s) \in \mathbb{B}$, $\lambda(s) \in \mathbb{R}$ depending twice continuously differentiably on some parameter $s \in [s_a, s_b]$. Of course the parameter, s, is quite arbitrary on each such branch and this fact is crucial in our study. In figures 1a and 1b we sketch, respectively, typical solution branches for problems (1.1) and (1.2). The "points" at λ_A and λ_B represent "simple" bifurcation while λ_D is a multiple bifurcation point. The "arcs" Γ_j in figure 1b for $j \neq 0$ represent the eigenspaces of A belonging to the eigenvalues λ_j while Γ_0 is the trivial solution. For simple eigenvalues the bifurcation is simple (i.e. a one parameter family of solutions branches off). At $\lambda = \lambda_C$ on branch Γ_1 in figure 1a we

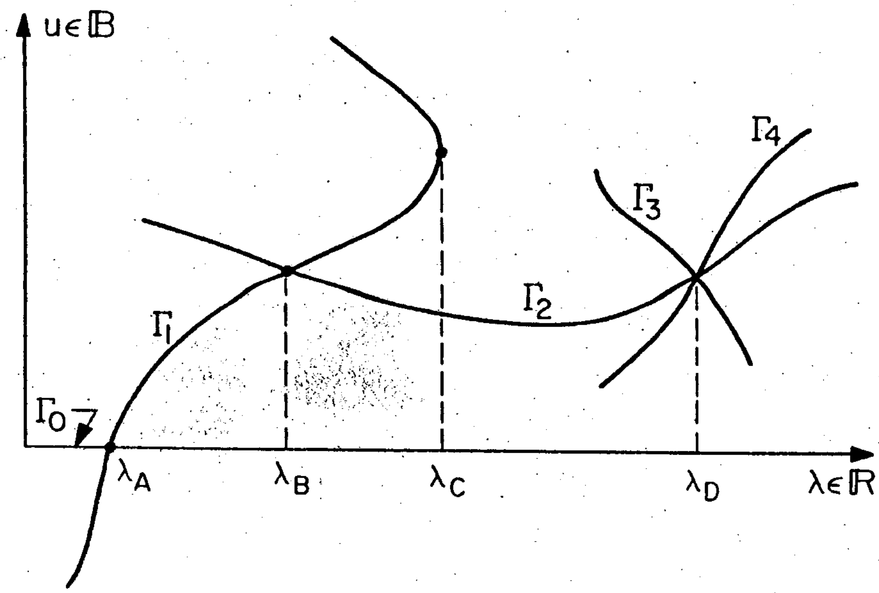

Figure 1a.

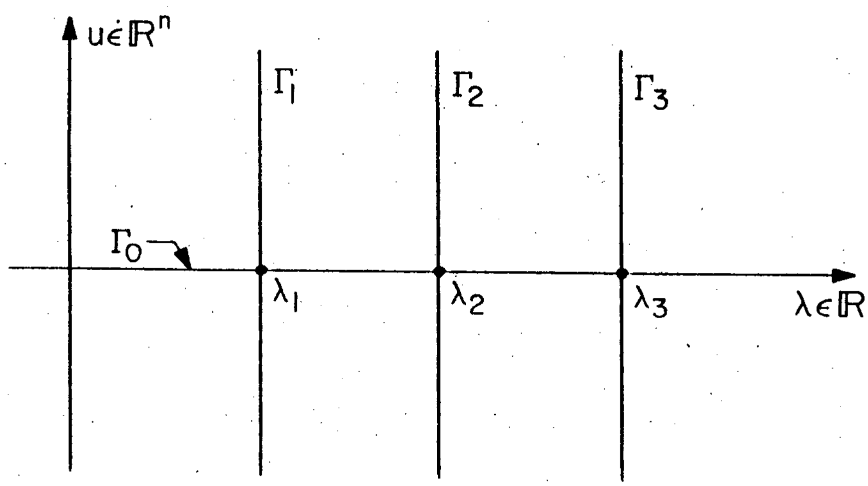

Figure 1b.

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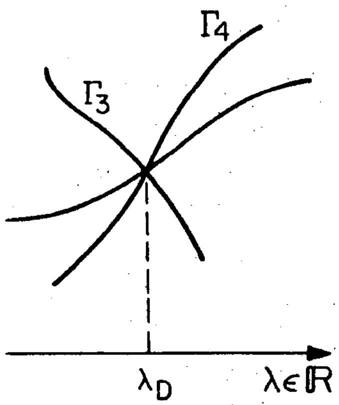
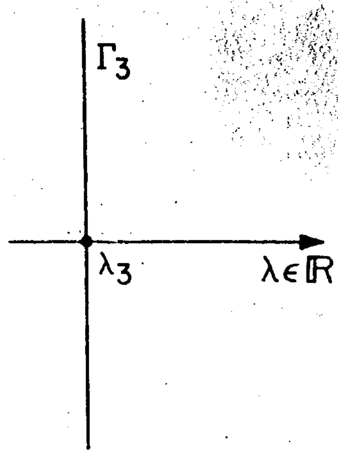
where:





have what is called a "limit point" in the applied mechanics literature. These present no difficulties analytically or computationally in our current theory although they seem to have been quite troublesome in the past [1, 12, 20]. Note that all the branches $\Gamma_1, \Gamma_2, \ldots$ in the eigenvalue problem are composed entirely of these limit points.

We assume that the basic problem is to compute large segments of solution branches of (1.1) including all branches bifurcating from each segment.

2. Parametrization and Continuation of Solution Arcs

The standard approach is almost invariably to use λ, one of the naturally occuring parameters of the problem, as the parameter defining solution arcs, $u(\lambda)$. Indeed if for some $\lambda = \lambda_0$ a solution $u = u_0$ of (1.1) is isolated, that is

$$G_u^0 \equiv G_u(u_0, \lambda_0) \tag{2.1}$$

is nonsingular, and if $G(u, \lambda)$ is C_1 in some ρ_0 - sphere about $[u_0, \lambda_0]$ then the implicit function theorem insures the existence of a unique smooth arc of solutions $u = u(\lambda)$ for $|\lambda - \lambda_0| < \rho_1$, say. Furthermore, with our assumed smoothness, it follows that $du(\lambda)/d\lambda$ exists and satisfies:

$$G_u(u(\lambda), \lambda)\frac{du}{d\lambda} = -G_\lambda(u(\lambda), \lambda) \tag{2.2}$$

Many procedures are now available for extending or approximating the solution branch through $[u_0, \lambda_0]$. The implicit function theorem suggests contraction mapping techniques while (2.2) suggests predictor-corrector continuation. A nice survey of the latter ideas (for finite dimensional problems) is given by Rheinboldt [18]. In particular we have frequently used [3] one step of Euler's method in (2.2) as a predictor:

$$u^0(\lambda + \delta\lambda) = u(\lambda) + \delta\lambda\,\frac{du(\lambda)}{d\lambda} \tag{2.3 a}$$

to supply the initial iterate for Newton's method to solve (1.1) at $\lambda + \delta\lambda$:

$$G_u^\nu\,\delta u^\nu(\lambda + \delta\lambda) = -G^\nu, \quad \nu = 0, 1, \ldots; \tag{2.3 b}$$

where:

$$(2.3)\ \mathrm{c)}\quad \begin{cases} G_u^\nu \equiv G_u(u^\nu(\lambda+\delta\lambda),\ \lambda+\delta\lambda)\ , \quad G^\nu \equiv G(u^\nu(\lambda+\delta\lambda),\ \lambda+\delta\lambda)\ , \\ u^{\nu+1}(\lambda+\delta\lambda) = u^\nu(\lambda+\delta\lambda) + \delta u^\nu(\lambda+\delta\lambda)\ . \end{cases}$$

It is not difficult to base existence proofs on such techniques provided the arc consists of isolated solutions.

All of the indicated continuation procedures may fail or encounter difficulties as a nonisolated solution is approached; that is a point $[u_0, \lambda_0]$ where G_u^0 is singular. We also call these singular points. As we shall see in §3 the above indicated continuation procedures could skip over some singular points but not over limit points as at $\lambda = \lambda_C$ in figure 1a. Also at bifurcation points some special procedures are required to switch from one branch to another. A simple analysis shows that the scheme (2.3) is incapable of tracing out any nontrivial solutions of the eigenvalue problem (1.2)!

To circumvent these difficulties we recall that the parametrization of solution arcs is at our disposal. Thus we are free to impose some additional constraint or normalization on the solution and we do this, quite generally, by replacing (1.1) by:

$$(2.4)\ \mathrm{a)}\qquad G(u,\lambda) = 0\ , \quad \mathrm{b)}\ N(u,\lambda,s) = 0\ .$$

Here $N : \mathbb{B} \times \mathbb{R}^2 \to \mathbb{R}$ and $s \in \mathbb{R}$ is the independent parameter on the solution arc. We shall show several choices for N which make s an approximation to "arclength" on the solution branch. Then, as we shall see, limit points essentially disappear, it is easy to jump over singular points or to compute them, relatively large steps in s can be taken and it is easy to switch branches at bifurcation points. Note that we are not simply changing the parameter in the problem as was first done in [12] or in another way in [1].

