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## *Report on Waves.* By J. Scott Russell, *Esq., M.A., F.R.S. Edin., made to the Meetings in* 1842 *and* 1843.

*Members of Committee* { Sir John Robison*, *Sec. R.S. Edin.*
J. Scott Russell, *F.R.S. Edin.*

A Provisional Report on this subject was presented to the Meeting held at Liverpool in 1838, and is printed in the Sixth Volume of the Transactions. That report was a partial one. It states that "the extent and multifarious nature of the subjects of inquiry have rendered it impossible to terminate the examination of all of them in so short a time; but it is their duty to report the progress which they have made, and the partial results they have already obtained, leaving to the reports of future years such portions of the inquiries as they have not yet undertaken."

The first of these subjects of inquiry is stated to have been "to determine the varieties, phænomena and laws of waves, and the conditions which affect their genesis and propagation."

It is this branch of the duty of the Committee which forms the subject of the present report. Ever since the date of that report, it has happened that the author of this has been so fully pre-occupied by inevitable duty, that it was not in his power to indulge much in the pleasures of scientific inquiry; and as the active part of the investigation necessarily devolved upon him, it was not practicable to continue the series of researches on the ample and systematic scale originally designed, so soon as he had anticipated, so that the former report has necessarily been left in a fragmentary state till now.

But I have never ceased to avail myself of such opportunities as I could contrive to apply to the furtherance of this interesting investigation. I have now fully discussed the experiments which the former report only registered. I have repeated the former experiments where their value seemed doubtful, I have supplemented them in those places where examples were wanting. I have extended them to higher ranges, and where necessary to a much larger scale. In so far as the experiments have been repeated and more fully discussed, they have tended to confirm the conclusions given in the former report, as well as to extend their application.

The results here alluded to are those which concern especially the velocity and characteristic properties of the solitary wave, that class of wave which the writer has called the great wave of translation, and which he regards as the primary wave of the first order. The former experiments related chiefly to the mode of genesis, and velocity of propagation of this wave. They led to this expression for the velocity in all circumstances,

$$v = \sqrt{g(h + k)},$$

$k$ being the height of the crest of the wave above the plane of repose of the fluid, $h$ the depth throughout the fluid in repose, and $g$ the measure of gravity. Later discussions of the experiments not only confirm this result, but are themselves established by such further experiments as have been recently instituted, so that this formerly obtained velocity may now be regarded as the phænomenon characteristic of the wave of the first order.

The former series of experiments also contained several points of research not published in the former report, because not sufficiently extended to be of

* I cannot allow these pages to leave my hands without expressing my deep regret that the death of Sir John Robison has suddenly deprived the Association of a zealous and distinguished office-bearer, and myself of a kind friend. In all these researches the responsible duties were mine, and I alone am accountable for them; but in forwarding the objects of the investigation I always found him a valuable counsellor and a respected and cordial cooperator.

the desired value. Among these were a series of observations on the actual motion of translation of particles of the fluid during wave transmission; these have since been completed and extended, and the results of the whole are now given.

The former report was inevitably a fragment. I have endeavoured to give to the present report a somewhat greater degree of completeness. For this purpose I have now incorporated under one general form all those results of the present as well as of all my former researches, which could contribute to the unity and completeness of the view of a subject so interesting and important. I have re-discussed my former experiments, combined them with the more recent observations, and thus, from a wider basis of induction, obtained results of greater generality. Until the date of these observations, there had been confounded together in an indefinite notion of waves and wave motion, phænomena essentially different,—different in their genesis, laws of propagation, and other characteristics. I have endeavoured, by a rigid course of examination, to distinguish these different classes of phænomena from each other. I have determined certain tests, by which these confused phænomena have been made to divide themselves into certain classes, distinguished by certain great characteristics. Contradictions and anomalies have in this process gradually disappeared; and I now find that all the waves which I have observed may be distinguished into four great orders, and that the waves of each order differ essentially from each other in the circumstances of their origin, are transmitted by different forces, exist in different conditions, and are governed by different laws. It is now therefore easy to understand how much has been hitherto added to the difficulty of this difficult subject, by confounding together phænomena so different. The characteristics, phænomena, and laws of these great orders I have attempted in the present report to determine and define.

The knowledge I have thus endeavoured to obtain and herein to set forth concerning these beautiful and interesting wave phænomena, is designed to form a contribution to the advancement of hydrodynamics, a branch of physical science hitherto much in arrear. But besides this their immediate design, these investigations of wave motion are fertile in important applications, not only to illustrate and extend other departments of science, but to subserve the purposes and uses of the practical arts. I have ascertained that what I have called the great wave of translation, my wave of the first order, furnishes a type of that great oceanic wave which twice a day brings to our shores the waters of the tide. This type enabled us to understand and explain by analogy many of the phænomena of fluvial and littoral *tides*, formerly anomalous (see Proceedings R.S. Ed., 1838); and thus do these wave researches contribute to the advancement of the theory of the tides, a branch of physical astronomy long stationary, but which has recently made rapid strides towards the same high perfection which other branches of predictive astronomy have long enjoyed, a perfection which we owe chiefly to Sir John W. Lubbock, to Mr. Whewell, and the co-operation of the British Association. It is the wave of the first order enumerated in this report which furnishes to us the model of a terrestrial mechanism, by means of which the forces primarily imparted by the sun and moon are taken up and employed in the transport of tidal waters to distant shores (see previous Reports of Brit. Ass.), and their distribution in remote seas and rivers, which they continue in succession to agitate long after the forces employed in the genesis of the wave have ceased to exist (see Report on Tides). This application of the phænomena of waves to explain the tides is not their only application to the advancement of other branches of science. The phænomena of *resistance of fluids* I have found to be intimately connected with those waves (see Phil. Trans. Edin.



1837). The resistance which the water in a channel opposes to the passage of a floating body along that channel depends materially on the nature of the great wave of the first order, which the floating body generates by the force which propels it, and its motion is materially affected by the genesis of waves also, of the second order, arising from the same cause. These waves are therefore important elements in the resistance of fluids, and acquaintance with their phænomena is essential to the sound determination and explanation of the motion of floating bodies. If to these two branches of science we add the useful arts, in which an accurate acquaintance with wave phænomena may be of practical value to the purposes of human life, we shall find that the improvement of *tidal rivers*, the construction of *public works* exposed to the action of waves and of tides, and the *formation of ships* (see Reports of Brit. Ass. *passim*), are among the most direct and necessary applications of this knowledge, which is indeed essential to the just understanding of the best methods of opposing the violence of waves, and converting their motion to our own uses. By a careful study of the laws and phænomena of waves, we are enabled to convert these dangerous enemies into powerful slaves. By such applications of our wave researches, we therefore extend our knowledge in conformity with the maxims of the illustrious founder of our inductive philosophy, who enjoins that we always study to combine with our *experimenta lucifera* such *experimenta fructifera*, that while science is advanced society may be advantaged.

### *The Nature of Waves and their Variety.*

When the surface of water is agitated by a storm, it is difficult to recognize in its tumultuous tossings, any semblance of order, law, or definite form, which the mind can embrace so as adequately to conceive and understand. Yet in all the madness of the wildest sea the careful observer may find some traces of method; amid the chaos of water he will observe some moving forms which he can group or individualize; he may distinguish some which are round and long, others that are high and sharp; he may observe those that are high gradually becoming acuminate and breaking with a foaming crest, and may notice that the motion of those which are small is short and quick, while the rising and falling of large elevations is long and slow. Some of the crests will advance with a great, others with a less velocity; and in all he will recognize a general form familiar to his mind as the form of the sea in agitation, and which at once distinguishes it from all other phænomena.

Just as the waters of a reservoir or lake when in perfect repose are characterized by a smooth and horizontal surface, so also does a condition of disturbance and agitation give to the surface of the fluid this form characteristic of that condition and which we may term the wave form. When any limited portion of the wave surface presents a defined figure or boundary, which appears to distinguish that portion of fluid visibly from the surrounding mass, our mind gives it individuality,—we call it *a wave*.

It is not easy to give a perfect definition of a wave, nor clearly to explain its nature so as to convey an accurate or sufficiently general conception of it. Persons who are placed for the first time on a stormy sea, have expressed to me their surprise to find that their ship, at one moment in the trough between two waves, with every appearance of instant destruction from the huge heap of waters rolling over it, was in the next moment riding in safety on the top of the billow. They discover with wonder that the large waves which they see rushing along with a velocity of many miles an hour, do not carry the floating body along with them, but seem to pass under the bottom of the ship without injuring it, and indeed with scarcely a perceptible effect in carrying the vessel out of its course. In like manner the observer near the shore perceives that pieces of wood, or any floating bodies immersed in the water near its surface, and the water in their vicinity, are not carried towards the shore with the rapidity of the wave, but are left nearly in the same place after the wave has passed them, as before. Nay, if the tide be ebbing, the waves may even be observed coming with considerable velocity towards the shore, while the body of water is actually receding, and any object floating in it is carried in the opposite direction to the waves, out to sea. Thus it is that we are impressed with the idea, that *the motion of a wave may be different from the motion of the water* in which it moves; that the water may move in one direction and the wave in another; that water may transmit a wave while itself may remain in the same place.

If then we have learned that a water wave *is not* what it seems, a heap of water moving along the surface of the sea with a velocity visible to the eye, it is natural to inquire what a wave really is; *what is wave-motion as distinct from water-motion?*

For the purpose of this inquiry let us take a simple example. I have a long narrow trough or channel of water, filled to the depth of my finger length. I place my hand in the water, and for a second of time push forward along the channel the water which my hand touches, and instantly cease from further motion. The immediate result is easily conceived; I have simply pushed forward the particles of water which I touched, out of their former place to another place further on in the channel, and they repose in their new place at rest as at first. Here is a final effect, and here my agency has ceased—not so the motion of the water; I pushed forward a given mass out of its place into another, but that other place was formerly occupied by a mass of water equal to that which I have forcibly intruded into its place; what then has become of the displaced occupant? it has been forced into the place of that immediately before it, and the occupant which it has dislodged is again pushed forward on the occupant of the next place, and thus in succession volume after volume continues to carry on a process of displacement which only ends with the termination of the channel, or with the exhaustion of the displacing force originally impressed by my hand, and communicated from one to another successive mass of the water. This process continues without the continuance of the original disturbing agency, and is prolonged often to great distances and through long periods of time. The continuation of this motion is therefore independent of the volition which caused it. It is a process carried on by the particles of water themselves obeying two forces, the original force of disturbance and the force of gravity. It is therefore a hydrodynamical phænomenon conformable to fixed law. I have now ceased to exercise any control over the phænomenon, but as I attentively watch the processes I have set a-going, I observe each successive portion of water in the act of being displaced by one moving mass of water, and in the act of displacing its successor. As the water particles crowd upon one another in the act of going out of their old places into the new, the crowd forms a temporary heap visible on the surface of the fluid, and as each successive mass is displacing its successor, there is always one such heap, and this heap travels apparently along the channel at that point where the process of displacement is going on, and although there may be only one crowd, yet it consists successively of always another and another set of migrating particles.

This *visible moving heap of crowding particles* is a true *wave*, the rapidity with which the displacement of one outgoing mass by that which takes its place, goes forward, determines the velocity with which the heap appears to move, and is called the *velocity of transmission* of the wave. The shape which the crowding of the particles gives to the surface of the water constitutes the *form of the wave*. The distance (in the direction of the transmission) along which the crowd extends, is called the *length or amplitude* of the wave. The number of particles which at any one time are out of their place, constitute the *volume of the wave*; the time which must elapse before particles can effect their translation from their old places to the new, may be termed the *period of the wave*. The *height of the wave* is to be reckoned from the highest point or crest to the surface of the fluid when in repose.

Such is the wave motion—very different is the *water motion*. Let us select from the crowd of water particles an individual and watch its behaviour during the migration. The progressive agitation first reaches it while still in perfect repose; the crowd behind it push it forward and new particles take its place. One particle is urged forward on that before it, and being still urged on from behind by the crowd still swelling and increasing, it is *raised* out of its place and *carried forward* with the velocity of the surrounding particles; it is urged still on until the particles which displaced it have made room for themselves behind it, and then the power diminishes. Having now in its turn pushed the particles before it along out of their place, and crowded them together on their antecedents, it is gradually left behind and finally *settles quietly down in its new place*. Thus then the *motion of migration of an individual particle* of water is very different from the *motion of transmission of the wave*.

The wave goes still forward along through the channel, but each individual water particle remains behind. The wave passes on with a continuous uninterrupted motion. The water particle is at rest, starts, rises, is accelerated, is slowly retarded, and finally stops still. The *range of the particle's motion* is short; its *translation* is interrupted and final. Its *vertical range* and *horizontal range are finite*. It describes an *orbit* or path during the *transit of the wave* over it, and remains for ever after at rest, unless when a second wave happens immediately to follow the first, when it will describe a second time *its path of translation*, passing through a series of new positions or *phases* during the *period of the wave*. The motion of the particle is not therefore like the apparent motion of the wave, either uniform or continuous. The motion of the water particles is a true motion of translation of matter from one place to another, with the velocity and range which the senses observe. But the wave motion is an ideal individuality attributed by the mind of the observer to a process of changes of relative position or of absolute place, which at no two instants belongs to the same particles in the same place. The water does not travel; the visible heap at no two successive instants is the same. It is the motion of particles which goes on, now at this place, now at that, having passed all the intermediate points. *It is the crowding motion alone which is transmitted.* This crowding motion transmitted along the water idealised and individualised is a true wave.

Wave propagation therefore consists in the transmission from one class of particles to another, of a motion differing in kind from the motion of transmission. *Wave motion* is therefore transcendental motion; motion in the second degree; the motion of motion—the transference of motion without the transference of the matter, of form without the substance, of force without the agent.

It is essential to the accurate conception and examination of waves, that this *distinction between the wave motion and the water motion* be clearly conceived. It has been well illustrated by the agitations of a crowd of people, and of a field of standing corn waving with the wind. If we stand on an eminence, we notice that each gust as it passes along the field bending and crowding the stalks, marks its course by the motion it gives to the grain, and the visible effect is like that of an agitated sea. The waving motion visibly travels across the whole length of the field, but the corn remains rooted to the ground; this illustration is as apt as old, being given to us in the Iliad, at the conclusion of the speech of Agamemnon, beginning Ὦ φίλοι, ἥρωες Δαναοί.

> Ὣς φάτο· . . . .
> Κινήθη δ' ἀγορή, ὡς κύματα μακρὰ θαλάσσης
> Πόντου Ἰκαρίοιο, τὰ μέν τ' Εὖρός τε Νότος τε
> Ὤρορ', ἐπαΐξας πατρὸς Διὸς ἐκ νεφελάων.
> Ὡς δ' ὅτε κινήσει Ζέφυρος βαθὺ λήϊον, ἐλθὼν
> Λάβρος, ἐπαιγίζων, ἐπί τ' ἠμύει ἀσταχύεσσιν·
> Ὣς τῶν πᾶσ' ἀγορὴ κινήθη.—Il. II. 144–149.

In the examination of the phænomena of waves, we have therefore two classes of elements for consideration, the elements of the wave motion and the elements of the water particle motion. We may first examine the *phænomena of a given wave-motion*, its range of transmission over the surface of the fluid, the velocity of that transmission, the form of the elevation, its amplitude, breadth, height, volume, period. We may next consider the *path which each water particle describes* during the wave transit; the *form* of that path, the *horizontal or vertical range* of the motion, the variation of the path with the depth, the relation of each *phase of the particle's orbit* to each portion of the corresponding wave length. By this examination I have found that there exist among waves groups of phænomena so different as to suggest their division into *distinct classes*. I find that the general form of waves is manifestly different, one kind of wave making its appearance in a form always wholly raised above the general surface of the fluid, and which we may call a *positive wave*, and so distinguish it from another form of wave which is wholly *negative*, or depressed below the plane of repose, while a third class are found to consist of both a negative and a positive portion. I find them propagated with extremely different velocities, and obeying different laws according as they belong to one or the other of these classes, the positive wave having *in a given depth of water a constant and invariable velocity*, while another class has a *velocity varying* according to other peculiarities, and *independent of the depth*. Some of them again are distinguished by always appearing alone as individual waves, and others as *companion phænomena* or *gregarious*, never appearing except in groups. In examining the paths of the water particles corresponding differences are observed. In some the water particles perform a *motion of translation* from one place to another, and effect a permanent and final change of place, while others merely change their place for an instant to resume it again; thus performing *oscillations* round their place of final repose. These waves may also be distinguished by the sources from which they arise, and the forces by which they are transmitted. One class of wave is a *motion of successive transference* of the whole fluid mass; a second, the *partial oscillation* of one part of it without affecting the remainder; a third, the propagation of an impulse by the *corpuscular forces* which determine the elasticity of the fluid mass; and a fourth, by the *capillary forces* uniting its molecules at the surface.

These classes, so various both in their origin, cause and phænomena, have not hitherto been sufficiently distinguished, but have either been unknown, or have been confounded with each other under the vague conception and general designation of wave motions. The following table is given as a *first approximation towards a classification of the phænomena of wave motion*. It comprehends all the waves which I have investigated, and sufficiently distinguishes them from each other. I find that water waves may be distributed into *four orders*. The *wave of translation* is the wave of the FIRST ORDER, and consists in a motion of translation of the whole mass of the fluid from one place to another, to another in which it finally reposes; its aspect is, a *solitary* elevation or a solitary hollow or cavity, moving along the surface with a *uniform* velocity; and hence it presents two species, *positive and negative*, and each of these may be found in a condition of *free* motion, or affected in form and velocity by the continual interference of a *force* of the same nature with that from which its genesis was derived. The wave of the SECOND ORDER is partly positive and partly negative, *each height having a companion hollow*, and this is the commonest order of visible water wave, being similar to the usual *wind waves*, in which the surface of the water visibly *oscillates* above and below the level of repose; these waves appear in *groups*; in some cases, as in running water, they may be *standing* elevations and depressions, and in others *progressive* along the surface, and like the waves of the first order, may be altered in form and velocity by the presence of a disturbing force, so as to differ from their phænomena when in a state of perfect freedom. The THIRD CLASS are met with under such conditions as agitate the fluid only to a very minute depth, and are determined by the same forces which in hydrostatics produce the phænomena of *capillary attraction*; and the FOURTH ORDER is that wave insensible to sight, which conveys the disturbance produced by a sonorous body through a mass of the fluid, and which is at once an index and a result of the molecular forces which determine the elasticity of the fluid. This classification has been adopted throughout the following paper.



TABLE I.

*System of Water Waves.*

| ORDERS. | FIRST. | SECOND. | THIRD. | FOURTH. |
|---|---|---|---|---|
| Designation. | Wave of translation. | Oscillating waves. | Capillary waves. | Corpuscular wave. |
| Characters... | Solitary. | Gregarious. | Gregarious. | Solitary. |
| Species ... | Positive.<br>Negative. | Stationary.<br>Progressive. | Free.<br>Forced. | |
| Varieties | Free.<br>Forced. | Free.<br>Forced. | | |
| Instances | The wave of resistance.<br>The tide wave.<br>The aërial sound wave. | Stream ripple.<br>Wind waves.<br>Ocean swell. | Dentate waves.<br>Zephyral waves. | Water-sound wave. |

An observer of natural phænomena who will study the surface of a sea or large lake during the successive stages of an increasing wind, from a calm to a storm, will find in the whole motions of the surface of the fluid, appearances which illustrate the nature of the various classes of waves contained in Table I., and which exhibit the laws to which these waves are subject. Let him begin his observations in a perfect calm, when the surface of the water is smooth and reflects like a mirror the images of surrounding objects. This appearance will not be affected by even a slight motion of the air, and a velocity of less than half a mile an hour ($8\frac{1}{2}$ in. per sec.) does not sensibly disturb the smoothness of the reflecting surface. A gentle zephyr flitting along the surface from point to point, may be observed to destroy the perfection of the mirror for a moment, and on departing, the surface remains polished as before; if the air have a velocity of about a mile an hour, the surface of the water becomes less capable of distinct reflexion, and on observing it in such a condition, it is to be noticed that the diminution of this reflecting power is owing to the presence of those minute corrugations of the superficial film which form waves of the *third order*. These corrugations produce on the surface of the water an effect very similar to the effect of those panes of glass which we see corrugated for the purpose of destroying their transparency, and these corrugations at once prevent the eye from distinguishing forms at a considerable depth, and diminish the perfection of forms reflected in the water. To fly-fishers this appearance is well known as diminishing the facility with which the fish see their captors. This first stage of disturbance has this distinguishing circumstance, that the phænomena on the surface cease almost simultaneously with the intermission of the disturbing cause, so that a spot which is sheltered from the direct action of the wind remains smooth, the waves of the third order being incapable of travelling spontaneously to any considerable distance, except when under the continued action of the original disturbing force. This condition is the indication of present force, not of that which is past. While it remains it gives that deep blackness to the water which the sailor is accustomed to regard as an index of the presence of wind, and often as the forerunner of more.

The second condition of wave motion is to be observed when the velocity of the wind acting on the smooth water has increased to two miles an hour. Small waves then begin to rise uniformly over the whole surface of the water; these are waves of the second order, and cover the water with considerable regularity. Capillary waves disappear from the ridges of these waves, but are to be found sheltered in the hollows between them, and on the anterior slopes of these waves. The regularity of the distribution of these secondary waves over the surface is remarkable; theybegin with about an inch of amplitude, and a couple of inches long; they enlarge as the velocity or duration of the wave increases; by and by conterminal waves unite; the ridges increase, and if the wind increase the waves become cusped, and are regular waves of the *second order*. They continue enlarging their dimensions, and the depth to which they produce the agitation increasing simultaneously with their magnitude, the surface becomes extensively covered with waves of nearly uniform magnitude.

How it is that waves of unequal magnitude should ever be produced may not seem at first sight very obvious, if all parts of the original surface continue equally exposed to an equal wind. But it is to be observed that it rarely occurs that the water is all equally exposed to equal winds. The configuration of the land is alone sufficient to cause local inequalities in the strength of the wind and partial variations of direction. By another cause are local inequalities rapidly produced and exaggerated. The configuration of the shores reflects the waves, some in one direction, some in another, and so deranges their uniformity. The transmission of reflected waves over such as are directly generated by the wind, produces new forms and inequalities, which, exposed to the wind, generate new modifications of its force, and of course, in their turn, give rise to further deviations from the primitive condition of the fluid. There are on the sea frequently three or four series of coexisting waves, each series having a different direction from the other, and the individual waves of each series remaining parallel to one another. Thus do the condition, origin, and relations of the waves which cover the surface of the sea after a considerable time, become more complex than at their first genesis.

It is not until the waves of the sea encounter a shallow shelving coast, that they present any of the phænomena of the wave of the first order (Report of 1838). After breaking on the margin of the shoal, they continue to roll along in the shallow water towards the beach, and becoming transformed into waves of the first order, finally break on the shore.



But the great example of a wave of the *first order*, is that enormous wave of water which rolls along our shores, bringing the elevation of high tide twice a day to our coasts, our harbours, and inland rivers. This great compound wave of the first order is not the less real that its length is so great, that while one end touches Aberdeen, the other reaches to the mouth of the Thames and the coast of Holland. Though the magnitude of this wave renders it impossible for the human eye to take in its form and dimensions at one view, we are able, by stationing numerous observers along different parts of the coasts, to compare its dimensions and to trace its progress at different points, and so to represent its phænomena to the eye and the mind on a small scale, as to comprehend its form and nature as clearly as we do those of a mountain range, or extensive country which has been mapped on a sheet of paper by the combination together of trigonometrical processes, performed at different places by various observers, and finally brought together and protracted on one sheet of paper.

As this great wave of the first order is not comprehended by the eye on account of its magnitude, so there is a wave of the *fourth order* which equally escapes detection from that organ, on account of its minuteness. By an undulation propagated among the particles of water, so minute as to be altogether insensible to the eye, and only recognised by an organ appropriate to that purpose, there is conveyed from one place to another the wave of sound. This wave, though invisible from its minuteness, is nevertheless of a nature almost identical with the wave of the first order. In air the sound wave is indeed the wave of the first order. It is only in liquids, when the measure of pressure of the fluid mass is different from the measure of the intercorpuscular force, that the phænomena of the wave of the first order is different from those of the fourth, and that we have one measure for the velocity of the water wave, and another for that of the sound wave. In a gaseous fluid, on the contrary, the measure of the pressure of the mass is also the measure of the intercorpuscular force, and the sound wave becomes identical with the air wave, the fourth order with the first.

## SECTION I.—WAVE OF THE FIRST ORDER.

### *The Wave of Translation.*

| | |
|---|---|
| Character ................ | Solitary. |
| Species ................ | Positive.<br>Negative. |
| Varieties ................ | Free.<br>Forced. |
| Instances ................ | Wave of Resistance.<br>Tidal Wave—Sound Wave. |

I believe I shall best introduce this phænomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phænomenon which I have called the Wave of Translation, a name which it now very generally bears; which I have since found to be an important element in almost every case of fluid resistance, and ascertained to be the type of that great moving elevation of the sea, which, with the regularity of a planet, ascends our rivers and rolls along our shores.

To study minutely this phænomenon with a view to determine accurately its nature and laws, I have adopted other more convenient modes of producing it than that which I have just described, and have employed various methods of observation. A description of these will probably assist me in conveying just conceptions of the nature of this wave.

*Genesis of the Wave of the First Order.*—For producing waves of the first order on a small scale, I have found the following method sufficiently convenient. A long narrow channel or box a foot wide, eight or nine inches deep, and twenty or thirty feet long (Plate I. fig. 1.), is filled with water to the height of say four inches. A flat board P (or plate of glass) is provided, which fits the inside of the channel so as to form a division across the channel where it is inserted.

*Genesis by Impulsion or Force horizontally applied.*—Let this plate be inserted vertically in the water close to the end A, and being held in the vertical position, be pressed forward slowly in the direction of X, care being taken that it is kept vertical and parallel to the end. The water now displaced by the plate P in its new position accumulates on the front of the plane forming a heap, which is kept there, being inclosed between the sides of the channel and the impelling plate. The amount thus heaped up is plainly the volume of water which has been removed by the advancing plane from the space left vacant behind it, and if the impulse increase, the elevation of displaced water will increase in the same quantity. When the water has reached the height $P_3$, let the velocity of impulsion be now gradually diminished as at $P_4$, until the plate is finally brought to rest as at $P_5$; the height of the water heaped on the front will diminish with the diminution of velocity as at $P_4$, and when brought to rest at $P_5$ it will be on the original level. The total height of the water does not however subside with the diminution of the impulsion, the crest $W_4$ retains the maximum height to which it had risen under the pressure of the plane at $P_3$, and moves horizontally forward; and the smaller elevation produced by the smaller pressure at $P_4$ down to $P_5$ moves forward after $W_5$. This elevation of the liquid, having a *crest*, or summit, or *ridge* in the centre of its length transverse to the side of the channel, continues to move along the channel in the direction of the original impulsion; from the crest there extends forward a curved surface, W*a*, forming the *face* of the *wave*, and a similar surface, W*w*, behind the crest is distinguished as its *back*. It is convenient to designate *a* as the *origin*, *w* as the *end* of the wave; and to designate the interval between *a* and *w*, the length of the wave in the direction of its transmission, its *amplitude*.

The kind of motion required for generating this wave in the most perfect way, that is, for producing a wave of given magnitude without at the same time creating any disturbance of a different kind in the water—this kind of motion may be given by various mechanical contrivances, but I have found that the dexterity of manipulation which experience bestows is perfectly sufficient for ordinary experimental purposes.

*Genesis by a Column of Fluid.*—This is a method of genesis, of considerable value for various experimental purposes, especially useful when waves of no great magnitude are required, and also when it is desirable to measure accurately volumes or forces employed in wave genesis. The same glass plate may be conveniently employed in this case as in the last, only it will now be used in the capacity merely of a sluice, and be supported by two small vertical slips fixed to the sides of the channel so as to keep it in the vertical position but to admit of its being raised vertically upwards as at G, Pl. XLVII. fig. 2. There is thus formed between the end of the channel G and the moveable plate $P_7$, a small generating reservoir $GP_7$. This is to be filled to any desired height with water, as from *w* to $P_7$, and the plate being drawn up, as at $P_9$, the water of the reservoir descends to *w*, the level of the water of the channel, and pushing forward and heaping up the adjacent fluid, raises a heap equal to the added volume on the surface of the water; and this elevation is in no respect sensibly different either in form or other phænomenon from that generated in the former method, provided the quantity of water added in the latter case be identical with the quantity of water displaced in the former case.

This method of genesis by fluid column affords a simple means of proving an elementary fact in this kind of wave motion. The fact is this, that while the volume of water in the wave is exactly equal to the volume of water added from the reservoir, it is by no means identical with it. I filled the reservoir with water tinged with a pink dye, which did not sensibly alter the specific gravity of the water. The column of water having descended as at K, and the wave having gone forward to $W_9$, the generating column remained stationary at K, thus indicating that the column of water had merely acted as a mechanical prime mover, to put in action the wave-propagating forces among the fluid, in the same way as had been formerly done by the power acting by the solid plate in the former case of genesis by impulsion. Thus is obtained a first indication that this wave exhibits *a transmission of force, not of fluid*, along the channel.

*Genesis by Protrusion of a Solid.*—The quantity of moving force required for the wave-genesis may be directly obtained by the descent of a solid weight. The solid at L (fig. 3.) may be a box of wood or iron, containing such weights as are desired, and suspended in such a manner as to be readily detached from its support. Its under surface should be somewhat immerged. On touching the detaching spring, the weight descends, and the water it displaces produces a wave of equal volume. If the weight and volume of solid thus immersed be equal to those of the water in the reservoir in the former case, it is found that the waves generated by the two methods are alike. It is expedient that the breadth and shape of this solid generator should be such as to fit the channel, as this removes some sources of disturbance. The results which are produced by this application of moving power are also convenient for giving measures of the mechanical forces employed in wave-genesis.

This method is especially convenient for the genesis of waves of considerable magnitude. With this view I erected a pyramidal structure of wood, capable of raising weights of several hundred pounds, over a pulley by means of a crane, and contrived to allow them to descend at will. This apparatus was adequate for the generation of waves in a channel three feet wide and three feet deep; and the same construction may be extended to greater dimensions.

*Transmission of Mechanical Power by the Wave.*—By the last two methods of genesis there is to be obtained a just notion of the nature of the wave of the first order as a vehicle for the transfer of mechanical power. By the agency of this wave the mechanical power which is employed in wave-genesis at one end of the channel, passes along the channel in the wave itself, and is given out at the other end with only such loss as results from the friction of the fluid. At one end, as of the channel G, fig. 4, there is placed the water, which, falling through a given height, is to generate the wave. At the other end, X, is a similar reservoir and sluice, open to the channel. When the wave has been generated as at K, and has traversed the length of the channel, it enters the receptacle X, and assuming the form marked at L, the sluice being suddenly permitted to descend, the column of water will be inclosed in the receptacle, and its whole volume raised above the level of repose nearly as at the first. The power expended in wave-genesis, having been transferred along the whole channel, is thus once more stored up in the reservoir at the other extremity. A part of this power is, however, expended *in transitu* by friction of the particles and imperfect fluidity, &c. When the channel is large, the sides and bottom smooth, the transmission of force may be accomplished with high velocity, at the rate of many miles an hour, to a distance of several miles.

*Re-genesis of Wave.*—In the channel AX, we have found the wave transmitted from A to X, and there the power of genesis transferred to the fluid column now stored up in the reservoir X. If we now repeat from the receptacle X the same process of genesis originally performed at G, elevating the sluice and allowing the fluid column to descend, it will again generate a wave similar to the first, only transmitted back in the opposite direction. This re-generated and re-transmitted wave may be again found in the primary reservoir of genesis as at G, and the same power, after having been transmitted twice through the length of the channel, be restored as at first in that channel, with only the small diminution of power lost *in transitu*. The process of re-genesis may now be repeated, as at first, and so on during any number of successive transmissions and re-transmissions.

*Reflexion of the Wave.*—This process of restoring the force employed in wave-genesis, and of re-genesis of the wave, may take place without the intervention of the sluices. The wave, on reaching the end of the channel G at $X_7$, becomes accumulated in the form of the curve $w\,x$. We have therefore the power of genesis now stored up in this water column, $w$ L $x$, above the level L, and in a state of rest. By means of a sluice we may detain it at that height for as long time as we please. But let us suppose we do not wish to detain it, but allow the water column to descend by gravity as at first, it generates the wave by again descending, and transmits it back towards G, as effectually as if the reservoir had been used, or as the genesis when first accomplished. By the same process of *laissez faire*, the power of genesis will be restored at G, a water column elevated, the fluid brought to rest and allowed again to descend, again to effect genesis of the wave, and again transmit the force along the channel through the particles of the wave. The wave is said to be reflected, and it is thus shown in reference to the wave of the first order, that the process called reflexion consists in a process of restoration of the power of genesis, and of re-genesis of the wave in an opposite direction. In this manner there is to be obtained an accurate view of the mechanical nature of the reflexion of the wave.

*Measure of the Power of Wave-Genesis.*—If we examine the process of wave-genesis as at K, fig. 2, we find that the change which has taken place after the wave-genesis and before, consists virtually in a different arrangement of the particles of a given volume of water. The given rectangular column of water A $P_{10}$ occupies after genesis the equal space A K. This, without regard to the paths in which the particles have proceeded to their new places, this descent is the final result and integral effect of the development of the power of the generating column. Take away from these two equal volumes of fluid the volume $g\,p$ common to both, and the remaining volumes $w$ P and $k$ are equal, and a given volume of water has effectively descended from $P\,G\,w$ into K $k\,p$, and $g_1$ and $g_2$ being the centres of gravity; the quantity of power developed is measured by the descent of the weight of water through a height $g_1\,g_2$, or through half the depth of the generating reservoir, and is of course capable of generating in any equal mass of fluid a velocity equal to that which is acquired by falling through a space equal to one-half the depth, reckoned from the top of the generating column to the bottom of the channel.



*Imperfect Genesis of the Wave.*—The wave may be said to have imperfect genesis, as far as the purposes of accurate experiment are concerned, when it is accompanied by other wave phænomena which interfere with it. The precautions necessary to perfect genesis appear to be these, that the volume of water should not widely differ from the volume of the wave it is proposed to generate, and that the height of the water should not greatly differ from that of the wave; and even these precautions are scarcely sufficient for the generation of a perfect solitary wave in a case where it is extremely high. The reason is obvious.

*Residuary Positive Wave.*—In a case of genesis where the precautions mentioned above are not observed, the following phænomenon is exhibited. If, as in the case fig. 6, the volume of the generating fluid considerably exceed (in consequence of the length of the generating reservoir) the length of the wave of a height equal to that of the fluid, the wave will assume its usual form W notwithstanding, and will pass forward with its usual volume and height; it will free itself from the redundant matter $w$ by which it is accompanied, leaving it behind, and this residuary wave, $w_2$, will follow after it, only with a less velocity, so that although the two waves were at first united in the compound wave, they afterwards separate, as at $W_2\,w$, and are more and more apart the further they travel.

*Disintegration of large Wave Masses.*—Thus also by increasing the length of the generating column, there may be generated any number of residuary waves, and it is a result of no little importance, to just conceptions of the nature of the wave of the first order, that it be not regarded as an arbitrary phænomenon deriving all its characters from the conditions in which it was at first generated, but that it is a phænomenon *sui generis*, assuming to itself that form and those dimensions under which alone it continues to exist as a wave. The existence of a moving heap of water of any arbitrary shape or magnitude is not sufficient to entitle it to the designation of a wave of the first order. If such a heap be by any means forced into existence, it will rapidly fall to pieces and become disintegrated and resolved into a series of different waves, which do not move forward in company with each other, but move on separately, each with a velocity of its own, and each of course continuing to depart from the other. Thus a large compound heap or wave becomes resolved into the principal and residuary waves by a species of spontaneous analysis.

*Residuary Negative Waves.*—There is a method of genesis the reverse of the last, which also produces residuary waves, but they are thus far the reverse of the last in form, as they have the appearance of *cavities* propagated along the surface of the still water in the channel, and they move *more slowly* than the positive wave: we may give them the appellation of residuary negative waves. When the elevation of the fluid in the reservoir is great in proportion to its breadth (reckoned as amplitude), the descending column of genesis communicates motion to a greater number of particles of water than its own, but with a less velocity; these go to form a wave which is larger in volume than the column of genesis, and therefore contribute to the volume of the wave some of the water which originally served to maintain the level of the fluid or surface of repose; this hollow is transferred like a hollow wave along the fluid, and there may exist several such waves, which I have called residuary negative waves. But these waves do not accompany the primary wave, nor have they the same velocity. See O, fig. 16.

It is of some importance to note, that these residuary phænomena of wave-genesis are *not companion phænomena* to the primary wave or positive wave of the first order. They will be separately considered at another time; meanwhile it is to be noted that these residuary phænomena accompany only the genesis of the wave, but do not attend the transmission, as they are rapidly left behind by the great primary solitary wave of the first order. Certain philosophers have fallen into error in their conceptions of these experiments by not sufficiently noting this distinction.

It is worth notice also, that besides these, many other modes of genesis have been employed; solids elevated from the bottom of the channel, vessels drawn along the channel, &c.; wherever a considerable addition is made to the height and volume of the liquid at any given point in the channel, a wave of the first order is generated, differing in no way from the former, except in such particulars as are hereinafter noticed.

*Motion of Transmission.*—The crest of the wave is observed to move along a channel which does not vary in dimension, with a *velocity sensibly uniform*, so that the velocity with which it is transmitted may be determined by simply measuring a given distance along the channel, and observing the number of seconds which may elapse during the transit from one end of the line to the other. This interval of time is sensibly equal for any equal space measured along the path, and hence we determine that the velocity of the wave transmission is sensibly uniform.

*Range of Wave Transmission.*—The distance through which a wave of the first order will continue to propagate itself, is so great as to afford considerable facility for accurate observation of its velocity. For accurate observations it is convenient to allow the early part of the range to escape without observation; for this purpose, that the primary wave, which is to be the subject of observation, may disembarrass itself of such secondary phænomena as frequently accompany its genesis, when that genesis cannot be accurately accomplished. A small part of the range is sufficient for this purpose, and the remainder is perfectly adapted for purposes of accurate observation, as it continues to travel along its path long after the secondary waves have ceased to exist. The *longevity of the wave of the first order*, and the facility of observing it, may be judged of from the following experiments, made in 1835–1837.

Ex. 1. A wave of the first order, only 6 inches high at the crest, had traversed a distance of 500 feet, when it was first made the subject of observation. After being transmitted along a further distance of 700 feet, another observation was noted, and it was observed still to have a height of 5 inches and to have travelled with a velocity of 7·55 miles an hour.

Ex. 2. A wave of the first order, originally 6 inches high, was transmitted through a distance of 3200 feet, with a mean velocity of 7·4 miles an hour, and at the end of this path still maintained a height of 2 inches.

Ex. 3. A wave 18 inches high, moving at the rate of 15 miles an hour, in a channel 15 feet deep, had still a height of 6 inches, having traversed the same space in 12 minutes.

Ex. 4. Among small experimental waves of the first order, in small channels, I have selected one, which, whose crest being 1·34 inch high, in a channel 5·10 inches deep, was transmitted through a range of 1360 feet, and still admitted of accurate observation.



These examples serve to convey an accurate idea of the longevity of a wave of the first order. And this longevity appears to increase with the depth and the breadth of the channel, and with the height of the wave crest.

*Degradation of the Wave of the First Order.*—In the progress of a wave of the first order, it is observed that its height diminishes with the length of its path; the velocity also diminishes with the diminution of height, though very slowly. This degradation of height is observed to go on more rapidly in proportion as the channel is narrow, shallow or irregular, and rough on the sides, and is diminished according as the channel is made smooth and regular in its form, or deep and wide. It is to be attributed to the imperfect fluidity of the water in some degree, but also to the adhesion of water to the sides. The particles of fluid near the sides and bottom are retarded in their motions, and the transmission takes place more slowly among them. The wave passes on, leaving in these particles a small quantity of the motion it had communicated, and of its force and volume, and in consequence of this there exists along the whole channel, over which the wave has passed, a residual motion or continuous residual wave, very small in amount, but still appreciable by accurate means of observation. The volume of the wave is thus diffused over a large extent along its path, where finally it has deposited the whole of its volume, and so disappears. This degradation is therefore the means by which the motion of a wave in an indefinite channel is gradually and slowly terminated. In the history of a solitary wave of the first order, the progress of this degradation is to be observed from the examination of Table II. column B, which gives the height of the wave as observed at every 40 feet along its path. In the first 200 feet this diminution amounts to about $\frac{1}{4}$ of the height at the commencement. At the end of the second 200 feet, the height is diminished by $\frac{4}{9}$ of the height at the commencement of that space. During the third space of 200 feet the degradation produced is nearly $\frac{1}{2}$ of the height of the wave; this appears to be the most rapid degradation, and in the next space of 200 feet it is little more than $\frac{1}{3}$; in the next, less than a third of the height at the beginning of that space. These successive heights are given graphically in Plate XLVIII. fig. 7.

*The Velocity of Transmission of the Wave of the First Order.*—The history of a single wave has sufficed to show us that the velocity with which its crest is transmitted along the channel is nearly that which a heavy body will acquire falling freely through a height equal to half the depth of the fluid. This is a very simple and important character in the phænomena of this wave, by which, when the depth of the channel is known, we may at once predict approximately the velocity of the wave of translation. The following are approximate numbers deduced from this conclusion, and which I find it convenient to recollect.

In a channel whose depth is $2\frac{1}{2}$ inches, the velocity of the wave is $2\frac{1}{2}$ feet per second.

In a channel whose depth is 15 feet, the velocity of the wave is 15 miles an hour.

In a channel whose depth is 90 fathoms, the velocity of the wave is 90 miles an hour.

These numbers are, however, only first approximations, for it is to be observed in reference to wave, Table II., that the wave, when its height is considerable, moves with greater velocity than when it is small. These numbers become accurate, if in the depth, the height of the wave be included.

*The Height of the Wave of the First Order, an element in its velocity.*—The height of the wave appears to enter as an element in its velocity, and to cause it to deviate from the simple formula A. Thus the velocity of the wave only coincides with the velocity assigned in Table II. when the height of the wave is inconsiderable.

I have found that this deviation is to be reconciled, without at all destroying the simplicity of the formula, by a very simple means. In order to obtain perfect accuracy, we have only to reckon the effective depth for calculation, from the ridge or crest of the wave instead of from the level of the water at rest; and having thus added to the depth of the water in repose, the height of the wave crest above the plane of repose, if we take the velocity which a heavy body would acquire in falling through a space equal to half the depth of the fluid (reckoning from the ridge of the wave to the bottom of the channel), that number accurately represents the velocity of transmission of the wave of the first order.

We have, therefore, for the velocity of the wave of the first order,

approximately $v=\sqrt{gh}$, . . . . . . . . . A.

accurately $v=\sqrt{g(h+k)}$, . . . . . . . B.

where $v$ is the velocity of transmission,

$g$ is the force of gravity as measured by the velocity which it will communicate in a second to a body falling freely $=32$,

$h$ is the depth of the fluid in repose,

$k$ is the height of the crest of the wave above the plane of repose.

The velocities of waves of the first order in channels of different depths are, therefore, as the square roots of the depth of these channels.

Nevertheless, when the height of one of the waves is considerable compared with the depth of the channel, a high wave in the shallower channel may move faster than a lower wave in a deeper channel; provided only the excess in height of the higher wave be greater than the difference of depth of the channels; in short, that wave will move fastest in a given channel whose crest is highest above the bottom of the channel, and in channels of different depths waves may be propagated with equal velocities, provided only the sum of the height of wave and standing depth of channel amount to the same quantity.

## Table II.

### *History of a Solitary Wave of the First Order, from observation.*

Depth of fluid in repose in the channel 5·1 inches.

Breadth of the channel 12 inches; the form rectangular.

Volume of generating column 445 cubic inches.

Column A is the observed height of the crest of the wave in inches above the bottom of the channel.

Column B is the observed height of the crest of the wave in inches above the surface of the water in repose.

Column C is the time in seconds occupied in traversing the distances in column D.

Column D is the spaces traversed by the wave in feet previous to each observation of time.

Column E is the velocity of the wave through each length of 40 feet deduced from observation.

Column F is the velocity deduced from the formula $\sqrt{g(h+k)}=v$.



| A. | B. | C. | D. | E. | F. | G. |
|---|---|---|---|---|---|---|
| 6·44 | 1·34 | 0·0 | 0 | 0·0 | 4·15 | − ·06 |
| 6·41 | 1·31 | 9·5 | 40 | 4·21 | 4·13 | − ·08 |
| 6·35 | 1·25 | 19·0 | 80 | 4·21 | 4·11 | + ·11 |
| 6·26 | 1·16 | 29·0 | 120 | 4·0 | 4·08 | + ·08 |
| 6·16 | 1·06 | 39·0 | 160 | 4·0 | 4·04 | + ·04 |
| 6·05 | 0·95 | 49·0 | 200 | 4·0 | 3·99 | − ·01 |
| 5·86 | 0·76 | 59·0 | 240 | 4·0 | 3·96 | − ·04 |
| 5·83 | 0·73 | 69·0 | 280 | 4·0 | 3·94 | + ·13 |
| 5·76 | 0·66 | 79·5 | 320 | 3·81 | 3·91 | − ·09 |
| 5·68 | 0·58 | 89·5 | 360 | 4·0 | 3·89 | + ·08 |
| 5·63 | 0·53 | 100·0 | 400 | 3·81 | 3·86 | + ·05 |
| 5·52 | 0·42 | 110·5 | 440 | 3·81 | 3·84 | + ·03 |
| 5·51 | 0·41 | 121·0 | 480 | 3·81 | 3·83 | + ·02 |
| 5·47 | 0·37 | 131·5 | 520 | 3·81 | 3·82 | + ·01 |
| 5·42 | 0·32 | 142·0 | 560 | 3·81 | 3·80 | − ·01 |
| 5·37 | 0·27 | 152·5 | 600 | 3·81 | 3·79 | − ·02 |
| 5·36 | 0·26 | 163·0 | 640 | 3·81 | 3·78 | − ·03 |
| 5·32 | 0·22 | 173·5 | 680 | 3·81 | 3·77 | − ·04 |
| 5·31 | 0·21 | 184·0 | 720 | 3·81 | 3·77 | + ·14 |
| 5·29 | 0·19 | 195·0 | 760 | 3·63 | 3·76 | − ·05 |
| 5·27 | 0·17 | 205·5 | 800 | 3·81 | 3·75 | + ·12 |
| 5·26 | 0·16 | 216·5 | 840 | 3·63 | 3·75 | + ·12 |
| 5·25 | 0·15 | 227·5 | 880 | 3·63 | 3·75 | − ·25 |
| 5·24 | 0·14 | 237·5 | 920 | 4·0 | 3·74 | + ·11 |
| 5·23 | 0·13 | 248·5 | 960 | 3·63 | 3·74 | + ·11 |
| 5·22 | 0·12 | 259·5 | 1000 | 3·63 | 3·73 | − ·08 |
| 5·20 | 0·10 | 270·0 | 1040 | 3·81 | 3·73 | + ·10 |
| 5·19 | 0·09 | 281·0 | 1080 | 3·63 | 3·73 | − ·08 |
| 5·19 | 0·09 | 291·5 | 1120 | 3·81 | 3·72 | + ·11 |
| 5·18 | 0·08 | 302·5 | 1160 | 3·61 | | |
| | | | | | | +1·36 |
| | | | | | | −0·84 |
| | | | | | Mean... | +0·52 |
| | | | | | | +0·018 |

*History of a solitary Wave of the First Order.*—In the accompanying table is given a history of the progress of a wave from its genesis through a range of 1160 feet, and during a period of 302 seconds. This wave was generated in the manner already described, by the addition of a volume of 445 cubic inches to the fluid at one extremity of the channel. The fluid in repose had a depth of 5·1 inches, and the wave generated had a height of 1·34 inch above the plane of repose, thus making the whole depth reckoned from the crest of the wave to the bottom of the channel $=5·1+1·34=6·44$ inches as the depth total. This, as successively observed, forms column A, and the simple height of the wave above the plane of repose forms column B. The height of the wave is recorded at successive distances of 40 feet, as recorded in column D, reckoning from the first observation, and the corresponding time of transit past the station of observation is given in column C. The column E gives the velocity between two successive stations as resulting from the observations C and D. In order to compare these observations with the formula $v=\sqrt{g(h+k)}$, $g$ is taken at the value 32·1908 feet, being the velocity required in one second by a body falling freely *in vacuo* in the latitude of Greenwich at the level of the sea, and $(h+k)$ is the number of inches in column A, reduced to decimals of a foot. The number resulting from these as the velocity per second which a heavy body will acquire in falling freely by gravity through a space equal to half the depth (reckoned from the crest of the wave), is that given in column F; with which the numbers in column E resulting from observation are compared, their excess or defect being set down with the signs + or − in column G.

We are thus enabled to compare the numbers given by observation E with the numbers given by formula F, and the result G shows that the coincidence is as close as the means of observation would admit. It was not possible with the chronometer then applied (although observations to fifths of a second have since been obtained) to depend upon accuracy to more minute intervals than half-seconds, and the differences in column G are precisely what we should have expected, being nearly alternately + and −, and being of nearly the same magnitude at both ends, and along the whole line of observation. The sum of the errors affected by the positive sign is +1·36, the sum of those affected with the negative sign −0·84, so that the whole of 29 observations give only an excess of +·52, or a mean excess of 0·018, showing a mean excess of velocity of the observation over the velocity assigned by the formula, of 0·018 of a foot per second, being less than $\frac{1}{200}$th of the whole. Hence we are warranted in assuming, that as far as the history of this wave is concerned, the velocity is accurately represented to within $\frac{1}{200}$th part by the formula $\sqrt{g(h+k)}=v$.

*Experiments on the Velocity.*—In order to determine the velocity of the wave of the first order with accuracy, a series of experiments have been made upon rectangular channels, extending from 1 inch in depth and a foot wide, to 12 feet wide and 6 feet deep. These experiments, forming a series of thirty different depths, are given in Table III. Column A contains the depth of the water, reckoned from the crest of the wave. Column B is the height of the crest of the wave above the level of the water in repose. Column C is the velocity of the wave as observed, and in column D is given the velocity due to half the depth in column A calculated by the formula $v=\sqrt{g(h+k)}$. Columns D and C are compared, and their difference given in E, from which it results that the formula represents the experiments to within a mean error of 0·007. The results of this table leave no room to doubt that, as far as observation can settle this point, the velocity is conclusively settled, and determined to be *that due by gravity through half the depth of the fluid, reckoned from the ridge of the wave.*



TABLE III.

*Determination of the Velocity of the Wave of the First Order, from observation.* (See Seventh Report of the British Association, and Researches on Hydrodynamics in the Philosophical Transactions of the Royal Society of Edinburgh, 1836.)

The form of the channels was rectangular.

The breadth of the channels varied from 12 inches to 12 feet.

Column A gives the depth of the channel in inches reckoned from the top of the wave.

Column B gives the height of the wave above the surface of the fluid in repose.

Column C is the velocity of the wave in feet per second, from observation.

Column D is the velocity of the wave calculated by formula B.

Column E is the difference between columns D and C.

| A. | B. | C. | D. | E. | A. | B. | C. | D. | E. |
|---|---|---|---|---|---|---|---|---|---|
| 1·0 | | | 1·63 | | 6·9 | 0·7 | 4·29 | 4·30 | +·01 |
| 1·05 | 0·05 | 1·64 | 1·67 | +·03 | 7·0 | | | 4·33 | |
| 1·30 | 0·15 | 1·84 | 1·86 | +·02 | 7·33 | 0·29 | 4·39 | 4·43 | +·04 |
| 1·62 | 0·32 | 2·06 | 2·08 | +·02 | 7·44 | 0·40 | 4·44 | 4·46 | +·02 |
| 2·0 | | | 2·31 | | 7·82 | 0·78 | 4·53 | 4·57 | +·04 |
| 2·19 | 0·29 | 2·30 | 2·42 | +·12 | 8·0 | 0·78 | 4·53 | 4·63 | +·10 |
| 3·0 | | | 2·83 | | 9·0 | | | 4·91 | |
| 3·10 | 0·16 | 2·87 | 2·88 | +·01 | 10·0 | | | 5·18 | |
| 3·23 | 0·15 | 2·99 | 2·94 | −·05 | 11·0 | | | 5·43 | |
| 3·84 | 0·92 | 3·24 | 3·21 | −·03 | 15·0 | | | 6·34 | |
| 3·9 | 0·96 | 3·33 | 3·23 | −·10 | 19·0 | | | 7·14 | |
| 3·97 | 0·81 | 3·26 | 3·26 | ·00 | 20·0 | | | 7·32 | |
| 4·0 | 0·19 | 3·33 | 3·27 | −·06 | 21·0 | | | 7·50 | |
| 4·08 | 0·13 | 3·24 | 3·30 | +·06 | 26·0 | | | 8·35 | |
| 4·20 | 0·13 | 3·33 | 3·35 | +·02 | 27·0 | | | 8·51 | |
| 4·31 | 0·24 | 3·40 | 3·40 | ·00 | 28·0 | | | 8·66 | |
| 4·49 | 0·42 | 3·46 | 3·47 | +·01 | 29·0 | | | 8·82 | |
| 4·61 | 0·74 | 3·52 | 3·51 | −·01 | 30·0 | | | 8·97 | |
| 4·75 | 0·8 | 3·52 | 3·56 | +·04 | 35·0 | | | 9·68 | |
| 5·0 | | | 3·66 | | 42·0 | 3·0 | 10·59 | 10·61 | +·02 |
| 5·20 | 0·10 | 3·73 | 3·73 | ·00 | 45·0 | | | 10·98 | |
| 5·25 | 0·15 | 3·72 | 3·75 | +·03 | 50·0 | | | 11·58 | |
| 5·61 | 0·57 | 4·05 | 3·88 | −·17 | 55·0 | | | 12·14 | |
| 5·82 | 0·72 | 3·90 | 3·95 | +·05 | 60·0 | | | 12·68 | |
| 6·0 | | | 4·01 | | 65·0 | | | 13·20 | |
| 6·47 | 0·27 | 4·14 | 4·16 | +·02 | 70·0 | | | 13·70 | |
| 6·74 | 0·54 | 4·32 | 4·25 | −·07 | 75·0 | 9·0 | 14·23 | 14·18 | −·05 |
| | | | | | | | | | +·66 |
| | | | | | | | | | −·54 |
| | | | | | | | | | +·12 |
| | | | | | | | | Mean.. | +·004 |

It appeared to me at one time matter of doubt, whether waves very low in height were not somewhat slower than the velocity of the formula, and those of a large size somewhat more rapid. To determine this point, Tables IV. and V. were prepared, the former consisting of larger waves, the latter of smaller. It can scarcely be said that these tables, which are arranged exactly as the previous one, establish any distinction in this respect.

To render the results of all these experiments still more appreciable, they are graphically laid down in Plate XLVIII., the stars representing the individual experiments, and the line the formula. The coincidence is satisfactory.

TABLE IV.

*Velocity of larger Waves.*

| A. | B. | C. | D. | E. |
|---|---|---|---|---|
| 1·62 | 0·32 | 2·06 | 2·08 | +·02 |
| 3·84 | 0·92 | 3·24 | 3·21 | −·03 |
| 3·9 | 0·96 | 3·33 | 3·23 | −·10 |
| 3·97 | 0·81 | 3·26 | 3·26 | ·00 |
| 4·49 | 0·42 | 3·46 | 3·47 | +·01 |
| 4·52 | 0·56 | 3·47 | 3·48 | +·01 |
| 4·61 | 0·74 | 3·52 | 3·51 | −·01 |
| 4·75 | 0·8 | 3·52 | 3·56 | +·04 |
| 5·61 | 0·57 | 4·05 | 3·88 | −·17 |
| 5·80 | 0·7 | 4·0 | 3·94 | −·06 |
| 5·82 | 0·72 | 3·90 | 3·95 | +·05 |
| 6·75 | 0·5 | 4·13 | 4·25 | +·12 |
| 6·86 | 0·61 | 4·21 | 4·28 | +·07 |
| 6·9 | 0·7 | 4·29 | 4·30 | +·01 |
| 7·82 | 0·78 | 4·53 | 4·57 | +·04 |
| 7·84 | 0·8 | 4·43 | 4·58 | +·15 |
| 7·87 | 0·83 | 4·53 | 4·59 | +·06 |
| 8·0 | 0·78 | 4·53 | 4·63 | +·10 |
| | | | | +·68 |
| | | | | −·37 |
| | | | | +·31 |
| | | | Mean.. | +·017 |

TABLE V.

*Velocity of smaller Waves.*

| A. | B. | C. | D. | E. |
|---|---|---|---|---|
| 1·01 | | | 1·63 | |
| 1·05 | 0·5 | 1·64 | 1·67 | +·03 |
| 1·30 | 0·15 | 1·84 | 1·86 | +·02 |
| 2·0 | | | 2·31 | |
| 2·19 | 0·29 | 2·30 | 2·42 | +·12 |
| 3·0 | | | 2·83 | |
| 3·10 | 0·16 | 2·87 | 2·88 | +·01 |
| 3·23 | 0·15 | 2·99 | 2·94 | −·05 |
| 4·00 | 0·19 | 3·33 | 3·27 | −·06 |
| 4·08 | 0·13 | 3·24 | 3·30 | +·06 |
| 4·20 | 0·13 | 3·33 | 3·35 | +·02 |
| 4·31 | 0·24 | 3·40 | 3·40 | ·00 |
| 5·0 | | | 3·66 | |
| 5·20 | 0·10 | 3·73 | 3·73 | ·00 |
| 5·25 | 0·15 | 3·72 | 3·75 | +·03 |
| 6·0 | | | 4·01 | |
| 6·40 | 0·15 | 4·04 | 4·14 | +·10 |
| 6·47 | 0·27 | 4·14 | 4·16 | +·02 |
| 6·74 | 0·54 | 4·32 | 4·25 | −·07 |
| 7·0 | | | 4·33 | |
| 7·33 | 0·29 | 4·39 | 4·43 | +·04 |
| 7·44 | 0·40 | 4·44 | 4·46 | +·02 |
| 8·0 | | | 4·63 | |
| | | | | +·47 |
| | | | | −·18 |
| | | | | +·29 |
| | | | Mean.. | +·018 |

*Wave of the First Order not formerly described.*—Although many distinguished philosophers from the time of Sir Isaac Newton have devoted themselves to the study of the theory of waves, I have not been able to discover in their works anything like the prediction of a phænomenon such as the wave of translation or the solitary wave of the first order. The waves of the second order, or gregarious oscillations, which make their appearance in successive groups, or long and recurring series, such oscillations of the surface of the water as we notice on the sea, or are excited when the quiescent surface of a lake is disturbed by dropping a stone, and which diffuse themselves in concentric circles around the centre of derangement; these have long been familiar to naturalists, and have been studied, though with comparatively little success, by philosophers. But I have not found the phænomenon, which I have called the wave of the first order, or the great solitary wave of translation, described in any observations, nor predicted in any theory of hydrodynamics.

Unquestionably the means of making such a prediction must have existed in any sound theory. It is, I think, pretty generally admitted that Lagrange was quite successful in stating the general equations of fluid motion; so that it was only necessary to obtain complete solutions of these equations to exhibit the formulæ of all motion consistent with the maintenance of continuity of the fluid and obedience to the laws of motion and pressure. After finding the general equations for the motion of incompressible fluids in the 'Mécanique Analytique,' part 2. sect. ix., Lagrange says, "Voilà les formules les plus générales et les plus simples pour la détermination rigoureuse du mouvement des fluides. La difficulté ne consiste plus que dans leur intégration;" and then he adds elsewhere, "malheureusement elles sont si rebelles, qu'on n'a pu jusqu'à présent en venir à bout que dans des cas très-limités." Indeed, ever since the publication of Euler's general formula for the motion of fluids in the Memoirs of the Academy of Sciences of Berlin, 1755, the whole phænomena of fluids in all conditions may be considered as having been represented. But the phænomena have remained there till now, locked up without any one to open, and amongst the rest I presume the wave of the first order.



There is one point, however, in which the analysis of M. Lagrange has appeared to make an approach to the representation of one of the phænomena peculiar to the wave of translation. In section xii. of part 2. of the 'Mécanique Analytique,' he investigates the propagation of vibrations in elastic fluids (like those of sound through the atmosphere), and obtains an equation

$$\frac{d^2\phi}{dt^2}=gh\left(\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dx^2}\right),$$

from which he afterwards deduces the well-known law that sound is propagated with a velocity (nearly) equal to that which is due to gravity, acting freely through a height equal to half the depth of the atmosphere (supposed homogeneous and of uniform density). And again, elsewhere he finds for the propagation of wave motion in a liquid in a channel with a level bottom, and a depth $a$, the equation

$$\frac{d^2\phi}{dt^2}=ga\left(\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dx^2}\right);$$

and from the similarity of this to the former equation, he argues as follows: "Ainsi comme la vîtesse de la propagation du son se trouve égale à celle qu'un corps grave acquerrait en tombant de la moitié de la hauteur de l'atmosphère supposée homogène, la vîtesse de la propagation des ondes sera la même que celle qu'un corps grave acquerrait en descendant d'une hauteur égale à la moitié de la profondeur de l'eau dans le canal."

Had this result been of the same general nature with the original equations from which it is deduced, we should have been able to assign to the analysis of M. Lagrange the honour of having predicted in 1815 the wave of the first order, never distinctly recognised by observation till 1834. Unhappily the nature of his investigation precludes us from doing so, and he goes on himself to admit that this conclusion will only apply to such waves as are infinitely small, and agitate the water to a very small depth below the surface. "On pourra toujours employer la théorie précédente, si on suppose que dans la formation des ondes l'eau n'est ébranlée et remuée qu'à une profondeur très-petite." The wave of the first order bears as its characteristics, the observed phænomena, that the agitation does extend below the surface to the very bottom of the channel, where it is quite as great as at the surface, and that its oscillations are large. The essential conditions of Lagrange's analysis being that the oscillation is minute, and that the agitation of the fluid is confined to the surface, we are precluded from the application of his formula to the wave of the first order.

I have been led to speak thus fully of M. Lagrange's solution, because his result is the only one that offers a tolerable approximation to the representation of the velocity of the wave of the first order. I do not find in the results obtained by M. Poisson in his 'Theory of Waves,' any result that represents the phænomena of this wave, although he shows that the solution of Lagrange cannot either mathematically or physically be applied to considerable depths. Nearly all of them seem to apply only to the phænomena of the fluid in the vicinity of the initial disturbance. The supposed method of genesis is one also which precludes the existence of the wave of the first order.

The greater part of the investigations of M. Poisson and of M. Cauchy under the name of wave theory, are rather to be regarded as mathematical exercises than as physical investigations; but an account of what has been accomplished in this way by them, and by M. Laplace, may be found in the excellent Reports of Mr. Challis in the Transactions of the British Association, and in the treatise of MM. Weber*.

* I think it right in this place to mention, with such distinction as I am able to bestow, a very valuable treatise on waves, which was published nearly twenty years ago in Leipsic, by the brothers Ernest H. Weber and William Weber, entitled 'Wellenlehre auf Experimente gegründet, oder über die Wellen tropfbarer Flüssigkeiten mit Anwendung auf die Schall- und Licht-Wellen, von den Brüdern Ernst Heinrich Weber, Professor in Leipzig und Wilhelm Weber in Halle. Mit 18 Kupfertafeln. Leipzig, bei Gerhard Fleischer, 1825.' The work is distinguished by more than the usual characteristics of German industry in the collection of materials, and contains nearly all that has ever been written on waves since the time of Newton, and as a book of reference alone is a valuable history of wave research. To this synopsis of the labours of others is appended a valuable series of experiments by the Messrs. Webers themselves, contrived with much ingenuity, and conducted with apparently a high degree of accuracy, designed to illustrate, extend, contradict or confirm the various theories that have been advanced. I have been disposed to regret that this excellent book did not reach me till long after my own researches had advanced far towards completion. But if it had done so, it might have diverted me from my own trains of research. As the subject now stands, it so happens that their labours and mine do not in the least degree supersede or interfere with each other. Our respective works may be rather reckoned as supplementary the one to the other, inasmuch as a great part of what they have done I have not attempted, and the most part of what I have done will not be found in any part of their work. Of the existence of my great solitary wave of the first order they were not aware, and although I am now able to recognise its influence on their results, yet owing to the nature of their experiments, it was not likely they should recognise its existence, much less could they examine its phænomena.

The following passages serve to show that the Messrs. Weber had never recognised the existence of my solitary wave of the first order. They say in Abschnitt IV. Art. 87,—

"Waves make their appearance as heights and hollows upon the surface of the liquid, one part being raised above the level surface, and another part sunk below it; hence the height may be called the wave-ridge, and the depression the wave-hollow. These wave-ridges and wave-hollows never come singly, but always connected with one another. This is the reason why we do not call the wave-ridge by itself alone *a wave*, nor the wave-hollow by itself alone a wave, but simply the two conjoined." Art. 89. "The sum of the breadths of one wave-ridge and of its companion wave-hollow, is called the breadth of a wave." Art. 101. "But never in nature appears a wave-ridge unconnected with a wave-hollow, nor in like manner any wave-hollow without its companion wave-ridge. Also from this reason it follows that we can never have, during wave-motion, a particle of the fluid moved forward in its path without immediately before or after having a contrary motion also; nor backwards, without also its path being reversed."

Their observations on the larger class of waves are ingeniously contrived, carefully observed, and faithfully recorded, but lose much of the value as the basis of calculation and of general laws from the following circumstances:—1st, the narrowness of the channel; that in which the greater number of observations was made, being only 6·7 lines wide; from this cause so great an influence was produced by the adhesion of the sides as seriously to interfere with the phænomena, which ought not therefore to be considered as the phænomena of perfectly free fluids; 2, the shortness of the channels; the longest having a depth of 2 feet and only 6 feet of length; in this case an observation of the wave of the first order was impossible; and when we add that the wave genesis was in general produced by the descent of a water column of great height, it was impossible that in the short period of wave transit the phænomena could attain a condition of uniformity favourable to accurate observation, one second and a fraction of a second being the whole period of an observation, and it being necessary to observe accurately to at least one-twentieth of a second, the results possess little value as measures of the phænomena. In my experiments we found that the first observations immediately after the wave genesis were the



Having ascertained that no one had succeeded in predicting the phænomenon which I have ventured to call the wave of translation, or wave of the first order, to distinguish it from the waves of oscillation of the second order, it was not to be supposed that after its existence had been discovered and its phænomena determined, endeavours would not be made to reconcile it with previously existing theory, or in other words, to show how it ought to have been predicted from the known general equations of fluid motion. In other words, it now remained to the mathematician to predict the discovery after it had happened, i. e. to give an *à priori* demonstration *à posteriori*.

*Theoretical Results subsequent to the publication of the Author's Investigations.*—Since the publication of my former observations on the wave of the first order, two attempts have been made to elicit from the wave theory, as developed by Poisson, &c., results capable of such physical interpretations as should represent the phænomena of that order.

The first of these investigations is that of Mr. Kelland in the Edinburgh Philosophical Transactions. This valuable and elegant investigation deduces theoretically, from the general equations of fluid motion, on the hypothesis of parallel sections, and of oscillations of the general form of the curve of sines, the following value for the velocity of a wave,

$$c^2=\frac{g}{a}\cdot\frac{e^{ah}-e^{-ah}}{e^{ah}+e^{-ah}}\div\left\{1-\varepsilon a\frac{e^{ah}-e^{-ah}}{e^{ah}+e^{-ah}}\right\}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad [C.]$$

$\varepsilon$ being the semi-elevation, $h$ the depth in repose, $\lambda$ the length of the wave, $c$ the velocity of transmission.

This expression gives values for the velocity of the wave which Mr. Kelland has himself compared with my experiments as follows:—

Theoretical value when $h=3{\cdot}97$ and $2\varepsilon=0{\cdot}53$, is $c=2{\cdot}8693$

Observed value $c=3{\cdot}38$,

showing the error in defect $=-\frac{1}{5\frac{1}{2}}$ or $-\frac{2}{11}$ of the whole theoretical velocity.

least accurate and the least valuable, and these are the *only* observations employed by MM. Weber in their larger wave observations. Further, as they did not recognise at all the possibility of the existence of the solitary wave of the first order, nor the difference of its phænomena from the negative waves, nor the distinction of waves into separate first and second orders, they have mingled together the observations and phænomena of both. Thus have they failed to recognise the existence of the law of the velocity which I have elicited.

Nevertheless, their observations are very valuable, and furnish interesting information to one already master of my observations. In their very deviations from the laws exhibited by my observations, they become instructive as manifesting and enabling us to measure the amount of those interfering influences which diminished the value of their experiments when taken by themselves. For this purpose I have taken some of their experiments and placed them beside the results of mine; the effects of adhesion to the sides, and of more or less perfect fluidity, are well manifested in the difference of the results. It is however to be remembered that in point of accuracy and precision, and also of weight, the shortness of period and path in their observations diminish their value.

These remarks, which I make with perfect deference, are designed to apply only to the large class of waves to which chiefly I have directed my attention; the observations on dropping waves, and all those made with reference to the phænomena of light and sound, are to be exempted from these remarks. I desire that my experiments should enhance rather than derogate from the value of those of my estimable predecessors, and I wish rather by these statements to make an apology to them for having arrived at different conclusions, by showing that the methods I chanced to light upon, and the circumstances in which I observed, were more favourable than those which they happened to employ. I only aspire to having brought to a more favourable conclusion what they had most meritoriously begun under circumstances less propitious; my having arrived at different conclusions is probably more owing to the chance of my being ignorant of their methods when I began, and alighting by chance upon better; for had I known of their elegant apparatus at first, it is not improbable that I should have been satisfied to adopt what so much ingenuity had contrived, and so failed to extend the subject beyond the conclusions they had attained.

Another example:

Theoretical value (when $h=1$ and $2e=0{\cdot}3$) $c=1{\cdot}547$
Observed value $c=1{\cdot}8$,

showing the error in defect $=-\frac{1}{6}$ of the whole theoretical velocity.

Again,

Theoretical value (when $h=7{\cdot}04$ and $2e=0{\cdot}89$) $c=4{\cdot}0$
Observed value $c=4{\cdot}6$,

showing the error in defect $=-\frac{1}{8}$ of the whole theoretical velocity.

I think it due to Mr. Kelland to say, that notwithstanding all the anxiety for success which naturally exists in the mind of one who has bestowed much time and talent on perfecting, as he has done, an elegant theory; he has not yielded to the temptation of twisting his theory to exhibit some apparent approximation to the facts, nor distorted the facts to make them appear to serve the theory, a proceeding not without precedent; but he has candidly stated the discrepancy, and says, "my solution can only be regarded as an approximation, nor does it very accurately agree with observation." This is a candour which cannot be too highly valued, and can only be justly appreciated by those who have, as I have, after working at a favourite theory, it may be for months and years, found it necessary to abandon it, and make the sacrifice for the sake of truth with readiness and candour.

Mr. Airy has followed Mr. Kelland over the same ground, in an elaborate paper on waves in the 'Encyclopedia Metropolitana,' published since the greater part of this Report was ready for the press. This paper I have long expected with much anxiety, in the hope that it would furnish a final solution of this difficult problem, or at least tend to reduce the number and extent of the unhappy discrepancies between the wave-prediction and the wave-phænomena, a hope justified by the reputation and position of the author, as well as by the clear views and elegant processes which characterize some of his former papers.

Mr. Airy has obtained for the velocity of a wave, an expression of a form closely resembling that which Mr. Kelland had previously obtained, viz.

$$c^2=\frac{g}{m}\cdot\frac{\varepsilon^{mk}-\varepsilon^{-mk}}{\varepsilon^{mk}+\varepsilon^{-mk}}. \quad \text{[D.]}$$

From the resemblance of this form of expression to the form previously given by Mr. Kelland, we are prepared for the conclusion that Mr. Airy has advanced in this direction little beyond his predecessor. And we accordingly find that a theory of the wave of the first order, accurately representing this characteristic phænomenon, is still wanting, a worthy object for the enterprise of a future wave-mathematician.

I have already stated that I have found, that by introducing the element of the wave's height into Lagrange's formula, I get the expression

$$v=\sqrt{g(h+k)},$$

and that I find it represent with great accuracy the characteristic velocity of the wave of the first order. As however Mr. Airy appears to intimate to his readers that his own formula is as close an approximation to my experiments as the nature of these experiments will warrant, I have thought it necessary to make a complete re-examination of my experiments, and to make a laborious comparison of the phænomena discussed after the best modern methods employed in inductive philosophy; the results of these discussions I have presented in a series of graphic representations, which will enable the reader at once to attain a sound conclusion on the question, whether the formula Mr. Airy has adopted, or that which I have always used, more truly represents the phænomena.



In the following table, E represents the velocity of the wave of the first order as taken from my observations by Mr. Airy himself. I have placed beside these results of experiments, the number given in column F, by the formula which I use to represent them. In the next four columns are Mr. Airy's numbers, calculated by himself, according to four different formulæ, which he appears here to have applied as a sort of tentative process for the purpose of selecting the one which should prove on trial least defective. I have next given five columns, which exhibit the results of comparing the phænomena of experiment with the results of the formulæ. The first of these columns represents the defects of my formula, the others those of Mr. Airy's.

The results of the first table are as follows:—

| | |
|---|---|
| The errors of Mr. Airy's first column amount to | 2635 |
| The errors of Mr. Airy's second column amount to | 1994 |
| The errors of Mr. Airy's third column amount to | 1674 |
| The errors of Mr. Airy's fourth column amount to | 1680 |
| The errors of mine amount to | 406 |
| The greatest error of Mr. Airy's first column is | 809 |
| The greatest error of Mr. Airy's second column is | 690 |
| The greatest error of Mr. Airy's third column is | 463 |
| The greatest error of Mr. Airy's fourth column is | 575 |
| The greatest error of mine is | 87 |

The results of the second table are as follows:—

| | |
|---|---|
| The errors of Mr. Airy's first column amount to | 6157 |
| The errors of Mr. Airy's second column amount to | 3350 |
| The errors of Mr. Airy's third column amount to | 3226 |
| The errors of Mr. Airy's fourth column amount to | 2274 |
| The errors of mine amount to | 447 |
| The greatest error of Mr. Airy's first column is | 911 |
| The greatest error of Mr. Airy's second column is | 689 |
| The greatest error of Mr. Airy's third column is | 473 |
| The greatest error of Mr. Airy's fourth column is | 480 |
| The greatest error of mine is | 122 |

TABLE VI.

*Small Waves.*

Column A is a mean height of wave crest.
" B the selected examples from which A is taken.
" C the depth of the fluid in repose.
" D the height of the wave.
" E the velocity of the wave observed.
(As taken from my experiments by Mr. Airy.)
" F the velocity of the wave as given by my formula.
" G the velocity of the wave as given by Mr. Airy's first formula.
" H the velocity of the wave as given by Mr. Airy's second formula.
" K the velocity of the wave as given by Mr. Airy's third formula.
" L the velocity of the wave as given by Mr. Airy's fourth formula.
" F′ the difference between observation and my formula.
" G′ the difference between observation and Mr. Airy's first formula.
" H′ the difference between observation and Mr. Airy's second formula.
" K′ the difference between observation and Mr. Airy's third formula.
" L′ the difference between observation and Mr. Airy's fourth formula.

| A. | B. | C. | D. | E. | F. | G. | H. | K. | L. |
|---|---|---|---|---|---|---|---|---|---|
| 1·075 | 1·05 and 1·10 | 1·000 | 0·075 | 1·670 | 1·697 | 1·629 | 1·689 | 1·803 | 1·747 |
| 1·3 | 1·3 | 1·150 | 0·150 | 1·810 | 1·867 | 1·744 | 1·854 | 2·057 | 1·958 |
| 3·17 | 3·09—3·23 | 2·963 | ·207 | 2·860 | 2·915 | 2·702 | 2·795 | 2·972 | 2·885 |
| 3·36 | 3·32 and 3·40 | 3·080 | ·280 | 2·960 | 3·002 | 2·747 | 2·869 | 3·099 | 2·986 |
| 4·16 | 4·0 —4·31 | 3·903 | ·256 | 3·310 | 3·340 | 3·016 | 3·114 | 3·300 | 3·208 |
| 5·34 | 5·20—5·5* | 5·088 | ·252 | 3·758 | 3·784 | 3·303 | 3·384 | 3·540 | 3·463 |
| 6·52 | 6·4 —6·65 | 6·220 | ·304 | 4·094 | 4·181 | 3·495 | 3·579 | 3·742 | 3·662 |
| 7·51 | 7·42—7·7 | 7·040 | ·474 | 4·406 | 4·488 | 3·597 | 3·716 | 3·943 | 3·831 |

Differences.

| F′. | G′. | H′. | K′. | L′. |
|---|---|---|---|---|
| +·027 | − ·041 | + ·019 | + ·133 | + ·077 |
| +·057 | − ·066 | + ·044 | + ·247 | + ·148 |
| +·055 | − ·158 | − ·065 | + ·112 | + ·025 |
| +·042 | − ·213 | − ·091 | + ·139 | + ·026 |
| +·030 | − ·294 | − ·196 | − ·010 | − ·102 |
| +·026 | − ·455 | − ·374 | − ·218 | − ·295 |
| +·087 | − ·599 | − ·515 | − ·352 | − ·432 |
| +·082 | − ·809 | − ·690 | − ·463 | − ·575 |
| +·406 | −2·635 | −1·931 | −1·043 | −1·404 |
| | | + ·063 | + ·631 | + ·276 |
| ·406 | −2·635 | 1·994 | 1·674 | 1·680 |

TABLE VII.