By introducing $x \in \mathbf{X} \equiv \mathbb{B} \times \mathbb{R}$ and $P : \mathbf{X} \times \mathbb{R} \to \mathbf{X}$ as

$$(2.5)\ \mathrm{a)}\qquad x \equiv [u, \lambda]\ , \quad P(x,s) \equiv \begin{pmatrix} G(u,\lambda) \\ N(u,\lambda,s) \end{pmatrix}$$

a solution arc of (1.1) or (2.4) is $x(s) \equiv [u(s), \lambda(s)]$ and it satisfies

$$(2.5)\ \mathrm{b)}\qquad P(x(s),\ s) = 0\ .$$

For fixed s a solution $x(s)$

$$(2.6)\qquad \mathcal{A}(s) \equiv P_x(x(s),\ s)$$

is nonsingular. Furthermore satisfies

$$(2.7)\qquad \mathcal{A}(s)\,\dot{x}(s)$$

Now continuation in s could continuation in λ. However t lations are clarified by the fa G_u is singular. Indeed thi

Lemma 2.8. Let $\mathbb{B}$ be a B operator $\mathcal{A} : \mathbb{B} \times \mathbb{R}^\nu \to \mathbb{B} \times \mathbb{R}$

$$\mathcal{A} \equiv \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \quad \text{where}$$

i) If A is nonsingular th

$$(2.8)\ \mathrm{a)}\qquad D - C^* A$$

ii) If A is singular and

$$(2.8)\ \mathrm{b)}\qquad \dim \mathcal{N}(A$$

then $\mathcal{A}$ is nonsingular iff:

$$(2.8)\qquad \begin{matrix} c_0)\ \ \dim \mathcal{R}(B) = \nu \\ c_2)\ \ \dim \mathcal{R}(C^*) = \end{matrix}$$

iii) If A is singular and

Proof. Not difficult to work o

Using Lemma 2.8 and ou prove a variety of bifurcation We only use in the present wor $\nu = 1$. Then given (2.8b) the c

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$N(u, \lambda, s)$

$(s), \lambda(s)]$ and it satisfies

For fixed s a solution $x(s)$ is isolated if

$$(2.6)\qquad \mathcal{A}(s) \equiv P_x(x(s), s) = \begin{pmatrix} G_u(u(s), \lambda(s)) & G_\lambda(u(s), \lambda(s)) \\ N_u(u(s), \lambda(s), s) & N_\lambda(u(s), \lambda(s), s) \end{pmatrix}$$

is nonsingular. Furthermore on a smooth arc $\dot{x}(s) \equiv dx(s)/ds$ satisfies

$$(2.7)\qquad \mathcal{A}(s)\,\dot{x}(s) = -\begin{pmatrix} 0 \\ N_s(u(s), \lambda(s), s) \end{pmatrix}$$

Now continuation in s could proceed in exact analogy with our prior continuation in λ. However the possible advantages of our reformulations are clarified by the fact that P_x can be nonsingular while G_u is singular. Indeed this is but a special case of the basic:

<u>Lemma 2.8.</u> <u>Let</u> $\mathbb{B}$ <u>be a Banach space and consider the linear operator</u> $\mathcal{A} : \mathbb{B} \times \mathbb{R}^{\nu} \to \mathbb{B} \times \mathbb{R}^{\nu}$ <u>of the form:</u>

$$\mathcal{A} \equiv \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \quad \underline{\text{where}} \quad \begin{cases} A : \mathbb{B} \to \mathbb{B}, & B : \mathbb{R}^{\nu} \to \mathbb{B}; \\ C^* : \mathbb{B} \to \mathbb{R}^{\nu}, & D : \mathbb{R}^{\nu} \to \mathbb{R}^{\nu}. \end{cases}$$

i) <u>If</u> A <u>is nonsingular then</u> $\mathcal{A}$ <u>is nonsingular iff:</u>

$$(2.8)\ \text{a)}\qquad D - C^* A^{-1} B \ \underline{\text{is nonsingular}}.$$

ii) <u>If</u> A <u>is singular and</u>

$$(2.8)\ \text{b)}\qquad \dim \mathcal{N}(A) = \operatorname{codim} \mathcal{R}(A) = \nu$$

<u>then</u> $\mathcal{A}$ <u>is nonsingular iff:</u>

$$(2.8)\qquad \begin{array}{ll} c_0)\ \dim \mathcal{R}(B) = \nu & c_1)\ \mathcal{R}(B) \cap \mathcal{R}(A) = 0, \\ c_2)\ \dim \mathcal{R}(C^*) = \nu, & c_3)\ \mathcal{N}(A) \cap \mathcal{N}(C^*) = 0. \end{array}$$

iii) <u>If</u> A <u>is singular and</u> $\dim \mathcal{N}(A) > \nu$ <u>then</u> $\mathcal{A}$ <u>is singular.</u>

<u>Proof.</u> Not difficult to work out; it will appear elsewhere, [10]. ∎

Using Lemma 2.8 and our reformulation of (1.1) it is easy to prove a variety of bifurcation theorems which we do not report here. We only use in the present work the special case of Lemma 2.8 with $\nu = 1$. Then given (2.8b) the conditions in (2.8c) simply reduce to

(2.9) $$B \notin \mathcal{R}(A) \quad \text{and} \quad C^* \notin \mathcal{R}(A^*) \; .$$

As a simple example of the use of our procedure consider the eigenvalue problem (1.2) subject to either of the normalizations

(2.10) a) $$N_0(u,\lambda,s) \equiv \lambda - s \quad ; \quad b) \quad N_1(u,\lambda,s) \equiv \|u\|_2^2 - s^2 \; .$$

Using $N_0 = 0$ the predictor-corrector continuation scheme generates the trivial solution branch, $u = 0$, λ = arbitrary, starting from any point on this branch. Alternatively using $N_1 = 0$ our scheme traces out the one dimensional eigenspace belonging to any simple eigenvalue $\lambda = \lambda_0$. Switching over from one of these normalizations to the other at a simple bifurcation point $(u,\lambda) = (0,\lambda_0)$ allows us to trace out the two branches through this point. Indeed this is the key to our method for switching branches at bifurcation points in the general case, see §5.

A somewhat natural parametrization of a solution branch $[u(s), \lambda(s)]$ is to use for s a form of arclength. That is, for some $\theta \in (0,1)$:

(2.10) c) $$N_\theta(u,\lambda,s) \equiv \theta\|\dot{u}(s)\|^2 + (1-\theta)\,|\dot{\lambda}(s)|^2 - 1 = 0 \; .$$

This form is not the most practical one to use, even for merely proving existence theorems and so we use approximations to it. Assuming a solution of (1.1) known say $[u,\lambda] = [u_0,\lambda_0]$ we set $[u_0,\lambda_0] = [u(s_0), \lambda(s_0)]$ and define over $s_0 \le s < s_1$:

(2.10) d) $$\begin{aligned} N_2(u,\lambda,s) \equiv {} & \theta\|u(s) - u(s_0)\|^2 + (1-\theta)\,|\lambda(s) - \lambda(s_0)|^2 \\ & - (s - s_0)^2 = 0 \; . \end{aligned}$$

Alternatively if in addition to $[u_0,\lambda_0]$ we know $[\dot{u}_0, \dot{\lambda}_0]$ satisfying (2.10c) at $s = s_0$ then we can use on $s_0 \le s < s_1$:

(2.10) e) $$\begin{aligned} N_3(u,\lambda,s) \equiv {} & \theta\,\dot{u}^*(s_0)[u(s) - u(s_0)] \\ & + (1-\theta)\,\dot{\lambda}(s_0)[\lambda(s) - \lambda(s_0)] - (s - s_0) = 0 \; . \end{aligned}$$

Here $\dot{u}^*(s_0) \in \mathbb{B}^*$ is the dual element to $\dot{u}(s_0)$ such that $\dot{u}^*(s_0)\,\dot{u}(s_0) = \|\dot{u}(s_0)\|^2$. (The existence of such an element is assured by the Hahn-Banach theorem.) We call N_2 or N_3 pseudo-arclength normalizations and examine some of their properties in §3. Previous attempts to use arclength as a parameter in solving nonlinear

algebraic systems have been

3. Continuation About Regula

We shall justify continuat N_3 on solution arcs compose points". Specifically let $[u_0,$ $[\dot{u}_0, \lambda_0]$ satisfy

(3.0) a) $G_u^0\,\dot{u}_0 + G_\lambda^0\,\dot{\lambda}_0$

Then we say that $[u_0, \lambda_0]$ is

(3.1) $G_u^0 \equiv G_u(u_0$

We call $[u_0, \lambda_0]$ a normal

place of (3.1) we have:

(3.2) a) $\dim \mathcal{N}(G_u^0) =$

Theorem 3.3. Let $[u_0, \lambda_0]$ limit solution. Let $G(u,\lambda)$ sphere about $[u_0, \lambda_0]$. Then $[\dot{u}(s_0), \dot{\lambda}(s_0)] \equiv [\dot{u}_0, \dot{\lambda}_0]$ and exists a unique smooth arc $N \equiv N_3$ on $|s - s_0| \le \rho$ for solution arc the Frechet deriv

Proof. All of these results theorem applied to (2.4) at singular. We first consider By (3.2b) in (3.0a) it then foll Now (3.2a) implies since $\dot{u}_0^*$ and (3.2) now yield with

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algebraic systems have been made in [13, 14].