*Large Waves.*

Columns A, B, C, &c. correspond to those in Table VI.

| A. | B. | C. | D. | E. | F. | G. | H. | K. | L. |
|---|---|---|---|---|---|---|---|---|---|
| 1·20 | 1·20 | 1·000 | 0·200 | 1·760 | 1·794 | 1·629 | 1·785 | 2·061 | 1·928 |
| 1·62 | 1·62 | 1·300 | ·320 | 2·060 | 2·083 | 1·858 | 2·072 | 2·446 | 2·267 |
| 2·19 | 2·19 | 1·900 | ·290 | 2·300 | 2·422 | 2·217 | 2·380 | 2·677 | 2·533 |
| 3·38 | 3·35—3·41 | 2·960 | ·420 | 3·010 | 3·010 | 2·701 | 2·887 | 3·225 | 3·061 |
| 3·55 | 3·5 —3·61 | 3·020 | ·532 | 3·080 | 3·085 | 2·724 | 2·954 | 3·368 | 3·168 |
| 3·83 | 3·69—3·97 | 3·007 | ·830 | 3·252 | 3·204 | 2·719 | 3·072 | 3·677 | 3·388 |
| 4·53 | 4·4 —4·75 | 3·910 | ·625 | 3·505 | 3·485 | 3·018 | 3·250 | 3·671 | 3·467 |
| 5·21 | 5·21 | 3·870 | 1·340 | 3·820 | 3·738 | 3·007 | 3·488 | 4·293 | 3·911 |
| 5·76 | 5·61—5·82 | 5·070 | 0·692 | 3·970 | 3·930 | 3·300 | 3·518 | 3·917 | 3·723 |
| 6·24 | 6·15—6·40 | 5·080 | 1·160 | 4·170 | 4·090 | 3·302 | 3·659 | 4·286 | 3·985 |
| 6·69 | 6·69—7·20 | 6·034 | 0·823 | 4·262 | 4·234 | 3·468 | 3·697 | 4·117 | 3·912 |
| 7·83 | 7·74—8·0 | 6·946 | 0·884 | 4·497 | 4·582 | 3·586 | 3·808 | 4·216 | 4·017 |

* Excluding 5·21.



Differences.

| F′. | G′. | H′. | K′. | L′. |
|---|---|---|---|---|
| +·034 | − ·131 | + ·025 | + ·301 | + ·168 |
| +·023 | − ·202 | + ·012 | + ·386 | + ·207 |
| +·122 | − ·083 | + ·080 | + ·377 | + ·233 |
| +·00 | − ·309 | − ·123 | + ·215 | + ·051 |
| +·005 | − ·356 | − ·126 | + ·288 | + ·088 |
| −·048 | − ·533 | − ·180 | + ·425 | + ·136 |
| −·020 | − ·487 | − ·255 | + ·166 | − ·038 |
| −·082 | − ·813 | − ·332 | + ·473 | + ·091 |
| −·040 | − ·670 | − ·452 | − ·053 | − ·247 |
| −·080 | − ·368 | − ·511 | + ·116 | − ·185 |
| −·028 | − ·794 | − ·565 | − ·145 | − ·350 |
| +·085 | − ·911 | − ·689 | − ·281 | − ·480 |
| −·298 | −6·157 | −3·233 | +2·747 | + ·974 |
| +·269 | | + ·117 | − ·479 | −1·300 |
| ·567 | 6·157 | 3·350 | 3·226 | 2·274 |

The conclusion which Mr. Airy deduces from this comparison is somewhat surprising, "we think ourselves fully entitled to conclude from these experiments that the theory (Mr. Airy's) is entirely supported"! This conclusion being so completely the opposite of that to which we should be led on the same grounds, it has appeared necessary to make a still more complete re-examination and discussion of all the experiments in our possession, to see whether from any or the whole of them there should appear to be any ground for a conclusion so contrary to the apparent phænomena.

I have, therefore, directed the whole of the experiments to be re-discussed*. They are graphically represented in the diagrams on Plates XLVIII. and XLIX., which, and the description, the reader is requested to examine carefully. The result of the whole is, that there is an irresistible body of evidence in favour of the conclusion that Mr. Airy's formulæ do not present anything like even a plausible representation of the velocity of the wave of the first order, and that the formula I have adopted does as accurately represent them as the inevitable imperfections of all observations will admit. It is deeply to be deplored that the methods of investigation employed with so much knowledge, and applied with so much tact and dexterity, should not have led him to a better result.

TABLE VIII.

*Re-discussion of the Observations by the Method of Curves.*

The observations of height and time were laid down on paper, as shown in Plate XLIX. (see description), each star representing an individual observation of height or time. The curves being drawn through among the observations, were taken to represent the *corrected observations*, and the *velocity* was then deduced from the corrected observation of *time* and *height*. The table consists of results of this process.

Column A gives the corrected depth in inches $(h+k)$ of my formula.

Column B gives the corrected time in seconds employed in describing 40 feet.

* For the accuracy and good faith with which these discussions were all conducted, I am indebted to my valued assistant Mr. I. Currie.

Column C gives the derived velocity of the wave.

Column D gives the characteristic number of the individual wave as observed (see former Report).

These results are compared with my formula in Plate XLIX.

| A. | B. | C. | D. |
|---|---|---|---|
| | s. | ft. | |
| 3·20 | 14·0 | 2·85 | 26 |
| 3·20 | 13·9 | 2·87 | |
| 3·20 | 13·7 | 2·92 | |
| 3·21 | 13·6 | 2·94 | |
| 3·22 | 13·5 | 2·96 | |
| 3·24 | 13·4 | 2·98 | |
| 3·27 | 13·3 | 3·00 | |
| 3·30 | 13·2 | 3·03 | |
| 3·35 | 13·1 | 3·05 | |
| 3·40 | 13·0 | 3·07 | |
| 3·45 | 12·9 | 3·10 | |
| 3·53 | 12·8 | 3·12 | |
| 3·61 | 12·7 | 3·15 | |
| 3·72 | 12·6 | 3·17 | |
| 3·84 | 12·5 | 3·20 | |
| 3·97 | 12·4 | 3·22 | |
| 3·97 | 12·5 | 3·20 | 2 |
| 4·00 | 12·3 | 3·25 | |
| 4·03 | 12·1 | 3·30 | |
| 4·07 | 12·0 | 3·33 | |
| 4·12 | 11·9 | 3·36 | |
| 4·17 | 11·9 | 3·36 | |
| 4·22 | 11·8 | 3·39 | |
| 4·28 | 11·8 | 3·39 | |
| 4·34 | 11·7 | 3·41 | |
| 4·42 | 11·7 | 3·41 | |
| 4·49 | 11·6 | 3·44 | |
| 4·04 | 12·2 | 3·27 | 3 |
| 4·06 | 12·1 | 3·30 | |
| 4·08 | 12·0 | 3·33 | |
| 4·11 | 11·9 | 3·36 | |
| 4·15 | 11·8 | 3·39 | |
| 4·20 | 11·8 | 3·39 | |

| A. | B. | C. | D. |
|---|---|---|---|
| | s. | ft. | |
| 4·24 | 11·8 | 3·39 | 3 |
| 4·30 | 11·7 | 3·41 | |
| 4·36 | 11·7 | 3·41 | |
| 4·43 | 11·7 | 3·41 | |
| 4·05 | 12·3 | 3·25 | 1 |
| 4·06 | 12·2 | 3·27 | |
| 4·08 | 12·2 | 3·27 | |
| 4·10 | 12·1 | 3·30 | |
| 4·12 | 12·0 | 3·33 | |
| 4·14 | 12·0 | 3·33 | |
| 4·17 | 11·9 | 3·36 | |
| 4·20 | 11·9 | 3·36 | |
| 4·23 | 11·9 | 3·36 | |
| 4·27 | 11·8 | 3·39 | |
| 4·32 | 11·8 | 3·39 | |
| 4·36 | 11·7 | 3·41 | |
| 4·42 | 11·7 | 3·41 | |
| 4·48 | 11·6 | 3·44 | |
| 4·26 | 12·2 | 3·27 | 8 |
| 4·28 | 12·1 | 3·30 | |
| 4·30 | 12·0 | 3·33 | |
| 4·32 | 11·9 | 3·36 | |
| 4·35 | 11·8 | 3·39 | |
| 4·38 | 11·7 | 3·41 | |
| 4·42 | 11·6 | 3·44 | |
| 4·46 | 11·5 | 3·47 | |
| 4·51 | 11·5 | 3·47 | |
| 4·57 | 11·4 | 3·50 | |
| 4·63 | 11·3 | 3·54 | |
| 4·70 | 11·3 | 3·54 | |
| 4·77 | 11·2 | 3·57 | |
| 4·85 | 11·2 | 3·57 | |
| 4·95 | 11·1 | 3·60 | |

| A. | B. | C. | D. |
|---|---|---|---|
| | s. | ft. | |
| 5·07 | 11·0 | 3·63 | 8 |
| 5·18 | 11·0 | 3·63 | 45 |
| 5·18 | 10·9 | 3·66 | |
| 5·19 | 10·9 | 3·66 | |
| 5·20 | 10·9 | 3·66 | |
| 5·21 | 10·8 | 3·70 | |
| 5·22 | 10·8 | 3·70 | |
| 5·23 | 10·8 | 3·70 | |
| 5·25 | 10·7 | 3·73 | |
| 5·27 | 10·7 | 3·73 | |
| 5·29 | 10·7 | 3·73 | |
| 5·31 | 10·7 | 3·73 | |
| 5·33 | 10·6 | 3·77 | |
| 5·35 | 10·6 | 3·77 | |
| 5·38 | 10·5 | 3·81 | |
| 5·41 | 10·5 | 3·81 | |
| 5·44 | 10·4 | 3·84 | |
| 5·48 | 10·4 | 3·84 | |
| 5·52 | 10·3 | 3·88 | |
| 5·57 | 10·3 | 3·88 | |
| 5·63 | 10·2 | 3·92 | |
| 5·70 | 10·2 | 3·92 | |
| 5·78 | 10·15 | 3·94 | |
| 5·87 | 10·1 | 3·96 | |
| 5·97 | 10·0 | 4·0 | |
| 6·10 | 10·0 | 4·0 | |
| 6·2 | 10·0 | 4·0 | |
| 6·29 | 9·95 | 4·02 | |
| 6·37 | 9·9 | 4·04 | |
| 6·43 | 9·8 | 4·08 | |
| 6·38 | 9·95 | 4·02 | 50 |
| 6·39 | 9·9 | 4·04 | |
| 6·40 | 9·8 | 4·08 | |

| A. | B. | C. | D. |
|---|---|---|---|
| | s. | ft. | |
| 6·41 | 9·8 | 4·08 | 50 |
| 6·42 | 9·8 | 4·08 | |
| 6·44 | 9·7 | 4·12 | |
| 6·46 | 9·7 | 4·12 | |
| 6·48 | 9·7 | 4·12 | |
| 6·51 | 9·7 | 4·12 | |
| 6·54 | 9·6 | 4·16 | |
| 6·57 | 9·6 | 4·16 | |
| 6·60 | 9·5 | 4·21 | |
| 6·63 | 9·5 | 4·21 | |
| 6·68 | 9·5 | 4·21 | |
| 6.73 | 9·5 | 4·21 | |
| 6·78 | 9·4 | 4·25 | |
| 6·83 | 9·4 | 4·25 | |
| 6·89 | 9·4 | 4·25 | |
| 6·95 | 9·4 | 4·25 | |
| 7·02 | 9·4 | 4·25 | |
| 7·19 | 9·3 | 4·30 | |
| 6·70 | 9·16 | 4·16 | 49 |
| 6·81 | 9·5 | 4·21 | |
| 6·95 | 9·4 | 4·25 | |
| 7·10 | 9·3 | 4·30 | |
| 7·30 | 9·2 | 4·34 | |
| 7·52 | 9·1 | 4·39 | |
| 7·66 | 9·0 | 4·44 | |
| 6·71 | 9·4 | 4·25 | 48 |
| 6·79 | 9·3 | 4·30 | |
| 6·92 | 9·2 | 4·34 | |
| 7·20 | 9·1 | 4·39 | |
| 7·54 | 9·0 | 4·44 | |
| 7·75 | 8·9 | 4·49 | |
| 7·82 | 8·8 | 4·54 | |

### TABLE IX.

*Velocity due to a Wave of the First Order,*

Obtained from the re-discussion of the experiments as described above.

Column A gives the depth in inches reckoned from the wave crest.

Column B gives the observed time of describing 40 feet, * the observations thus marked being over half that space.

Column C gives the observed velocity.

Column D is a reference to the ordinal number of the wave observed.

The close approximation of these velocities of observation with the numbers of the formula, proves at once the accuracy of the one and the truth of the other.



| A. | B. | C. | D. | A. | B. | C. | D. | A. | B. | C. | D. | A. | B. | C. | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | s. | ft. | | | s. | ft. | | | s. | ft. | | | s. | ft. | |
| 1·5 | 10·1* | 1·98 | 35 | 4·0 | 12·5 | 3·20 | 3 | 4·5 | 11·5 | 3·47 | 15 | 5·5 | 10·5 | 3·80 | 46 |
| 2·0 | 8·7* | 2·30 | 36 | 4·0 | 11·8 | 3·39 | 25 | 4·5 | 12·0 | 3·33 | 17 | 6·0 | 9·9 | 4·04 | 43 |
| 2·5 | 7·2* | 2·77 | 36 | 4·0 | 6·0* | 3·33 | 37 | 4·5 | 11·7 | 3·42 | 19 | 6·0 | 10·0 | 4·0 | 45 |
| 2·5 | 8·4* | 2·38 | 35 | 4·0 | 6·1* | 3·27 | 38 | 4·5 | 12·3 | 3·25 | 23 | 6·0 | 10·0 | 4·0 | 46 |
| 3·0 | 7·5* | 2·66 | 40 | 4·0 | 6·0* | 3·33 | 39 | 5·0 | 11·1 | 3·60 | 8 | 6·5 | 9·5 | 4·21 | 46 |
| 3·0 | 7·4* | 2·70 | 41 | 4·0 | 6·2* | 3·22 | 40 | 5·0 | 11·4 | 3·50 | 9 and 10 | 6·5 | 9·8 | 4·08 | 49 |
| 3·5 | 13·7 | 2·92 | 25 | 4·5 | 11·5 | 3·47 | 1 | 5·0 | 10·9 | 3·67 | 15 | 6·5 | 9·7 | 4·12 | 50 |
| 3·5 | 12·8 | 3·12 | 26 | 4·5 | 11·6 | 3·45 | 2 | 5·0 | 10·7 | 3·73 | 17 | 7·0 | 9·3 | 4·3 | 46 |
| 3·5 | 6·4* | 3·12 | 37 | 4·5 | 12·0 | 3·33 | 4 | 5·0 | 10·9 | 3·67 | 19 | 7·5 | 9·2 | 4·35 | 53 |
| 3·5 | 6·2* | 3·22 | 38 | 4·5 | 11·8 | 3·39 | 5 | 5·0 | 11·2 | 3·57 | 22 | 7·5 | 9·0 | 4·44 | 55 |
| 3·5 | 6·5* | 3·07 | 40 | 4·5 | 11·4 | 3·50 | 6 | 5·0 | 11·1 | 3·60 | 23 | 8·0 | 8·9 | 4·49 | 51 |
| 3·5 | 6·5* | 3·07 | 41 | 4·5 | 11·5 | 3·47 | 7 | 5·5 | 11·0 | 3·63 | 9 and 10 | 8·0 | 8·7 | 4·6 | 52 |
| 3·5 | 12·8 | 3·12 | 42 | 4·5 | 11·5 | 3·47 | 8 | 5·5 | 10·6 | 3·77 | 43 | 8·0 | 8·6 | 4·65 | 54 |
| 4·0 | 12·3 | 3·25 | 2 | 4·5 | 11·5 | 3·47 | 13 | 5·5 | 10·3 | 3·88 | 45 | 8·0 | 8·9 | 4·49 | 55 |

*The Magnitude and Form of the Wave of the First Order.*—This is one of the subjects to which, since the date of the former Report, I have devoted a good deal of attention. The exact determination of the dimensions and form of the wave, although at first sight it may seem simple enough, is not without peculiar difficulties. When it is observed that the two extremities of the wave are vertices of curves of very small curvature tangent to the plane of repose, it will be understood how difficult it is to detect the place of contact with precision. A variety of methods have been tried: reflexion of an image from the surface, tangent points applied to the surface so as to be observed simultaneously at both ends of the wave, and the self-registration of a float moved by the wave have all been tried with various success. On the whole, however, the most perfect observations have been obtained by a very simple autographic method, in which it was contrived that the wave should leave its own outline delineated on the surface without the intervention of any mechanism*. The method was simply this: a dry smooth surface was placed over the surface of the water in the channel, with such arrangements that it could be moved along with the velocity of transmission of the wave, and at the instant of observation it was pushed vertically down on the wave, and raised out again without sensibly disturbing the water; the surface when brought out, brought with it a moist outline of the wave, which was immediately traced by pencil, and afterwards transferred to paper. I have given a few of these autographic types of the wave in Plates L. and LI., the engravings being precise copies of the lines as drawn by the wave itself.

Another method of obtaining an autographic representation of waves of the first order was this. Two waves were generated at opposite ends of the same channel at given instants of time, so that by calculating their velocities they should both reach a given spot at the same instant; here a prepared surface was placed, and as one passed over the other it left a beautiful outline of the excess in height of each point of one wave above the summit height of the other. These forms are not identical with those of the same wave moving along a plane surface, but as true registers of actual phænomena they are interesting.

The results of all my observations on this subject are as follows:—

That the wave of the first order has a definite *form and magnitude* as much characteristic of it as the uniform velocity with which it moves, and

* I find that I am not the first person who employed an apparatus of this sort. MM. Webers employed a powdered surface to register the form of agitated mercury, the fluid rubbing off the powder.

depending like that velocity only on the depth of the fluid and the height of the wave crest.

That this wave-form has its surface wholly raised above the level of repose of the fluid. This is what I mean to express by calling this wave *wholly positive*. I apply the word negative to another kind of wave whose surface exhibits a depression below the surface of repose. The wave-proper of the first order is wholly positive.

The simple elementary wave of the first order assumes a definite *length equal to about six times the depth of the fluid below the plane of repose*. When the height of the wave is small the length does not sensibly differ from that of the circumference of a circle whose radius is the depth of the fluid; or $h$ being the depth of the fluid in repose, the length of the wave is represented by the quantity $2\pi h$, $\pi$ being the number 3·14159, we may use the notation,

$$\lambda=2\pi h \quad . \quad . \quad . \quad . \quad . \quad . \quad E.$$

The length, therefore, increases with the depth of the fluid directly, being equal to about 6·28 times the depth. The length does not, like the velocity of the wave, increase with the height of the wave in a given depth of fluid. On the contrary, the length appears to diminish as the height of the wave is increased, and the length of the wave when thus corrected is

$$\lambda=2\pi h-\alpha \quad . \quad . \quad . \quad . \quad F.$$

the value of $\alpha$ will be afterwards examined.

The form of the wave surface when not large is a surface of single curvature, the curvature being in the longitudinal and vertical planes alone, and the curve is the curve of sines, or rather of versed sines, the horizontal ordinates of which vary as the arc and the vertical ordinates, as the versines of a circle whose radius is the depth of the fluid in repose, $2\pi h$ being the length of the wave, and $\frac{2\pi}{m}$ an arc of that circle $=\theta$. We have for the equation of the wave curve,

$$x=h\theta$$
$$y=\tfrac{1}{2}k.\ \text{versin}\ \theta \quad . \quad . \quad . \quad . \quad G.$$

the height of the wave being denoted by $k$, reckoned above the plane of repose of the surface of the fluid.

The height of the wave above the surface of the water in repose may increase till it be equal to the depth of the fluid in repose. When it approaches this height it becomes acuminate, finally cusped, and falls over breaking and foaming with a white crest. The limits of the wave height are, therefore,

$$k=0, \text{ and } k=h \quad . \quad . \quad . \quad K.$$

that is to say, the height of the wave may increase from 0 to $k$, but can never exceed a *height* above the level of repose equal to the *depth* of the fluid in repose; that is, the height total reckoned from the bottom is never greater than twice the depth of the fluid in repose.

*The absolute Motions of each Water-Particle during Wave-Transmission.*—This is one of the subjects on which, prior to last Report, I had not made a sufficient number of observations to enable me to make a full report. The methods I had employed for such observations as I had then already made were the observation of the motions of small particles visible in the water of the same, or nearly the same specific gravity with water, or small globules of wax connected to very slender stems, so as to float at required depths. The motions of these were observed from above, on a minutely divided surface on the bottom of the channel, and from the side through glass windows, themselves accurately graduated, the side of the channel opposite to the window being covered with lines at distances precisely equal to those on the window and similarly situated. These methods are the only methods of observation I have found it useful to employ, but I have now increased the number and variety of the observations sufficiently to enable me to adduce the conclusions hereinafter following, as representing the phænomena as far as their nature will admit of accurate observation.



It is characteristic of waves that the *apparent motion visible on the surface* of the water is of one species, while the *absolute motion of the individual particles* of the water is very different. In reference to all the species of waves this is true, both as regards the velocity and nature of the motion; nevertheless the one is the immediate cause or consequence of the other. In the case of the wave of the first order, the visible motion of the wave form along the surface of the water may be called the *motion of transmission*, the actual motion of the particles themselves is to be distinguished as the *motion of translation*.

We infer the motions of the individual wave particles from those of visible small bodies floating in the water; any minute particle floating on the surface will sufficiently indicate the motion of the water particles about it, and the motion of deeper particles may be conveniently observed in the case of waves of the first order, by using the little globules of wax already mentioned; these small globules may be so made as to float permanently at any given depth, yet they will be visibly affected by very minute forces.

In this way the following observations were made:

*Absolute Motion of Translation.*—The phænomenon of translation characteristic of the wave of the first order, and which we have used as its distinguishing appellation, is to be observed as follows. Floating globules, as already described, being placed in the fluid, and their positions being noted with reference to the sides and bottom of the channel, let a wave of the first order be transmitted along the fluid; it is found that the effect of this transmission is to lift each of the floating particles, and similarly, therefore, the water particles themselves, out of their positions, and to transfer them permanently forward to new positions in the channel, and in these new positions the particles are left perfectly at rest, as in their original places in the channel.

The measure or range of translation is just equal to that which would result from increasing the column of water in the channel behind the wave by a given quantity, and diminishing the column anterior to the particles by the same quantity, that quantity being equal to the volume of the wave. That is to say, *the range of translation is simply equal to the space in length of the channel which the volume of the wave would occupy* on the level of the water in repose.

The *total effect* of having transmitted a wave of the first order along a channel, is to have moved successively every particle in the whole channel forward, through a space equal to the volume of the wave divided by the water-way of the channel.

*Parallelism of Translation.*—If the floating spherules before mentioned be arranged in repose in one vertical plane at right angles to the direction of transmission, and carefully observed during transmission, it will be noticed that the particles remain in the same plane during transmission and repose in the same place after transmission.

It is further found, as might be anticipated from the foregoing observations, that a thin solid plane transverse to the direction of transmission, and so poised as to float in that position, does not sensibly interfere with the motion of translation or of transmission.

*The Range of Horizontal Translation is equal at all Depths.*—Vertical excursions are performed by each particle of fluid simultaneously with the horizontal translation. These diminish in extent with the distance from the bottom when they become zero.

*The Path of each Water Particle during Translation lies wholly in a Vertical Plane.*—It may be observed by means of the glass windows already mentioned, its surface being graduated for purposes of measurement. The path is so rapidly described that I do not think any measurements of time which I have made, nor even of paths is *minutely* correct. The following observations are such as a practised eye with long experience and much pains has made out.

When a wave of the first order in transmission makes a transit over floating particles in a given transverse plane, the observations are as follows. All the particles begin to rise, scarcely advancing; they next advance as well as rise; they cease to rise but continue advancing; they are retarded and come to rest, descending to their original level. The path appears to be an ellipse whose major axis is horizontal and equal to the range of translation; the semi-minor axis of the elliptic path is equal to the height of the wave near the surface, and diminishes directly with the depth.

The results of these observations are, therefore, as follows:—representing by $b$ the breadth of the channel, by $h$ the depth of the fluid, by $\alpha$ the range of translation, and by $v$ the volume of water employed in forming the waves; we have for every particle throughout the breadth and depth of the fluid

$$\alpha=\frac{v}{bh} \quad . \quad . \quad . \quad . \quad . \quad (\text{L.})$$

which everywhere measures the horizontal range of translation.

The range of vertical motion of each particle at the surface during translation being everywhere

$$y=k \quad . \quad . \quad . \quad . \quad . \quad (\text{M.})$$

we have for the vertical range $y'$ of any other particle at a depth $h'$ below the surface,

$$y'=\frac{h'}{h}.k \quad . \quad . \quad . \quad . \quad (\text{N.})$$

being directly as the height of the particle in repose above the bottom of the channel.

Also throughout the whole period of translation we have the height of a particle of the surface above its place of repose represented by

$$y=\tfrac{1}{2}k \text{ versin } \theta \quad . \quad . \quad . \quad . \quad . \quad (\text{O.})$$

and the height of any other particle in the same vertical plane at the same place represented by

$$y'=\tfrac{1}{2}\frac{h'k}{h} \text{ versin } \theta \quad . \quad . \quad . \quad . \quad (\text{P.})$$

The whole of these results are united in the following Table of wave phænomena.

TABLE X.

*Phænomena of Wave of the First Order.*

Let $c$ be the velocity of wave transmission;
$h$ the depth of fluid in repose;
$k$ the height of wave-crest above surface of repose;
$b$ breadth of channel;



$v$ the volume of fluid constituting the wave;
$g$ the measure of gravity;
$\alpha$ the horizontal range of translation;
$\lambda$ the wave length or amplitude;
$\theta$ an arc $\frac{2\pi}{m}$, $m$ being an arbitrary number;
$\psi$ the arc whose sine $=\frac{1}{2}$ versed sine of $\theta$;
$\pi$ the number 3·1416;
$\pi'$ the circumference of an ellipse whose axes are given;
$x$ and $y$ horizontal and vertical ordinates of wave-curve;
$x'$ and $y'$ horizontal and vertical ordinates of translation-path;
$h'$ the height of a particle in repose, above the bottom of the channel.

Then we have

(1.) For velocity of wave transmission,

$$c=\sqrt{g(h+k)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{B.}$$
$$=\sqrt{gh} \text{ nearly, when } h \text{ is small} \quad . \quad . \quad . \quad . \quad \text{A.}$$

(2.) For the wave length,

$$\lambda=2\pi h-\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{F.}$$
$$=2\pi h \text{ nearly, when } k \text{ is small} \quad . \quad . \quad . \quad . \quad \text{E.}$$

(3.) For the range of translation,

$$\alpha=\frac{v}{bh} \text{ always,}$$
$$=\pi k \text{ when } k \text{ is small, } =2\,k \text{ nearly when } k \text{ is large} \quad \text{L.}$$

(4.) For the wave form,

$$x=h\theta-x'=h\theta, \text{ when } k \text{ is small}$$
$$y=\tfrac{1}{2}k. \text{ versin } \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{G}'$$

(5.) For the path of translation,

$$\left.\begin{array}{l} x'=\alpha \text{ versin } \psi \\ y'=\tfrac{1}{2}k \text{ versin } \theta \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{O}'.$$

and below the surface at $h'$, $y'=\frac{h'k}{2h}$. versin $\theta$. . . . . . . . . . P.

(6.) The limits of the value of $k$ are as follows:—
Inferior limit $k=0$, and $k=h$ superior limit . . . . . . . . K.

(7.) The range of vertical motion of a particle during translation being $y=k$ at the surface; the range of vertical motion of any other particle at the height $h$ above the bottom is

$$y'=\frac{h'}{h}k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{N}'.$$

*Geometrical Representation of the Wave of the First Order.*—These data enable to approximate to the exact conception of the motions of the wave particles, and the relations which the wave form and the particle path bear to each other. We may thus construct a geometrical representation of the wave motion, which, however, is to be carefully distinguished from a physical determination of its phænomena.

Let us then endeavour to follow the motion of a given particle on the surface of the fluid during the wave form transmission.

Let us take D E for the depth of the fluid. (Plate LII. fig. 3.)

Let us take C D for the height of the wave.

Let us mark off $d$ D $d'$ = the circumference of the circle of which D E is the radius = 6·2832 × D E. Let also semicircles be described on $c\,d$ and on $c'\,d'$ each equal to C D. Let the semicircles $c\,d$ and $c'\,d'$ and the distances

$d$ D and D $d'$ be divided into the same number of equal parts. Let there be drawn through each division of the circles horizontal lines, and through each division of the wave lengths let there be drawn perpendiculars, meeting successively the horizontal lines in 9, 8, 7, 6, 5, 4, 3, 2, 1,—these will be points in the *curve of versed sines*, that is of the (approximate) form of the wave. If, therefore, we conceive the wave-form to move horizontally and uniformly along the line $d$ D $d'$, and at the same time a particle of water on the surface to rise successively to the heights 1, 2, 3, 4, 5, and fall vertically to 6, 7, 8, 9, on the diameters $c\,d$ and $c'\,d'$, then the place of the particle will always coincide with the wave curve.

This is the same form (only wholly positive) which Laplace assigns to the tide wave in the 'Mécanique Céleste,' tom. iii. liv. iv. chap. iii. Art. 17. "Concevons un cercle vertical, dont la circonférence en partant du point le plus bas, expriment les temps écoulés depuis la basse; les sinus verses de ces arcs, seront les hauteurs de la mer, qui correspondent à ces temps." Or as he says elsewhere, "Ainsi, la mer en s'élevant, baigne en temps égal, des arcs égaux de cette circonférence." So if we imagine a circular disc placed vertically so as to touch the surface of the water in repose, the passing wave will in successive equal times cover equal successive arcs of the circumference.

The wave is of this form when its height is small, and the deviation increases with the increase of height.

*Vertical Motion of each Particle.*—No more then is necessary to the exhibition of the wave curve than that every particle of the surface of the water should be made to rise and fall successively, according to the increase and decrease of the versed sines of the circle of height. Let us follow the motion of a single particle. Draw $c'\,d'$ a vertical diameter of the wave circle, suppose C $e\,f\,g\,h\,c'$ the successive places of the wave crest in successive equal intervals of time, 1, 2, 3, 4, 5, 6, 7, 8, 9, successive versed sines on $c\,d$ and $c'\,d'$ of equal arcs of the wave circle. When the wave centre is at C, the particle is at $d'$. When the wave centre is at $e$, the particle has risen to 1. When the wave centre has reached $f$, the water particle has risen to 2. When the wave has advanced to $g\,h\,c'$, &c., the water particle has risen to 3, 4, 5, &c.; and if every successive particle along the surface be conceived to perform successively a similar series of vertical motions, the surface of the water will present to the eye the visible moving wave form. Such is the simplest geometrical mode of exhibiting to the eye and of conceiving wave motion of the first order; it approximately represents the form of a wave of the first order whose height is small.

*Horizontal Motion of each Particle.*—This mode of representing the wave motion is inaccurate, in so far as it does not take account of the horizontal motion, which must of necessity accompany the vertical elevation of the water. Water being an inelastic fluid, any vertical column of the liquid can only have its length increased by a diminution of its horizontal dimension. It is necessary, therefore, to represent or conceive this horizontal motion as well as the vertical motion.

The horizontal range of motion of the wave is necessarily determined by the volume of the wave. The water which forms the wave is added to the given volume in which the wave is formed, at its posterior extremity, and thence displaces a new volume of water which goes to displace the volume of the wave in the next portion of the channel. Thus the volume of water which occupied the space A$'$ B$'$ $b$ $d$ before the transit of the wave (see Plate LII. fig. 4), occupies only the length A B $b$ $d$ during the wave transit, and it now consists of the rectangle A B $b$ $d$, together with the volume of the wave A C $d$, which volume is equal to the volume A B B$'$ A$'$ by which it is re-



placed; and this happens successively in every point of the fluid. The horizontal range of motion is thus equal to the volume of water employed to form the wave.

While, therefore, the front of the wave is transmitted from A$'$ to $d$, the water particle A$'$ is transferred to A. The same particle is also raised and depressed through the height of the wave. These motions in the vertical and horizontal plane are simultaneous. It is required to represent accurately these motions: take $c\,d =$ the height of the wave, A A$' =$ the range of translation: describe an ellipse whose major axis is the range of translation, and whose semi-minor axis is the height of the wave: describe the wave circle $d$ 1, 2, 3, 4, $c$, and having divided as formerly its circumference into equal parts, draw the horizontal ordinates 11, 22, 33, 44, &c., as in fig. 3, and let the curve of versed sines A$'$ C$'$ $d$ be drawn as in fig. 3, then will the curve A$'$ 8, 7, 6, C$'$ 4, 3, 2, 1, $d$, represent the wave curve, the vertical motion only being considered. But at the same time that the particle rises and falls through 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 on the diameter $c\,d$, and in the curve of versed sines, the particle A$'$ will advance to A, through A$'$ 1, 2, 3, 4, 5, 6, 7, 8, 9, A. Thus every point in the curve will have to be advanced forward in the direction of translation in order to represent the actual form of the wave. This is done in fig. 4, and also for a larger wave in fig. 5. While the wave rises to 1, 2, 3, 4, C$'$, &c., it also advances simultaneously at each point by the quantity A$'$ 1, A$'$ 2, A$'$ 3, A$'$ 4, A$'$ 5, A$'$ 6, A$'$ 7, &c., and thus the wave A$'$ C$'$ $d$ becomes transformed in both figures into A C $d$. This curve represents the form of the wave as corrected for the horizontal translation. Thus are reconciled to each other the apparently diverse motions of the particle, by one of which it describes the observed sinuous wave surface, and by the other the semiellipse of its path of translation.

Finally, as the motions of translation are equal and simultaneous throughout all particles situated in the same vertical line, the path of translation of each particle is an ellipse having the same major axis with that of the particle on the surface, but having its minor axis less in proportion to its distance from the surface of the liquid in repose. (See Plate XLVII. fig. 5.)

Hence, when the wave is not large, the amplitude of the particle path or range of translation is 3·1416 times the height of the wave; this quantity gradually diminishes as the height increases, and becomes nearly 2· when the height approaches the limit of equality with the height of the wave. But near this limit it is not capable of accurate observation.

*Mechanism of the Wave.*—The study of the phænomena of the translation of water particles during the transit of a wave is peculiarly valuable, as affording us the means of correctly conceiving the real nature of wave transmission of the first order; it therefore deserves great attention.

We perceive, in the first place, that the vertical arrangement of the water particles is not deranged by wave transmission; that is, if we conceive the whole fluid in repose to be intersected by transverse vertical planes, thin, and of the specific gravity of water, these planes will retain their parallelism during transmission and will not affect that transmission.

We may therefore accurately conceive the whole volume of water as reposing in rectangular vessels, each of them formed between two successive vertical thin moveable planes, and bounded by the two sides and bottom of the channel, and above by the plane of repose. The water in each of these elementary vessels undergoes in successive instances the same change as each of the others preceding it, and therefore we may direct our attention to one individual among them.

Let us study the manner in which wave motion is originally communicated to and through each of these elementary columns of fluid.

For this purpose it may be well to recur to the original mode of wave ge-nesis (Plate XLVII. fig. 5.). A vertical generating plane P is inserted in the fluid, and forms one of the vertical boundaries of one of the elementary water columns. $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ $\genfrac{}{}{0pt}{}{bc}{\beta\gamma}$ $\genfrac{}{}{0pt}{}{cd}{\gamma\delta}$ $\genfrac{}{}{0pt}{}{de}{\delta\varepsilon}$ $\genfrac{}{}{0pt}{}{ef}{\varepsilon\zeta}$ $\genfrac{}{}{0pt}{}{fg}{\zeta\theta}$ $\genfrac{}{}{0pt}{}{gh}{\eta\theta}$ &c. A moving force is applied to P, and the plane communicates to the water column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ that pressure; now this water column is bounded on its anterior surface by a similar vertical plane (of water particles) $\genfrac{}{}{0pt}{}{b}{\beta}$ in a state of rest, and the effect of this pressure is two-fold, to raise the water column above the level to the height due to the velocity of P, and to diminish the breadth of the column in proportion to the increase of length. Such is the immediate effect of pressure on the plane P. Let us now consider the second (water) plane $\genfrac{}{}{0pt}{}{b}{\beta}$; it has now behind it a column of water pressing it forward with a velocity due to its height above the level of repose: it is therefore pressed forward, *à tergo*, just as the plane P originally was pressed forward, only its moving force is measured by the pressure of the column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ with a given height above the plane of repose. In all respects the water column $\genfrac{}{}{0pt}{}{bc}{\beta\gamma}$ is now in the condition which in the previous moment we found the column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$. Let us now return to $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ which is pressed by the plane P with a pressure not only equal to that which raised it to its former height, but with an accelerating force which raises it still higher, and communicates to it a velocity due to that greater height, and also diminishes its breadth in proportion to the increment in height. This new height in the column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ is a new increment of pressure on the vertical water plane $\genfrac{}{}{0pt}{}{b}{\beta}$, which in its turn presses the water column $\genfrac{}{}{0pt}{}{bc}{\beta\gamma}$ in the same manner, with a pressure due to the new height of the water column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$, raises its height to that due to this pressure, and gives it a corresponding velocity. The third water column $\genfrac{}{}{0pt}{}{cd}{\gamma\delta}$ is now in similar circumstances to those of its predecessor $\genfrac{}{}{0pt}{}{bc}{\beta\gamma}$ at the preceding instant of time, and is pressed by the plane $\genfrac{}{}{0pt}{}{c}{\gamma}$ with a force due to the height of $\genfrac{}{}{0pt}{}{bc}{\beta\gamma}$, and the plane $\genfrac{}{}{0pt}{}{c}{\gamma}$ now moves forward, raises the height of $\genfrac{}{}{0pt}{}{cd}{\gamma\delta}$, and diminishes proportionally its breadth. The same process continues during the acceleration of the original plane P until it ceases to be further accelerated, and now the whole anterior half of the wave has been generated, and the column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ is moving with the velocity due to its elevation above the level, or the height due to the crest of the wave, having passed successively through each of the successive conditions of the columns before it. The force acting on P, *à tergo*, is now to be diminished; the pressure back upon



surface, arising from the height of $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$, tends to retard the motion of P, nd as the accelerating force is diminished the retardation increases, the hole action of the column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ being continually to retard the plane P; and the diminution of force take place in the same succession as the original ncrements, the diminution of the velocity of P will take place in a manner imilar to that of its original increase, and it will finally be brought to rest hen the column $\genfrac{}{}{0pt}{}{ab}{\alpha\beta}$ has regained its level.

The same succession of conditions takes place in the plane which separates ny two successive elementary columns; first of all the posterior surface of the lane is pressed by a higher column than itself, tending to increase its height nd increased velocity, and having reached the maximum, the anterior surface s thereafter pressed by a water column of greater height than the posterior surface, retarding its velocity, and finally bringing it into a state of rest. Thus the forces and motion of each elementary plane are repetitions of the forces and motions of the original disturbing plane by which the wave was generated.