3. Continuation About Regular and Limit Points.

We shall justify continuation procedures using the normalization N_3 on solution arcs composed of "regular points" or "normal limit points". Specifically let $[u_0, \lambda_0]$ be a solution of (1.1) and let $[\dot u_0, \dot\lambda_0]$ satisfy

(3.0) a) $\quad G_u^0\, \dot u_0 + G_\lambda^0\, \dot\lambda_0 = 0$, b) $\|\dot u_0\|^2 + |\dot\lambda_0|^2 > 0$.

Then we say that $[u_0, \lambda_0]$ is a <u>regular solution (point)</u> if in addition:

(3.1) $\quad G_u^0 \equiv G_u(u_0, \lambda_0)$ is nonsingular .

We call $[u_0, \lambda_0]$ a <u>normal limit solution (point)</u> if (3.0) holds but in place of (3.1) we have:

(3.2) a) $\quad \dim \mathcal{N}(G_u^0) = \operatorname{codim} \mathcal{R}(G_u^0) = 1$, b) $G_\lambda^0 \notin \mathcal{R}(G_u^0)$.

<u>Theorem 3.3. Let $[u_0, \lambda_0]$ be either a regular solution or a normal limit solution. Let $G(u, \lambda)$ have two continuous derivatives in some sphere about $[u_0, \lambda_0]$. Then with $[u(s_0), \lambda(s_0)] \equiv [u_0, \lambda_0]$, $[\dot u(s_0), \dot\lambda(s_0)] \equiv [\dot u_0, \dot\lambda_0]$ and $\dot u^*(s_0)$ as defined after (2.10e) there exists a unique smooth arc of solutions $[u(s), \lambda(s)]$ of (2.4) using $N \equiv N_3$ on $|s - s_0| \le \rho$ for some sufficiently small $\rho > 0$. On this solution arc the Frechet derivative $\mathcal{A}(s)$ of (2.6) is nonsingular.</u>

<u>Proof</u>. All of these results will follow from the implicit function theorem applied to (2.4) at $[u, \lambda, s] = [u_0, \lambda_0, s_0]$ if $\mathcal{A}(s_0)$ is nonsingular. We first consider $[u_0, \lambda_0]$ to be a normal limit solution. By (3.2b) in (3.0a) it then follows that $\dot\lambda_0 = 0$ and hence $\dot u_0 \in \mathcal{N}(G_u^0)$. Now (3.2a) implies since $\dot u_0^* \dot u_0 \neq 0$ that $\dot u_0^* \notin \mathcal{R}(G_u^{0\,*})$. This result and (3.2) now yield with $\nu = 1$ in Part ii) of Lemma 2.8 that

$$\mathcal{A}(s_0) = \begin{pmatrix} G_u^0 & G_\lambda^0 \\ \theta \dot u_0^* & (1-\theta)\dot\lambda_0 \end{pmatrix}$$

is nonsingular [use the form in (2.9)].

Next let $[u_0, \lambda_0]$ be a regular solution. If $\dot\lambda_0 = 0$ then by (3.0a) and (3.1), $\dot{u}_0 = 0$. This contradicts (3.0b) so $\dot{\lambda}_0 \neq 0$ at a regular point. Then (3.0a) and (3.1) imply

$$\dot{u}_0/\dot{\lambda}_0 = -(G_u^0)^{-1} G_\lambda^0 .$$

Now by Part i) of Lemma 2.8. $\mathcal{A}(s_0)$ above is nonsingular iff

$$(1-\theta)\dot{\lambda}_0 - \theta\, \dot{u}_0^*(G_u^0)^{-1} G_\lambda^0 \neq 0 .$$

That is, using the above, iff

$$[\theta\, \dot{u}_0^* \dot{u}_0 + (1-\theta)\dot{\lambda}_0^2]/\dot{\lambda}_0 = [\theta \|\dot{u}_0\|^2 + (1-\theta)\, |\dot{\lambda}_0|^2]/\dot{\lambda}_0 \neq 0 .$$

Since $\dot{\lambda}_0 \neq 0$ and $\|\dot{u}_0\| \neq 0$ it follows that $\mathcal{A}(s_0)$ is nonsingular. ∎

Clearly any smooth branch of solutions composed of regular points or normal limit points can be determined using, say, Euler-Newton continuation on (2.4) with the normalization $N \equiv N_3$. We could easily justify the normalization $N = N_2$ for smooth arcs since on them:

$$u(s) - u(s_0) = \dot{u}(s_0)(s - s_0) + \mathcal{O}(|s - s_0|^2) ,$$
$$\lambda(s) - \lambda(s_0) = \dot{\lambda}(s_0)(s - s_0) + \mathcal{O}(|s - s_0|^2) .$$

When using (2.10e) over a sequence of intervals $[s_0, s_1]$, $[s_1, s_2]$, ... it is a good policy to impose the arclength condition

$$\theta\|\dot{u}(s)\|^2 + (1-\theta)\, |\dot{\lambda}(s)|^2 = 1$$

periodically, say at each joint, $s = s_k$. The resulting arc is then only piecewise smooth in s, with jump discontinuities in the length of the tangent vector $[\dot{u}(s), \dot{\lambda}(s)]$ at $s = s_k$. Specifically if $[\dot{u}(s_k^-), \dot{\lambda}(s_k^-)]$ is the limit as $s \uparrow s_k$ then we use on $[s_k, s_{k+1}]$

$$[\dot{u}(s_k), \dot{\lambda}(s_k)] = c[\dot{u}(s_k^-), \dot{\lambda}(s_k^-)] ,$$

$$c^{-2} = \theta\|\dot{u}(s_k^-)\|^2 + (1-\theta)\, |\dot{\lambda}(s_k^-)|^2 .$$

This renormalization allows more uniform steps in s to be taken during the continuation process.

4. Continuation Past Singular Points.

A solution $x(s) = [u(s), \lambda(s)]$ of (2.4) is said to be singular or a singular point if $\mathcal{A}(s)$ in (2.6) is singular. We will consider smooth arcs of solutions $x(s)$ for $s_a \leq s \leq s_b$ on which only $x(s_0)$ for some

$s_0 \in (s_a, s_b)$ is singular. Under mild smoothness conditions various continuation procedures can "jump" over the singular point in going from s_a to s_b. A simple way to do this is by using Euler-Chord or Euler-Newton continuation, as we proceed to show.

We assume, as stated above, that

(4.0) $\quad \mathcal{A}(s) \equiv P_x(x(s), s)$ is nonsingular for $s \in [s_a, s_b] - \{s_0\}$.

Then the tangent, $\dot{x}(s_a)$, is uniquely defined by

(4.1) a) $$\mathcal{A}(s_a)\,\dot{x}(s_a) = -P_s(x(s_a), s_a) \,,$$

and an approximation to $x(s)$ is

(4.1) b) $$x^0(s) \equiv x(s_a) + [s - s_a]\,\dot{x}(s_a) \;.$$

Using this approximation we consider the chord (or special Newton) method:

(4.2)
$$\text{a)}\quad \mathcal{A}^0(s) \equiv P_x(x^0(s), s) \,,$$
$$\text{b)}\quad \mathcal{A}^0(s)\left[x^{\nu+1}(s) - x^{\nu}(s)\right] = -P(x^{\nu}(s), s) \,, \quad \nu = 0, 1, \ldots .$$

We could also try Newton's method, with $y^0(s) \equiv x^0(s)$:

(4.3)
$$\left.\begin{aligned} &\text{a)}\quad \mathcal{A}^{\nu}(s) \equiv P_x(y^{\nu}(s), s) \\ &\text{b)}\quad \mathcal{A}^{\nu}(s)\left[y^{\nu+1}(s) - y^{\nu}(s)\right] = -P(y^{\nu}(s), s) \end{aligned}\right\} \quad \nu = 0, 1, \ldots .$$

To get convergence we need only show that $x^0(s)$ is in the appropriate domain of attraction about $x(s)$ and that the $\mathcal{A}^{\nu}(s)$ are all nonsingular. We do this in

Theorem 4.4. Let $x(s)$ be a twice continuously differentiable arc of solutions of (2.5) on $[s_a, s_b]$. Let (4.0) hold and for some positive functions $K(s)$, $\mathcal{X}(s)$ and $\rho(s)$ defined on $[s_a, s_b]$:

(4.1)
$$\text{a)}\quad \|P_x(y, s) - P_x(x(s), s)\| \le K(s)\,\|y - x(s)\| \;\forall\, y \text{ in } \|y - x(s)\| \le \rho(s) \;;$$
$$\text{b)}\quad \max_{s_a \le t \le s} \|\ddot{x}(t)\| \le \mathcal{X}(s) \;.$$

For $s \in [s_a, s_b] - \{s_0\}$ define

(4.4) c) $$M(s) \equiv \|\mathcal{A}^{-1}(s)\| ,$$

<u>and for some positive</u> $\theta(s) < 1/3$ <u>define</u>

(4.4) d) $$r(s) \equiv \min\left[\rho(s), \frac{\theta(s)}{M(s)\,K(s)}\right] .$$

<u>Then if</u>

(4.4) e) $$|s - s_a|^2 \,\mathcal{K}(s) \leq 2r(s) , \quad s \neq s_0 ,$$

<u>the Chord iterates</u> $x^\nu(s) \to x(s)$ <u>with geometric convergence factor</u>

$$\frac{2\theta(s)}{1 - \theta(s)} .$$

<u>Proof</u>. It easily follows from (4.1) and (4.4) that

$$\|x^0(s) - x(s)\| \leq \tfrac{1}{2}|s - s_a|^2 \,\mathcal{K}(s) .$$

Then (4.4d, e) imply that $\|x^0(s) - x(s)\| \leq r(s)$. Now (4.4a, c) yield

$$\|\mathcal{A}^{-1}(s)[\mathcal{A}^0(s) - \mathcal{A}(s)]\| \leq M(s)\,K(s)\,r(s) \leq \theta(s) .$$

Thus the Banach Lemma insures that $\mathcal{A}^0(s)$ is nonsingular with

$$\|\mathcal{A}^0(s)^{-1}\| \leq \frac{M(s)}{1 - \theta(s)} .$$

We can now define

$$H(y, s) \equiv y - \mathcal{A}^0(s)^{-1}\,P(y, s)$$

and for all y, z in $\|y - x(s)\| \leq r(s)$:

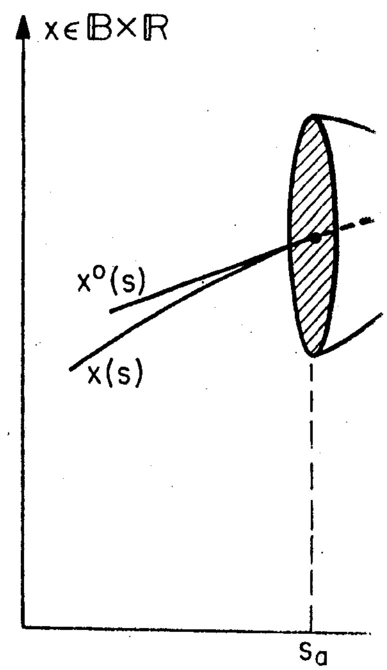
$$\|H(y, s) - H(z, s)\| = \|\mathcal{A}^0(s)^{-1}[\mathcal{A}^0(s)(y - z) - (P(y, s) - P(z, s)]\| ,$$

$$\leq \frac{M(s)}{1 - \theta(s)}\,K(s)\,2r(s)\,\|y - z\| ,$$

$$\leq \frac{2\theta(s)}{1 - \theta(s)}\,\|y - z\| .$$

Since $\theta(s) < 1/3$ we get that $H(y, s)$ is contracting on $\|y - x(s)\| \leq r(s)$ and the theorem follows.

Note that $M(s)$ must become unbounded and thus $r(s) \to 0$ as $s \to s_0$. Our result thus uses the fact (see figure 2) that there is a cone about $\dot{x}(s)$ with vertex at $x(s_0)$ and interior to this cone the chord method converges. To jump over a singular point the tangent to $x(s)$ at $x(s_a)$ need only penetrate the cone for some $s > s_0$ for which (4.4e) holds. Clearly if the curvature of the solution arc is not



‖ ,

$\frac{\theta(s)}{A(s)\,K(s)}]$.

(s) , $s \neq s_0$,

metric convergence factor

. 4) that

$|^2\ \mathcal{K}(s)$.

r(s). Now (4. 4a, c) yield

$K(s)\ r(s) \leq \theta(s)$.

) is nonsingular with

$\frac{)}{s)}$.

(y, s)

z) - (P(y, s) - P(z, s)] ‖ ,

(s) ‖y-z‖ ,

.

ntracting on ‖y-x(s)‖ ≤ r(s)

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figure 2) that there is a
interior to this cone the
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e of the solution arc is not

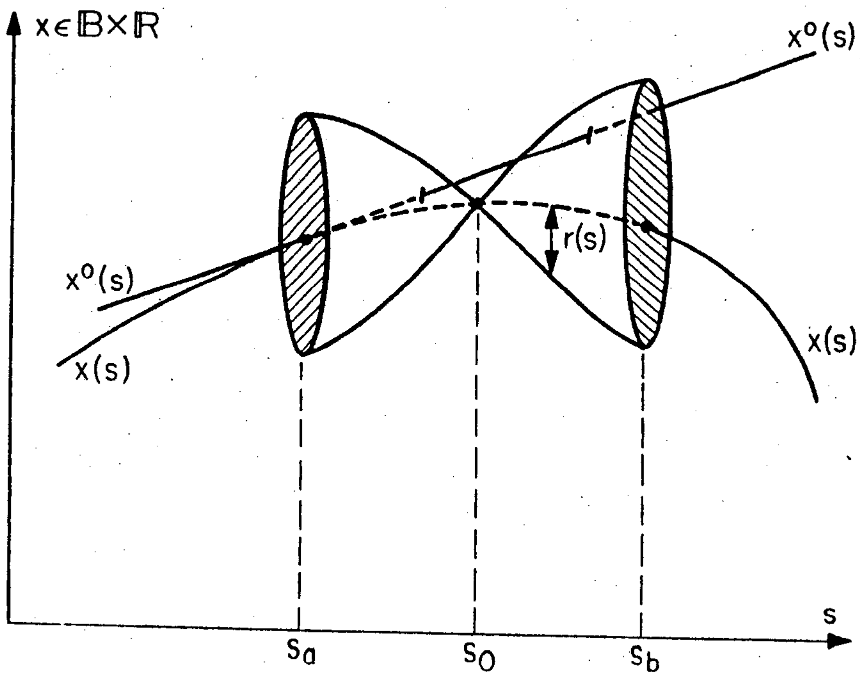

Figure 2

too great over $[s_a, s_b]$ this can be achieved.

Newton's method can also be shown to converge in the same cone. However Newton's method may even converge in a much larger region, including a cylindrical tube about $x(s)$. Thus the singular solution $x(s_0)$ can be determined directly in such cases not only by bisection. Unfortunately the status of the convergence of Newton's method at singular points is not completely clear at the present time. The main idea and no doubt the behavior to be expected in many cases is explained in the basic paper of Rall [16]. But the details of a proof with reasonable sufficient conditions seem to be lacking. Progress in this direction has recently been made by Reddien [17].

5. Switching Branches at Bifurcation Points.

Bifurcation points are solutions at which two or more smooth branches of solutions of (1.1) have non-tangential intersections. In particular they are singular points, say $[u_0, \lambda_0]$, at which:

$$(5.0)\quad \begin{aligned} &a) && \dim \mathcal{N}(G^0_u) = \operatorname{codim} \mathcal{R}(G^0_u) = m\ , \\ &b) && G^0_\lambda \in \mathcal{R}(G^0_u)\ . \end{aligned}$$

From (5.0a) we have the existence of elements $\phi_j \in \mathbb{B}$ and $\psi^*_j \in \mathbb{B}^*$ such that:

$$(5.1)\quad \left.\begin{aligned} \mathcal{N}(G^0_u) &= \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_m\} \\ \mathcal{N}(G^{0*}_u) &= \operatorname{span}\{\psi^*_1, \psi^*_2, \ldots, \psi^*_m\} \end{aligned}\right\} \psi^*_i \phi_j = \delta_{ij}\ ;\ i, j = 1, 2, \ldots, m.$$

In addition (5.0b) implies the existence of a unique element $\phi_0 \in \mathbb{B}$ such that:

$$(5.2)\qquad G^0_u \phi_0 + G^0_\lambda = 0\ ;\ \psi^*_j \phi_0 = 0\ ,\ 1 \le j \le m\ .$$

Let $[u(s), \lambda(s)]$ be any smooth branch of (1.1) through the bifurcation point, say with $u(s_0) = u_0$, $\lambda(s_0) = \lambda_0$. Then since

$$(5.3)\ a)\qquad G^0_u \dot{u}(s_0) + G^0_\lambda \dot{\lambda}(s_0) = 0$$

it follows from (5.1)-(5.2) that

$$(5.3)\ b)\qquad \dot{u}(s_0) = \sum_{j=0}^{m} \alpha_j \phi_j$$

where

$$(5.3)\ c)\qquad \alpha_0 = \dot{\lambda}(s_0)\ ;$$

We get by differentiation of (

$$(5.4)\qquad G^0_u \ddot{u}_0 = -[G^0_{uu} \dot{u}($$

$$+ G^0_{\lambda\lambda} \dot{\lambda}($$

Since $G^0_u \ddot{u}(s_0) \in \mathcal{R}(G^0_u)$ and
right side of (5.4) is also in
this term and so using (5.3)
$\{\alpha_0, \ldots, \alpha_m\}$ must satisfy

$$(5.5)\ a)\qquad \sum_{j=1}^{m}\sum_{k=1}^{m} a_{ijk}\,\alpha_j\alpha_k + 2$$

where

$$(5.5)\ b)\qquad a_{ijk} \equiv \psi^*_i G^0_{uu} \phi_j$$

$$c_i \equiv \psi^*_i (G^0_{uu}$$

Thus the tangent, $[\dot{u}(s_0)$,
the bifurcation point $[u_0, \lambda_0]$
fy (5.5). Conversely if at a s
hold and (5.5) has $r \ge 2$ dist
bifurcation point with at least
tersecting there. Essentially
important special case $m = 1$,
(5.5) reduce to the single qua

$$(5.6)\ a)\qquad a_{111}\alpha_1^2 + 2$$

If $[\alpha_0, \alpha_1]$ is one nontrivial
is distinct provided

$$(5.6)\ b)\qquad a_{111}\, c$$

If $[u_1(s), \lambda_1(s)]$ is a smooth

† Since (5.5a) is homogeneous
multiplicative constant. Root
scalar multiples of each other

where

(5.3) c) $$\alpha_0 = \dot{\lambda}(s_0) \ ; \ \alpha_j = \psi_j^* \, \dot{u}(s_0) \ , \quad 1 \leq j \leq m \ .$$