The power employed in wave genesis is therefore expended in raising to a height equal to the crest of the wave, each successive water column; each water column, again descending, gives out that measure of power to the next in succession, which it thus raises to its own height. The time employed in raising a given column to this height, and in its descent and communication of its own motion to the next in succession, constitutes the period of a wave, and the number of such columns undergoing different stages of the process at the same time measures the length of a wave.

During the anterior half of the wave the following processes take place. The generating force communicates to the adjacent column through its posterior bounding plane, a pressure; this pressure moves the posterior plane forward, the water in the column is thereby raised to the height due to the velocity, and the pressure of this water column communicates to the anterior bounding plane also a velocity and a pressure in the same direction; therefore the accelerating force produces a given motion of translation in the whole column a height of column due to that velocity, and an approximation of the anterior forces of the column to each other; these are all the forces and the motions concerned in the matter. The motive power thus stored during the anterior half of the wave is restored in the latter half wave length thus: the column raised to its greatest height presses on both its posterior and anterior surface, on the anterior surface it presses forward the anterior column, tending to sustain its velocity and maintain its height; on the posterior column its pressure tends to oppose the progress and retard the velocity of the fluid in motion, and thus retarding the posterior and accelerating the anterior surface, widens the space between its own bounding planes until it repose once more on the original level.

*The Wave a Vehicle of Power.*—The wave is thus a receptacle of moving power, of the power required to raise a given volume of water from its place in the channel to its place in the wave, and is ready to transmit that power through any distance along that channel with great velocity, and to replace it at the end of its path. In doing this the motion of the water is simple and easily understood, each column is diminished in horizontal dimension and increased proportionally in vertical dimension, and again suffered to regain its original shape by the action of gravity. There is no transference of individual particles through, between and amongst one another, so as to produce collisions, or any other motions which impair moving force; the particles simply glide for the moment over each other into a new arrangement, and retire back to their places. Thus the wave resembles that which we may conceive to pass along an elastic column, each slice of which is squeezed into a thinner slice, and restored by its elastic force to its original bulk, only in the water wave the force which restores the force of each water column is gravity, not elasticity.

To conceive accurately of the forces which operate in wave transmission, and of the *modus operandi*, to understand how the primary moving force acts on the column of fluid in repose, how this force is distributed among the particles, to distinguish the relative and absolute motions of the particles and the nature of the transmission of the form, and to understand how the force operates in at once propagating itself and restoring completely to rest those particles which form the vehicle of its transmission, is a study of much interest to the philosopher. To show how under a given form and outline of wave, in a given time, all and each of the individual particles of water obeying every one its own impulse and that of those around it, and subject to the laws of gravity and of the original impulse, shall describe its own path without interfering with another's, and shall unite in the production of an aggregate motion consistent with the continuity of the mass and with the laws of fluid pressure,—this is a problem which belongs to the mathematician, which has hitherto proved too arduous for the human intellect, and which we have thus endeavoured to facilitate and promote by the study of the absolute forms and phænomena of the waves themselves, and by the determination of the actual paths and motions of the individual particles of water.

*The Negative Wave of the First Order.*—The negative wave is a phænomenon whose place among waves it is somewhat difficult to assign. Its phænomena partake of those of the first order. But in its genesis and propagation it is always attended by a train of following phænomena of the second order.

The genesis of the negative wave of the first order is effected under conditions precisely the reverse of those of the positive wave. A solid body, $Q_2$ Q, (Plate LII. figs. 7, 8), is withdrawn from the water of the reservoir at one extremity, a cavity is created, and this cavity, $W_1$, is propagated along the surface of the water under a defined figure.

The velocity of the negative wave in a shallow channel is nearly that which is due to the depth calculated from the lowest part of the wave (as in the positive from the highest), but in longer waves it is sensibly less than that velocity. In Plate XLVIII. fig. 5 the observations are compared with this formula, from which they exhibit considerable deviations. Table XI. is a collection of negative waves observed in a small rectangular channel, and Table XII. contains others made in a triangular channel, both being made under the same conditions as the positive waves already given.

### Table XI.

*Observations on the Velocity of Negative Waves of the First Order.—In a rectangular channel 12 inches wide.*

Col. A is the depth of the fluid reckoned in inches from the lowest point of the wave.

Col. B is the depth of the wave reckoned below the surface of repose.

Col. C is the number of seconds observed while the wave described the space given in column D in feet.

Col. E is the resulting velocity.



Col. F gives the velocities due to the depth, calculated by the formula $c = \sqrt{g(h-k)}$.

Col. G are the differences between observation and the formula.

| A. | B. | C. | D. | E. | F. | G. |
|---|---|---|---|---|---|---|
| ·915 | —·085 | 9·0 | 14·62 | 1·62 | 1·56 | — ·06 |
| ·925 | —·075 | 9·5 | 14·62 | 1·53 | 1·57 | + ·04 |
| ·93 | —·07 | 16·5 | 21·08 | 1·27 | 1·58 | + ·31 |
| ·935 | —·065 | 12·0 | 20·0 | 1·66′ | 1·58 | — ·08 |
| ·96 | —·04 | 14·5 | 20·0 | 1·38 | 1·60 | + ·22 |
| ·965 | —·035 | 15·0 | 21·08 | 1·40 | 1·60 | + ·20 |
| ·97 | —·03 | 14·0 | 20·5 | 1·46 | 1·61 | + ·15 |
| 1·0 | | | | | 1·63 | |
| 2·0 | | | | | 2·31 | |
| 3·0 | | | | | 2·83 | |
| 3·3 | —·8 | 5·5 | 14·62 | 2·65 | 2·97 | + ·32 |
| 3·4 | —·7 | 6·0 | 14·62 | 2·43 | 3·02 | + ·59 |
| 3·495 | —·605 | 8·0 | 21·08 | 2·63 | 3·08 | + ·45 |
| 3·603 | —·497 | 13·5 | 41·08 | 3·04 | 3·10 | + ·06 |
| 3·71 | —·39 | 6·5 | 20·0 | 3·07 | 3·15 | + ·08 |
| 3·745 | —·355 | 7·0 | 20·0 | 2·85 | 3·16 | + ·31 |
| 3·77 | —·33 | 10·83′ | 33·3′ | 3·07 | 3·18 | + ·11 |
| 4·0 | | | | | 3·27 | |
| 4·365 | —·735 | 4·25 | 14·62 | 3·44 | 4·42 | — ·02 |
| 4·575 | —·525 | 6·0 | 20·0 | 3·33′ | 3·50 | + ·17 |
| 4·6 | —·5 | 6·25 | 21·08 | 3·37 | 3·51 | + ·14 |
| 4·625 | —·475 | 7·5 | 20·0 | 2·66 | 3·52 | + ·86 |
| 4·75 | —·35 | 5·25 | 20·0 | 3·81 | 3·57 | — ·24 |
| 5·0 | | | | | 3·66 | |
| 6·0 | | | | | 4·01 | |
| 7·0 | | | | | 4·33 | |
| | | | | | | +4·01 |
| | | | | | | —0·40 |
| | | | | | | +3·61 |
| | | | | | Mean | +0·19 |

### Table XII.

*Observations on the Velocity of Negative Waves of the First Order.—In a triangular channel with sides sloping at 45°.*

Cols. A, B, C, D, E, F and G, as in the preceding table.

Col. H is the ratio of defective velocity on the whole.

Col. F″ is taken, not from the formula like F, but from observed positive waves in the same channel of the same height.

Col. G″ contains the differences between F″ and E.

| A. | B. | C. | D. | E. | F. | G. | H. | F″. | G″. |
|---|---|---|---|---|---|---|---|---|---|
| 8·7 | −0·7 | 29·8 | 100· | 3·35 | 3·41 | + ·06 | ·0179 | 3·27 | − ·08 |
| 8·8 | −0·6 | 92·4 | 315·5 | 3·41 | 3·43 | + ·02 | ·0058 | 3·31 | − ·10 |
| 8·9 | −0·5 | 62·8 | 215·5 | 3·43 | 4·45 | + ·02 | ·0058 | 3·35 | − ·08 |
| 9·0 | | | | | 3·47 | | | | |
| 10·0 | | | | | 3·66 | | | | |
| 11·0 | | | | | 3·84 | | | | |
| 11·6 | −0·9 | 29·2 | 100· | 3·41 | 3·94 | + ·53 | ·1554 | 3·78 | + ·37 |
| 12·0 | | | | | 4·01 | | | | |
| 13·0 | | | | | 4·17 | | | | |
| 14·0 | | | | | 4·33 | | | | |
| 15·0 | | | | | 4·48 | | | | |
| 16·0 | | | | | 4·63 | | | | |
| 16·8 | −1·7 | 22·2 | 100· | 4·50 | 4·74 | + ·24 | ·0533′ | 4·50 | ·00 |
| 17·0 | −1·5 | 22·0 | 100· | 4·54 | 4·77 | + ·23 | ·0506 | 4·55 | + ·01 |
| 17·4 | −1·1 | 21·6 | 100· | 4·62 | 4·83 | + ·21 | ·0454 | 4·67 | + ·05 |
| 18·0 | −0·5 | 21·6 | 100· | 4·62 | 4·91 | + ·29 | ·0627 | 4·84 | + ·22 |
| 19·0 | | | | | 5·04 | | | | |
| 20·0 | | | | | 5·18 | | | | |
| 21·0 | | | | | 5·30 | | | | |
| 22·0 | | | | | 5·43 | | | | |
| 23·0 | | | | | 5·55 | | | | |
| 24·0 | | | | | 5·67 | | | | |
| 24·5 | −1·5 | 19·0 | 100· | 5·26 | 5·73 | + ·47 | ·0893 | 5·55 | + ·29 |
| 24·7 | −1·3 | 18·9 | 100· | 5·29 | 5·75 | + ·46 | ·0869 | 5·60 | + ·31 |
| 24·8 | −1·2 | 18·6 | 100· | 5·38 | 5·76 | + ·38 | ·0706 | 5·62 | + ·24 |
| 25·0 | −1·0 | 18·6 | 100· | 5·38 | 5·78 | + ·40 | ·0743 | 5·67 | + ·29 |
| | | | | | | +3·31 | ·7180 | | +1·78 |
| | | | | | Mean | +0·275 | ·0598 | Mean | − ·26 |
| | | | | | | | | | +1·52 |
| | | | | | | | | Mean | + ·126 |

The horizontal translation of water particles in the negative wave presents considerable resemblance to the corresponding phænomenon in the positive wave. All the particles of water in a given vertical plane move simultaneously with equal velocities backwards in the opposite direction to the transmission, and repose in their new planes, at the end of the translation; with this modification, however, that this state of repose is much disturbed near the surface by those secondary waves which follow the negative wave, but which do not sensibly agitate the particles considerably removed from the surface. (See Plate LII. fig. 9.) The path is the ellipse of the positive wave inverted.

The following measures may be useful. In a rectangular channel 4 inches deep in repose and 8 inches wide, a volume of 72 cubic inches is withdrawn; the depth of the negative wave below the plane of repose is $\frac{5}{8}$ths of an inch deep, the translation throughout the lower half-depth is $2\frac{1}{4}$ inches, and diminishes from the half-depths upwards, settling finally at the surface at $1\frac{5}{4}$ inch from the original position of the superficial particle.

The form of surface of the anterior half of the negative wave resembles closely the posterior half of a positive wave of equal depth, but the posterior half of the negative wave passes off into the anterior form of a secondary wave which follows it.

After translation the superficial particles continue to oscillate, as shown in Plate LII. figs. 9, 10, in the manner hereafter to be described, as a phænomenon of the train of secondary waves.

The characteristics of this species of wave of the first order are,—

(1.) That it is negative or wholly below the level of repose.



(2.) That it is a wave of translation, the direction of which is opposite to the direction of transmission.

(3.) That its anterior form is that of the positive wave reversed.

(4.) That the path of translation is nearly that of the positive wave reversed.

(5.) That its velocity is, in considerable depths, sensibly less than that due by gravity to half the depth reckoned from the lowest point, or the velocity of a positive wave being the same total height.

(6.) That it is not solitary, but always carries a train of secondary waves.

It is important to notice that the positive and negative waves do not stand to each other in the relation of companion phænomena. They cannot be considered in any case as the positive and negative portions of the same phænomena, for the following reasons:—

(1.) If an attempt be made to generate or propagate them in such manner that the one shall be companion to the other, they will not continue together, but immediately and spontaneously separate.

(2.) If a positive wave be generated in a given channel and a negative wave behind it, the positive wave moving with the greater velocity, rapidly separates itself from the other, leaving it far behind.

(3.) If a positive wave be generated and transmitted behind a negative wave, it will overtake and pass it.

(4.) Waves of the secondary class which consist of companion halves, one part positive and the other negative, have this peculiarity, that the positive and negative parts may be transmitted across and over each other without preventing in any way their permanence or their continued propagation. It is not so with the positive and negative waves of the first order.

(5.) If a positive and negative wave of equal volume meet in opposite directions, they neutralize each other and both cease to exist.

(6.) If a positive wave overtake a negative wave of equal volume, they also neutralize each other and cease to exist.

(7.) If either be larger, the remainder is propagated as a wave of the larger class.

(8.) Thus it is nowhere to be observed that the positive and negative wave coexist as companion phænomena.

These observations are of importance for this reason, that it has been supposed by a distinguished philosopher that the positive and the negative wave might be corresponding halves of some given or supposed wave.

*On some Conditions which affect the Phænomena of the Wave of the First Order.*—It has not appeared in any observations I have been able to make on the subject, that the wave of the first order retains the stamp of the many peculiarities that may be conceived to affect its origin. In this respect it is apparently different from the waves of sound or of colour, which bear to the ear and the eye distinct indications of many peculiarities of their original exciting cause, and thus enable us to judge of the character of the distant cause which emitted the sound or sent forth the coloured ray. It is not possible always to form an accurate judgement from the phænomena of the wave of the first order, of the nature of the disturbing cause, except in peculiar and small number of cases.

I have not found that waves generated by impulse by a fluid column of given and very various dimension, by immersion of a solid body of given figure, by motion in given velocity or in different directions; I have not found in the wave obtained by any of the many means any peculiarity, any variation either of form or velocity, indicating the peculiarity of the original. In one respect therefore the wave of translation resembles the sound wave; that all waves travel with the velocity due to half the depth, whatever be the nature of their source.

In one respect alone does the origin of the wave affect its history. Its volume depends on the quantity of power employed in its genesis, and on the distance through which it has travelled. A great and a little wave at equal distances from the source of disturbance, arise from great or little causes, but it is impossible to distinguish between a small wave which has travelled a short distance, and one which, originally high, has traversed a long space.

This however does not apply to compound waves of the first order, hereafter to be examined.

*Form of Channel.—Its Effect on the Wave of Translation.*—The conditions which affect the phænomena of the wave of translation are therefore to be looked for in its actual circumstances at the time of observation rather than in its history. The form and magnitude of the channel are among the most important of these circumstances. Thus a change in depth of channel immediately becomes indicated to the eye of the observer by the retardation of the wave, which begins to move with the same velocity as if the channel were everywhere of the diminished depth, that is, with the velocity due to the depth. Thus in a rectangular channel $4\frac{1}{2}$ feet deep, the wave moves with a velocity of 12 feet per second, and if the channel become shallower, so as to have only 2 feet depth, the change of depth is indicated by the velocity of the wave, which is observed now to move only with the velocity of 8 feet per second; but if the channel again change and become 8 feet deep, the wave indicates the change by suddenly changing to a velocity of 16 feet per second.

*Length of Wave an Index of Depth.*—In like manner, a wave which in water 4 feet deep is about 8 yards long, shortens on coming to a depth of 2 feet to a length of 4 yards, and extends itself to 16 yards long on getting into a depth of 8 feet. This extension of length is attended with a diminution of height, and the diminution of length with an increase of height of the wave, so that the change of length and height attend and indicate changes of depth.

In a rectangular channel whose depth gradually slopes until it becomes nothing, like the beach of a sea, these phænomena are very distinctly visible: the wave is first retarded by the diminution of depth, shortens and increases in height, and finally breaks when its height approaches to equality with the depth of the water. The limit of height of a wave of the first order is therefore a height above the bottom of the channel equal to double the depth of the water in repose. If we reckon the velocity of transmission as that due to half the total depth, and the velocity of translation as that due to the height of the wave, it is manifest that when the height is equal to the depth these two are equal, but that if the height were greater than this, the velocity of individual particles at the crest of the wave would exceed the velocity of the wave form; here accordingly the wave ceases, the particles in the ridge of the wave pass forward out of the wave, fall over, and the wave becomes a surge or broken foam, a disintegrated heap of water particles, having lost all continuity.

In like manner does the gradual narrowing of the channel affect the form and velocity of the wave, but its effects are by no means so striking as where the depth is diminished. The narrowing of the channel increases the height of the wave, and the effect of this is most apparent when the height is considerable in proportion to the depth; the velocity of the wave increases in proportion as the increase of height of the wave increases the total depth; but with this increase of depth, the length of the wave also increases rapidly, and it does not break so early as in the case of the shallowing of the water. Its phænomena are only visibly affected to the extent in which a change of depth is produced in the channel, by the volume of water added to the channel taking the velocity and form peculiar to that increased depth.



TABLE XIII.

*Observed Heights of a Wave in Channel of variable Breadth.—Depth 4 inches.*

| | A. Breadth 12 in. Height of wave. | B. Breadth 6 in. Height of wave. | C. Breadth 3 in. Height of wave. |
|---|---|---|---|
| | in. | in. | in. |
| I. | 2·0 | 2·4 | 3·3 |
| II. | 2·0 | 2·4 | 3·6 |
| III. | 2·0 | 2·55 | 3·3 |
| IV. | 1·5 | 2·5 | 3·5 |
| V. | 1·5 | 2·35 | 3·25 |
| VI. | 1·25 | 2·0 | 2·5 |
| VII. | 1·0 | 1·3 | 2·0 |
| VIII. | 0·25 | 0·3 | 0·4 |

These numbers appear to indicate that the increase of height does not widely differ from the hypothesis, that the height of a given wave in a channel of variable width is inversely as the square root of the breadth.

Thus, the inverse square roots of the breadths are as 1·73, 2·45 and 3·47, and the mean heights of the first five experiments are 1·8 2·45 3·39.

In the first five experiments the velocity observed was 4·25 feet per second. The velocity due by gravity to half the total depth $4+2·45$ inches is 4·15 feet per second; and as the range of the wave was only 17 feet, and the time was only observed to half-seconds, these numbers coincide well enough to bear the conclusion that the velocity does not considerably differ from that due to the wave of the same mean height in a parallel channel of the same depth.

TABLE XIV.

*Observations in a Channel of variable Depth.—Diminution of depth from 4 inches to 0· in a length of 17 feet.*

| | A. Height of wave in a depth of 4 in. | B. Height of wave breaking in depth (C). | C. Depth of water where wave (B) broke. | D. Time of traversing 17 ft. | E. Velocity in feet per sec. |
|---|---|---|---|---|---|
| | in. | in. | in. | s. | |
| I. | 4·0* | 4·0* | 4·0 | 5·5 | 3·09 |
| II. | 3·7* | 3·7* | 3·7 | 5·5 | 3·09 |
| III. | 3·4* | 3·4* | 3·4 | 5·5 | 3·09 |
| IV. | 2·5 | 2·7 | 2·7 | 5·5 | 3·09 |
| V. | 2·0 | 2·4 | 2·4 | 5·5 | 3·09 |
| VI. | 1·8 | 2·2 | 2·2 | 5·5 | 3·09 |
| VII. | 1·5 | 2·0 | 2·1 | | |
| VIII. | 1·3 | 1·9 | 1·9 | | |
| IX. | 1·25 | 1·9 | 1·9 | | |
| X. | 1·2 | 1·7 | 1·7 | | |
| XI. | 1·1 | 1·4 | 1·4 | 6·0 | 2·83 |
| XII. | 1·0 | 1·2 | 1·2 | | |
| XIII. | 0·8 | 0·8 | 1·1 | 6·5 | 2·6 |
| XIV. | 0·5 | 0·7 | 0·9 | 7·0 | 2·4 |
| XV. | 0·2* | 0·2* | 0·2 | 7·5 | 2·0 |

Hence we find that the numbers representing depths in column C may be regarded as the limits of those in column B, that the depth of the fluid below the level of repose is equal to the greatest height which a wave can attain at that point, and at that height the wave breaks.

* These numbers are interpolated; the numbers in column D are waves not observed on the identical waves in the first three columns, but are others of nearly equal heights, in identical conditions.

The time occupied by the largest class of wave is 5·5 seconds, and the corresponding *mean* velocity is 3·09 feet per second; this is the velocity due to a depth of 3·6 inches, but the depth total at the one end of the channel is nearly double this quantity, diminishing to 0 at the end. The time in which the wave in a shelving channel passes along the whole length, is therefore nearly equal to the time in which a wave would travel the same distance if the channel were uniformly of a depth equal to the mean depth of the channel, reckoning in both cases from the top of the wave. In these cases the height of the wave is large. Let us take a small height of wave as Ex. XIV.; there we have also in this case the mean depth reckoned from the top of the wave =2·2, the velocity in a channel of that uniform depth =2·4, and the time 7s·08. These experiments are sufficiently accurately represented if we take for the velocity of the wave in the sloping channel that of a wave in a channel having a uniform depth equal to the mean depth of the channel, reckoned as usual from the top of the wave.

If therefore we are to calculate the time in which a wave will traverse a given distance $q$, to the limit of the standing water-line, after it has begun to break on a sloping beach, we have, the height at breaking being $h$ = the standing depth of the water at the breaking-point,

$$t=\frac{q}{\sqrt{g(h+h)}} \text{ and } v=\sqrt{g(h+h)}.$$

Ex. A wave 3 feet high breaking in water 3 feet deep, on a sloping shore at a distance of 60 feet from the edge of the water, would traverse that space in about 6 seconds, for

$$t=\frac{60}{32·3\sqrt{}}=\frac{60}{9·82}=6 \text{ seconds nearly.}$$

By repeated observations I have ascertained that waves break whenever their height above the level of repose becomes equal very nearly to the depth of the water.

The gradual retardation of the velocity of waves breaking on a sloping beach, as they come into shallower water, is rendered manifest in the closer approximation of the waves to each other as they come near the margin of the water. Vide *et seq.*

It may be observed also that the height of the wave does increase, but very slowly (before breaking), as the depth diminishes; thus in VII., a height of 1·8 in a depth of 4 inches becomes 2·2 in 2 inches depth, and in XII. a height of 1 inch in a depth of 4 inches becomes a depth of 1·2 inch only 1·2 inch high. The increase of height is therefore very much slower than the inverse ratio of the depth, or than the inverse ratio of the square of the depth.

*Form of Transverse Section of Channel.*—We have seen that in a given rectangular channel, the volume of the wave, its height and the depth being given, no peculiarity of origin or other condition sensibly affects its actual phænomena. But it becomes of importance to know whether the form of a given channel, its volume being given, will affect the phænomena of the wave of the first order; for example, whether in a channel which is semicircular on the bottom, or triangular, but holding a given quantity of water, the wave would be affected by the form of the channel, the volume or cross section remaining unchanged.

Considering this question *à priori*, we might form various anticipations. We might expect in a channel in which the depth of transverse section varies, that as its depth is greatest at one point, suppose the middle, and less at the sides, the wave might move with the velocity due to the middle or greatest depth; or we might expect that it would move with the velocity simply due to the mean depth, that is, with the same velocity as in a rectangular channel of a depth equal to the mean depth of the channel; or we might expect that each portion of the wave would move with a velocity due to the depth of that part of the channel immediately below each part of the wave, and so each part passing forward with a velocity of its own, have a series of waves, each propagating itself with an independent velocity, and speedily becoming diffused, and so a continued propagation of a wave in such circumstances would become *impossible* from disintegration; and instead of a single large wave we should have a great many little ones. Or, finally, we might have a perfect wave moving with a velocity, the mean of the velocities which each of these elementary waves might be supposed to possess.



I soon found that the propagation of a single wave, *i. e.* one of which all the parts should have a given common velocity, was *possible* in a channel whose depth at different breadths is variable; that the wave does not necessarily become disintegrated; that its parts do not move with the different velocities due to the different depths of the different parts of the channel, but that the entire wave does (with certain limits) move with such velocity as if propagated in a channel of a rectangular form, but of a less depth than the greatest depth of the channel of variable channel.

It became necessary therefore to determine the depth of a rectangular channel equivalent to the depth of a channel of variable transverse section; to determine, for example, in a channel of triangular section ▽, the depth of rectangular channel in which a wave would be propagated with equal velocity. In this case the simple arithmetical mean depth of the channel is *half of the depth in the middle.* But on the other hand, if we calculate the velocity due to each point of variable depth, and take the mean of these velocities, we shall find a mean velocity such as would be due to a wave in a rectangular channel *two-thirds of the greatest depth.*

In the first series of experiments I made on this subject, I conceived that the results coincided sufficiently well with the latter supposition; but they were on so small a scale, that the errors of observation exceeded in amount the differences between the quantities to be determined, and the results did not establish either. Mr. Kelland arrived at the opposite conclusion, his theoretical investigations indicating the former result. I examined the matter afresh, and after an extensive series of experiments, have established beyond all question the fact, that the velocity in a triangular channel is that due by gravity to one-fourth of the maximum depth. Although therefore the absolute velocity assigned by Mr. Kelland's investigations deviates widely from the true velocity, yet he has assigned the true relation between the velocities in the triangular and the rectangular channel; and if therefore we take the absolute velocity which I have determined for the rectangular channel, and deduce from it the relative velocity which Mr. Kelland has assigned to the triangular form, we obtain a number which is the true velocity of the wave in a ▽r channel.

TABLE XV.

*Observations on the Wave of the First Order in triangular Channels.*

The sides of the channels are planes, and slope at an angle with the horizon = 45°.

Col. A is the observed depth of the channel in the middle, reckoned from the crest of the wave.

Col. B is the height of the wave taken as the mean between the observations at the beginning and end of the experiment.

Col. C is the observed time in seconds occupied by the wave in describing the distance in column D.

Col. D is the space in feet described by the wave during each observation.

Col. E is the velocity resulting from these observations.

Col. F is the velocity due by gravity to $\frac{1}{4}$ of the depth of the fluid, $v=\sqrt{\frac{1}{2}g\,(h+k)}$.

Col. G is the velocity due by gravity to $\frac{2}{3}$ of the depth of the fluid, $v=\sqrt{\frac{2}{3}g\,(h+k)}$.

Cols. H and K show the difference between Cols. F and G and the observations, and the result in favour of F.

| A. | B. | C. | D. | E. | F. | G. | H. | K. |
|---|---|---|---|---|---|---|---|---|
| in. | in. | | | | | | | |
| 4·15 | 0·15 | 36·5 | 80·0 | 2·19 | 2·35 | 2·72 | + ·16 | + ·53 |
| 4·23 | 0·22 | 33·0 | 80·0 | 2·42 | 2·38 | 2·75 | − ·04 | + ·33 |
| 4·32 | 0·31 | 31·0 | 75·5 | 2·43 | 2·40 | 2·78 | − ·03 | + ·35 |
| 4·38 | 0·37 | 47·0 | 115·5 | 2·46 | 2·42 | 2·79 | − ·04 | + ·33 |
| 4·71 | 0·70 | 13·5 | 35·5 | 2·62 | 2·51 | 2·90 | − ·11 | + ·28 |
| 4·81 | 0·80 | 29·5 | 75·5 | 2·57 | 2·54 | 2·93 | − ·03 | + ·36 |
| 4·86 | 0·85 | 14·0 | 35·5 | 2·53 | 2·55 | 2·95 | + ·02 | + ·42 |
| 5·29 | 0·18 | 31·0 | 80·0 | 2·58 | 2·66 | 3·07 | + ·08 | + ·49 |
| 5·44 | 0·33 | 45·5 | 120·0 | 2·63 | 2·70 | 3·11 | + ·07 | + ·48 |
| 5·55 | 0·44 | 58·0 | 160·0 | 2·75 | 2·72 | 3·15 | − ·03 | + ·40 |
| 5·59 | 0·48 | 30·0 | 80·0 | 2·66 | 2·73 | 3·16 | + ·07 | + ·50 |
| 5·99 | 0·88 | 12·0 | 35·5 | 2·95 | 2·83 | 3·27 | − ·12 | + ·32 |
| 6·01 | 0·90 | 24·5 | 71·0 | 2·89 | 2·84 | 3·29 | − ·05 | + ·40 |
| 6·18 | 0·14 | 28·0 | 80·0 | 2·85 | 2·87 | 3·32 | + ·02 | + ·47 |
| 6·26 | 0·21 | 55·5 | 160·0 | 2·88 | 2·89 | 3·34 | + ·01 | + ·46 |
| 6·38 | 0·34 | 14·0 | 40·0 | 2·85 | 2·92 | 3·37 | + ·07 | + ·52 |
| 6·44 | 1·33 | 12·0 | 35·5 | 2·95 | 2·93 | 3·39 | − ·02 | + ·44 |
| 6·52 | 0·48 | 26·5 | 80·0 | 3·02 | 2·95 | 3·41 | − ·07 | + ·39 |
| 6·78 | 0·74 | 35·0 | 111·0 | 3·17 | 3·01 | 3·48 | − ·16 | + ·31 |
| 7·10 | 0·60 | 26·5 | 80·0 | 3·02 | 3·08 | 3·56 | + ·06 | + ·54 |
| 7·12 | 0·08 | 39·5 | 120·0 | 3·03 | 3·09 | 3·56 | + ·06 | + ·53 |
| 7·15 | 0·11 | 78·5 | 240·0 | 3·05 | 3·09 | 3·57 | + ·04 | + ·52 |
| 7·16 | 0·12 | 52·5 | 160·0 | 3·04 | 3·10 | 3·58 | + ·06 | + ·54 |
| 7·21 | 0·17 | 26·5 | 80·0 | 3·02 | 3·11 | 3·59 | + ·09 | + ·57 |
| 7·36 | 0·32 | 26·5 | 80·0 | 3·02 | 3·14 | 3·62 | + ·12 | + ·60 |
| 7·51 | 0·47 | 25·0 | 80·0 | 3·20 | 3·18 | 3·66 | − ·02 | + ·46 |
| 7·53 | 0·47 | 24·0 | 80·0 | 3·33 | 3·17 | 3·67 | − ·16 | + ·34 |
| 10·0 | 0·75 | 55·4 | 215·5 | 3·89 | 3·66 | 4·23 | − ·23 | + ·34 |
| 10·5 | 1·1 | 41·94 | 166·0 | 3·95 | 3·75 | 4·33 | − ·20 | + ·38 |
| 11·0 | 1·44 | 31·2 | 123·1 | 3·94 | 3·84 | 4·43 | − ·10 | + ·49 |
| 14·5 | 2·0 | 48·36 | 215·5 | 4·45 | 4·41 | 5·09 | − ·04 | + ·64 |
| 15·0 | 2·58 | 26·46 | 119·25 | 4·50 | 4·48 | 5·18 | − ·02 | + ·68 |
| 15·5 | 3·1 | 22·2 | 100·0 | 4·50 | 4·56 | 5·26 | + ·06 | + ·76 |
| 19·0 | 0·35 | 19·8 | 100·0 | 5·06 | 5·04 | 5·83 | − ·02 | + ·77 |
| 19·5 | 0·87 | 19·5 | 100·0 | 5·13 | 5·11 | 5·90 | − ·02 | + ·77 |
| 20·0 | 1·35 | 25·66 | 138·5 | 5·40 | 5·18 | 5·98 | − ·22 | + ·58 |
| 20·5 | 1·85 | 28·8 | 157·75 | 5·48 | 5·24 | 6·05 | − ·24 | + ·57 |
| 21·0 | 2·36 | 24·93 | 138·5 | 5·55 | 5·30 | 6·13 | − ·25 | + ·58 |
| 21·5 | 2·8 | 17·8 | 100·0 | 5·61 | 5·36 | 6·20 | − ·25 | + ·59 |
| 26·0 | 1·5 | 35·8 | 215·5 | 6·02 | 5·90 | 6·82 | − ·12 | + ·80 |
| 26·5 | 1·95 | 22·46 | 138·5 | 6·16 | 5·96 | 6·88 | − ·20 | + ·72 |
| 27·0 | 2·12 | 20·7 | 128·87 | 6·22 | 6·01 | 6·95 | − ·21 | + ·73 |
| 27·5 | 2·4 | 21·73 | 138·5 | 6·37 | 6·07 | 7·01 | − ·30 | + ·64 |
| 28·0 | 3·12 | 20·45 | 128·75 | 6·29 | 6·13 | 7·07 | − ·16 | + ·78 |
| 28·5 | 3·03 | 15·93 | 100·0 | 6·27 | 6·18 | 7·14 | − ·09 | + ·87 |
| 29·0 | 3·02 | 15·8 | 100·0 | 6·33 | 6·23 | 7·20 | − ·10 | + ·87 |
| 29·5 | 2·5 | 15·68 | 100·0 | 6·37 | 6·29 | 7·26 | − ·08 | + ·89 |
| 30·0 | 2·77 | 15·6 | 100·0 | 6·41 | 6·34 | 7·32 | − ·07 | + ·91 |
| 30·5 | 2·25 | 15·6 | 100·0 | 6·41 | 6·39 | 7·38 | − ·02 | + ·97 |
| 31·0 | 2·5 | 15·8 | 100·0 | 6·33 | 6·44 | 7·44 | + ·11 | + 1·11 |
| 31·5 | 3·0 | 15·26 | 100·0 | 6·55 | 6·50 | 7·50 | − ·05 | + ·95 |
| | | | | | | | −2·77 | +29·27 |



No great number of experiments has been made on channels of other forms of variable depth, such as have been made coinciding with those in the triangular channel, so far as to show that we may take the simple arithmetical mean depth as the depth of the rectangular channel of a wave of equal velocity, and so in general reckon the mean depth as

$$h=\frac{1}{x}\int y\,dx,$$

or

$$v=\left(\frac{g}{x}\int y\,dx\right)^{\frac{1}{2}}.$$

The form of transverse section does not therefore affect the velocity of the wave otherwise than as it becomes necessary to use the mean depth as the argument in calculating it, and not the maximum depth.

*The Form of Channel affects the Form of the Wave as well as its Velocity.*—When the channel is very broad the wave ceases to have a velocity, it loses unity of character, and each part of it moves along the channel independent of the velocity of the other, and with the velocity due to the local depth of the channel. Where the water is shallow the wave becomes sensibly higher and shorter, and when the difference of depth is not considerable, the wave is found to increase in height so as to give in the shallow part a velocity equal to that in the narrow part. When the channel is narrow in proportion to its depth, this unity of propagation exists without sensible difference of velocity towards the side, and without very great difference in height at the sides. In a channel of the form of a right-angled and isosceles triangle, with the hypothenuse upwards and horizontal, it is visible to the eye that the wave is somewhat longer and lower in the middle, but higher and shorter at the sides, but that it retains most perfect unity of form and velocity, and moves along unbroken with the velocity due to the mean depth. The same figure with the angle at the bottom increased so that each side has a slope of one in four, still contains a single wave propagated with a single velocity, being that due to half the depth, but breaks at the shallow side, becoming disentegrated in form though not in velocity.

In a channel 12 inches wide, 5 inches deep on one side, and 1 inch deep on the other, the following observations were made:—

*Height of the Wave.*

| Deep side. | Shallow side. |
|---|---|
| in. | in. |
| 2·00 | 2·50 |
| 1·50 | 2·50 |
| 1·20 | 2·00 |
| 0·75 | 1·20 |
| 0·75 | 1·20 |
| 0·75 | 1·00 |
| 0·50 | 1·00 |
| 0·25 | 0·50 |
| 0·25 | 0·40 |
| 0·25 | 0·40 |

*On the Incidence and Reflexion of the Wave of the First Order.*—When a wave of the first order encounters a solid plane at right angles to the direction of its propagation, it is wholly reflected and is thrown back in the opposite direction with a velocity equal to that in which it was moving before impact, remaining in every respect unchanged, excepting in direction of motion. This process may be repeated any number of times without affecting any of the wave phænomena excepting the direction of motion.

When the angle which the ridge of the incident wave makes with the solid plane is small, that is, when the direction of propagation does not deviate much from the perpendicular to the plane, the wave undergoes total reflexion, and the angles of reflexion and of incidence are equal, as in the case of light.

When the deviation of the direction of propagation from the perpendicular is considerable, the reflexion ceases to be total. At 45° the reflected wave is sensibly less than the incident wave.

When the ridge of the wave is incident at about 60° from the plane surface, and the direction of the ridge only diverges about 30° from a perpendicular to the plane, reflexion ceases to be possible. A remarkable phænomenon is exhibited which I may be allowed to designate the *Lateral Accumulation* and *Non-Reflexion* of the wave. It is to be understood by considering the effect of supposed reflexion; this would be to double over upon itself a part of the wave moving in nearly the same direction; the motions of translation of the particles being compounded will give a resultant at right angles to the plane, and will also give a wave of greater magnitude and a translation of greater velocity. By these means accumulation of volume and advancement of the ridge in the vicinity of the obstacle take place; as represented in the diagram.

These phænomena are accurately represented in Plate LIII., as observed in a large shallow reservoir of water.

*On the Lateral Diffusion and the Lateral Accumulation of the Wave of the First Order.*—When a wave of the first order has been generated in a narrow channel, and is propagated into a wider one, it becomes of some importance to know whether and how this wave will affect the surface of the larger basin into which it is admitted. It is known that common surface waves of the second order diffuse themselves equably in concentric circles round the point of disturbance. How is the great primary wave diffused?

## Table XVI.

*Observations on the Lateral Diffusion of the Wave of the First Order, generated in a narrow Channel and transmitted into a wide Reservoir.*

The apparatus employed for this purpose is exhibited in Plate LIV. figs. 1 and 2. T was a tank 20 feet square, filled to the depth of 4 inches; the chamber C, fig. 2, was 12 inches square, in which the wave was generated by impulse for the first five experiments, in all subsequent to which C was enlarged in width to 2 feet, as shown in fig. 1. The line marked A, figs. 1 and 2, was a wooden bar, in which were inserted at intervals of 6 inches, sharp pieces of pencil, projecting downwards to the surface of the water; the numbers of which, reckoning from the side of the tank outwards, are contained in the first vertical column of numerals, the Roman numerals in this table denoting the number of the experiment. The bar being placed parallel to the side of the tank at C, and distant from it 12 feet, consequently distant 9 feet from the mouth of the channel, whose length is 3 feet; the distance from its under edge to the surface of the still water was carefully measured, and when the wave had passed, and before its reflexion, the bar was removed, the distances from its under edge to the highest marks on the pencils were put down in column A of the table, and the absolute height of the wave itself, obtained by subtracting these figures from the statical level, was put down in column B.