We get by differentiation of $G(u(s), \lambda(s)) = 0$ at $s = s_0$:

(5.4) $$G_u^0 \, \ddot{u}_0 = -\,[\, G_{uu}^0 \, \dot{u}(s_0)\dot{u}(s_0) + 2G_{u\lambda}^0 \, \dot{u}(s_0)\,\dot{\lambda}(s_0) + G_{\lambda\lambda}^0 \, \dot{\lambda}(s_0)\,\dot{\lambda}(s_0)] - G_\lambda^0 \, \ddot{\lambda}(s_0) \ .$$

Since $G_u^0 \, \ddot{u}(s_0) \in \mathcal{R}(G_u^0)$ and $G_\lambda^0 \in \mathcal{R}(G_u^0)$ the bracketed term on the right side of (5.4) is also in $\mathcal{R}(G_u^0)$. Then $\psi_i^*[\ldots] = 0$ must hold for this term and so using (5.3) it follows that the $m+1$ scalars $\{\alpha_0, \ldots, \alpha_m\}$ must satisfy the quadratic system:

(5.5) a) $$\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ijk} \, \alpha_j \alpha_k + 2 \sum_{j=1}^{m} b_{ij} \, \alpha_j \, \alpha_0 + c_i \, \alpha_0^2 = 0 \ , \quad 1 \leq i \leq m \ ;$$

where

(5.5) b) $$a_{ijk} \equiv \psi_i^* \, G_{uu}^0 \, \phi_j \phi_k \ , \quad b_{ij} \equiv \psi_i^* [\, G_{uu}^0 \, \phi_0 + G_{u\lambda}^0]\, \phi_j \ , \quad 1 \leq j, \ k \leq m \ ;$$

$$c_i \equiv \psi_i^* \, (G_{uu}^0 \, \phi_0 \phi_0 + 2G_{u\lambda}^0 \, \phi_0 + G_{\lambda\lambda}^0) \ .$$

Thus the tangent, $[\dot{u}(s_0), \dot{\lambda}(s_0)]$, to every smooth branch through the bifurcation point $[u_0, \lambda_0]$ must have the form (5.3b, c) and satisfy (5.5). Conversely if at a solution $[u_0, \lambda_0]$ of (1.1) conditions (5.0) hold and (5.5) has $r \geq 2$ distinct nontrivial roots† then $[u_0, \lambda_0]$ is a bifurcation point with at least r smooth solution branches of (1.1) intersecting there. Essentially this result is proven in [11]. In the important special case $m = 1$, the algebraic bifurcation equations (5.5) reduce to the single quadratic

(5.6) a) $$a_{111} \, \alpha_1^2 + 2b_{11} \, \alpha_1 \, \alpha_0 + c_1 \, \alpha_0^2 = 0 \ .$$

If $[\alpha_0, \alpha_1]$ is one nontrivial root of this quadratic then the other root is distinct provided

(5.6) b) $$a_{111} \, \alpha_1 + b_{11} \, \alpha_0 \neq 0 \ .$$

If $[u_1(s), \lambda_1(s)]$ is a smooth branch of solutions through the

† Since (5.5a) is homogeneous a root is determined only to within a multiplicative constant. Roots are said to be distinct if they are not scalar multiples of each other.

bifurcation point with tangent $[\dot{u}_1(s_0),\ \dot{\lambda}_1(s_0)]$ determined by $[\alpha_0, \alpha_1]$ in (5.3b, c) with $m = 1$ then the condition (5.6b) can be written as:

$$\text{(5.6) c)} \qquad \psi_1^*[G_{uu}^0\, \dot{u}_1(s_0) + G_{u\lambda}^0\, \dot{\lambda}_1(s_0)]\,\phi_1 \neq 0 \ .$$

This is essentially the form of the bifurcation condition given by Crandall and Rabinowitz [5]. However it is seldom pointed out that this condition insures the existence of two distinct roots of a quadratic†; Rheinboldt [18] makes this quite explicit for the class of problems he treats.

<u>Method I.</u> An obvious way to determine branches bifurcating at $[u_0, \lambda_0]$ is to determine several distinct roots of (5.5), use them in (5.3b, c) to construct several distinct tangent vectors $[\dot{u}_k(s_0),\ \dot{\lambda}_k(s_0)]$, $k = 1, 2, \ldots,$ then use each of these tangents in $N_3 = 0$ of (2.10e) and proceed as previously indicated. For $m = 1$, simple bifurcation, similar devices have been used and suggested in analytical and perturbation studies [7, 18, 21]. Rheinboldt [18] uses an approximation to the second root in some of his numerical methods to predict a point on the "second" branch. This idea can be used quite generally to obtain approximations to the coefficients $\{a_{ijk}, b_{ij}, c_i\}$ in (5.5) if the ϕ_j and ψ_j^* are known (or sufficiently well approximated). Thus we define:

$$\text{(5.7)}\quad
\begin{aligned}
&\text{a)} \quad a_{ijk}(\epsilon) \equiv \psi_i^* \frac{1}{\epsilon}[G_u(u_0 + \epsilon\phi_j, \lambda_0) - G_u^0]\,\phi_k \ , \\
&\text{b)} \quad b_{ij}(\epsilon) \equiv \psi_i^* \frac{1}{\epsilon}\{[G_u(u_0 + \epsilon\phi_j, \lambda_0) - G_u^0]\,\phi_0 \\
&\qquad\qquad + [G_\lambda(u_0 + \epsilon\phi_j, \lambda_0) - G_\lambda^0]\} \\
&\text{c)} \quad c_i(\epsilon) \equiv \psi_i^* \frac{1}{\epsilon}\{[G_u(u_0 + \epsilon\phi_0, \lambda_0) - G_u^0]\,\phi_0 \\
&\qquad\qquad + 2[G_\lambda(u_0 + \epsilon\phi_0, \lambda_0) - G_\lambda^0] \\
&\qquad\qquad + [G_\lambda(u_0, \lambda_0 + \epsilon) - G_\lambda^0]\} \ .
\end{aligned}$$

† In bifurcation from the trivial solution $[u_1(s),\ \lambda_1(s)] = [0, s]$ it follows that $\phi_0 \equiv 0$, since $G_\lambda^0 = 0$. Hence $c_i \equiv 0$ and the quadratic (5.6a), in the case $m=1$ of simple bifurcation, has the two solutions: $\alpha_1/\alpha_0 = 0$, $\alpha_1/\alpha_0 = -2b_{11}/a_{111}$. Clearly $b_{11} \neq 0$ if (5.6c) holds since $\dot{u}_1(s_0) = 0$.

Clearly $\{a_{ijk}(\epsilon),\ b_{ij}(\epsilon),\ c_i(\epsilon$
bove scheme avoids the need
tives.

<u>Method II.</u> There are other
the coefficients or roots of (5
through the bifurcation point
$[\dot{u}(s_0),\ \dot{\lambda}(s_0)]$ can also be a
is simply to seek solutions o
but displaced from the bifurca
the tangent.

For example in the case
solution branch $[u_1(s),\ \lambda_1(s$
$s = s_0$ given by (5.3b, c). An
spanned by $[\phi_1, 0]$ and $[\phi_0,$
α_1 and α_0 replaced by:

(5.8) a) $\hat{\alpha}_1 = \alpha_0(1 +$

Then we seek solutions in the

(5.8) b) $u_2 = u_1(s_0)$
$\lambda_2 = \lambda_1(s_0)$

These are to satisfy:

(5.9) $G($
$N(u_2,\ \lambda_2) \equiv (\hat{\alpha}$

We use Newton's method to s
initial estimate $(v^0, \eta^0) = (0,$
large so that the scheme does
solution.

<u>Method III.</u> Another way to d
known branch $[u(s),\ \lambda(s)]$ at
tence theory, using iterations
idea we consider simple bifurc
in the form

$(s_0)]$ determined by
condition (5.6b) can be writ-

$s_0)]\phi_1 \neq 0$.

tion condition given by
is seldom pointed out that
distinct roots of a quadra-
licit for the class of pro-

ranches bifurcating at
roots of (5.5), use them in
gent vectors $[\dot{u}_k(s_0), \dot{\lambda}_k(s_0)]$,
nts in $N_3 = 0$ of (2.10e)
$m = 1$, simple bifurcation,
sted in analytical and per-
18] uses an approximation
al methods to predict a point
used quite generally to
$\{a_{ijk}, b_{ij}, c_i\}$ in (5.5) if
well approximated). Thus

$\lambda_0) - G_u^0]\phi_k$,

$, \lambda_0) - G_u^0]\phi_0$.

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$, \lambda_0) - G_u^0]\phi_0$

$) - G_\lambda^0]$

$_\lambda(u_0, \lambda_0 + \epsilon) - G_\lambda^0]\}$.

$u_1(s), \lambda_1(s)] = [0, s]$ it fol-
$= 0$ and the quadratic (5.6a),
the two solutions: $\alpha_1/\alpha_0 = 0$,
5c) holds since $\alpha_1(s_0) = 0$.

Clearly $\{a_{ijk}(\epsilon), b_{ij}(\epsilon), c_i(\epsilon)\} \to \{a_{ijk}, b_{ij}, c_i\}$ as $\epsilon \to 0$. The above scheme avoids the need for determining second Fréchet derivatives.