In the diagrams, Plate LIV., the waves are laid down from the line A A, and at horizontal intervals of one-tenth of an inch, corresponding to the relative positions of the points at which they were observed. In figs. 1 and 2, an approximate mean is given of the waves generated in the large and small channels, each line at the bar A indicating a height of one-tenth part of an inch.

| | I. | | II. | | III. | | IV. | | V. | | VI. | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | A. | B. | A. | B. | A. | B. | A. | B. | A. | B. | A. | B. |
| 1 | 2·3 | ·375 | 2·15 | ·525 | 2·05 | ·625 | 2·1 | ·575 | ·8 | ·7 | ·3 | 1·2 |
| 2 | 2·4 | ·275 | 2·175 | ·5 | | | 2·15 | ·525 | ·85 | ·65 | ·4 | 1·1 |
| 3 | 2·45 | ·225 | 2·225 | ·45 | 2·15 | ·525 | 2·2 | ·475 | ·95 | ·575 | ·5 | 1·0 |
| 4 | 2·45 | ·225 | 2·2 | ·475 | 2·15 | ·525 | 2·15 | ·525 | ·8 | ·725 | ·4 | 1·225 |
| 5 | 2·45 | ·225 | 2·225 | ·45 | 2·15 | ·525 | 2·15 | ·525 | ·85 | ·675 | ·4 | 1·225 |
| 6 | 2·45 | ·225 | 2·3 | ·375 | 2·2 | ·475 | 2·2 | ·475 | ·925 | ·612 | ·45 | 1·075 |
| 7 | 2·475 | ·2 | 2·275 | ·4 | 2·2 | ·475 | 2·25 | ·425 | ·925 | ·625 | ·5 | 1·035 |
| 8 | 2·5 | ·175 | 2·35 | ·325 | 2·25 | ·425 | 2·275 | ·4 | 1·25 | ·5 | ·625 | ·925 |
| 9 | 2·55 | ·125 | 2·4 | ·275 | 2·3 | ·375 | 2·35 | ·325 | 1·05 | ·5 | ·625 | ·925 |
| 10 | 2·55 | ·125 | 2·35 | ·325 | 2·3 | ·375 | 2·3 | ·375 | 1·0 | ·55 | ·65 | ·9 |
| 11 | 2·6 | ·075 | 2·4 | ·275 | 2·3 | ·375 | 2·325 | ·35 | 1·1 | ·45 | ·7 | ·85 |
| 12 | 2·6 | ·075 | 2·425 | ·25 | 2·35 | ·325 | 2·375 | ·3 | 1·05 | ·512 | ·825 | ·75 |
| 13 | 2·6 | ·075 | 2·425 | ·25 | 2·35 | ·325 | 2·35 | ·325 | 1·075 | ·487 | ·8 | ·775 |
| 14 | 2·6 | ·075 | 2·45 | ·225 | 2·375 | ·3 | 2·4 | ·275 | 1·1 | ·487 | ·8 | ·775 |
| 15 | 2·65 | ·025 | 2·45 | ·225 | 2·375 | ·3 | 2·4 | ·275 | 1·1 | ·475 | ·9 | ·675 |
| 16 | 2·65 | ·025 | 2·45 | ·225 | 2·4 | ·275 | 2·4 | ·275 | 1·1 | ·475 | ·85 | ·735 |
| 17 | 2·65 | ·025 | 2·45 | ·225 | 2·4 | ·275 | 2·4 | ·275 | | | ·85 | ·75 |

This table shows in column B, how the height of the wave diminishes as it spreads out from the line of original direction in which it was generated. Lateral diffusion therefore takes place, but with a great diminution of height of the wave.

This phænomenon is of importance in reference especially to the law of diffusion of the tides, in such situations as where they enter the German Sea through the English Channel, and the Irish Sea through St. George's Channel. It enables us to account for the great inequality of tides in the same locality. It likewise furnishes an analogy by which we may explain some of the hitherto anomalous phænomena of sound.

*Axis of Maximum Displacement of the Wave of the First Order.*—That a wave of the first order, on entering a large sheet of water, does not diffuse itself equally in all directions around the place of disturbance (as do the waves of the second order produced by a stone dropped in a placid lake), but that there is in one direction *an axis* along which it maintains the greatest height, has the widest range of translation, and travels with greatest velocity, viz. in the direction of the original propagation as it emerged from the generating reservoir, is a phænomenon which I have further confirmed by a number of experiments. This phænomenon is of importance, especially if we take the wave of the first order, the same (as I think I have established) as type of the tide wave of the sea and of the sound wave of the atmosphere. I determined this in the simplest way. I filled a reservoir which has a smooth flat bottom and perpendicular sides some 20 feet square, to a depth of 4 inches with water. In a small generating reservoir only a foot wide, I generated a wave of the first order. A circle was drawn on the bottom of the large basin, and of course visible through the water, having its centre at the place of disturbance, and divided into arcs of 30°, 45°, 60° and 90°, on which observers were placed, and the heights of the same wave, as observed at the points, is given in the accompanying table.

### Table XVII.

*Observations on the Diffusion of the Wave of the First Order round an Axis of original Transmission.*

The observations were made upon the wave at various points in circles of 9 and 15 feet radius, described from the outer extremity of the side of the channel C, as shown in Plate LIV. fig. 3. The depth of the water when at rest was taken at the various points, and these being subtracted from the absolute height to which the wave attained in its transit, gave the amounts which are contained in the lower part of the table, the absolute heights from which these are deduced being given immediately above in columns marked thus, A, B, C, D, E, while the deducted heights are distinguished thus, A′, B′, C′, D′, E′. Experiments VII. to XV. were made in the 9 feet circle, and the remainder in that of 15 feet radius. It will be observed that in the latter set there are two columns which are headed zèro, but it must be remembered that the one in brackets contains observations which were made at the 9 feet distance along the axis and the remainder on the outer circle.

Fig. 3 contains the approximate ratio of the height of the wave at different points in the circumference of the circles expressed by lines concentric to the circles, each of which denotes the tenth part of an inch.

The observations are laid down accurately in the diagrams, where the lines A B and C D represent the circumference of the quadrants of the observed circles. Upon these lines the true heights of the wave are measured upwards at their respective points of observation, and a curve drawn through these, representing the mean of the wave's height. From these and from a numerical discussion of the observations, it appears that the height of the wave at 0° being 1, its height at the remaining points will be $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$, and $\frac{1}{10}$, or taking integral numbers to express the ratio, it will stand thus, 30, 15, 12, 10, 3. And from a discussion of the whole of the experiments it is found that the height of the wave is inversely as the distance from the centre.

Fig. 4 shows the appearance of the wave upon which these observations were made.

| | A. | B. | C. | D. | E. | | G. | H. | K. | L. | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0°. | 30°. | 45°. | 60°. | 90°. | | (0°.) | 0°. | 30°. | 60°. | 90°. |
| VII. | 4·5 | | 4·15 | | 4·05 | XVI. | | 4·25 | 4·3 | | 4·4 |
| VIII. | 4·625 | | 4·35 | | 4·05 | XVII. | | 4·125 | 4·5 | | |
| IX. | 4·875 | | 4·4 | | 4·05 | XVIII. | | 4·25 | 4·5 | | |
| X. | 4·5 | 4·35 | | | 4·05 | XIX. | | 4·25 | 4·4 | | |
| XI. | 4·325 | 4·3 | | | 4·05 | XX. | | 4·25 | 4·3 | | |
| XII. | 4·5 | 4·3 | | | 4·05 | XXI. | | 4·25 | | 4·3 | |
| XIII. | 4·5 | 4·2 | | | | XXII. | | 4·375 | | 4·3 | |
| XIV. | 4·75 | | | 4·3 | 4·05 | XXIII. | | 4·1 | | 4·3 | |
| XV. | 4·5 | | | 4·25 | 4·05 | XXIV. | | 4·1 | | 4·25 | |
| | A′. | B′. | C′ | D′. | E′. | | G′. | H′. | K′. | L′. | M′. |
| VII. | 1·0 | | ·3 | | ·1 | XVI. | | ·75 | ·3 | | ·1 |
| VIII. | 1·125 | | ·5 | | ·1 | XVII. | ·7 | ·625 | ·5 | | |
| IX. | 1·375 | | ·55 | | ·1 | XVIII. | | ·75 | ·5 | | |
| X. | 1·0 | ·55 | | | ·1 | XIX. | | ·75 | ·4 | | |
| XI. | ·825 | ·5 | | | ·1 | XX. | 1·3 | ·75 | ·3 | | |
| XII. | 1·0 | ·5 | | | ·1 | XXI. | 1·5 | ·75 | | ·25 | |
| XIII. | 1·0 | ·4 | | | | XXII. | | ·875 | | ·25 | |
| XIV. | 1·25 | | | ·4 | ·1 | XXIII. | 1·1 | ·6 | | ·25 | |
| XV. | 1·0 | | | ·35 | ·1 | XXIV. | 1·15 | ·6 | | ·2 | |



Thus it was determined that along the axis of maximum intensity, the height of the wave there being the greatest, there was a corresponding acceleration of the wave motion. On each side of this axis the magnitude of the wave diminishes rapidly, being at 30° diminished to $\frac{1}{2}$, and at 60° to $\frac{1}{3}$ of its height along the axis, and as this diminution was attended with a corresponding retardation of propagation, so the ridge of the wave became somewhat elliptical, having for its major axis the axis of maximum intensity of the wave. At right angles to the principal axis of propagation the wave is scarcely sensible, a height of one-tenth part of that in the axis being the greatest that was observed; and that indeed was, in the circumstances of observation, scarcely sensible.

*Concluding Remarks and Application.*—There are several great applications of our knowledge of waves of the first order, which give value to that knowledge beyond that which belongs to truth for its own sake. The phænomena of the wave of translation are so beautiful and regular, that as a study of nature it possesses a high interest. The velocity of the wave is one of the great constants of nature, and is to the phænomena of fluids what the pendulum is to solids, a connecting link between time and force; as a phænomenon of hydrodynamics, it furnishes one of the most elegant and interesting exercises in the calculus of the wave mathematics.

But besides its importance in these aspects, there are others in which it is capable of being regarded, each of which gives it value both in art and in science:—

1. The wave of the first order is to be regarded as a vehicle for the transmission of mechanical force (geological application).
2. The wave of the first order is an important element in the calculation and phænomena of resistance of fluids (form of ships, canals, &c.).
3. The wave of the first order is identical with the great oceanic wave of the tide (improvement of tidal rivers).
4. The water-wave of the first order presents some analogy to the sound wave of the atmosphere (phænomena of acoustics).

### Table XVIII.

*The Velocity of the Wave of the First Order*, calculated for various depths of the fluid in a channel of uniform depth, extending a depth from 0·1 of an inch to 100 feet.

Column A contains the depths of the fluid in decimal parts of an inch.
Column B the corresponding velocities in feet per second.
Column C gives the depth in inches.
Column D the corresponding velocities in feet per second.
Column F gives the depths in feet.
Column G the corresponding velocities in feet per second.
Columns of Differences, E and H, will assist in extending the table.

| A. Value of $h+c$ in inches. | B. Value of $\sqrt{g(c+h)}$ in feet per sec. | C. Value of $h+c$ in inches. | D. Value of $\sqrt{g(c+h)}$ in feet per sec. | E. First difference. | F. Value of $h+c$ in feet. | G. Value of $\sqrt{g(c+h)}$ in feet per sec. | H. First difference. |
|---|---|---|---|---|---|---|---|
| 0·0 | 0·0000 | 0·0 | 0·000 | | 0·0 | 0·000 | |
| ·1 | 0·5179 | 1·0 | 1·637 | | 1·0 | 5·674 | |
| ·2 | 0·7325 | 2·0 | 2·316 | | 2·0 | 8·024 | |
| ·3 | 0·8971 | 3·0 | 2·836 | | 3·0 | 9·827 | |
| ·4 | 1·0359 | 4·0 | 3·275 | | 4·0 | 11·347 | |
| ·5 | 1·1581 | 5·0 | 3·662 | | 5·0 | 12·687 | |
| ·6 | 1·2687 | 6·0 | 4·011 | | 6·0 | 13·898 | |
| ·7 | 1·3703 | 7·0 | 4·333 | | 7·0 | 15·011 | |
| ·8 | 1·4649 | 8·0 | 4·632 | | 8·0 | 16·047 | |
| ·9 | 1·5538 | 9·0 | 4·913 | | 9·0 | 17·021 | |
| 1·0 | 1·6378 | 10·0 | 5·179 | | 10·0 | 17·942 | |
| ·1 | 1·7178 | 11·0 | 5·432 | 253 | 11·0 | 18·817 | 875 |
| ·2 | 1·7942 | I. 12·0 | 5·673 | 241 | 12·0 | 19·654 | 837 |
| ·3 | 1·8674 | 13·0 | 5·905 | 231 | 13·0 | 20·457 | 803 |
| ·4 | 1·9379 | 14·0 | 6·128 | 222 | 14·0 | 21·229 | 772 |
| ·5 | 2·0060 | 15·0 | 6·343 | 215 | 15·0 | 21·974 | 745 |
| ·6 | 2·0717 | 16·0 | 6·551 | 207 | 16·0 | 22·695 | 721 |
| ·7 | 2·1355 | 17·0 | 6·753 | 201 | 17·0 | 23·393 | 698 |
| ·8 | 2·1974 | 18·0 | 6·948 | 195 | 18·0 | 24·071 | 678 |
| ·9 | 2·2576 | 19·0 | 7·139 | 190 | 19·0 | 24·731 | 660 |
| 2·0 | 2·3163 | 20·0 | 7·324 | 185 | 20·0 | 25·374 | 643 |
| ·1 | 2·3735 | 21·0 | 7·505 | 180 | 21·0 | 26·000 | 626 |
| ·2 | 2·4293 | 22·0 | 7·682 | 176 | 22·0 | 26·612 | 612 |
| ·3 | 2·4839 | 23·0 | 7·854 | 172 | 23·0 | 27·210 | 598 |
| ·4 | 2·5373 | II. 24·0 | 8·023 | 168 | 24·0 | 27·796 | 586 |
| ·5 | 2·5896 | 25·0 | 8·189 | 165 | 25·0 | 28·368 | 572 |
| ·6 | 2·6409 | 26·0 | 8·351 | 162 | 26·0 | 28·930 | 562 |
| ·7 | 2·6913 | 27·0 | 8·510 | 159 | 27·0 | 29·481 | 551 |
| ·8 | 2·7405 | 28·0 | 8·666 | 156 | 28·0 | 30·023 | 542 |
| ·9 | 2·7891 | 29·0 | 8·820 | 153 | 29·0 | 30·554 | 531 |
| 3·0 | 2·8368 | 30·0 | 8·970 | 150 | 30·0 | 31·076 | 522 |
| ·1 | 2·8834 | 31·0 | 9·118 | 149 | 31·0 | 31·589 | 513 |
| ·2 | 2·9299 | 32·0 | 9·265 | 147 | 32·0 | 32·095 | 505 |
| ·3 | 2·9753 | 33·0 | 9·408 | 143 | 33·0 | 32·593 | 497 |
| ·4 | 3·0200 | 34·0 | 9·550 | 141 | 34·0 | 33·083 | 490 |
| ·5 | 3·0641 | 35·0 | 9·689 | 139 | 35·0 | 33·566 | 480 |
| ·6 | 3·1076 | III. 36·0 | 9·827 | 137 | 36·0 | 34·042 | 476 |
| ·7 | 3·1505 | 37·0 | 9·962 | 135 | 37·0 | 34·512 | 470 |
| ·8 | 3·1928 | 38·0 | 10·096 | 133 | 38·0 | 34·976 | 464 |
| ·9 | 3·2337 | 39·0 | 10·228 | 131 | 39·0 | 35·434 | 458 |
| 4·0 | 3·2756 | 40·0 | 10·358 | 130 | 40·0 | 35·883 | 449 |
| ·1 | 3·3164 | 41·0 | 10·487 | 128 | 41·0 | 36·329 | 446 |
| ·2 | 3·3566 | 42·0 | 10·614 | 127 | 42·0 | 36·771 | 442 |
| ·3 | 3·3963 | 43·0 | 10·740 | 125 | 43·0 | 37·205 | 434 |
| ·4 | 3·4356 | 44·0 | 10·864 | 124 | 44·0 | 37·635 | 430 |
| ·5 | 3·4744 | 45·0 | 10·987 | 122 | 45·0 | 38·060 | 425 |
| ·6 | 3·5128 | 46·0 | 11·108 | 121 | 46·0 | 38·481 | 421 |
| ·7 | 3·5508 | 47·0 | 11·229 | 120 | 47·0 | 38·897 | 416 |
| ·8 | 3·5884 | IV. 48·0 | 11·347 | 118 | 48·0 | 39·308 | 411 |
| ·9 | 3·6225 | 49·0 | 11·464 | 117 | 49·0 | 39·716 | 408 |
| 5·0 | 3·6623 | 50·0 | 11·581 | 116 | 50·0 | 40·119 | 403 |
| ·1 | 3·6988 | 51·0 | 11·696 | 115 | 51·0 | 40·518 | 399 |
| ·2 | 3·7348 | 52·0 | 11·810 | 114 | 52·0 | 40·913 | 395 |
| ·3 | 3·7704 | 53·0 | 11·923 | 113 | 53·0 | 41·304 | 391 |
| ·4 | 3·8056 | 54·0 | 12·035 | 112 | 54·0 | 41·693 | 389 |
| ·5 | 3·8405 | 55·0 | 12·146 | 111 | 55·0 | 42·079 | 386 |
| ·6 | 3·8758 | 56·0 | 12·256 | 110 | 56·0 | 42·458 | 379 |
| ·7 | 3·9101 | 57·0 | 12·365 | 109 | 57·0 | 42·834 | 376 |
| ·8 | 3·9441 | 58·0 | 12·473 | 108 | 58·0 | 43·209 | 375 |
| ·9 | 3·9778 | 59·0 | 12·580 | 107 | 59·0 | 43·580 | 371 |



Table XVIII. continued.

| A. Value of $h+c$ in inches. | B. Value of $\sqrt{g(c+h)}$ in feet per sec. | C. Value of $h+c$ in inches. | D. Value of $\sqrt{g(c+h)}$ in feet per sec. | E. First difference. | F. Value of $h+c$ in feet. | G. Value of $\sqrt{g(c+h)}$ in feet per sec. | H. First difference. |
|---|---|---|---|---|---|---|---|
| 6·0 | 4·0120 | V. 60·0 | 12·686 | 106 | 60·0 | 43·948 | 368 |
| ·1 | 4·0451 | 61·0 | 12·791 | 105 | 61·0 | 44·315 | 367 |
| ·2 | 4·0779 | 62·0 | 12·895 | 104 | 62·0 | 44·678 | 363 |
| ·3 | 4·1105 | 63·0 | 12·998 | 103 | 63·0 | 45·037 | 359 |
| ·4 | 4·1434 | 64·0 | 13·101 | 103 | 64·0 | 45·392 | 355 |
| ·5 | 4·1755 | 65·0 | 13·203 | 102 | 65·0 | 45·745 | 353 |
| ·6 | 4·2074 | 66·0 | 13·305 | 102 | 66·0 | 46·095 | 350 |
| ·7 | 4·2390 | 67·0 | 13·406 | 101 | 67·0 | 46·442 | 347 |
| ·8 | 4·2710 | 68·0 | 13·506 | 100 | 68·0 | 46·786 | 344 |
| ·9 | 4·3021 | 69·0 | 13·605 | 99 | 69·0 | 47·127 | 341 |
| 7·0 | 4·3333 | 70·0 | 13·704 | 99 | 70·0 | 47·467 | 340 |
| ·1 | 4·3640 | 71·0 | 13·801 | 97 | 71·0 | 47·805 | 338 |
| ·2 | 4·3958 | VI. 72·0 | 13·897 | 96 | 72·0 | 48·142 | 337 |
| ·3 | 4·4251 | 73·0 | 13·993 | 96 | 73·0 | 48·477 | 335 |
| ·4 | 4·4551 | 74·0 | 14·088 | 95 | 74·0 | 48·809 | 332 |
| ·5 | 4·4850 | 75·0 | 14·183 | 95 | 75·0 | 49·137 | 328 |
| ·6 | 4·5152 | 76·0 | 14·277 | 94 | 76·0 | 49·462 | 325 |
| ·7 | 4·5447 | 77·0 | 14·371 | 94 | 77·0 | 49·786 | 324 |
| ·8 | 4·5740 | 78·0 | 14·464 | 93 | 78·0 | 50·108 | 322 |
| ·9 | 4·6031 | 79·0 | 14·556 | 92 | 79·0 | 50·429 | 321 |
| 8·0 | 4·6325 | 80·0 | 14·648 | 92 | 80·0 | 50·748 | 319 |
| ·1 | 4·6612 | 81·0 | 14·739 | 91 | 81·0 | 51·061 | 317 |
| ·2 | 4·6898 | 82·0 | 14·830 | 91 | 82·0 | 51·376 | 315 |
| ·3 | 4·7182 | 83·0 | 14·921 | 91 | 83·0 | 51·689 | 313 |
| ·4 | 4·7470 | VII. 84·0 | 15·011 | 90 | 84·0 | 52·000 | 311 |
| ·5 | 4·7761 | 85·0 | 15·100 | 89 | 85·0 | 52·309 | 309 |
| ·6 | 4·8040 | 86·0 | 15·189 | 89 | 86·0 | 52·616 | 307 |
| ·7 | 4·8318 | 87·0 | 15·277 | 88 | 87·0 | 52·921 | 305 |
| ·8 | 4·8586 | 88·0 | 15·364 | 87 | 88·0 | 53·224 | 303 |
| ·9 | 4·8860 | 89·0 | 15·451 | 87 | 89·0 | 53·526 | 302 |
| 9·0 | 4·9134 | 90·0 | 15·537 | 86 | 90·0 | 53·827 | 301 |
| ·1 | 4·9404 | 91·0 | 15·623 | 86 | 91·0 | 54·126 | 299 |
| ·2 | 4·9678 | 92·0 | 15·709 | 86 | 92·0 | 54·423 | 297 |
| ·3 | 4·9946 | 93·0 | 15·794 | 85 | 93·0 | 54·719 | 296 |
| ·4 | 5·0213 | 94·0 | 15·879 | 85 | 94·0 | 55·014 | 295 |
| ·5 | 5·0479 | 95·0 | 15·963 | 84 | 95·0 | 55·307 | 293 |
| ·6 | 5·0746 | VIII. 96·0 | 16·047 | 84 | 96·0 | 55·597 | 290 |
| ·7 | 5·1011 | 97·0 | 16·130 | 83 | 97·0 | 55·886 | 289 |
| ·8 | 5·1275 | 98·0 | 16·212 | 82 | 98·0 | 56·172 | 286 |
| ·9 | 5·1538 | 99·0 | 16·293 | 81 | 99·0 | 56·455 | 283 |
| 10·0 | 5·1792 | 100·0 | 16·373 | 80 | 100·0 | 56·737 | 282 |

## Section II.—Waves of the Second Order.

### *Oscillating Waves.*

| | |
|---|---|
| Character ............... | Gregarious. |
| Species ............... | Stationary.<br>Progressive. |
| Varieties ............... | Free.<br>Forced. |
| Instances............... | Stream ripple.<br>Wind waves.<br>Ocean swell. |

*The Standing Wave of Running Water.*—Among oscillating waves of the second order, I know none more common or more curious than the standing wave of running water. I begin the account of my examination of waves of the second order, because it is that species which appears to me to be the most easy to be conceived, because it presents the closest analogy to the ordinary known phænomena of wave motion, and because, although most frequently exhibited to the eye of the common gazer, it has not, as far as I know, ever been made the subject of accurate observation.

If the surface of a running stream be examined as it runs with an equal velocity along a smooth and even channel, its surface will present no remarkable feature to the eye, although it is known by accurate observation that the surface of the water is higher above the level in the middle or deep part than at the sides of the channel. On the bottom of the channel let there be found a single large stone; this interruption, although considerably below the surface of the water, will give indication of its presence by a change of form visible on the surface of the water. An elevation of surface will be visible, not immediately above it, but in its vicinity. Simultaneous with the appearance of this protuberance, there will appear a series of others lower down the stream. These form a group of companion phænomena, are waves of the second order, oscillatory, and of the standing species, their place remaining fixed in the water, while the water particles themselves continue to flow down with the stream. For examples see Pl. LV.

This species of wave is especially deserving of the notice both of the mathematician and of the natural philosopher, for this cause especially, that the apparent motions of the water are in this case identical with the actual paths of individual particles; each particle on the surface actually describes the path apparent on the surface; the outline of the surface of the water is the true path of a particle during its progress down the stream. It does not exhibit like other waves the form merely, a form very different from the true motion of the water particles, nor does it exhibit the motion of a motion, nor do the particles themselves remain behind while they transmit forward the wave. The particles are themselves translated along the fluid in the paths which form the apparent outline of the fluid.

In this respect, therefore, this wave appears to me important as presenting a case of transition from ordinary fluid motion to wave motion.

I found by observation on a mountain stream that waves $3\frac{1}{2}$ feet long rise in water moving at the rate of $3\frac{1}{2}$ feet per second.

Also, that waves 2 feet long rose in water moving at $2\frac{1}{2}$ feet per second.

These numbers coincide with those given in Table XXI. from which the following approximate numbers are deduced. These numbers will enable an observer to judge of the velocity of a stream by inspection of the waves on the surface.

| | | | | | | |
|---|---|---|---|---|---|---|
| The length of wave being | | 1 inch, | the velocity of the stream per second is | | | $\frac{1}{2}$ foot. |
| ” | ” | *3 inches, | ” | ” | ” | *1 foot. |
| ” | ” | 1 foot, | ” | ” | ” | $1\frac{3}{4}$ feet. |
| ” | ” | $1\frac{1}{4}$ feet, | ” | ” | ” | 2 feet. |
| ” | ” | 2 feet, | ” | ” | ” | $2\frac{1}{4}$ feet. |
| ” | ” | *$3\frac{1}{2}$ feet, | ” | ” | ” | *$3\frac{1}{2}$ feet. |
| ” | ” | 6 feet, | ” | ” | ” | $4\frac{1}{2}$ feet. |
| ” | ” | 7 feet, | ” | ” | ” | 5 feet. |
| ” | ” | 10 feet, | ” | ” | ” | 6 feet. |
| ” | ” | *30 feet, | ” | ” | ” | *10 feet. |

This Table is given for convenience of reference to observers, and it is useful and easy to recollect the velocities corresponding to 3 inches, $3\frac{1}{2}$ feet, and 30 feet. By these means it will be easy for observers to verify or correct these numbers.



These waves are very peculiar in this respect, that they exhibit little or no tendency to lateral diffusion; the breadth of a wave does not apparently exceed the length of a wave, and is often much smaller. When a stream enters a large pool, its path across the pool is marked by these waves very distinctly, and the diminishing length of the waves accompanies the diminishing velocity of the stream, and at the same time indicates the extreme slowness with which diffusion takes place.

*The motion of the particles of water*, as observed by a body floating on the surface, is this, the motion is retarded at the top of each wave and accelerated in the bottom, thus oscillating about the mean motion of the stream. The motion, as far as it can be observed by bodies floating near the surface, is a simple combination of a circular with a rectilineal motion. The disturbing body, the stone at the bottom, gives to the particles which pass over it the motion of eddy as indicated, Plate LV. fig. 2, and this being continued downwards, and combined with the rectilineal motion of the particles, presents the cycloidal form of the wave.

If we conceive a uniform revolving motion in a vertical plane communicated to a particle of water, the centre of the circle of revolution being at the same time carried uniformly along the horizontal line, Plate LVI., then the path of the particle having these two motions is marked out by the cycloidal line 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 joining these points, and if every successive particle of the fluid have the same motions communicated to it, the simultaneous places of successive particles will give the line 1, 2, 3, 4, 5, 6, 7, 8, &c. as the form of the surface of the fluid. It is to be observed that at A and C the direction of the motion of revolution is opposite to the motion of transference, and ∴ the absolute velocity of the particle is diminished by the oscillating motion, while at B and D it is increased by an equal amount, and in the intermediate positions 3 and 9 it is neither increased nor diminished. It is also to be observed, that when the motion of the water in the direction of transference is slowest (*i. e.* when the motion of oscillation is opposite to the motion of transference), the transverse section of moving fluid is greatest, and when the motion of transference and of oscillation coincide, and the motion is quickest in the direction of transference, the transverse section of the fluid is greatest. Thus we see how during a change of form the dynamical equilibrium of the fluid may be unchanged.

The fluid may thus be conceived as moving with varying velocity along a channel of variable section, its upper surface being conformable to the outline of the wave. Hence we might infer that a rigid channel of varying area, of the form of this standing wave, would not interfere with the free motion of the fluid.

And hence it may follow, that when the area of a pipe conveying fluid is to undergo a change, the best form of pipe or channel is indicated by the form of this wave. Thus the velocity has undergone a change between 0 and 4 which the form of a close pipe might render permanent.

In the examples already given, a solid impediment has generated the waves on the surface of the fluid. At the confluence of streams I have observed the same waves generated by the oblique action of one current on another meeting it in a different direction.

The height and hollow of the fluid and the change of velocity are to be regarded as reciprocally the cause and effect each of the other. The obstacle first retards the velocity of the fluid, so as to accumulate it above the obstacle, the water rises to a height due to this diminished velocity, and as all the particles of the stream pass through this area of the stream with a diminished velocity, the area of transverse section must be increased at this point; thus the elevation of surface, enlargement of section, diminution of velocity above the obstacle are its necessary consequences of that obstacle. Again, below the obstacle the accumulation above generates an additional velocity due to that height in addition to the mean motion of the stream; the same volume of water which passed through the large area, with its increased section and diminished velocity, being now a higher velocity, is transferred through the smaller area which allows its transmission. Thus the constant volume passing down the stream varies its velocity with the conservation of its forces by means of a varying area of transference; and thus we are enabled to conceive how the observed form of the surface becomes at once possible and necessary to the transmission of the fluid under the action of the disturbing force.

I am not aware that this species of standing wave in moving water has ever before been made the subject of philosophical examination. But I conceive that its study is highly important, especially in a theoretical view, as the means of conveying sound elementary conceptions of wave motion, as exhibiting the transition from the phænomena of water currents to those of water waves, as the intermediate link between motions of the first degree and motions of the second degree, and as affording a basis from which we may commence, with some prospect of success, the application of the known principles and laws of motion to the investigation of the difficult theory of waves.

*Moving Waves of the Second Order—Sea Waves.*—It is not difficult to pass from the conception of standing waves in running water to the conception of running waves in standing water. Let us first conceive the waves in Plate LV. to be formed in water running in the direction there indicated from right to left, with a given mean motion, and a given motion of uniform circular oscillation; and next let us conceive the whole water channel and waves to be transferred uniformly in the opposite direction with a velocity equal to the mean velocity of transference; then the absolute motion of transference of the water will become nothing: the waves formerly standing are now moved in the opposite direction with a velocity equal to the former mean velocity of the running stream, and the motion of oscillation remains. Thus, the running water becoming still, the waves become moving waves, and if we reverse the hypothesis once more, and conceive the waves which move with a given velocity to exist in water which has a motion of transference with equal velocity in the opposite direction, it is manifest that these waves running up the stream as fast as the waters run down, the wave-crests remain fixed in place. Thus then the same oscillating phænomenon which in standing water gives moving waves, will give in moving water standing waves; taking for granted always that the motions of oscillation are such as to be possible, consistent with the nature of the fluid, and independent of the common mean motion of the fluid; a condition equally essential to the possibility of the wave motion and of our conceptions of it.

I have been able accurately to observe the phænomena of wave motion in still water, the waves being of the second order and gregarious, under the following circumstances:—

1. I have drawn a body through the water with a uniform motion, and have observed the group of waves which follow in its wake.

2. I have propagated the negative wave of the first order, and observed the group of waves which follow in its wake.

I have not observed in the results of these two methods any distinction of form, velocity, or other character.



The form under which these waves appear has already been exhibited in Plate LII. figs. 9 and 10, and equally in Plate LV. figs. 1, 2, 3, and in Plate LVI. fig. 1.

I have made a series of observations by dragging a body through the water, the results of which are given in the following Table. I first made preparatory observations to find whether the form of body or depth of channel made any change on the phænomenon. I found that larger bodies and higher velocities made higher waves, but that the length and velocity of the wave were unchanged by either the form of body, or the depth of the channel, or the height of the wave. I observed that when the waves became high and broke, the elevation above the mean level was 6 inches, and the depression below it 2 inches, making a height total of 8 inches; this was at a velocity of 6·25 feet per second. Immediately behind the body dragged through the water, the mean level appears to be considerably lowered.

I examined the motion of oscillation of these waves by means of small floating spherules. Waves of the second order having a total height of half an inch, in water 4 inches deep made by a negative wave, were accompanied by motion in a circle of half an inch diameter at the surface, and the particles below described also circles which rapidly decreased in diameter and at 3 inches deep ceased to be sensible; the waves were about one foot long.

## Table XIX.

*Observations on the Length and Velocity of Waves of the Second Order.*

Column A the order and number of the experiments.

Column B the number of seconds in which the waves were transmitted along 100 feet.

Column C the aggregate length in feet of the number of waves in Column D.

Column D the number of waves extending to the length in Column C.

Column E the length in feet of one wave from crest to crest.

Column F the velocity in feet per second given by experiment.

These results are the means of many experiments, differing from each other not more than the examples preceding them, which have been given in detail as a fair specimen.

| A. | B. | C. | D. | E. | F. | A. |
|---|---|---|---|---|---|---|
| I. | 33·2 | 26·5 | 10 | 2·65 | 3·01 | I. |
| II. | 33·2 | 26·5 | 10 | 2·65 | 3·01 | II. |
| III. | 31·6 | 25· | 8½ | 2·94 | 3·16 | III. |
| IV. | 31·8 | 25· | 8½ | 2·94 | 3·14 | IV. |
| V. | 31·8 | 25· | 8½ | 2·94 | 3·14 | V. |
| VI. | 30·4 | 25· | 8 | 3·125 | 3·29 | VI. |
| VII. | 29·6 | 25· | 7⅔ | 3·26 | 3·37 | VII. |
| VIII. | 29·6 | 25· | 7⅔ | 3·26 | 3·37 | VIII. |
| IX. | 28·0 | 25· | 7 | 3·57 | 3·57 | IX. |
| X. | 28·4 | 25· | 7 | 3·57 | 3·51 | X. |
| XI. | 28·0 | 25· | 7 | 3·57 | 3·57 | XI. |
| XII. | 28·0 | 25· | 7 | 3·57 | 3·57 | XII. |
| XIII. | 28·0 | 25· | 7 | 3·57 | 3·57 | XIII. |
| XIV. | 28·0 | 25· | 7 | 3·57 | 3·57 | XIV. |
| †XV.–XVII. | 26·8 | 25· | 6½ | 3·84 | 3·72 | XV.–XVII. |
| †XVIII.–XXII. | 26·0 | 25· | 6 | 4·18 | 3·84 | XVIII.–XXII. |
| †XXIII.–XXVI. | 24·0 | 25· | 5 | 5·00 | 4·16 | XXIII.–XXVI. |
| †XXVII.–XXXIV. | 21·6 | 25· | 4 | 6·25 | 4·62 | XXVII.–XXXIV. |

As these waves appear in groups, their velocity and lengths are easily observed and measured. I have reckoned as many as a dozen such waves in a group all about the same magnitude, so that the aggregate length of the first six was sensibly equal to the length of the second group of six. The method of observation was this: a given distance was marked off along one side of the channel; an observer marked the instant at which the first of a group of secondary waves arrived at a given point, while another observer at the further end of the given distance counted the number of waves as they passed, and marked the point at which the last had arrived when the signal was given that the first wave had reached the other station; thus it was observed that in a group of waves moving over 100 feet in 28 seconds, there were seven comprehended in a distance of 25 feet, whence

$\frac{28}{100}$ = 3·57 feet per second for the velocity of the wave, and

$\frac{25}{7}$ = 3·57 feet as the length of the wave.

Also, since the wave passes along 3·57 feet its own length in one second, its length divided by the velocity gives 1 second as the period of one complete oscillation.

The *velocity* of the wave of the second order, the *length* from the crest of one wave to the crest of the next, or from hollow to hollow, and the time of passing from one crest to another, called the *period* of the wave; these are the principal elements for observation.

These elements are calculated for the convenience of observers in the Table XXI. It will also be observed that the circles which represent the oscillatory motion of the water particles (Plate LVI.), showing the Wave Motion of the Second Order, diminish very rapidly with the increasing depth of the particles below the surface of the water at the lowest part of the wave. By my observations I found that in high waves at a depth = $\frac{1}{3}$rd of a wave length, the range of oscillation of the particles is only about $\frac{1}{30}$th of that of particles on the surface*.

* I have here to express the favourable opinion which I have formed of a wave theory given to the world by M. Franz Gerstner, so early as 1804*, and reprinted in the work of the MM. Weber, to whom I am indebted for my acquaintance with this theory. Gerstner's theory is characterized by simplicity of hypothesis, precision of application, its conformity with the phænomena, and the elegance of its results. It is not without faults, yet I cannot agree with the Messrs. Weber, nor with MM. Professors Mollweide and Mobius, in the precise opinion at which they arrive, although I confess I could wish that he had assumed as an hypothesis the doctrine which in (14.) he deduces as a conclusion from hypotheses less firmly established than this conclusion, unless indeed we should esteem it an argument in favour of his hypothesis, that it conducts him directly to a conclusion of well-known truth. Neither do I find that his hypotheses are so much at variance with the actual conditions of the waves I have observed, as they appear to be in MM. Weber's view of their own experiments. The calculations of M. Gerstner are applied primarily to a kind of standing oscillation. But it does not appear to me that his calculations ought to be applied in any way to the standing oscillations which M. Weber reckons to be their closest representation. In M. Gerstner's first part of the work the wave form is standing, wave oscillation is circular, the fluid is in motion, and the particle paths are identical with the lines which indicate the form of the wave. I conceive, therefore, that the wave which he has examined, and the conditions of its genesis, find a perfect representative in my standing waves of the second order, in running water, which I have represented in Plates LV. and LVI. From this hypothesis it is not difficult to arrive at the moving wave of standing water, for if we conceive the whole channel moved horizontally along in an opposite direction with a velocity equal to the horizontal velocity of transference, the particles will then be relatively at rest, the cycloidal waves become moving forms, the particle paths stationary circles, and the motion of transmission of the wave equal and opposite to the former mean horizontal

* Theorie der Wellen. Prague, 1804.



One observation which I have made is curious. It is, that in the case of oscillating waves of the second order, I have found that the motion of propagation of the whole group is different from the apparent motion of wave transmission along the surface; that in the group whose velocity of oscillation is as observed 3·57 feet per second, each wave having a seeming velocity of 3·57, the whole group moves forward in the direction of transmission with a much slower velocity. The consequence of this is a difficulty in observing these waves (especially such as are raised by the wind at sea), namely, that as the eye follows the crest of the wave, this crest appears to run out of sight, and is lost in the small waves in which the group terminates. The termination of these groups in a series of waves becoming gradually smaller and smaller, yet all continuous with the large wave, is curious and leads to a curious conclusion. It is plain that if these large waves are moving with the same velocity as the small ones, this result would be inconsistent with the other experiments. But if we conceive each to be transmitted with the velocity due to its breadth, we shall have the velocity of oscillation varying from point to point in the same group of waves, but it will be impossible always to measure this velocity directly as it may be continually changing. There is to be observed, therefore, this distinction in a group of waves of the second order, between the velocity of individual wave transmission and the velocity of aggregate wave propagation.

I have not found it possible to measure this velocity of aggregate propagation of a group of waves, from want of a point to observe. If I fix my eye upon a single wave, I follow it along the group, and it gradually diminishes and then disappears; I take another and follow it, and it also disappears. My eye, in following a wave crest, follows the visible velocity of transmission merely. After one or two such observations, I find that the whole group of

motion of transference of the particles. In short, they become moving waves of the third order, the common waves of the sea.

From M. Gerstner's investigations we obtain the following results, for oscillating waves which correspond to our second order:—

1. Waves of the same amplitude are described in equal times independently of their height. (This corresponds with the results of our experiments.)

2. Waves are transmitted with velocities which vary as the square roots of their amplitudes.

3. The waves on the surface are of the cycloidal form, always elongated, never compressed; the common cycloid being the limit between the possible and impossible, the continuous and the broken wave.

4. The particle paths in the standing waves of running water are cycloids, which on the surface are identical with the wave form, and below the surface have the same character with the wave lines of the surface, the height of the waves only diminishing with the increase of depth.

5. The particle paths of moving waves in standing water are circles corresponding to the circles of height of the cycloidal paths; the diameters of these circles of vertical oscillation diminish in depth as follows. Let 0, $u$, $2u$, $3u$, &c. be depths increasing in arithmetical progression, then $b$, $b\,\varepsilon^{-\frac{u}{a}}$, $b\,\varepsilon^{-\frac{2u}{a}}$, $b\,\varepsilon^{-\frac{3u}{a}}$, which decrease in geometrical proportion, are the ratio of the diminishing diameters of vertical oscillation. Thus, if 0, $\frac{1}{2}a$, $\frac{2}{2}a$, $\frac{3}{2}a$, &c. be depths, $a$, 0·6065 $a$, 0·3679 $a$, 0·2231 $a$, 0·1353 $a$, are the ranges.

6. The forms of these paths and the circles of oscillation are shown in Plate X. fig. 1, which has been drawn with geometrical accuracy from the data of M. Gerstner's theory, and it is at the same time the most accurate representative I am able to give of my observations on the wave of the second order.

7. The period of wave oscillation is $t = \pi\sqrt{\frac{2a}{g}}$.

8. The velocity of wave propagation is $v = \sqrt{2ag}$, $a$ being the radius of the wave cycloid generating circle.

9. It follows that the length of a pendulum isochronous with the wave is less than the wave length in the ratio of the diameter of a circle to its semi-circumference. Newton made these equal. These last three results are inconsistent with my observations on transmission.

oscillations has been transferred along in the direction of transmission with a velocity comparatively slower; but I have not been able to measure this velocity of propagation of the wave motion from one place to another.

We have already seen that the velocity assigned by Mr. Kelland and Mr. Airy falls much short of that of the wave of the first order, to which they have thought their results were to be applied. Their results are much nearer to that of the secondary wave, so that it may be questioned whether they should not have applied their results to that rather than the other. Thus by comparing Table XXI. with Table XVIII., it will be found that while the velocity of a wave of the first order, about 6 feet long, is from 5·5 to 8 feet per second, according to the height, that of a wave of the second order is only 4·62 feet, which is much nearer to their results. There remains however this difficulty, that high and low waves of the second order of equal length have equal velocities.

*On Observations of the Waves of the Sea.*—The chief difficulty in obtaining accurate measures of sea waves consists in this fact, that the surface is seldom covered with a uniform series of equidistant equal waves, but with several simultaneous groups of different magnitude or in different directions. If there exist more groups than one, the resulting apparent motion of the surface will be extremely different from the motion of either, and may be apparently in an opposite direction from that of the actual motion of the individual series themselves.

Besides the coexistence of different series of waves, we have the difficulty arising from the fact already mentioned, that a difference exists between the velocity of transmission and the velocity of propagation. From this it results, that after the eye has followed the apparent ridge of a wave, moving with a given velocity of transmission, it will outrun the velocity of propagation, and the wave will appear to cease. This I have continually observed at sea. The eye follows a large wave and suddenly it ceases to pass on, but on looking back we find it making once more an appearance on the same ground along which we formerly traced its ridge; this arises from the cause just mentioned.

But there are still many occasions on which tolerable observations may be made, and the best will be such as are least complicated by separate systems. The best observations of this kind I have been able to obtain were made for the Committee of the British Association, by the Queen's Harbour-master at Plymouth, William Walker, Esq., who has paid much attention to this subject. He observed the waves as they traversed a space of about half a mile, between two buoys, noting the time of passing, and also the number of waves in the distance between the buoys, whose distance was accurately known. He remarks that in counting the number of waves, great difficulty was found in following a single wave along this space. In fact, as we have already shown, a wave will be often found to fall behind its expected place.

The resulting velocities got from Mr. Walker's experiments are very various. But on taking out of the others all those which are mentioned by Mr. Walker as having causes of uncertainty, I found those which remained very close to those given in Table XXI.

The following is the Table of observations on sea waves.

Distance traversed about half a mile; depth 40 to 50 feet.



TABLE XX.

*Observations on the Length and Velocity of Waves of the Second Order.—In the Sea.*

| | Wave length, feet. | Vel. per sec. feet. | Vel. per hour. miles. | Height of wave in feet above mean level, | Remarks at the time of Observations. |
|---|---|---|---|---|---|
| I. | 110·5 | 20·2 | 11·9 | $2\frac{3}{4}$ | A fresh breeze blowing. |
| II. | 175·0 | 34·3 | 20·3 | $2\frac{1}{2}$ | Waves not easily traced. |
| III. | 302· | 37·0 | 21·9 | 4 | High seas overtake smaller ones. |
| IV. | 345· | 37·0 | 21·9 | $4\frac{1}{2}$ | These waves came down channel. |
| V. | 306· | 37·0 | 21·9 | $4\frac{1}{2}$ | Long low swell. |
| VI. | 408· | 41·2 | 24·2 | $4\frac{1}{2}$ | Small waves merged in large ones. |
| VII. | 442· | 41·8 | 24·7 | 27 | Height of wave correctly measured, they break in 5 and 6 fathoms water. |
| VIII. | 450· | 44·7 | 26·5 | ? | Strong S.W. wind. |
| IX. | 460· | 46·0 | 27·2 | ? | Waves running high and breaking. |
| X. | 345· | 46·0 | 27·2 | 5 | Long low swell. |
| XI. | 394· | 38·3 | 22·7 | 5 | Waves generated by wind of yesterday. |
| XII. | 345· | 41·5 | 24·5 | 4 | Waves crowd near the beach. |
| XIII. | 306· | 36·8 | 21·6 | irregular. | Shifting wind. |
| XIV. | 460· | 42·5 | 25·2 | regular. | Easterly winds. |

Of these there are five which coincide with my observations and with my tables, Nos. XIX. and XXI.; and it is curious that these five are those which are made in the most unexceptionable circumstances. No. II. has the remark that the waves are not easily traced. No. III. has a mixture of waves, which always causes great confusion and difficulty of observation. No. V. and No. X. are long and low, and therefore not easily traced, and so on; but Nos. I., IV., VII., XI., XIV., are unexceptionable, and are compared with my formula in the following Table:—

| | Length of wave observed. feet. | Velocity of wave observed. feet per sec. | Velocity of wave calculated. |
|---|---|---|---|
| I. | 110·5 | 20·2 | 19·5 |
| IV. | 345· | 37·0 | 35· |
| VII. | 442· | 41·8 | 40· |
| XII. | 394· | 38·3 | 37· |
| XIV. | 460· | 42· | 40·* |

We may therefore continue to use Table XXI. for the velocity of sea waves, unless we obtain further and decisive experiments to the contrary. It does not appear that sea waves present any characteristic to distinguish them from other oscillating waves of the second order which I have experimentally examined.

It also follows that these waves coincide with my observations, that the depth of water is the limit of the height of waves; see No. VII., where waves 27 feet high, break in water of 5 to 6 fathoms.

How it happens that individual large waves should ever arise in the surface of a large sea, uniformly exposed to the action of the wind, is not very obvious. Thus much is plain—that if a wave, greater than those around it, be generated by a local inequality of the wind, or by one of the moving whirlpools which we know to be so common, *that wave* will be increased continually by the presence of other waves coexisting with it, for when these other waves are crossing the top of this larger wave, they are suddenly exposed to increased force by the obstruction they present to the wind, and being cusped in form by the coincidence of the crests, they are in a position of delicate equilibrium easily deranged; and the derangement producing a breaking of the wave, the disintegrated fragments of the smaller wave detached from it, leave it smaller, and increase by an equal quantity the magnitude of the larger.

This exaggeration of an individual wave or group is increased by the phænomenon already noticed, that the velocity of wave *transmission* may be very different from the velocity of wave *propagation*. A large wave of the sea remaining in a state of much slower motion than the motion of wave transmission, being traversed by another series of different velocity, exposes them successively on its summit to the increased action of the wind to disintegration, thus making them tributary to its own further accumulation; such phænomena I have often noticed at sea; the wave appears to over-run itself, and the wave behind *seems* to take its place and acquire the magnitude and form it has appeared to lose; but it is the same wave which remains behind it, and its motion is merely a deception, or rather it is as explained in a preceding paragraph.

The final destruction of the waves of the sea, as they expend their strength and conclude their existence on the rocks and sands of the shore, is a subject of interesting study and observation. The sea-shore after a storm is a scene of great grandeur; it presents an instance of the expenditure of gigantic forces, which impress the mind with the presence of elemental power as sublime as the water-fall or the thunder. It is peculiarly instructive to watch these waves as they near the shore: long before they reach the shore they may be said to feel the bottom as the water becomes gradually more shallow, for they become sensibly increased in height; this increase goes on with the diminution of depth and a diminution of length likewise as the wave becomes sensible; finally, the wave passes through the successive phases of cycloidal form, as in Plate LVI., and becoming higher and more pointed, reaching the limit of the cycloid, assumes a form of unstable equilibrium, totters, becomes crested with foam, breaks with great violence, and continuing to break, is gradually lessened in bulk until it ends in a fringed margin on the sea-shore.

But there are a variety of questions to be determined concerning this shore wave or breaking surf. Why and how does it break? What happens after it begins to break? What are the relative levels of the waves and of the water? What is the mean level of the sea, and what sort of waves are breakers?

It is not at first obvious what form the mean level of the sea will assume on a sloping beach sea-ward on which heavy breakers are rolling. It is plainly not level; the action of the wind is known to heap the water up on it. The impetus of the waves also must raise it to some height due to their velocity and force. Hence the mean surface of the sea will form a slope upwards towards the sea-shore; and this slope will form a continual and uniform current of water outwards towards the sea, except when it is directly opposed by the action of the wave in the opposite direction.

There is a phænomenon of some importance in breaking waves, to which I have directed attention; it is this, that the wave of the second order disappears, and that a wave of the first order takes its place. It is to be observed as follows:—In waves breaking on a shore, I have observed a phænomenon which is curious and not without importance. The wave of the second order may disappear, and a wave of the first order take its place. The conditions in which I have noticed this phænomenon are as follows. One of the common sea waves, being of the second order, approaches the shore, consisting as usual of a negative or hollow part, and of a positive part elevated above



the level; and as formerly noticed, this positive portion gradually increases in height, and at length the wave breaks, and the positive part of the wave falls forward into the negative part, filling up the hollow. Now we readily enough conceive that if the positive and the negative part of a wave were precisely equal in height, volume, and velocity, they would, by uniting, exactly neutralize each other's motion, and the volume of the one filling the hollow of the other, give rise to smooth water; but in approaching the shore the positive part increases in height, and the result of this is, to leave the positive portion of the wave much in excess above the negative. After a wave has first been made to break on the shore, it does not cease to travel, but if the slope be gentle, the beach shallow and very extended (as it sometimes is for a mile inwards from the breaking-point, if the waves be large), the whole inner portion of the beach is covered with positive waves of the first order, from among which all waves of the second order have disappeared. This accounts for the phænomenon of breakers transporting shingle and wreck, and other substances shorewards after a certain point; at a great distance from shore, or where the shores are deep and abrupt, the wave is of the second order, and a body floating near the surface is alternately carried forward and backward by the waves, neither is the water affected to a great depth; whereas nearer the shore, the whole action of the wave is inwards, and the force extends to the bottom of the water and stirs the shingle shorewards; hence the abruptness also of the shingle and sand near the margin of the shore where the breakers generally run.

I have observed this most strikingly exemplified in Dublin Bay after a storm: there is a locality peculiarly favourable to the study of breaking waves above Kingston, where over an extent of several miles there is a broad, flat, sandy beach, varying in level very slightly and slowly. Waves coming in from the deep sea are first broken when they approach the shallow beach in the usual way; they give off residuary waves, which are positive; these are wide asunder from each other, are wholly positive, and the space between them, several times greater than the amplitude of the wave, are perfectly flat, and in this condition they extend over wide areas and travel to great distances. These residuary positive waves evidently prove the existence, and represent the amount of the excess of the positive above the negative forces in the wind wave of the second order. See Plate XLIX. fig. 7.

### TABLE XXI.

*Length, Period and Velocity of Transmission of Waves of the Second Order.*

A the length of the waves (observed) in feet.
B the period of the waves in seconds.
C the velocity of the waves in feet per second (by observation).
D the velocity of the waves in feet per second, calculated by formula.

| A. | B. | C. | D. | A. | A. | B. | C. | D. | A. |
|---|---|---|---|---|---|---|---|---|---|
| 0·01 | ·053 | | ·1889 | 0·01 | 8 | 1·496 | | 5·344 | 8 |
| 0·05 | ·118 | | ·4224 | 0·05 | 9 | 1·587 | | 5·667 | 9 |
| 0·1 | ·167 | | ·5975 | 0·1 | 10 | 1·670 | | 5·975 | 10 |
| 0·25 | | 1·00 | | | 20 | 2·366 | | 8·45 | 20 |
| 0·3 | ·290 | | 1·034 | 0·3 | 30 | 2·90 | | 10·34 | 30 |
| 0·5 | ·374 | | 1·336 | 0·5 | 40 | 3·34 | | 11·95 | 40 |
| 0·7 | ·443 | | 1·580 | 0·7 | 50 | 3·74 | | 13·36 | 50 |
| 1·0 | ·529 | | 1·889 | 1·0 | 100 | 5·29 | | 18·89 | 100 |
| 1·2 | ·579 | | 2·070 | 1·2 | 110 | | 20· | 19·5 | |
| 1·5 | ·648 | | 2·314 | 1·5 | 200 | 7·48 | | 26·72 | 200 |
| 1·7 | ·690 | | 2·463 | 1·7 | 300 | 9·16 | | 32·73 | 300 |
| 2·0 | ·748 | | 2·672 | 2·0 | 345 | | 37· | 35· | |
| 2·2 | .781 | | 2·802 | 2·2 | 394 | | 38· | 37· | |
| 2·4 | ·820 | | 2·927 | 2·4 | 400 | 10·58 | | 37·78 | 400 |
| 2·65 | ·862 | 3·01 | 3·075 | 2·65 | 442 | | 42· | 40· | |
| 2·94 | ·907 | 3·15 | 3·240 | 2·94 | 460 | | 42· | 40· | |
| 3·00 | ·916 | | 3·273 | 3·00 | 500 | 11·83 | | 42·25 | 500 |
| 3·12 | ·934 | 3·29 | 3·338 | 3·12 | 1,000 | 16·70 | | 59·75 | 1,000 |
| 3·26 | ·955 | 3·37 | 3·411 | 3·26 | 2,000 | 23·66 | | 84·5 | 2,000 |
| 3·57 | 1·000 | 3·57 | 3·57 | 3·57 | 3,000 | 29·0 | | 103·4 | 3,000 |
| 3·84 | 1·038 | 3·72 | 3·702 | 3·84 | 4,000 | 33·4 | | 119·5 | 4,000 |
| 4·00 | 1·058 | | 3·778 | 4·00 | 5,000 | 37·4 | | 133·6 | 5,000 |
| 4·18 | 1·095 | 3·84 | 3·909 | 4·18 | 10,000 | 52·9 | | 188·9 | 10,000 |
| 4·50 | 1·122 | | 4·008 | 4·50 | 20,000 | 74·8 | | 267·2 | 20,000 |
| 4·70 | 1·147 | | 4·096 | 4·70 | 30,000 | 91·6 | | 327·3 | 30,000 |
| 5·00 | 1·183 | 4·16 | 4·225 | 5·00 | 40,000 | 105·8 | | 377·8 | 40,000 |
| 6·00 | 1·296 | | 4·628 | 6·00 | 50,000 | 118·3 | | 422·5 | 50,000 |
| 6·25 | 1·323 | 4·62 | 4·724 | 6·25 | 100,000 | 167·0 | | 597·5 | 100,000 |
| 6·5 | 1·349 | | 4·817 | 6·5 | 500,000 | 374·0 | | 1336·0 | 500,000 |
| 7 | 1·400 | | 4·999 | 7 | 1,000,000 | 529·0 | | 1889·0 | 1,000,000 |



## SECTION III.—WAVES OF THE THIRD ORDER.

### *Capillary Waves.*

Character............ Gregarious.
Varieties ............ { Forced. Free.
Instances............ { Dentate waves. Zephyral waves.

*Capillary Waves.*—If the point of a slender rod or wire, being wet, be inserted in a reservoir of water perfectly still, to a minute depth, say one-tenth part of an inch below the surface of repose, it is known that the surface of the water will visibly rise in the vicinity of this wire, being highest in the immediate vicinity of the wire, and gradually diminishing until it cease to be sensible. I have examined this elevation by reflected rays from the surface, and I find that this elevated mass does not sensibly rise from the surface at more than an inch distance from the centre of the rod, the rod itself being one-sixteenth of an inch in diameter.

This statical phænomenon belongs to a well-known class of phænomena, which have been experimentally examined by many philosophers, and successfully explained by Dr. Thomas Young and Laplace, and recently investigated very fully and completely by M. Poisson, in his profound work entitled, 'Nouvelle Théorie de l'Action Capillaire,' Paris, 1831. An admirable Report on the present state of our knowledge of the phænomena of capillary attraction will be found in the Transactions of the British Association, vol. ii. All that it is necessary for my present purpose to advert to on this subject is, that the phænomena of elevation of fluids by capillary attraction, are chiefly due to the condition of tension of the superficial particles of the water under the influence of a force acting on these superficial particles at insensible distances only, or by physical contact or adhesion. These superficial particles form a chain, or catenary, or lintearian curve, one end supported by the immediate adhesion of one extremity to the solid body at a given height above the water, the other end lying on the surface of the water, the underlying particles being suspended immediately by their mutual adhesion to this superficial film. M. Poisson especially has shown that "capillary phænomena are due to molecular action, modified by a particular state of compression of the fluid at its superficies." I have been thus particular for the purpose not only of explaining my meaning in a future article, but also to justify a term which I am desirous of introducing here as an expression not only convenient, but also philosophically sound. I have called the phænomena noticed in this section *Capillary Waves*, because they appear to me to present themselves exclusively in the thin superficial film which forms the bounding surface of the free liquid, and which is already recognised in the known hydrostatical phænomena of capillary attraction, and which if I may be allowed, I will call the *capillary film*.

By capillary waves I therefore designate a class of hydrodynamical phænomena, which exhibit themselves when particles of water are put in motion under the action of such forces as when at rest produce the usual hydrostatical capillary phænomena. Let the slender rod already alluded to, as supporting a capillary column, bounded by a concave surface of revolution, be moved horizontally along the surface of the fluid with a velocity of one foot per second, and we shall have exhibited to us all the beautiful phænomena represented in Plate LVII. In order to produce these phænomena, it is only necessary that the slender rod touch the surface without descending to any sensible depth; and the depth to which it descends in no sensible manner affects the phænomenon. I have called these phænomena capillary waves.

*Free Capillary Waves.*—If the point of a rod sustaining a capillary column be suddenly raised, so as to allow the capillary film to remain without support, it descends and propagates through the capillary film an undulation which diffuses itself in every direction circular-wise, in a small group of about half a dozen visible waves which soon become insensible. Or if a very slender silk fibre, stretched horizontally along the surface of the water, be first wetted, and made to sustain a long strip of the capillary film, and then suddenly withdrawn, leaving a ridge of unsupported fluid, waves parallel to this are generated, which remain longer visible, are short and narrow at first, and becoming longer and flatter, at first about a quarter of an inch in amplitude from ridge to ridge, and about half a dozen in number, they become an inch in amplitude about the time when they are last visible; their *longevity* does not exceed *twelve or fifteen seconds*, and their *visible range eight or ten feet.*

These latter are what I designate the *free* capillary waves; the former class, shown in Plate LVII., existing under the continued influence of the disturbing force, may be called the *forced* species of this order of wave. As forced waves, and while under the influence of the exciting body, they may apparently attain great velocity; but if the disturbing body be suddenly removed, they immediately expand backwards from the place where they were crowded by the solid point, and becoming all of nearly equal breadth, move forward together as free waves for twelve or fifteen seconds, at a rate of $8\frac{1}{2}$ inches per second.

*Forced Capillary Waves.*—I have already stated that if a slender rod or wire, one-sixteenth of an inch in diameter, be inserted, after having been wetted, into water in repose, there will be raised all round this rod a column of fluid by the action of the capillary forces, as indicated at figure 2, Plate LVI. I have stated that this surface may be observed by reflexion to extend on every side about an inch, forming a circular elevation, bounded by a surface of revolution round the axis of the rod as a centre; the line which divides the elevated from the level surface being a circle of two inches in diameter. When this rod is moved horizontally along the surface of the fluid, the form of the elevated mass changes; before the disturbing point the extent of elevation diminishes, and the outline of the capillary volume of fluid sustained by the cylinder ceases to be a figure of revolution, becoming distorted as at fig. 3. At a velocity of about eight inches per second, the capillary volume has taken the bifurcate form, fig. 6, and a small wave, *b b*, about an inch broad, is visible before the disturbing point, and a ridge, *a a*, begins to manifest itself, diverging from the disturbing body; at about ten inches per second there become visible distinctly three waves, the disturbing body being in the middle of the first *a*, and the sum of the length of waves *b* and *c*, being about an inch. At higher velocities than this, the waves increase rapidly in number, diminish in amplitude, and extend out in length, spreading into the form indicated in Plate LVII., which is formed at a velocity of 60 feet per minute, or of 12 inches per second.

As the velocity increases, the following changes are to be observed:—

1. The waves diminish in amplitude from ridge to ridge; that is to say, denominating the wave in which is the disturbing body ridge *a*, and the others in succession before the point *b c d*, &c. the first space of an inch forward, in the direction of motion contains at a velocity of 12 inches per second, or 60 feet per minute, besides *a*, 3 ridges *b c d*; at 65 feet per second, 4 ridges *b c d e*; at 72 feet per second there are in the first inch formed five ridges *b c d e f*, and so on. This crowding of the ridges with the velocity is given in the following Table:—



TABLE XXII.

*Observations on the Velocity, Distance, and Divergence of Waves of the Third Order.*

Column A contains the time in which the disturbing body, a wire of one-sixteenth of an inch in diameter, was drawn with a uniform motion along distances of 12 feet each; each experiment being frequently repeated.

B and C are the corresponding velocities of the disturbing body.

D, E, F are the number of complete waves, reckoning from hollow to hollow, contained in each successive inch from the centre of the disturbing wire, formed in the direction of the motion of the wire.

G. The numbers in this column are measures of the divergence of the first wave from the path of the exciting wire, measured at 25 inches behind that wire, and of course these numbers are tangents to the radius 25 for the angle of divergence.

H contains the angles deduced from the numbers in G.

*Observations on the Capillary Waves.*

See Plate LVII.

| | A. | B. | C. | D. | E. | F. | G. | H. |
|---|---|---|---|---|---|---|---|---|
| | Time of describing 12 feet. | Velocity in feet per sec. | Velocity in feet per min. | No. of waves observed before the disturbing body, in first inch. | in second inch. | in third inch. | Tang. of angle to radius 25. | Angle of crest of first wave *aa*, with direction of disturbing bodies. |
| | s. | | | | | | | ° |
| I. | 12 | 1 | 60 | 3 | 4 | 5 ? | 25 | 45 |
| II. | 11 | $1\frac{1}{11}$ | 65 | 4 | 5 | 6 ? | 21 | 40 |
| III. | 10 | $1\frac{2}{10}$ | 72 | 5 | 6 | 7 ? | 17 | 34 |
| IV. | 9 | $1\frac{3}{9}$ | 80 | 6 | 7 | | 14 | 29 |
| V. | 8 | $1\frac{4}{8}$ | 90 | 7 | 8 | | 11 | 24 |
| VI. | 7 | $1\frac{5}{7}$ | 103 | 8 | | | 9 | 20 |
| VII. | 6 | 2 | 120 | 9 | | | 8 | 18 |
| VIII. | 5 | $2\frac{1}{5}$ | 132 | | | | 7 | |
| IX. | 4 | 3 | 180 | | | | 6 | |

The crowding of the ridges is not the only phænomenon that accompanies the increase of velocity of the moving point; the first wave, that whose ridge is in the focus, scarcely differs from a straight line, and the angle which it makes with the path of the disturbing point, diminishes with the increase of velocity; the divergence of the first wave from the path of the exciting body is given in another column by an observation of the distance of the wave from that path at a given distance behind the body. These numbers show that the velocity of the wave, taken at right angles to the ridge, is nearly that of the free wave. This angle therefore becomes an index of the relation of the velocity of disturbance to the velocity of wave propagation.

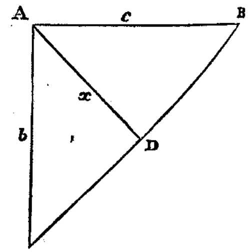
The form of the wave ridges appears to be nearly that of a group of confocal hyperbolas, the exciting body being in the focus.

I have found the numbers given in columns C, D and E, to be determined by the velocity of the disturbing body, and quite independent of its size and form. But while I have found the number of ridges in an inch at a given velocity to be thus invariable, I have not found the number of inches over which these vibrations range to be equally invariable. At a velocity of 100 feet per minute, they may sometimes be observed advancing only over the first two inches before the point; then suddenly the vibrations will spread out, not increasing in magnitude but in number to thirty or forty, extending along many inches in advance of the disturbing point, and covering ten or twelve square feet with an extension of the representation in Plate LVII. Then suddenly without apparent cause, they will subside and become visible only as a thin narrow belt, comprising the two or three waves nearest the disturbing body, and as suddenly will again spread out over the surface of the water. The play of this beautiful symmetrical system of confocal hyperbolas is a phænomenon not inferior in beauty to some of the exquisite figures exhibited by polarising crystals. I have found that the purity of the water had much to do with the extent and range of this phænomenon; that any small particles loading at a few points the capillary film was sufficient to derange the propagation of these waves, and prevent their distribution over a wide range; but I have not found that the agitation of the water at all affected the formation of these waves.

It is perhaps of importance to state that when these forced waves were being generated, I have suddenly stopped or withdrawn the disturbing point, that the first wave immediately sprang back from the others, showing that it had been in a state of compression—that the ridges became parallel, and moving on at the rate of $8\frac{1}{2}$ inches per second, disappeared in about 12 seconds.

The manner in which the divergence of the ridge passes through the point of disturbance is shown in the annexed diagram. A B is the path of disturbance, the disturbing point being in B; a rod B A is 25 inches long; B C is the diverging wave ridge; a graduated rod A C projects from A B at the point A, 25 inches behind B, on which are observed the distance of the wave from A along A C, registered in Col. $b$, Table XXIII.

If a body move with a given velocity along a known line A B, the side A C being measured at right angles to the line of direction A B, and cutting, in C, the line B C which represents the ridge of the wave proceeding from the moving body B; it is required to find the velocity of the wave in the line A D perpendicular to its ridge.




As the triangle A B C is right-angled, $\sin B = \frac{\sin A \times b}{\sqrt{b^2+c^2}}$;

and since the triangle A B D is right-angled, $x = \frac{\sin B \times c}{\sin D}$;

hence, the time being the same as that in which A B is described, the velocity is at once obtained.

Table I. contains some observations which were made with a view to the investigation of the ratio subsisting between these velocities. The sides and angles are indicated by the same letters which are used in the diagram.

$c$, its velocity, and $b$ being given; $x$, its velocity, and B were calculated by means of the preceding formulæ.



TABLE XXIII.

*Comparison of Experiments on the Divergence due to given Velocities of Genesis.*

Column $c$ is the constant measure in inches taken along the path of genesis A B in the figure; the adjacent column is the velocity of genesis along A B in inches per second.

Column $b$ is the length A C, measured by observation in a direction at right angles to A B.

Column $x$ is the length of $x$ deduced from the measure $b$, and the adjacent column shows the corresponding deduced velocity of the wave at right angles to its ridge.

Column B shows the angles of divergence given by these observations.

Column $b'$ and B′ are numbers corresponding to $b$ and B obtained from the supposition that the velocity of the wave in a direction at right angles to its own ridge is constant and precisely equal to the velocity of the free wave, viz. $8\frac{1}{2}$ inches per second. The deviations of $b'$ and B′ from $b$ and B were chiefly due to disturbance of the fluid produced by the apparatus employed in genesis.

| $c$. | Velocity in inches per sec. | $b$. | $x$. | Velocity in inches per sec. | B. | $b'$. | B′. |
|---|---|---|---|---|---|---|---|
| | | | | | ° ′ ″ | | ° ′ ″ |
| 25 | 12 | 25 | 17·67 | 8·49 | 45 0 0 | 25·0 | 45 0 0 |
| 25 | 13 | 21 | 16·07 | 8·37 | 40 0 1 | 21·60 | 40 49 48 |
| 25 | 14·4 | 17 | 14·05 | 8·10 | 34 13 7 | 18·27 | 36 10 26 |
| 25 | 16.0 | 14 | 12·16 | 7·79 | 29 7 46 | 15·67 | 32 5 23 |
| 25 | 18·0 | 11 | 10·05 | 7·28 | 23 42 55 | 13·38 | 28 9 32 |
| 25 | 20·6 | 9 | 8·44 | 6·98 | 19 44 43 | 11·31 | 24 21 24 |
| 25 | 24·0 | 8 | 7·61 | 7·31 | 17 43 22 | 9·46 | 20 44 27 |
| 25 | | 7 | 6·74 | | 15 38 35 | | |
| 25 | | 6 | 5·83 | | 13 29 44 | | |

Various considerations induced the acceptance of a constant velocity along $x$ of 8·5 inches per second. The deviations from it in the increasing velocities are due principally to the disturbance of the fluid by the peculiar method of genesis in that instance employed as most convenient. On this assumption the values of $b$ were calculated by the following formula and placed in the column $b'$, and the values of the angle B found in this manner are written under B′.

In the triangle A B D, $\sin B = \frac{\sin D \times x}{c}$;

whence in the triangle A C D, $b = \frac{\sin D \times x}{\sin C}$.

From what has been said, it follows that there can be no difficulty in calculating the velocity of a body or current from the divergence of the capillary wave.

Let $b$ represent the amount of divergence per foot, the time in which a foot will be described, and consequently the velocity per second, can be obtained by the formulæ which were first given; thus, finding the length of $x$, and its velocity being known, the absolute time occupied can at once be found, which time is that in which the moving body traverses one foot. In Table II., columns A, B, contain the divergence of the wave expressed in inches per foot, and the corresponding velocity in inches per second.

## Table XXIV.

*For determining the Velocity of Currents or Moving Bodies by Observation of Divergence.*

Column A gives the divergence from the path of disturbance measured at right angles to the path, in inches per foot of distance from the disturbing point.

Column B gives the corresponding velocity in inches per second, measured along the direction of the stream or the path of the disturbing point.

Column C contains the angle, which may be observed, at which the wave passes off from the disturbing point, and gives in degrees its divergence from the direction of the stream or the path of the disturbing point.

Column D gives the velocity in inches per second corresponding to the angles in C.

| A. | B. | C. | D. |
|---|---|---|---|
| 12 | 12·0 | 60° | 9·81 |
| 11 | 12·62 | 55 | 10·37 |
| 10 | 13·49 | 50 | 11·09 |
| 9 | 14·16 | 45 | 12·02 |
| 8 | 15·38 | 40 | 13·22 |
| | 17·0 | 35 | 14·82 |
| 6 | 19·08 | 30 | 17·0 |
| 5 | 22·10 | 25 | 20·12 |
| 4 | 27·0 | 20 | 24·85 |
| 3 | 35·13 | 15 | 32·84 |
| 2 | 51·77 | 10 | 48·94 |
| 1 | 102·33 | 5 | 97·51 |
| | | 1 | 487·10 |

When the angle of divergence is given, the process is facilitated, as one of the equations used in the previous case has for its sole object the finding of that angle; in Table II., columns C, D, contain the velocities in inches per second corresponding to the given angle of divergence.

Waves of a similar description with those I have here examined, appear first to have been noticed by M. Poncelet, in the course of the valuable experiments made by him and M. Lesbros, which are published in their 'Mémoire sur la dépense des orifices rectangulaires verticaux à grandes dimensions présenté à l'Académie Royale des Sciences,' 16th November, 1829. In a notice in the 'Annales de Chimie et de Physique,' vol. xlvi. 1831. "Sur quelques phénomènes produits à la surface libre des fluides, en repos ou en mouvement, par la présence des corps solides qui y sont plus ou moins plongés, et spécialement sur les ondulations et les rides permanentes qui en résultant," M. Poncelet gives the following description of the phænomena.

"Lorsqu'on approche légèrement l'extrémité d'une tige fine, formée par une substance solide quelconque, de la surface supérieure d'un courant d'eau bien réglé ou constant, il se forme aussitôt à cette surface une quantité de rides proéminentes, enveloppant de toutes parts le point de contact de la tige et du fluide, et présentant l'aspect d'une série de courbes paraboliques qui s'envelopperaient les uns les autres, et auraient pour axe de symmétrie, ou pour grand axe commun, un droit passant par le point dont il s'agit, et dirigée dans le sens même du courant en ce point. L'extrémité inférieure de la tige occupe le sommet de la première parabole intérieure, qui sert comme de limite commun à toutes les autres; le nombre des rides paraît d'ailleurs être infini, et elles sont disposées entre elles à des intervalles distincts qui croissent avec leur distance au point du contact . . . . . . les rides sont parfaitement immobiles et invariables de forme tant que l'état de repos de la tige et de mouvement du courant n'est pas changé; de plus, au lieu de persister plus ou moins après que cette tige a été enlevée, le phénomène disparait brusquement, et à l'instant où le liquide abandonne l'extrémité inférieure de la tige, à laquelle il n'est plus retenue vèrs la fin qu'en vertu de l'adhérence . . . . . . le phénomène s'opere essentiellement à la surface supérieure du fluide.



"...... quand le courant se trouve limité per des parois plus ou moins voisines de la tige, et parallèles à la direction générale des filets fluides, le phénomène des rides se reproduit de la même manière et avec des circonstances sensiblement identiques à celles qui auraient lieu si ces parois n'existaient pas, ou si la masse du fluide était indéfinie; c'est-à-dire que la disposition, la forme et les dimensions des rides sont sensiblement les mêmes, à cela près qu'elles se trouvent brusquement coupées ou interrompues par les parois solides qui limitent le courant comme on le voit représenté, sans éprouver d'ailleurs aucune sorte d'inflexion, de déviation ou de réflexion; l'action de la paroi n'ayant d'autre effet ici que de soulever, à l'ordinaire la surface générale du niveau du fluide . . . . ! ! . . . . le phénomène des rides se manifeste également à l'entour des corps de dimensions plus ou moins grandes, si ce n'est que ces rides s'étendent plus au loin, sont plus larges, plus saillantes, et forment par conséquent des courbes moins déliées et moins distinctes . . . . . . soit que l'on considère les ondulations dans un même profil, soit que l'on considère les ondulations qui se correspond dans des profils différens ou qui appartiennent aux mêmes rides l'amplitude de ces ondulations, c'est-à-dire leur hauteur verticale sera autant moindre, et l'intervalle qui les sépare d'autant plus grand, que les points auxquels elles appartiennent se trouveront plus éloignés . . . . . ces différens systèmes se superposent exactement aux points de leur rencontres mutuelles sans que leur forme soit aucunement altérée . . . . . l'examen attentif de ces changemens de forme et de position des rides produites à la surface d'un courant quelconque par la presence d'un point fine, serait donc très-propre à faire juger, au simple coup d'œil, de l'état même du mouvement en chacun des points de cette surface, et pour chacun des instants successifs où l'on viendrait l'observer . . . . mais cela suppose qu'on a fait à l'avance; une étude beaucoup trop compliquée et trop délicate pour que nous ayons pu jusqu'ici nous en occuper. . . . . on trouve, 1°, que les rides sont imperceptibles quand sa vitesse est moyennement au dessous de 25 c. per seconde; 2°, qu'elles sont d'autant plus déliées d'autant plus distinctes que la vitesse est plus grande; 3°, que le nombre des rides se multiplie aussi à mesure que la vitesse du courant augmente, surtout aux environs du point du contact de la tige; 4°, que les longues branches des rides se réservent de plus en plus. . . . quand la vitesse surpasse 5 ou 6 mètres les differens rides paraissent se réduire à une seule . . . . ce phénomène est telle (in standing water) qu'on croirait volontiers que le déplacement de la tige n'a d'autre effet que de pousser les rides en avant d'elle et d'un mouvement commun sur la surface immobile."

These are mere points of difference between these observations and my own, which I am disposed to attribute to the peculiarities of condition in which the observations of M. Poncelet were made. His observations appear chiefly to have been made in currents, where it was of course impossible to secure uniformity of motion over the whole surface.

## SECTION IV.—WAVES OF THE FOURTH ORDER.

### *The Corpuscular Wave.*

### *The Sound Wave of Water.*

This order of wave I have denominated the corpuscular wave, because the motions by which it is propagated are so minute as to escape altogether direct observation, and it is only by mathematical *à priori* investigation and indirect deductions from phænomena, that we come to recognise its existence as a true physical wave. The motions by which it is propagated are so minute, that it is only by supposing a change in the form of the molecules of the liquid, or of their density, if conceived to be in contact, or an instantaneous and infinitesimal change in the minute distances of the molecules from each other, that the existence of such a wave can be conceived to be possible.

I have not examined this wave by any experiments of my own, and indeed I find that labour to be perfectly unnecessary, for there has been kindly transmitted to me by M. Colladon, a communication of his to the Academy of Sciences, which has been printed in the fifth volume of the 'Mémoires des Savans Étrangères,' in which there is given in great detail, an account of a complete and most satisfactory determination of the elements of this question.

Newton's approximate determination of the velocity of sound in the atmosphere was followed by that of Dr. Young and M. Laplace, who effected a similar approximation for water and other liquids, and finally the complete solution was satisfactorily given by M. Poisson, the velocity being determined both for solids and liquids by the formula

$$c=\sqrt{\frac{\mathrm{P}k}{\mathrm{D}\varepsilon}},$$

where D is the density of the substance, $k$ the length of a given column, and $\varepsilon$ the small diminution of length caused by a given pressure P.