Method II. There are other devices which avoid the need to evaluate the coefficients or roots of (5.5). This assumes that one branch through the bifurcation point has been determined. Then the tangent $[\dot{u}(s_0), \dot{\lambda}(s_0)]$ can also be assumed known on this branch. The idea is simply to seek solutions on some subset "parallel" to the tangent but displaced from the bifurcation point in some direction "normal" to the tangent.

For example in the case $m = 1$ of simple bifurcation the known solution branch $[u_1(s), \lambda_1(s)]$ has the tangent at the bifurcation, $s = s_0$ given by (5.3b, c). An "orthogonal" to this tangent in the plane spanned by $[\phi_1, 0]$ and $[\phi_0, 1]$ is also given by (5.3b, c) but with α_1 and α_0 replaced by:

(5.8) a) $$\hat{\alpha}_1 = \alpha_0(1 + \|\phi_0\|^2), \quad \hat{\alpha}_0 = -\alpha_1 \|\phi_1\|^2 .$$

Then we seek solutions in the form:

(5.8) b) $$\begin{aligned} u_2 &= u_1(s_0) + \epsilon[\hat{\alpha}_0\phi_0 + \hat{\alpha}_1\phi_1] + v , \\ \lambda_2 &= \lambda_1(s_0) + \epsilon\hat{\alpha}_0 + \eta . \end{aligned}$$

These are to satisfy:

(5.9) $$\begin{aligned} G(u_2, \lambda_2) &= 0 , \\ N(u_2, \lambda_2) &\equiv (\hat{\alpha}_0\phi_0^* + \hat{\alpha}_1\phi_1^*)v + \hat{\alpha}_0\eta = 0 . \end{aligned}$$

We use Newton's method to solve (5.9) for $v \in \mathbb{B}$ and $\eta \in \mathbb{R}$ with the initial estimate $(v^0, \eta^0) = (0, 0)$. Here ϵ must be taken sufficiently large so that the scheme does not return to $(u_1(s_0), \lambda_1(s_0))$ as the solution.

Method III. Another way to determine a branch bifurcating from a known branch $[u(s), \lambda(s)]$ at $s = s_0$ is to apply a constructive existence theory, using iterations, say as in [8, 11]. To sketch the basic idea we consider simple bifurcation and seek the bifurcated branch in the form

(5.10) a) $$u = u_1(\sigma) + \epsilon[\phi_1 + v], \quad \psi_1^* v = 0 ;$$
$$\lambda = \lambda_1(\sigma) .$$

Then (1.1) is written, using (5.10a), as

(5.10) b) $$G_u^0 v = G_u^0 v - \frac{1}{\epsilon} G(u_1(\sigma) + \epsilon[\phi_1 + v], \lambda_1(\sigma)) \; ; \; \psi_1^* v = 0 .$$

To insure that the right hand side is in $\mathcal{R}(G_u^0)$ we try to pick $\sigma = s$ such that $h(s;\epsilon, v) = 0$ where

(5.10) c) $$h(s;\epsilon, v) \equiv \begin{cases} \psi_1^*[G_u^0 v - \frac{1}{\epsilon} G(u_1(s) + \epsilon[\phi_1 + v], \lambda_1(s))], & \epsilon \neq 0 , \\ \psi_1^*[G_u^0 v - G_u(u_1(s), \lambda_1(s))[\phi_1 + v] , & \epsilon = 0 . \end{cases}$$

It easily follows that $h(s_0;0,0) = 0$ and

(5.10) d) $$h_s^0 \equiv h_s(s_0, 0, 0) = -\psi_1^*[G_{uu}^0 \dot u_1(s_0) + G_{u\lambda}^0 \dot\lambda_1(s_0)]\phi_1 .$$

Thus as in (5.6), $h_s(s_0;0,0) \neq 0$ and so the implicit function theorem yields a root, $s = \sigma(\epsilon, v)$, of $h(s;\epsilon,v) = 0$. We use this root in (5.10b) and then, by contraction maps, it is easily shown that (5.10b) has a unique solution $v = v(\epsilon)$ for $|\epsilon|$ sufficiently small [10].

The main difficulty in applying the above procedure is in solving $h(s;\epsilon, v) = 0$ for s at each iterate $v = v^\nu$, say. Of course if λ occurs linearly in the problem and it is used as the parameter, s, then this is trivial. But when λ occurs nonlinearly as it must for secondary bifurcation, modifications must be introduced. Several have been proposed in [4, 6, 15, 19]. For example given the ν^{th} iterate, (σ^ν, v^ν), we could use the chord method to define $\sigma^{\nu+1}$ as in

(5.11) a) $$m^0 \sigma^{\nu+1} = m^0 \sigma^\nu - h(\sigma^\nu;\epsilon, v^\epsilon) , \quad m^0 \equiv \psi_1^* B ;$$

and then $v^{\nu+1}$ is obtained from

(5.11) b) $$G_u^0 v^{\nu+1} = G_u^0 v^\nu - \frac{1}{\epsilon} G(u_1(\sigma^\nu) + \epsilon[\phi_1 + v^\nu], \lambda_1(\sigma^\nu))$$
$$- B[\sigma^{\nu+1} - \sigma^\nu] \; ; \; \psi_1^* v^{\nu+1} = 0 .$$

Applying ψ_1^* to the right hand side above we see that it is in $\mathcal{R}(G_u^0)$. Furthermore with the choice

(5.12) a) $$B = B^0 \equiv [G_{uu}^0 \dot u_1(s_0) + G_{u\lambda}^0 \dot\lambda_1(s_0)] \phi_1$$

it follows from (5.10d) that $m^0 = h_s^0 \neq 0$. There is no difficulty in showing convergence of the above scheme. To avoid the evaluation of

second derivatives we can

(5.12) b) $B = B(\epsilon) \equiv G$
$+ G_\lambda($

Clearly $B(\epsilon) = B^0 + \mathcal{O}(\epsilon)$
proceed as before. This
it is justified for finite dim
the solution branch $[u_1(s), \lambda$
lic" approximation: †

(5.13) $\hat u_1(s) = u_1(s_0) +$
$\hat\lambda_1(s) = \lambda_1(s_0)$

the procedure still converg

Method IV. A final method
a simple eigenvalue is to us
of the Crandall and Rabinow
again seek solutions of the

(5.14) a) $g(v, s;\epsilon) \equiv \begin{cases} \frac{1}{\epsilon} G(u_1 \\ G_u(u_1(\end{cases}$

b) $N(v, s;\epsilon) \equiv \psi_1^* v$.

Now we note that

(5.15) a) $g(0, s_0;0)$

and the Frechét derivative

(5.15) b) $\mathcal{A}^0 = \frac{\partial(g,}{\partial(v,}$

where B^0 is given in (5.12
it follows that $\mathcal{A}^0$ is nonsin
theorem shows that

(5.15) c)

† In [19] the parameter $s \equiv$
generalization can be shown

second derivatives we can use the trick in (5.7) and take:

$$(5.12)\ \ b)\qquad B \equiv B(\epsilon) \equiv G_u(u_1(s_0) + \epsilon\phi_1, \lambda_1(s_0))\,\dot{u}_1(s_0) + G_\lambda(u_1(s_0) + \epsilon\phi_1, \lambda_1(s_0))\,\dot{\lambda}_1(s_0)\ .$$

Clearly $B(\epsilon) = B^0 + \mathcal{O}(\epsilon)$ so that $\psi_1^* B(\epsilon) = h_s^0 + \mathcal{O}(\epsilon)$ and the proofs proceed as before. This modification is due to Rheinboldt [19] where it is justified for finite dimensional problems. He also shows that if the solution branch $[u_1(s), \lambda_1(s)]$ in (5.11) is replaced by the "parabolic" approximation: †

$$(5.13)\qquad \begin{aligned} \hat{u}_1(s) &= u_1(s_0) + (s-s_0)\,\dot{u}_1(s_0) + \tfrac{1}{2}(s-s_0)^2\,\ddot{u}_1(s_0) \\ \hat{\lambda}_1(s) &= \lambda_1(s_0) + (s-s_0)\,\dot{\lambda}_1(s_0) + \tfrac{1}{2}(s-s_0)^2\,\ddot{\lambda}_1(s_0) \end{aligned}$$

the procedure still converges to a bifurcated solution.

Method IV. A final method for determining the bifurcating branch at a simple eigenvalue is to use a technique based on a modification [10] of the Crandall and Rabinowitz [5] proof of bifurcation. Thus we again seek solutions of the form (5.10a) and define

$$(5.14)\ \ a)\quad g(v, s; \epsilon) \equiv \begin{cases} \dfrac{1}{\epsilon} G(u_1(s) + \epsilon[\phi_1 + v], \lambda_1(s)) & , \ \epsilon \neq 0\ ; \\ G_u(u_1(s), \lambda_1(s))[\phi_1 + v] & , \ \epsilon = 0\ ; \end{cases}$$

$$b)\quad N(v, s; \epsilon) \equiv \psi_1^* v\ .$$

Now we note that

$$(5.15)\ \ a)\qquad g(0, s_0; 0) = 0\ ,\quad N(0, s_0; 0) = 0$$

and the Frechét derivative at $(v, s; \epsilon) = (0, s_0; 0)$ is

$$(5.15)\ \ b)\qquad \mathcal{A}^0 = \left.\frac{\partial(g, N)}{\partial(v, s)}\right|_{(0, s_0; 0)} = \begin{pmatrix} G_u^0 & B^0 \\ \psi_1^* & 0 \end{pmatrix}$$

where B^0 is given in (5.12a). If (5.6c) holds then by our Lemma 2.8 it follows that $\mathcal{A}^0$ is nonsingular. Now the usual implicit function theorem shows that

$$(5.15)\ \ c)\qquad \begin{cases} g(v, s; \epsilon) = 0\ , \\ N(v, s; \epsilon) = 0\ , \end{cases}$$

† In [19] the parameter $s \equiv \lambda$ is employed but the above indicated generalization can be shown to work with no additional difficulties.

has a smooth solution $(v(\epsilon), \sigma(\epsilon))$ for each $|\epsilon| \leq \epsilon_0$ and using this solution in (5.10a) yields the bifurcating branch of solutions.

In solving (5.15c) we never use $\epsilon = 0$ so that even when employing Newton's method second derivatives need not be computed. Further we can use the device of Rheinboldt and replace $[u_1(s), \lambda_1(s)]$ in (5.10a) and (5.14a) by $[\hat{u}_1(s), \hat{\lambda}_1(s)]$ of (5.13). The implicit function theorem still holds as above and now we get a solution $(\hat{v}(\epsilon), \hat{\sigma}(\epsilon))$ of the modified (5.15c) to use in the modified form of (5.10a). The rigorous justification of this procedure is straightforward and obviously the method can be made constructive, [10].

6. Numerical Methods

To apply the previously indicated procedures we must assume that stable, convergent numerical methods are known for approximating the solutions of the linearized problems which arise in Newton's method or in the chord method. Indeed even more is required to rigorously justify the numerical methods used in switching branches at a bifurcation point. We must be assured that bases for $\mathcal{N}(G_u^0)$ and $\mathcal{N}(G_u^{0*})$ can be accurately determined. For the case of simple bifurcation this is not very difficult since the theory of numerical methods for computing eigenfunctions belonging to simple eigenvalues for broad classes of linear operators is well developed. Indeed the only works thus far to justify numerical methods at bifurcation points consider simple bifurcation from a tirvial branch, [2, 22, 23]. We shall not present a general convergence theory here but rather indicate the practical aspects in actually carrying out our procedures.

Basically the problem (2.4) is discretized in some form which we indicate by:

(6.1) a) $$G_h(u_h, \lambda_h) = 0 , \quad N_h(u_h, \lambda_h, s) = 0 .$$

Here (u_h, λ_h) represents the approximation to (u, λ) on the net or family of nets which is parametrized by h. If (2.4) has a smooth isolated solution then under modest assumptions on the consistency of $[G_h, N_h]$ with $[G, N]$ and on the stability and Lipschitz continuity of the linearized difference operators, say $\mathcal{A}_h$, the general theory in [9] assures us that (6.1) has a unique solution which can be computed

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Of course we solve (6.4a, b)
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ALGORITHM

i) Using Euler-Newton

by Newton's method. The bulk of the computations occur in this case and we write the difference operators linearized about (u_h, λ_h) say in the form:

$$(6.2) \qquad \mathcal{A}_h \equiv \begin{pmatrix} A_h & b_h \\ c_h^* & d_h \end{pmatrix} .$$

From the comparison with $\mathcal{A}$ in (2.6) we see that A_h is a form of difference approximation to G_u. Indeed A_h is in general a matrix of (large) order h^{-m} when the basic problem is formulated in E^m. Similarly b_h and c_h^* are column and row vectors approximating G_λ and N_u, respectively, while d_h is a scalar approximating N_λ.

The basic computational problem is to solve linear systems in the form

$$(6.3) \qquad \mathcal{A}_h \begin{pmatrix} \delta u_h \\ \delta\lambda_h \end{pmatrix} = \begin{pmatrix} r_h \\ \rho_h \end{pmatrix} .$$

To do this we need only determine y_h and z_h satisfying

$$(6.4) \text{ a)} \qquad A_h y_h = b_h , \quad \text{b)} \; A_h z_h = r_h ;$$

and then:

$$(6.4) \text{ c)} \quad \delta\lambda_h = (c_h^* z_h + \rho_h)/(d_h - c_h^* y_h) , \quad \text{d)} \; \delta u_h = z_h - \delta\lambda_h y_h .$$

Of course we solve (6.4a, b) by some Gaussian elimination procedure with some form of pivoting to get (neglecting the permutations for clarity of presentation)

$$(6.4) \text{ e)} \qquad A_h = L_h U_h .$$

Since the bulk of the computations occur in determining the factorization (6.4e) it follows that our normalization procedure costs very little extra effort. Now we sketch an algorithm for generating "all" the solution branches of (1.1) using the indicated techniques and assuming that only simple bifurcation occurs.

ALGORITHM

i) Using Euler-Newton continuation generate an approximate solution arc: $x_{1h}(s) \equiv (u_{1h}(s), \lambda_{1h}(s))$, skipping over any "singular" points that are encountered.

ii) Return to the neighborhood of each singular point and locate it accurately (i.e. use false position or bisection to determine the zero, s_{0h}, of $\det A_h(s)$). In particular simple roots or more generally odd order roots are easily sensed by the sign change in $\det A_h$. However one must remember to account for the row or column interchanges in the LU-decomposition.

iii) Test for limit point or bifurcation at each singularity. To do this we need an approximation ψ_{1h}^* to ψ_1^*, the null vector of $G_h^{0\,*}$. We do this as we also compute ϕ_{1h} an approximation to ϕ_1 the null vector of G_h^0. This is easily and efficiently done by inverse iteration. In fact if we are really close to a singular point then it suffices to use, say:

$$(6.5)\qquad \text{a)}\quad A_h\phi_h = \phi_h^0\,,\qquad \text{b)}\quad A_h^*\psi_h = U_h^*L_h^*\psi_h = \phi_h\;;$$

where $\phi_h^0 = \delta u_h$ is the last correction in the Newton scheme used to compute $u_h(s_{0h})$. As we already have the LU-decomposition in (6.4) these calculations are not costly.

The test is in the form:

$$(6.6)\qquad \psi_{1h}^*\, b_h \begin{cases} \doteq 0\,, & \text{seek bifurcation}\;;\\[4pt] \neq 0\,, & \text{a limit point.}\end{cases}$$

iv) To switch over to a bifurcating branch we must also compute $[\dot u_{1h}(s), \dot\lambda_{1h}(s)]$, an approximation to the tangent to the solution branch, $x_{1h}(s)$, at the point s_{0h} best approximating the bifurcation point. However, this will have been computed in step i) or ii) if, as we assume, the normalization N_3 of (2.10c) has been employed. Then we can easily determine ϕ_{0h} an approximation to ϕ_0 of (5.7) as follows. We set

$$(6.7)\qquad \text{a)}\quad \alpha_{0h} = \dot\lambda_{1h}(s_{0h})\,,\qquad \text{b)}\quad \alpha_{1h} = \psi_{1h}^*\,\dot u_{1h}(s_{0h})/\psi_{1h}^*\,\phi_{1h}\,,$$

and then compute

$$(6.7)\qquad \text{c)}\qquad \phi_{0h} = \frac{1}{\alpha_{0h}}\,[\dot u_{1h}(s_{0h}) - \alpha_{1h}\,\phi_{1h}]\,.$$

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Now to use Method I w
by:

(6.8) a) $a_{11h}(\delta) = \psi_{1h}^*$

b) $b_{11h}(\delta) = \psi_{1h}^*\,\frac{1}{\delta}$

- A

Then we approximate the o

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where $N_h(\cdot)$ is taken as:

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$_h^* \dot u_{1h}(s_{0h})/\psi_{1h}^* \phi_{1h}$.

$\alpha_{1h}\phi_{1h}]$.

[Of course if $\alpha_{0h} = 0$ we cannot use this simple procedure. In its place we must use a solution of (6.4a) at $s = s_{0h}$ and subtract a multiple of ϕ_{0h} so that $\psi_{1h}^* \phi_{0h} = 0$.]

Now to use Method I we approximate $a_{111}(\delta)$ and $b_{11}(\delta)$ of (5.7b) by:

(6.8)

a) $$a_{111h}(\delta) = \psi_{1h}^* \frac{1}{\delta}[A_h(u_{1h}(s_{0h}) + \delta\phi_{1h}, \lambda_{1h}(s_{0h})) - A_h(u_{1h}(s_{0h}), \lambda_{1h}(s_{0h}))]\phi_{1h}$$

b) $$b_{11h}(\delta) = \psi_{1h}^* \frac{1}{\delta}[A_h(u_{1h}(s_{0h}) + \delta\phi_{1h}, \lambda_{1h}(s_{0h})) - A_h(u_{1h}(s_{0h}), \lambda_{1h}(s_{0h}))]\phi_{0h} + \psi_{1h}^* \frac{1}{\delta}[b_h(u_{1h}(s_{0h})+\delta\phi_{1h}, \lambda_{1h}(s_{0h})) - b_h(u_{1h}(s_{0h}), \lambda(s_{0h}))] .$$

Then we approximate the other root of (5.6a) by $[\bar\alpha_{0h}, \bar\alpha_{1h}]$ where

(6.8) c) $$\bar\alpha_{1h}/\bar\alpha_{0h} = -\left(\frac{\alpha_{1h}}{\alpha_{0h}} + \frac{2b_{11h}(\delta)}{a_{111h}(\delta)}\right) .$$

The tangent to the bifurcated branch is approximated by:

(6.9) $$\dot u_{2h}(s_{0h}) = \bar\alpha_{1h}\,\phi_{1h} + \bar\alpha_{0h}\,\phi_{0h} , \quad \dot\lambda_{2h}(s_{0h}) = \bar\alpha_{0h} .$$

Using (6.9) in the normalization N_3 we simply return to step i) and generate the bifurcating branch.