For the determination of the velocity of the sound wave in water, MM. Colladon and Sturm undertook a series of experiments on the compression of liquids, conducted with very ingenious apparatus, and observed and discussed with much accuracy; by this means they obtained values for the quantities P, $k$ and $\varepsilon$, from which the velocity of sound should be theoretically determined.

They obtained for the water of the lake of Geneva the following quantities:—

Assuming $\mathrm{D}=1,\ k=1{,}000{,}000,$

they found $\varepsilon=48{\cdot}66,$

and $\mathrm{P}=(0^{\mathrm{m}}{\cdot}76).g.m=(0^{\mathrm{m}}{\cdot}76).(9{\cdot}8088).(13{,}544),$

whence $c=1437{\cdot}8$ mètres,

being the theoretical velocity per second of the sound wave in water.

A very elegant apparatus was next employed for the direct determination by experiment of the truth of this result. Two stations were taken on the lake of Geneva, the mean depth of water lying between them being about seventy fathoms, and the distance between the stations was carefully determined to be 13,487 mètres, or 14,833 yards, about eight miles and a half, lying between the towns Rolle and Thonon. At one end of this station a large bell was suspended at a depth of five or six fathoms below the surface of the water, and struck by mechanism so contrived, as at the instant of striking to explode a small quantity of gunpowder, and so indicate (during a dark night) to the observer, eight miles off, the instant at which the bell was struck. This sound was distinctly heard by a sort of ear-trumpet lowered in the water at the other end, and so the observations made.



The mean time occupied in propagating the sound from one station to the other as thus determined, was nine seconds and a half, or more precisely 9·4 seconds, giving for the velocity of sound by direct experiment

$$c=\frac{13487}{9{\cdot}4}=1435 \text{ mètres},$$

the actual velocity of the sound wave thus being found to differ from the theoretical by not three mètres per second.

The velocity of transmission of the wave of the fourth order in water is therefore in English measure about 1580 yards per second, being about one-half more rapid than the velocity of sound through the atmosphere.

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## DESCRIPTION OF THE PLATES.

### PLATE XLVII.

*Genesis and Mechanism of the Wave of Translation.*—Order I.

Fig. 1. *Genesis by impulsion.*—A X is the bottom of a long rectangular channel filled with water to a uniform depth; P a thin plate inserted vertically in the fluid and fitting the internal surface of the channel. It is moved forward from A towards X through the successive positions $\mathrm{P}_1$, $\mathrm{P}_2$, $\mathrm{P}_3$, $\mathrm{P}_4$, $\mathrm{P}_5$, and heaping up the water before it generates a wave of the first order $\mathrm{W}_4$, $\mathrm{W}_5$, which is transmitted along the channel as at $\mathrm{W}_5$, $\mathrm{W}_6$ to $\mathrm{W}_8$, &c., being transmitted with uniform velocity as a great solitary wave, and leaving the water behind it in repose at the original level.

Fig. 2. *Genesis by a column of fluid.*—In the same channel the moveable plate $\mathrm{P}_7$ is fixed so as with the end and sides of the channel to form reservoir A G $\mathrm{P}_4$, containing a column of water G W, raised above the surface of repose of the water in the channel. $\mathrm{P}_7$ is suddenly raised as at $\mathrm{P}_8$ and $\mathrm{P}_9$; the column descends, presses forward the column anterior to P, and raises the surface, generating a wave of translation, which is transmitted along the channel as before. After genesis the volume $g_1$ reposes on the level $g_2$, the water in the channel having been translated forwards from P to $kk$; every particle of water in the channel has during the transmission of the wave been translated towards X through a horizontal distance equal to P$k$.

Fig. 3. *Genesis by protrusion of a solid.*—$\mathrm{L}_1$ is a solid suspended at the end of the channel, its inferior surface slightly immersed in the fluid. It is suddenly detached, descends, displaces the adjacent fluid, and generates a wave of translation as in the foregoing methods.

Fig. 4 exhibits the phænomena of genesis, transmission and regenesis, or reflexion of the wave of translation.

Fig. 5 exhibits in four diagrams the motions of individual wave particles during wave transmission. The *first* diagram represents by arrows the simultaneous motions of the particles in different portions of the same wave at successive points in its length. At the front of the wave the particles $a$, $c$, $e$, $g$, taken at equal depths below the surface, are at rest. The wave-length is divided into ten equal parts: at the first the motion is chiefly upwards, and very slightly forwards; at the second, less upwards and more forwards; at the third, still less upwards and still more forwards, and so on; the inclination of the path diminishing to the middle of the wave, where the velocity is greatest and the direction quite horizontal. Behind this part of the wave the particles are to be seen descending more and more with a motion gradually retarded, and at the hinder extremity of the wave they are in repose, as at the front. These motions of the particles of water are rendered visible by minute particles of any kind mixed with the water and nearly of the same specific gravity. Such are the simultaneous motions of the successive particles at different stations along the same wave, as observed in a channel by glass windows placed in the sides and carefully graduated in small squares for the purpose of observation, the side of the channel opposite to the window being similarly graduated. The *second* diagram represents the paths of four particles described during the whole period of transmission of a wave. The wave is transmitted from A towards X. The anterior extremity of the wave finds one particle at $a$ and carries it forward through an ellipse to $b$, where it is left by the end of the wave: the same wave translates the particle $c$ vertically below $a$ through its elliptical path and leaves it at $d$ vertically below $b$, and in like manner $e$ and $g$ are transferred to $f$ and $h$. All these paths are semi-ellipses (as nearly as it is possible to observe them), and are of the same major axis; but the semi-minor axis is at the surface equal to the height of the wave-crest, and diminishes with the distance from the bottom of the channel, where it is nil. The *third* diagram exhibits the phænomena of vertical sections during wave transmission: small globules of greater specific gravity than water are suspended at different depths by means of long slender stalks of less specific gravity. These globules are arranged while the water is in repose, in vertical planes at equal distances along the fluid. These vertical planes are, by transmitting the wave, made to approach each other, but still retaining their verticality without sensible disturbance. At the middle of the wave-length they are brought closest, and at the hinder extremity they recede and settle down at their original depth. The *fourth* diagram shows the change of the position of points in the same horizontal planes during wave transmission, particles vertically equidistant in repose remaining equidistant during wave transmission.

Fig. 6. *Genesis of compound waves.*—The first diagram represents the genesis by a large low column of fluid of a compound or double wave of the first order, which immediately breaks down by spontaneous analysis into two, the greater moving faster and altogether leaving the smaller. The second diagram represents the genesis by a high small column of fluid of a positive and negative wave, which soon separate, the positive wave travelling more rapidly, leaving altogether the residuary negative wave. The negative wave is further noticed in another Plate. $W_1$ is the positive and $w_1$ the residuary positive or negative wave as generated. $W_2$ and $w_2$ represent them separated by propagation.

## Plate XLVIII.

### *Discussion of Observations on the Velocity of Waves.*—Order I.

Fig. 1. Comparison of the observations marked by stars with the formula B. indicated by the parabola A B, of which A X is the axis, parallel to which are measured abscissæ I., II., III., &c., representing the depth of the fluid in inches, the corresponding velocities being represented by ordinates A 1, A 2, A 3, A Y, &c. at right angles to A X. The manner in which the curve passes through among the stars, shows the close approximation of the results of individual experiments to the formula B adopted to represent them. These are taken from the Table V.

Fig. 2 exhibits a similar comparison for waves of a larger size than the former. See Table IV.

Figs. 3 and 4 show a comparison with the observations, marked by stars as before, with formulæ proposed by Mr. Airy, shown as dots connected by dotted lines, and with the formulæ employed by the author, shown by a continuous black line A B. The eye at once decides whether the black line or the dots and dotted line most nearly coincides with the stars. See Tables VI. and VII.

Fig. 5 exhibits a similar comparison of the velocity of negative waves, as observed in a rectangular channel along A B, and in a triangular channel as shown along A B′. The stars show that the velocity falls below that which the formulæ would assign as due to the depth, especially in the triangular channel. See Tables XI. and XII.

Fig. 6 exhibits the general results of experiments on velocity; the horizontal spaces indicate depths of five inches to each, and the velocities in per second are represented by the vertical spaces which represent each the velocity of one foot per second in transmission of the wave. A B is the line of the formula for a rectangular channel, see Table III.; and A B′ for a triangular channel, see Table XV.

Fig. 7. A X is the surface of water four inches deep; B X represents the successive heights of a wave as referred to in Table II.

## Plate XLIX.

### *Rediscussion of the Experiments on Velocity.—By the Method of Curves.*

Fig. 1. BC, DE, FG, &c. are lines drawn by the eye through the observations of heights of waves shown by the stars; similar lines were drawn through the corresponding observations of velocity. These waves were taken as representing the experiments cleared of errors of observation; they were then collected and laid down in fig. 2.

Fig. 2. A B is the line given by the formulæ employed by the author to represent the velocity of the wave of the first order; the stars are the observations freed in some measure from errors of observation as described above.

## Plate XLIX. (continued.)

### *Effects of Form of Channel on the Wave.*—Order I.

*Note.*—In a rectangular channel on a level plane the crest of the wave is a horizontal line, parallel to the bottom.

Fig. 3. The section across a channel; $a\,w$ the surface of the water in repose; $a\,d = 4$ inches; $w\,e = 1$ inch; A $a$ the height of the wave-crest $= 1\frac{1}{2}$ inch; B $w$ the height on the shallow side $= 2\frac{1}{2}$ inches.

Fig. 4. A B $d$ the cross section of a triangular channel, A B the crest of the wave, $a\,w$ the level of the water in repose; the angle A $d$ B $= 90°$.

Fig. 5. B $c\,d$ a slope of 1 in 3, being the cross section of a channel $c\,d\,f$; A B the crest of the wave breaking on the sides, where the height of the wave becomes equal to the depth of the water.

Fig. 6. Cross section of another form of channel.

Fig. 7. The sea-beach near Kingstown and Dublin. Common sea waves, $W_1$, $W_2$, $W_3$, $W_4$, $W_5$, $W_6$, break on the ridge $d$, where their height is equal to the depth of the still water. They generate small waves of the first order, $w_1$, $w_2$, $w_3$, $w_4$, $w_5$, &c., which are propagated through the still,

shallow water to great distances, and the intervals between them are left level and in repose.

## Plate L.

### *Waves of the First Order—Drawn by themselves.*

These eight waves are of the natural size, being mere transcripts of the outline of a wave left on a dry surface. The four lower outlines in the Plate were obtained by inserting a dry surface, moved horizontally with a uniform velocity equal to that of the wave, and instantly removing it. The moist outline left by the wave was copied on tracing paper, and transferred without change to the copper-plate. Another method produced the four upper outlines, which were obtained by passing under the wave to be observed another wave transmitted in the opposite direction. These outlines are not therefore to be regarded as copies of a wave, but as transcripts of the outline left by the passage of one wave over another; the crests of both describe horizontal straight lines on the side of the channel, but as every point of one may be regarded as passing over the crest of the other, there is a moist outline left on the side of the channel at the crossing, which outline is simply transferred to the copper, as in the four upper waves. Where a dotted line occurs a blank was left in the outline, which is filled up by the eye. The depth of the water was 2 inches, and the parallel lines in the figure are at 1 inch apart and serve as a scale.

## Plate LI.

These waves are taken in the same manner, but have been reduced from the original outlines to a smaller scale—smaller than the original in the ratio of 2 to 3. The horizontal lines are $\frac{2}{3}$rds of an inch apart, which represent an inch on the full size. The four lowest are taken from waves in water 2 inches deep on a sloping beach, parallel to $g$ X, $k$ X, $l$ X and $m$ X, with an inclination of 1 in 12. The four next are imperfect or compound waves taken from the outline left by passing another in the opposite direction. The two highest are taken in the same way, one of them in the act of breaking.

## Plate LII.

### *The Wave of the First Order.*

Fig. 1 represents the genesis of a compound wave by impulsion of the plate with a variable force and velocity, which variations have produced corresponding variations on the wave form. After propagation the wave breaks down by spontaneous analysis; the higher part moves forward, as shown by the dotted line, and ultimately leaves the rest behind, so that after the lapse of a considerable period the compound wave is resolved into single separate waves, each moving with the velocity due to the depth.

Fig. 2 represents the phænomena resulting from genesis by a long, low column of water. Instead of genesis of a compound wave, as in the former case of impulsion, the added mass sends off a series of single waves, the first being the greatest: these however do not remain together, but speedily separate, as shown in the dotted lines, and become the further apart the longer they travel.

Figs. 3, 4, 5 and 6 give geometrical approximations to the representation of the wave form and phænomena. In fig. 3, $d$ D $d$ is the length of a small wave divided into ten equal parts; $c\,d$ is equal to the height of the wave, on which a circle is described, and of which the circumference is also divided into ten equal parts. Through these equal divisions of the circle are drawn horizontal lines, which are intersected by vertical lines from each of the divisions of the straight line $d\,d$, as shown in the figure. A continuous line, passing through these points of intersection, has for its vertical ordinates the versed sines of the arcs of the circle, while its abscissæ are proportional to the arcs themselves. Such a line is the curve of versed sines, and gives a first approximation to the form of the wave of the first order.



Fig. 4 gives a second approximation to the form and the representation of the phænomena of the wave of the first order. A D $d$ is taken equal to the length of the wave in the first approximation = 6·28 times the depth of the fluid in repose; on $d\,c$ = the height of the wave, a circle is described and divided into equal arcs as formerly, and thus a dotted line, A C $d$, is formed as before, being the first approximation to the wave form. These equal arcs being taken to represent equal times, the versed sines also represent the rise and fall of the surface of the wave during equal successive intervals of time. But hitherto we have neglected the motion of translation, the horizontal transference of each vertical column of fluid in the direction of wave transmission simultaneous with the vertical motion.

Take the length A to A′, such that A A′ × A B shall $= \frac{V}{b} =$ the volume of water generating the wave divided by the breadth of the fluid. This length, A B, in a small wave will be about three times the height of the wave. Take A A′ as the major axis of an ellipse, of which the minor axis is C D or $c\,d$, the height of the wave. Let the horizontal lines through the equal arcs of the small circle $c\,d$ be extended to pass through the ellipse A A′, and from the points of division let fall perpendiculars on A A′ on the points 1, 2, 3, 4, 5, 6, 7, 8, 9, then the lines on the axis A A′, viz. A 1, A 2, A 3, A 4, A 5, A 6, A 7, A 8, A 9, A A′ represent the amount of horizontal transference effected during the same time, in which a given particle on the surface is rising and falling through the versed sines of the equal arcs, viz. $d$ 1, $d$ 2, $d$ 3, $d$ 4, $d$ 5, $d$ 6, $d$ 7, $d$ 8, $d$ 9, $d\,d$. Let us now effect this horizontal transference on each point of the surface on the first wave A C $d$, by advancing the point 1 horizontally through a distance equal to A 1; 2 through a distance A 2; 3 through a distance $a$ 3, and so on, and we shall get a curve A′ C′ $d$, which closely represents the form of the wave, and also its phænomena of horizontal translation = throughout the whole depth to A 1, A 2, A 3, A 4, A 5, &c.

Fig. 5 is obtained in the same way as fig. 4, only for a larger wave; where the height is nearly equal to the depth of the fluid, the ellipse is nearly a semicircle. The same ellipse represents also the absolute path of a particle on the surface during wave transmission. Ellipses of the same major axes, but having their minor axes diminishing with their distance from the bottom of the channel, will represent the simultaneous motions of particles below the surface.

Fig. 6 shows a single particle path, and three successive positions of the wave outline in regard to it. The figures 1, 2, 3, 4, 5, &c., give the simultaneous positions of the particle referred to the wave surface, and the same particle referred to the path of the particle. When at 1, 2, 3, 4, 5, &c. in the orbit, the particle is also at 1, 2, 3, 4, 5, &c. in the wave surface. Thus the points which succeed each other towards the right on the path, succeed towards the left on the wave form.

Figs. 7 and 8 represent the genesis of the negative wave of the first order. A solid Q 2 reposes suspended in the fluid, and is suddenly raised out of it. A negative wave is generated and propagated along the channel, as W 1 in figs. 8, 9 and 10. This negative wave of the first order, if it encounter a positive wave of the first order, of equal volume, does not pass over it, but they neutralize each other and are annihilated. If unequal, their difference, positive or negative, alone remains, and is propagated as a wave of the first order.

Figs. 9 and 10 record observations, showing that although the negative wave is in its own order solitary, yet that its existence is the cause of genesis of a group of gregarious waves, or waves of oscillation of the second order; W 1 is a negative wave of the first order: W 1, W 2, W 3, &c. are all waves of the second order. The curved arrows in fig. 9 show the semi-elliptical path of the particles during the transmission of the negative wave. After which, during the transmission of the waves of the second order, the particles describe circles, continually diminishing in diameter as the waves gradually subside.

## Plate LIII.

### *Waves of the First Order.—Reflexion, Non-reflexion and Lateral Accumulation.*

In this Plate a wave of the first order, $W_1$ R, is represented as incident upon a vertical plane surface immovable at R; *i. e.* the ridge of the wave forms a given angle $R_1$ W A. After impact at R the wave is reflected, so that the angle of reflexion is equal to the angle of incidence; and when the angle of direction of transmission is great (*i.e.* when the angle of the ridge with the surface is small, not greater than 30°), the reflexion is complete in angle and in quantity. When the angle of direction of transmission diminishes (*i.e.* when the ridge of the wave makes an angle greater than 30°), the angle of reflexion is still equal to the angle of incidence, but the reflected wave is less in quantity than the incident wave. The magnitude of the reflected wave diminishes as the angle of incidence diminishes, until at length, when the angle of the ridge of the wave is within 15° or 20° of being perpendicular to the plane, reflexion ceases, the size of the wave near the point of incidence and its velocity rapidly increases, and it moves forward rapidly with a high crest at right angles to the resisting surface. Thus at different angles we have the phænomena of total reflexion, partial reflexion, and non-reflexion and lateral accumulation; phænomena analogous in name, but dissimilar in condition from the reflexion of heights, &c.

## Plate LIV.

### *Lateral Diffusion of the Wave of Translation round an Axis.*

Figs. 1, 2, 3 and 4 represent a large rectangular reservoir of water filled to an uniform depth with water. It is 20 feet square. From a chamber C in one corner a wave of the first order was transmitted in the direction W 1, W 2; and the observations made which appear in the figures.

In fig. 4 the aspect of the phænomenon is represented. The wave is propagated in the direction of original propagation, which we shall call its axis, with a gradual diminution of its height according to the length of its path along the axis. The observations are probably not yet sufficiently numerous to determine accurately the law of diminution. From this axis the wave spreads on every side. At right angles to the axis of propagation the height of the wave is scarcely sensible, and the diminution of magnitude is very rapid as the line of direction diverges from the axis. The wave is also propagated faster in the direction of the primary axis than in any other direction, so that the wave-crest is elliptic and elongated in that direction.

In fig. 3 the heights of a wave are marked by lines. Each line along W w and W 2 *w* represents one-tenth of an inch in height of the wave; so that the height of the wave is indicated to the eye by the number of lines. These observations are made on concentric circles.

In figs. 1 and 2 the same kind of observations is represented, only along straight lines.



## Plate LV.

### *Waves of the Second Order.—Standing Waves in Running Water.*

The forms of the waves in these figures are the same as those in figs. 9, 10 of Plate LII., being all cycloidal; with this difference only, that the waves in Plate LII. were moving along the standing water with a uniform velocity, while those in Plate LV. are standing in the running water. The generating course in this case is a large obstacle or large stone in the running stream. On this the water impinges; it is heaped up behind it; it acquires a circular motion which is alternately coincident with and opposed to the stream; the water having once acquired this circular oscillating motion in a vertical direction retains it, the water is alternately accumulated and accelerated, and thus standing waves are formed, as shown in figs. 1 and 2.

Figs. 3 and 4 exhibit a remarkable case of the coexistence in one stream of two sets of waves moving with velocities differing in about the proportion of two to three. On one side of a stream there projected a ledge of rock M, over which fell a thin sheet of water into a large pool, nearly still, without generating any sensible wave. On the opposite side a deep violent current was running round the obstacle with great rapidity. The middle part of the channel was occupied by a large boulder, over which also a stream flowed, generating standing waves with a smaller velocity. These waves are also remarkable for non-diffusion, as they will preserve their visible identity to a great distance without being dissipated.

## Plate LVI.

### *Waves of the Second Order.—Their Mechanism.*

All the waves of the second order, whether standing waves in running water or travelling waves in standing water, exhibit the forms of the curves B A B C D in fig. 1. These are cycloids, having for their base the rectilineal distance A C, and for their height the corresponding circles. In the case of standing waves in running water these cycloids represent the actual paths of individual particles of water in the running stream, as shown in Plate LV. In the case of travelling waves in standing water, the circles represent the paths described by the individual particles of water, and the cycloids the visible moving surface presented to the eye. The motion of oscillation in the *upper* half of the circle is in running water, *opposite* to the motion of the stream, and in standing water is in the same direction as the visible motion of transmission of the waves. The figure shows the rapid diminution of the motion of oscillation with the depth. I am indebted for this figure to M. Gerstner, whose theory it illustrates, and I have given it because I find it represent my own observations as correctly as any figure of my own could do. I have only found it necessary in reconstructing his figure to clear it of some slight inaccuracies. The shaded parts on the left show the different forms which given portions of water successively assume during wave motion. The circular orbits are divided into equal portions, numbered 1, 2, 3, 4, &c., to show that the particles of water are in those points of the circles at the same instants the corresponding particles are at the points 1, 2, 3, 4, &c. of the cycloidal paths.

PLATE LVI. (continued.)

*Waves of the Third Order.—Capillary Waves.*

Fig. 2 represents a slender rod inserted in standing water, raising around it by capillary attraction a circular portion of the surface of the fluid. A slow motion gives it the form represented in fig. 3, and more rapid motions, but all of less than a foot per second, give it the forms in figs. 4, 5 and 6; at the velocity of one foot per second the phænomena become those represented in Plate LVII.

PLATE LVII.

*Waves of the Third Order.—Capillary Waves.*

This Plate gives a plan and section on one-half of the natural size of the group of capillary waves generated by a disturbing rod one-sixteenth of an inch in diameter, moving along the surface with a uniform velocity. The section is taken in the direction of the motion of disturbance from A to X, and the same letters refer to the ridges of the same waves in both plan and section. The velocity is one foot per second.

## PROVISIONAL REPORTS AND NOTICES OF PROGRESS IN SPECIAL RESEARCHES ENTRUSTED TO COMMITTEES AND INDIVIDUALS.

### *On the Marine Zoology of Corfu and the Ionian Islands. By* Capt. PORTLOCK, *R.E., F.R.S.*

THE author presented a Report of the progress which he had made in the above research by dredging the sea-bed and registering the results of this operation. The spaces at present investigated are of small extent, but the author is preparing to enter on wider areas, in the expectation of presenting hereafter an arranged summary of his observations. [A Committee has been appointed to cooperate with Capt. Portlock.]

### *On Captive Balloons.*

DR. ROBINSON stated that he must still report progress, for in a course of experiments so new to him and his coadjutors, they had found it necessary occasionally to vary arrangements which at first seemed satisfactory. In particular, the plan of having the telegraphic wires separate from the moorings of the balloon has been changed, and a single cord, *wormed* as sailors call it with copper wire, is substituted. This, besides being more manageable, will permit a greater elevation to be attained. In one of the trials the balloon received a trifling injury, which however was easily repaired. Dr. Robinson thinks that no serious difficulties are now likely to occur.

### *Report of the Dredging Committee for* 1844.

THIS Report consisted of two parts; first, of the records of a series of dredging operations conducted round the coasts of Anglesea in September 1844, by Mr. M'Andrew and Prof. E. Forbes, exhibiting the distribution of the marine animals procured in various depths down to thirty fathoms, and the state of the sea-bed in the localities explored. Among the more interesting facts recorded in these papers were the following:—Rolled specimens of *Purpura lapillus*, a shell which lives only above low-water mark, were found in twenty-eight to thirty fathoms water on the gravelly bed of a line of current at the distance of eight miles from the nearest shore. In the same line of current it was found that the few mollusca which lived there, such as *Modiolæ* and *Limæ*, had constructed nests or protecting cases of pebbles bound together by threads of byssus; and one species, the *Modiola discrepans*, had made its nest of the leaf-like expansions of *Flustra foliacea* cemented together.

The attention of the dredgers was directed among other subjects to the distribution of *Serpulæ*, and the results of their researches were confirmatory of the statements recently advanced by Dr. Philippi of Cassel, namely, that no dependence could be placed even as to the genus on the shell of a *Serpula*, perfectly similar shells being constructed by animals of different genera. Thus they found all the *Serpulæ* of a particular form in twelve fathoms water to be a species of *Eupomatus*, whilst exactly similar shells in twenty fathoms proved to be the habitation of a species of the genus, wanting opercula, of which *Serpula tubularia* is the type. All the triangular *Serpulæ* they met with were *Pomatoceros tricuspis*. In twelve fathoms, at the entrance of the Menai Straits, they dredged the shell of *Helix aspersa*, the common snail, covered with barnacles and *Serpulæ*, and inhabited by a hermit crab.

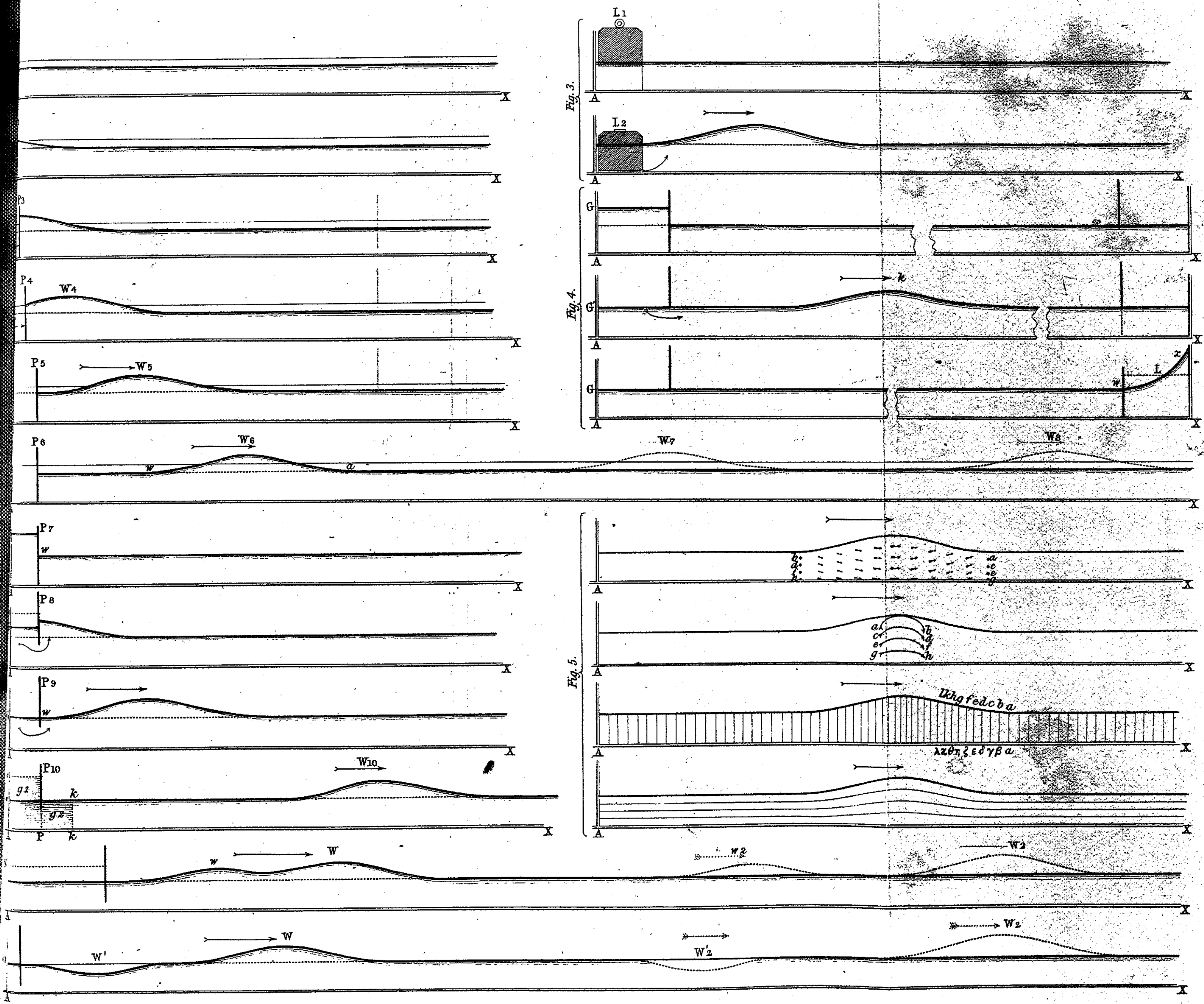
The second part consisted of a series of dredging observations on the Irish coast, drawn up by Mr. Hyndman.

### *On the Hourly Meteorological Observations carried on at Inverness, at the Expense of the British Association, by* Mr. Thomas Mackenzie, *a Provisional Report was presented by* Sir D. BREWSTER.

### *On the Forms of Ships.*

MR. SCOTT RUSSELL reported that the Committee on the form of ships had now completed their labours; that the whole of the tables of the experiments and all the drawings of the forms of the ships were now ready for publication. These tables were so voluminous, and the plates required for illustration were so numerous and expensive, that the question of publication was likely to be attended with some difficulty; but a committee, consisting of the President of the Royal Society, the Dean of Ely, Colonel Sabine and Mr. Taylor, had been appointed for the purpose of making the necessary arrangements. He had now to communicate to the meeting an important addition which had been made to these experiments during the past year. The members of this Section were aware that the former experiments made by the Committee comprehended vessels of many forms and various sizes, from the length of a few inches to ships of 2000 tons displacement; but in all these experiments direct mechanical means of propulsion had been employed and not the force of the wind, and they were therefore regarded as applicable to steam-vessels rather than to sailing ships. During last year, however, most satisfactory experiments had been made in which the propelling force was the wind acting on the sails of the vessel of the open sea. The circumstances in which this experiment originated displayed in a striking manner the advantages conferred by an Association like this on the districts which it visited. The two gentlemen who had conducted this experiment were both Irishmen,—one, Dr. Corrigan of Dublin, having become acquainted through the last meeting in Cork with the experiments of this Association, determined, in building a pleasure-boat, to carry out the principles which had been established by those experiments, and to have his vessel built on that form which was pointed out by these experiments as the *form of least resistance*; he accordingly built

# WAVES_Order I. The Great Wave of Translation



British Association for 1841.
Plate 48.

# WAVES_Order I._Discussion of Experiments.

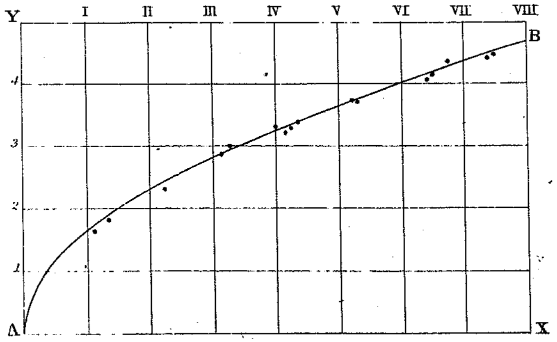


Fig. 1.

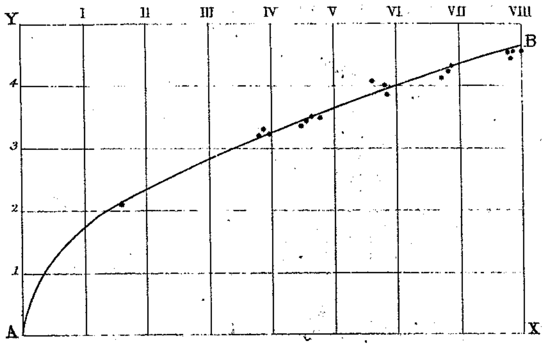


Fig. 2.

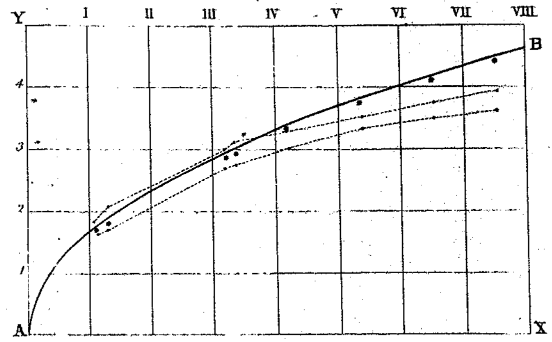


Fig. 3.

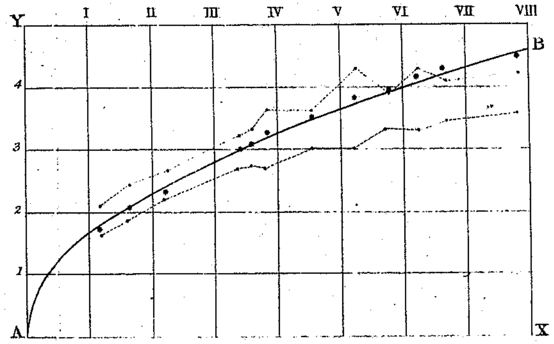


Fig. 4.

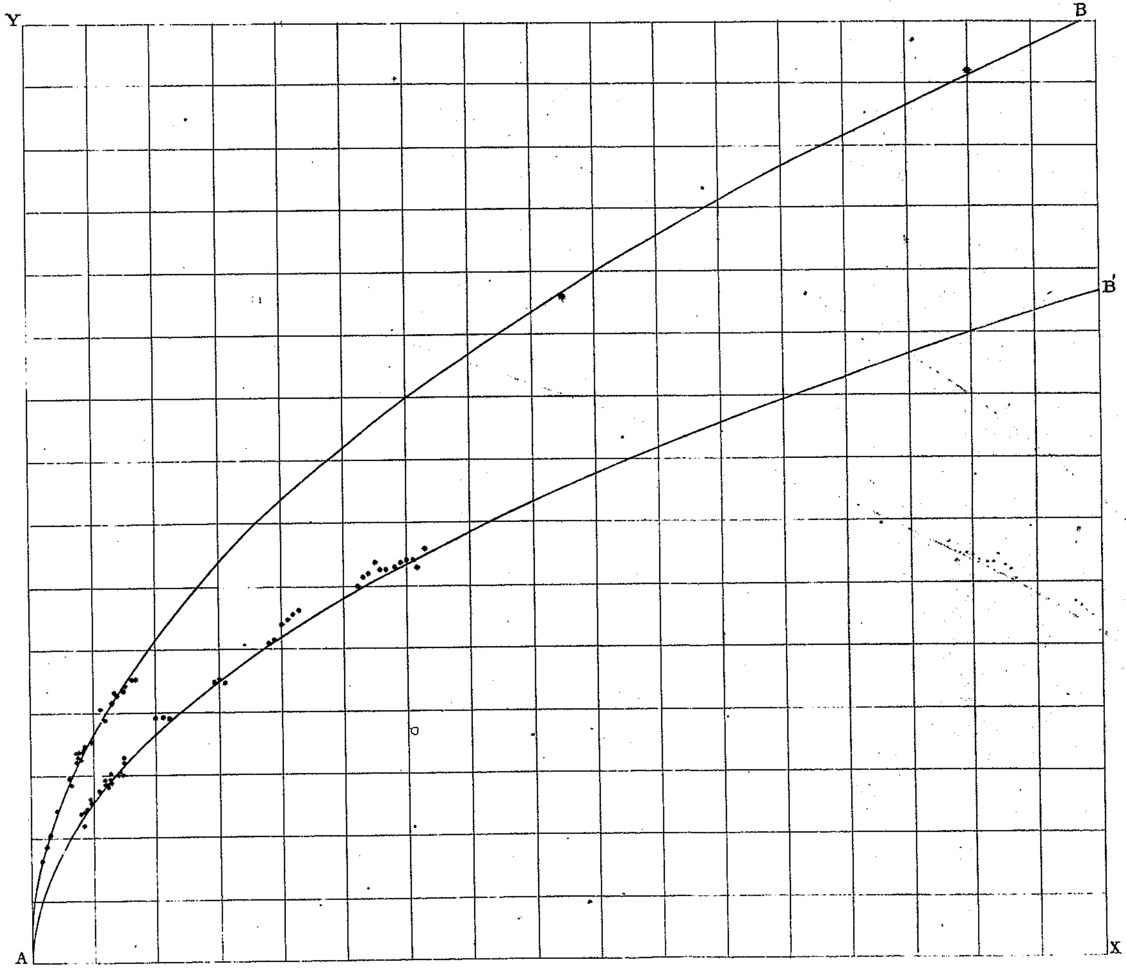


Fig 6.

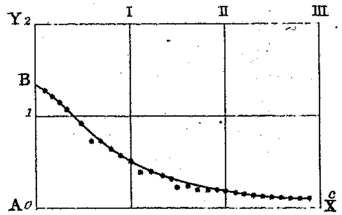


Fig. 7.

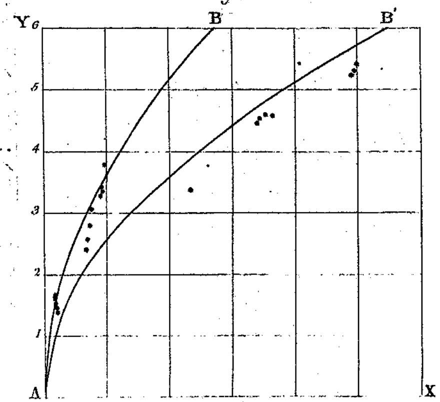


Fig. 5.

# WAVES — Order I.

## Re discussion of the Experiments.

Fig. 1.