To use Method II we proceed as before but do not bother with (6.8). Rather we approximate $[\hat\alpha_0, \hat\alpha_1]$ of (5.8a) by:

(6.10) a) $$\hat\alpha_{0h} = -\alpha_{1h}\|\phi_{1h}\|^2 , \quad \hat\alpha_{1h} = \alpha_{0h}(1 + \|\phi_{0h}\|^2) .$$

Then we seek a solution of (6.1) in the form:

(6.10) b) $$u_h = u_{1h}(s_{0h}) + \epsilon[\hat\alpha_{0h}\phi_{0h} + \hat\alpha_{1h}\phi_{1h}] + v_h , \quad \lambda_h = \lambda_{1h}(s_{0h}) + \epsilon\,\hat\alpha_{0h} + \eta_h ;$$

where $N_h(\cdot)$ is taken as:

(6.10) c) $$N_h(u_h, \lambda_h) \equiv (\hat\alpha_{0h}\phi_{0h}^* + \hat\alpha_{1h}\phi_{1h}^*)v_h + \hat\alpha_{0h}\,\eta_h .$$

Once a solution $[v_h, \eta_h]$ is obtained a new tangent vector is computed and we return to step i) using the normalization N_3. Indeed the above indicated computations involve but minor modifications from the procedure of step i).

Method III has been discussed in more detail by Rheinboldt [19]. It has also been used for bifurcation from the trivial solution in [15, 22, 23].

To our knowledge the new Method IV has not yet been used in actual calculations. However it is in the process of being tested at the present time.

7. A Simple Example.

We have used the procedures of §6 on several examples of the form:

(7.1) a) $$u_{xx} + f(x, u;\lambda) = 0 \quad , \quad u(0) = u(1) = 0 \quad ,$$

where

(7.1) b) $$f(x, u;\lambda) \equiv 2\, q(\lambda) + \pi^2 \lambda\, p(u - q(\lambda)x(1-x)) \quad .$$

If $p(0) = 0$ then a solution of (7.1) is given by

(7.2) $$u_1(x, \lambda) = q(\lambda)\, x(1-x) \quad , \quad \lambda = \text{arb} \quad .$$

The linearized problem about $u_1(x)$ is

(7.3) $$\phi_{xx} + \pi^2 \lambda p_u(0)\phi = 0 \quad , \quad \phi(0) = \phi(1) = 0 \quad .$$

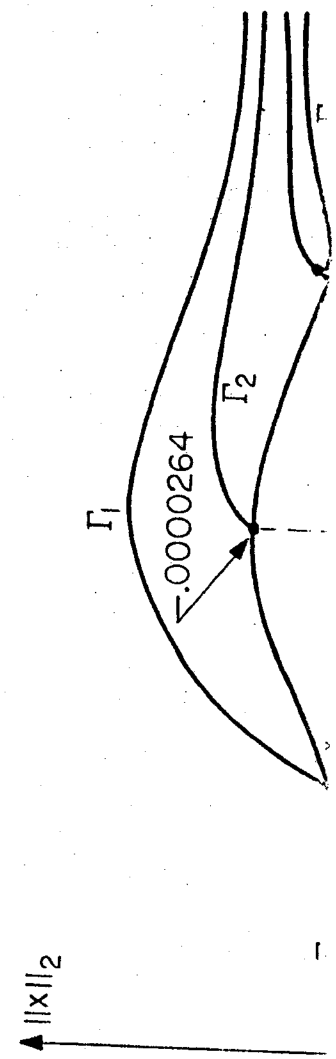
Thus if $p_u(0) = 1$ the eigenvalues of (7.3) are

(7.4) $$\lambda_k = k^2 \quad , \quad k = 1, 2, \ldots \quad .$$

We have used several choices for $q(\lambda)$ and $p(z)$ but we show here, in figure 3, the computed results for the choice

(7.5) $$q(\lambda) \equiv \lambda^2 e^{-\lambda/2} \quad , \quad p(z) \equiv z + z^2 \quad .$$

The difference scheme used was the Collatz Mehrstellenverfahren which is $\mathcal{O}(h^4)$ accurate and the net spacing was $h = 1/20$. The procedures of §6 were applied using the pseudoarclength normalizations N_2 and N_3 and Method II was used to switch branches at bifurcation points. (The norm, $\|x\|_2$ was allowed to go "negative" if it went to zero along a branch; thus Γ_1 is a smooth curve on figure 3.) The only initial guess required was $[u, \lambda] \equiv [0, 0]$ employed near $\lambda = 0$ to start the Euler-Newton continuation on the branch Γ_0 which is the



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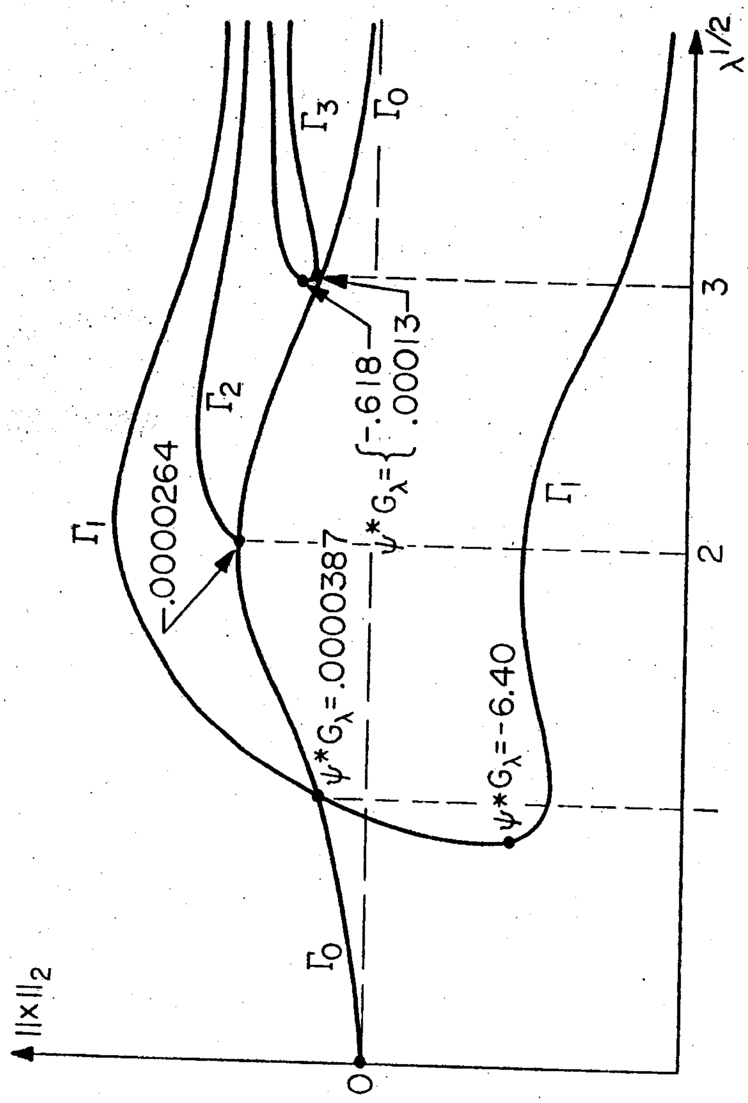

Figure 3

basic solution $u_1(x, \lambda)$. This branch was easily computed and sign changes in $\det A_h(\lambda)$ were noted near the first three eigenvalues. Upon refining the location of these potential bifurcation or limit points, using false position, the eigenvector approximations ϕ_h and ψ_h were computed. The test $\psi^* G_\lambda = 0$ indicated bifurcation at each point and Method II easily switched to the branch Γ_1, Γ_2 or Γ_3 bifurcating from the corresponding point $\lambda^{\frac{1}{2}} = 1, 2$ or 3. These branches were extended in either direction by simply changing the sign of the ϵ used in defining the normal vector in Method II. Then continuation generated the branch. On two of these extensions, Γ_1 and Γ_3, new zeros of $\det A_h(\lambda)$ were found but they failed the necessary test for bifurcation. Indeed we see in figure 3 that they are simple limit points. The branch Γ_2 as shown in figure 3 is actually covered twice but it does not show on the figure due to symmetry. There is a fundamental difference between bifurcations from odd and even "eigenvalues" but our scheme for computing has no difficulties with either case.

We also started our procedure at a remote point on the bifurcated branch Γ_1. It of course located the basic solution Γ_0 as a bifurcation from this branch and then proceeded to find the remainder of the branches in figure 3. To completely automate our procedure we would have to devise step control techniques to allow optimum steps in the arclength parameter, s. Also net selection, variable order (via deferred corrections or Richardson extrapolation) and accuracy tests should be included. However further testing with Methods I-IV should be carried out before general purpose codes are seriously contemplated. Furthermore since the bulk of the computations occur in the continuation process the choice of Euler-Newton must be reconsidered. Rheinboldt [18] has initiated serious studies of this question.

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