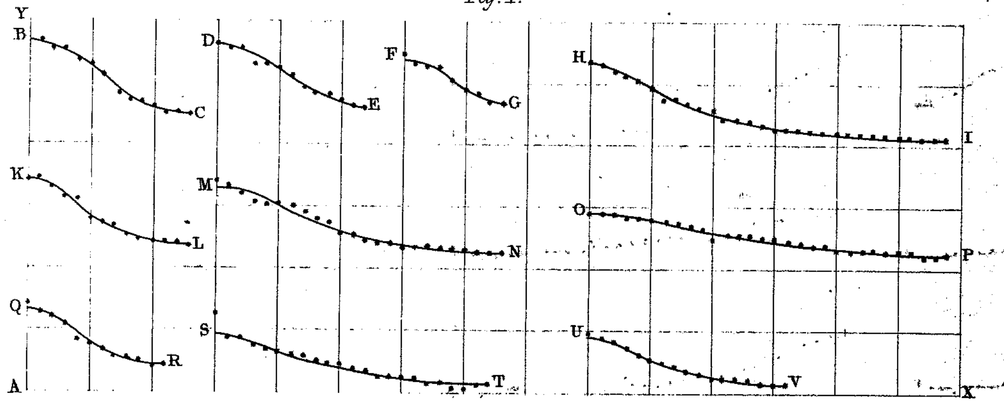
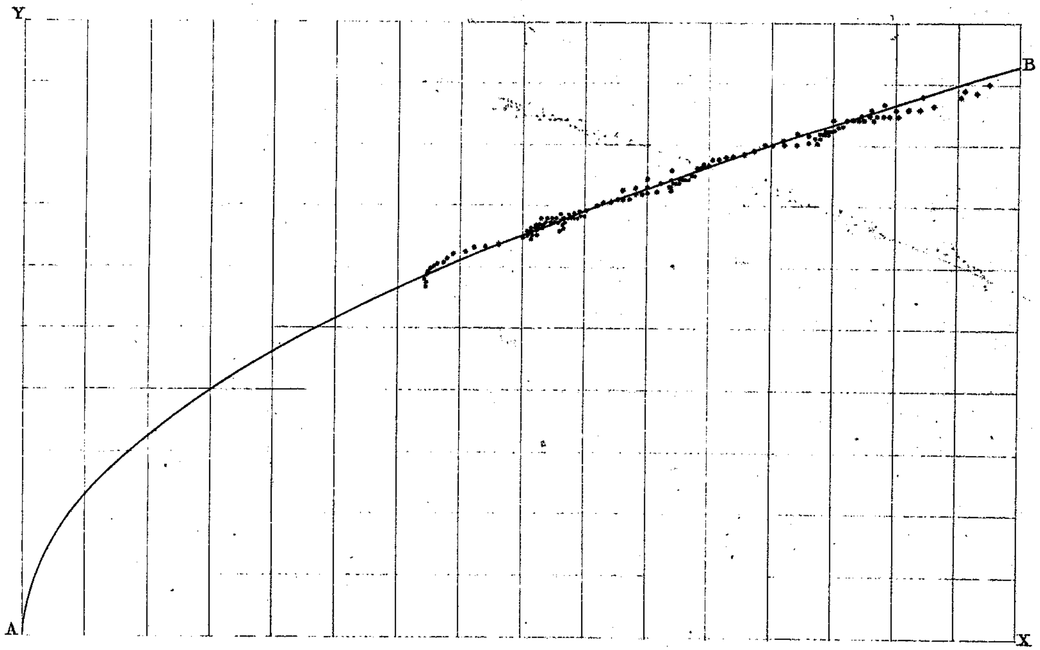



Fig. 2.




## Effects of form of Channel.

Fig. 3.

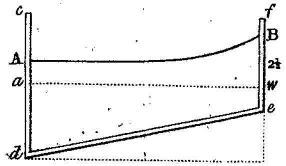



Fig. 4.

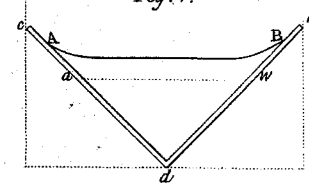



Fig. 5.

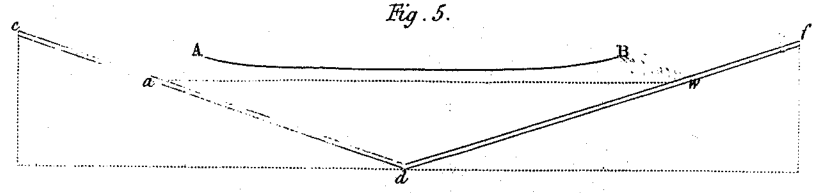
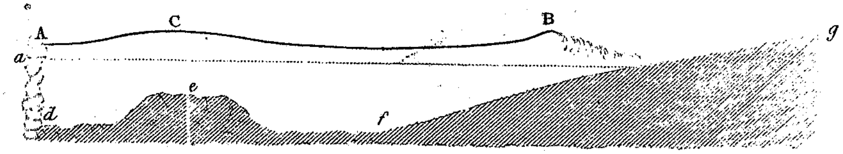
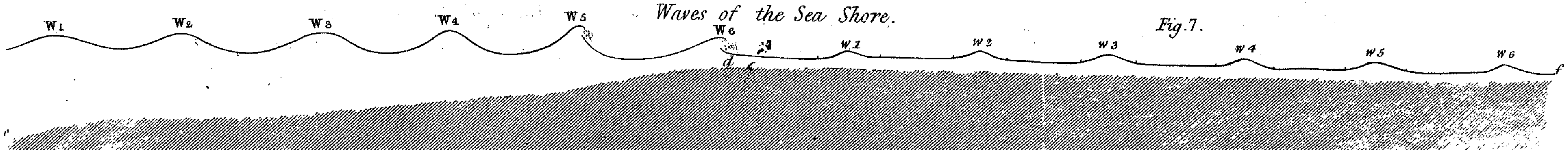



Fig. 6.



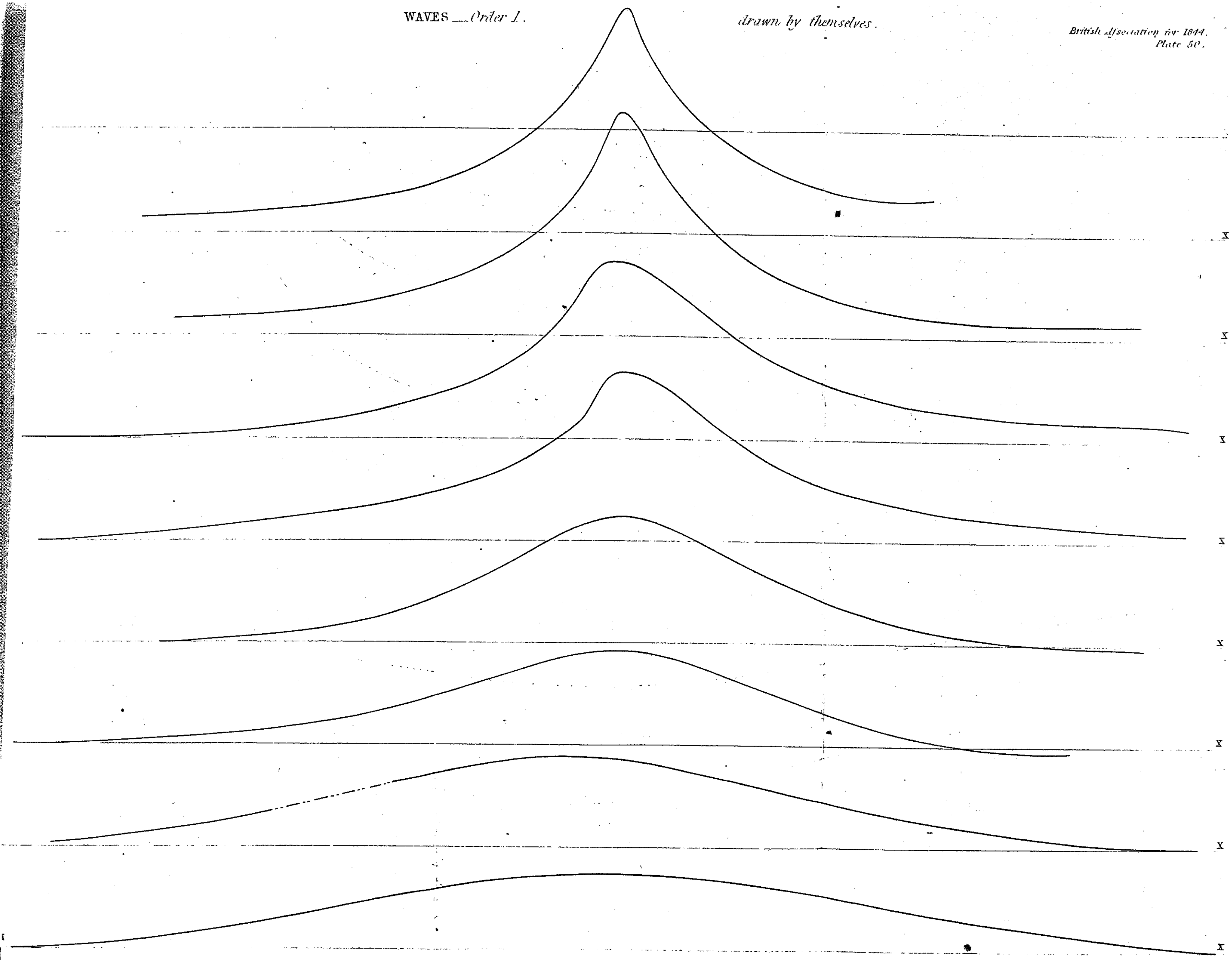

## Waves of the Sea Shore.

Fig. 7.



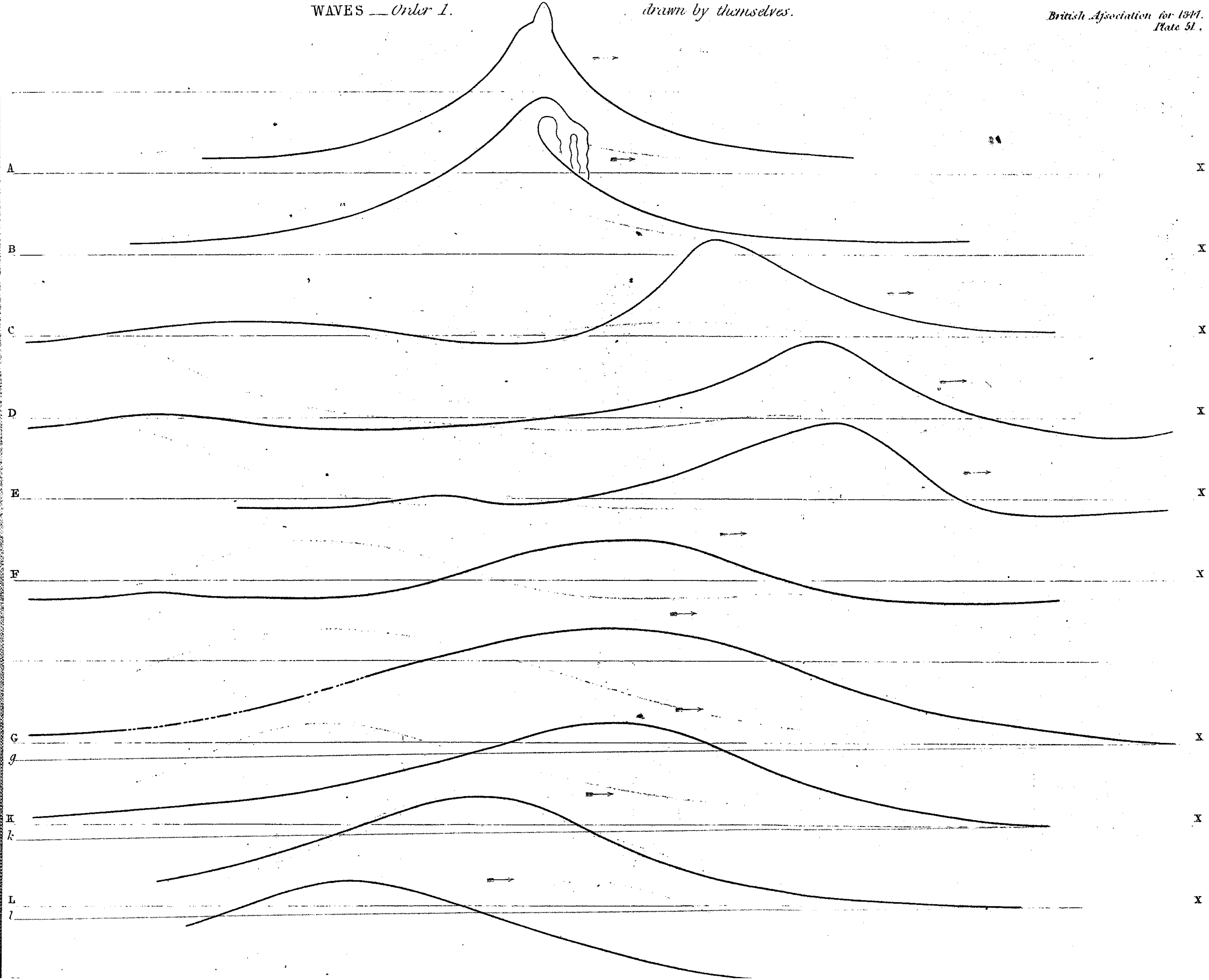
WAVES — Order 1. drawn by themselves.

British Association for 1844.
Plate 50.

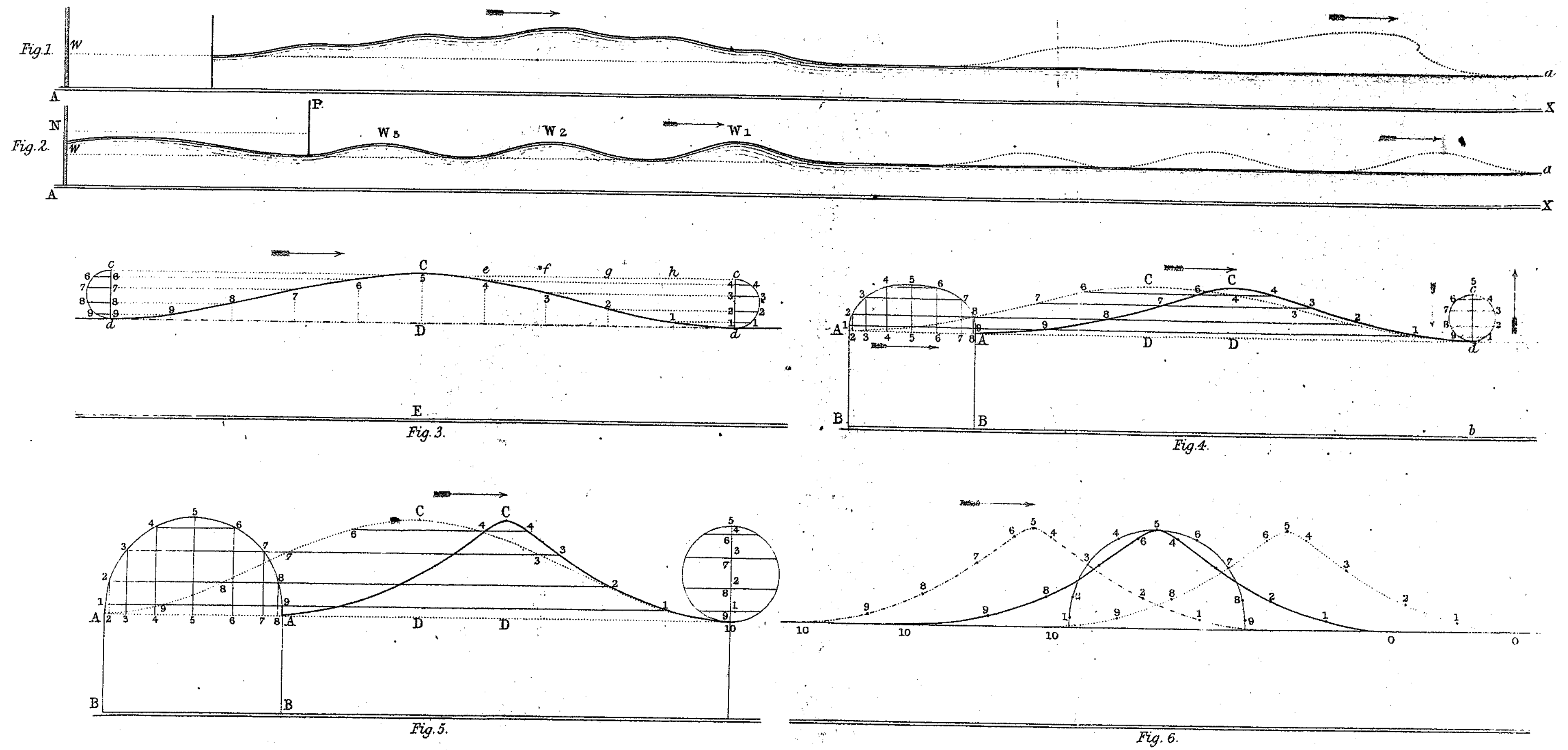


WAVES — *Order I.* *drawn by themselves.*

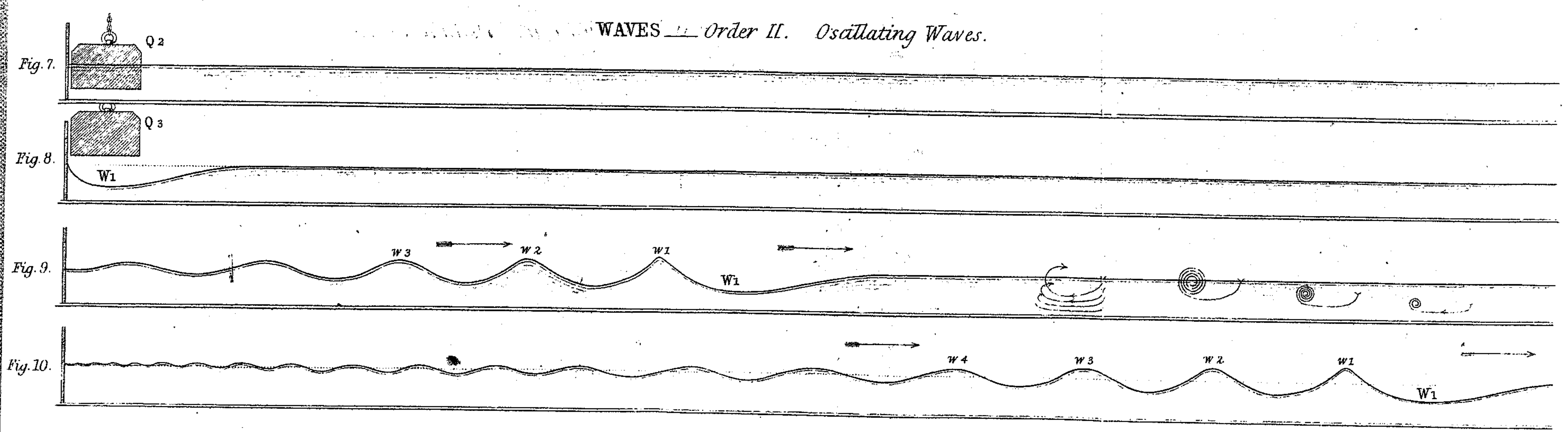




## WAVES — Order I.

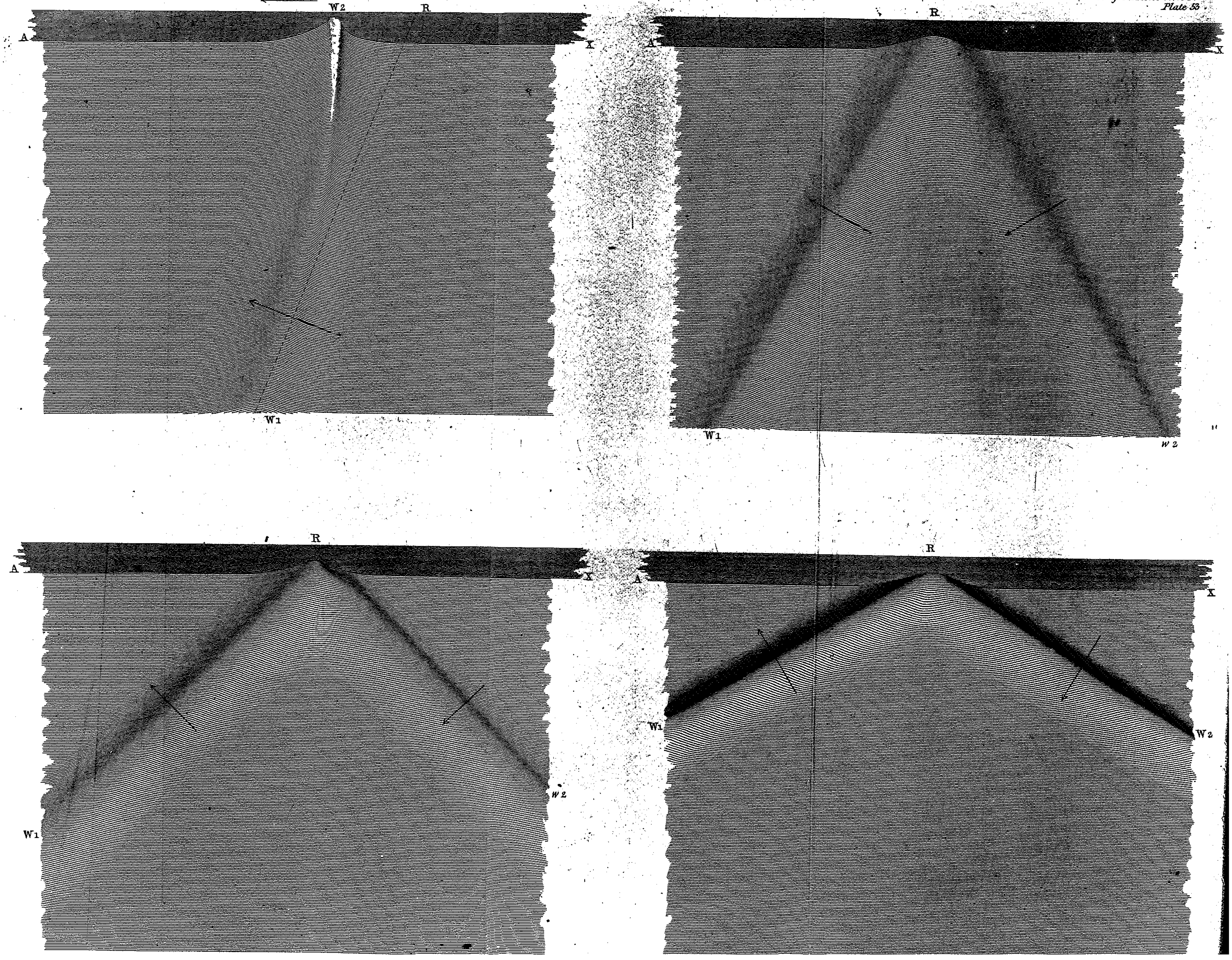



## WAVES — Order II. Oscillating Waves.



WAVES — Order I. Reflexion & lateral accumulation.

British Association for 1844. Plate 53.



WAVES — Order I.

Fig. 1.

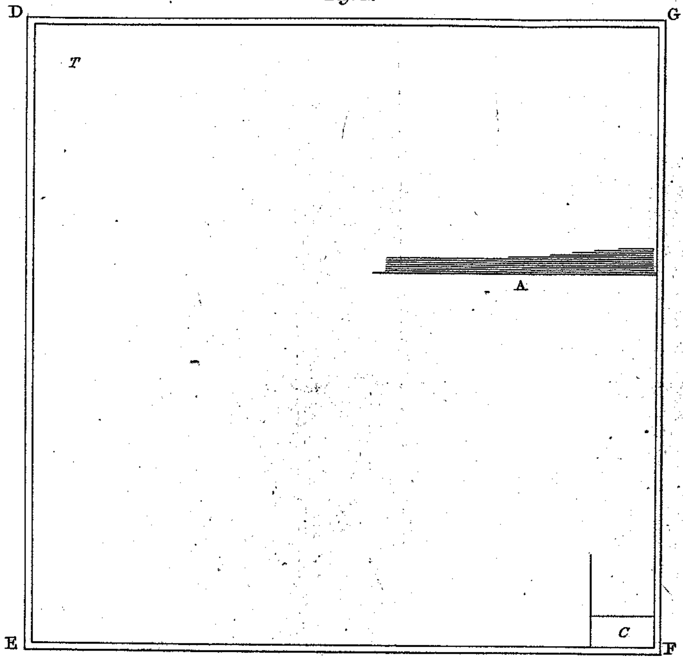



Fig. 3.

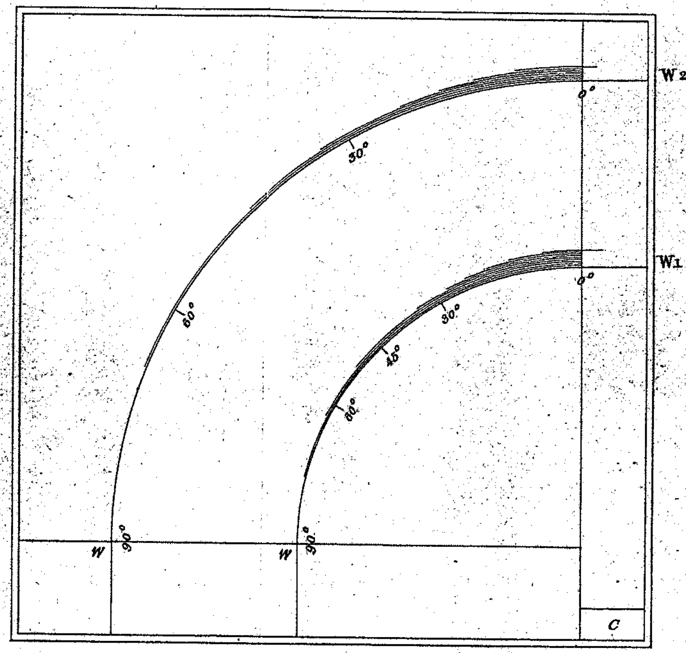



Fig. 2.

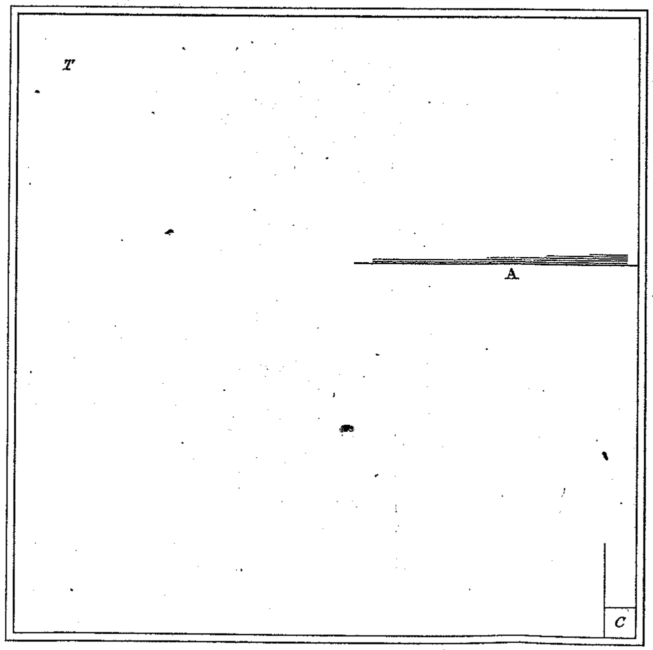
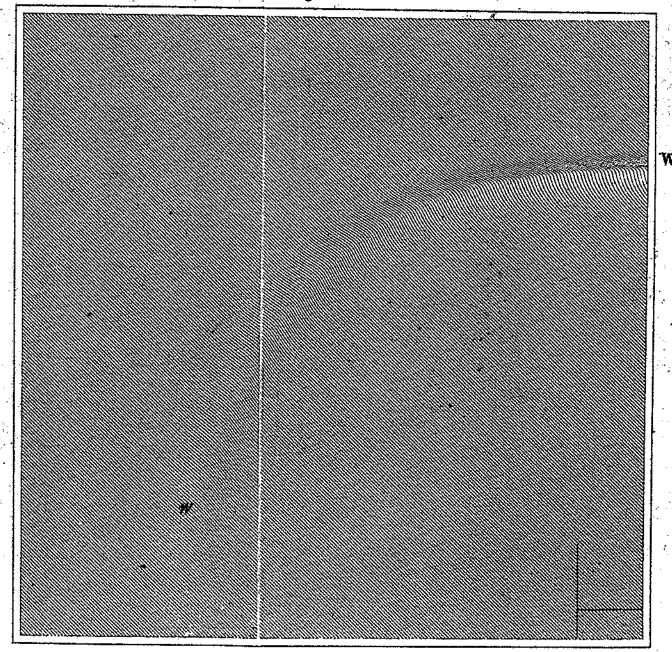



Fig. 4.



# WAVES — Order II. — Standing Waves.

Fig. 1.

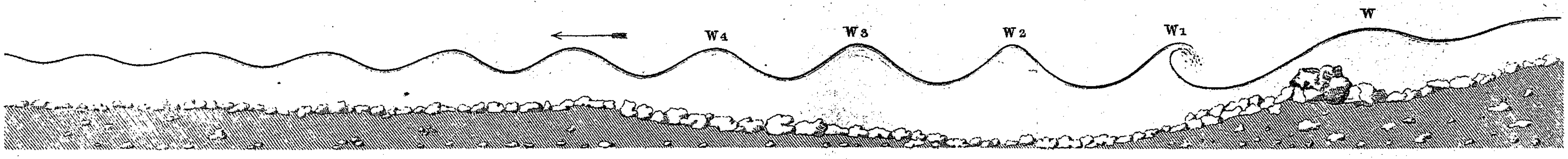



Fig. 2.

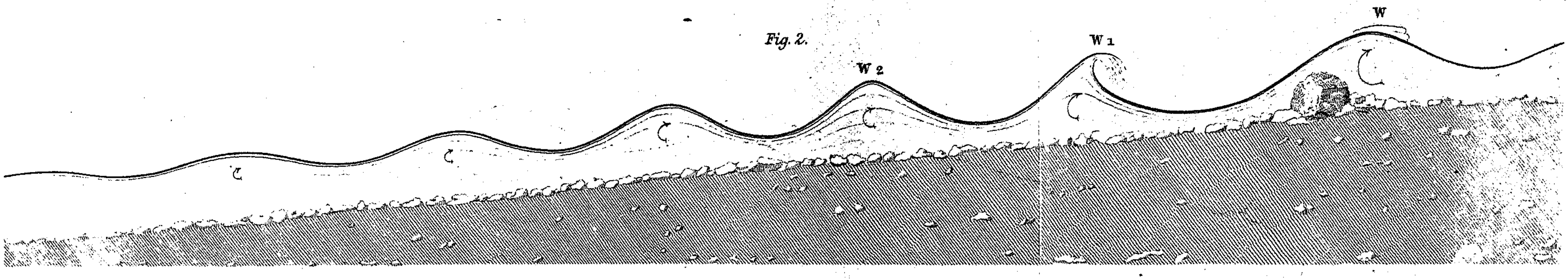



Fig. 3

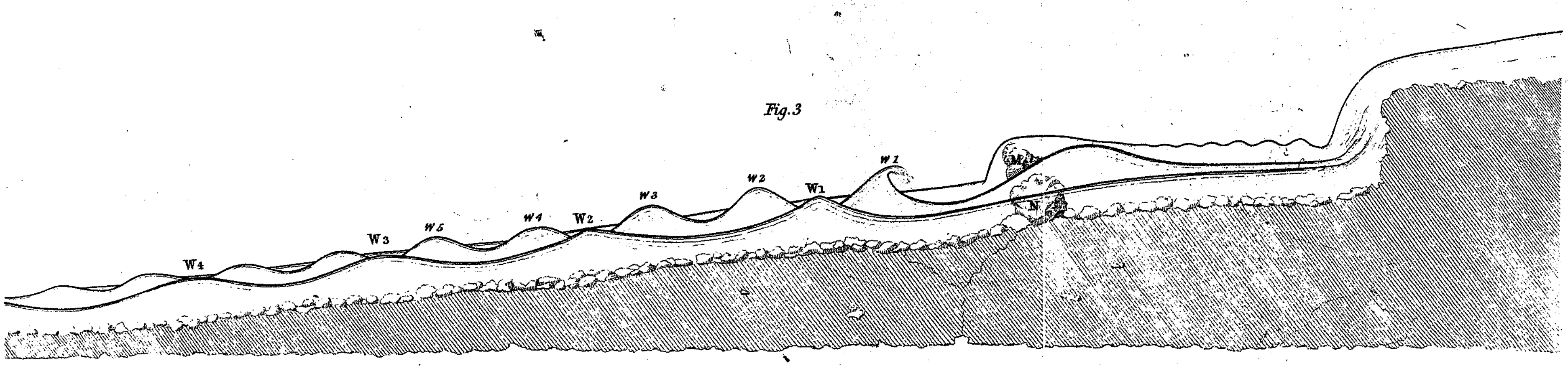
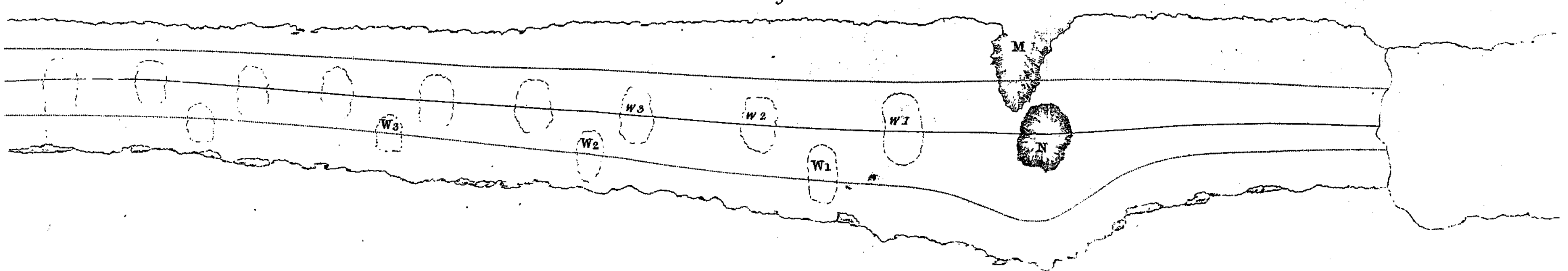



Fig. 4.



# WAVES ___ Order II.

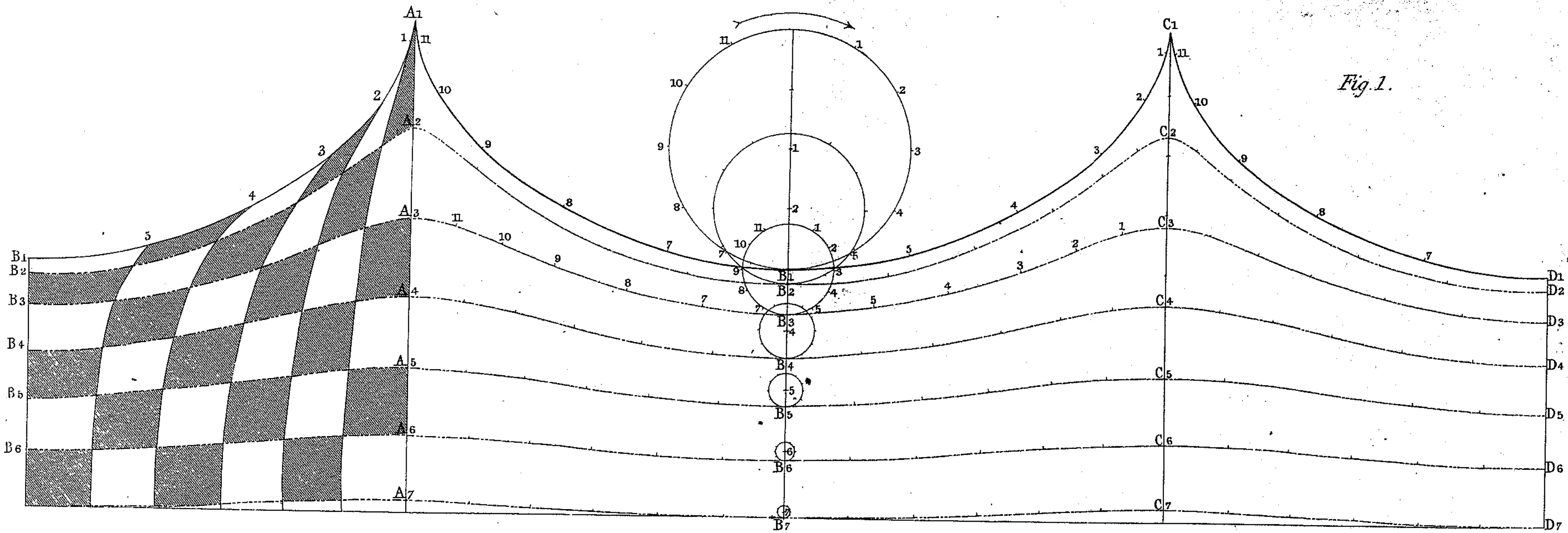



Fig. 1.

# WAVES ___ Order III.

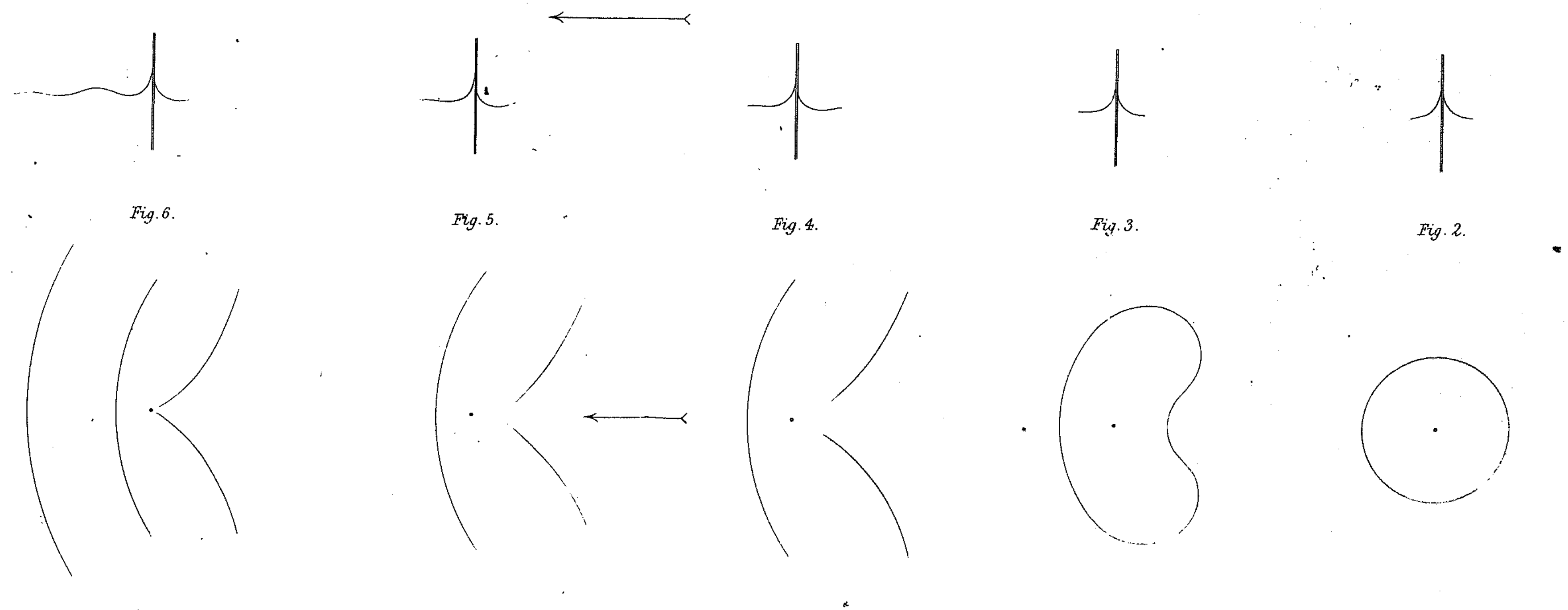


Fig. 6. Fig. 5. Fig. 4. Fig. 3. Fig. 2.

British Association for 1844.
Plate 57.

WAVES.... Order III. Capillary Waves.