Differentiability and strict convexity of the stable norm

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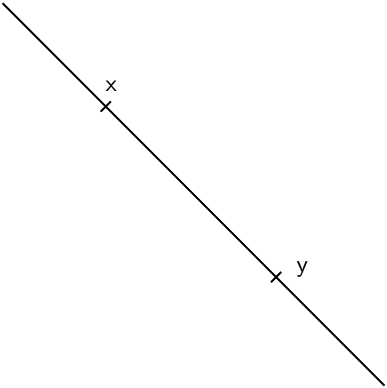
Introduction

The shortest path between two points is the straight line



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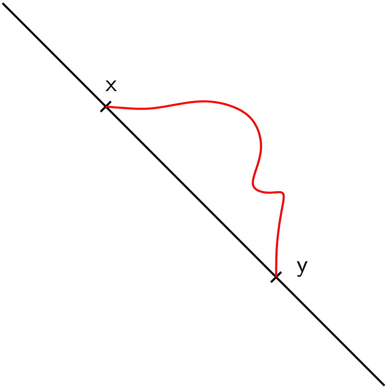
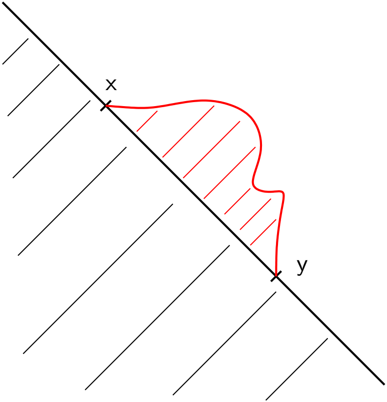
The shortest path between two points is the straight line



Introduction

The shortest path between two points is the straight line

$\Rightarrow$ half-spaces are local minimizers of the perimeter



Setting of the problem

We consider $F(x,p) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ s.t.:

- $F(\cdot, p)$ is $\mathbb{Z}^d$-periodic
- $F(x, \cdot)$ is convex one-homogeneous and smooth on $\mathbb{S}^{d-1}$
- $F(x, \cdot) - \delta|\cdot|$ is still convex (i.e. F is elliptic).

We will consider interfacial energies:

$$\int_{\partial E} F(x,\nu)d\mathcal{H}^{d-1}$$

where ν is the internal normal to E.

Definition

We say that E is a Class A Minimizer if $\forall R > 0$, $\forall (E \Delta F) \subset B_R$,

$$\int_{\partial E \cap B_R} F(x,\nu) \le \int_{\partial F \cap B_R} F(x,\nu).$$

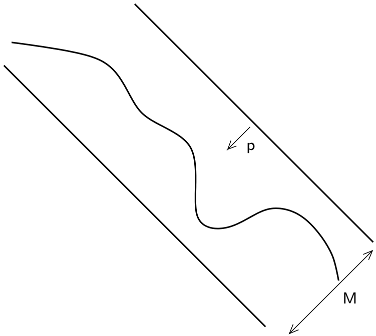
Existence of Plane-Like minimizers

Theorem (Caffarelli-De La Llave '01)

$\exists M > 0$ s.t. $\forall p \in \mathbb{S}^{d-1}$, there exists a Class A Min. E with

$$\{x \cdot p > M\} \subset E \subset \{x \cdot p > -M\}$$

$\Rightarrow$ E is a plane-like minimizer.



The Stable Norm

Definition

For $p \in \mathbb{S}^{d-1}$ let

$$\varphi(p) := \lim_{R\to\infty} \frac{1}{\omega_{d-1} R^{d-1}} \int_{\partial E \cap B_R} F(x,\nu)$$

where E is any PL in the direction p and ω_{d-1} is the volume of the unit ball of $\mathbb{R}^{d-1}$. Extend then φ by one-homogeneity to $\mathbb{R}^d$.

Question: What are the qualitative properties of φ? Strict convexity? Differentiability?

Relation with other works

- Codimension 1 analogue of the Weak KAM Theory for Hamiltonian systems (Aubry-Mather...)
- In the non-parametric setting, works of Moser, Bangert and Senn
- In the parametric setting, related works of Auer-Bangert and Junginger-Gestrich

The cell formula

Proposition (Chambolle-Thouroude '09)

$$\varphi(p) = \min \left\{ \int_{\mathbb{T}} F(x, p + Dv(x)) \;:\; v \in BV(\mathbb{T}) \right\}$$

and for every minimizer u and every $s \in \mathbb{R}$,

$$\{u + p \cdot x > s\}$$

is a plane-like minimizer.

Let $X := \{z \in L^\infty(\mathbb{T}) \,/\, F^*(x, z(x)) = 0 \; a.e. \;\; \mathrm{div}\, z = 0\}$ then

$$\varphi(p) = \sup_{z \in X} \left(\int_{\mathbb{T}} z \right) \cdot p$$

thus if $C := \{\int_{\mathbb{T}} z \,/\, z \in X\}$, C is a closed convex set and

$$\varphi(p) = \sup_{\xi \in C} \xi \cdot p$$

$\Rightarrow \varphi$ is the support function of C.

Structure of the subdifferential of p

$$\partial\varphi(p) = \{\xi \;/\; \xi \in C \text{ and } \xi \cdot p = \varphi(p)\}$$

$\Rightarrow \varphi$ is differentiable at p iff

$\forall z_1, z_2 \in X$ with $\int_{\mathbb{T}} z_i \cdot p = \varphi(p)$,

$$\int_{\mathbb{T}} z_1 = \int_{\mathbb{T}} z_2.$$

Calibrations

Definition

We say that $z \in X$ is a calibration in the direction p if

$$\int_{\mathbb{T}} z \cdot p = \varphi(p).$$

Proposition

For every calibration z and every minimizer u,

$$\int_{\mathbb{T}} z \cdot (Du + p) = \int_{\mathbb{T}} F(x, Du + p) \, (= \varphi(p) \,) \, .$$

Calibration of a set

Definition

We say that $z \in X$ calibrates a set E if

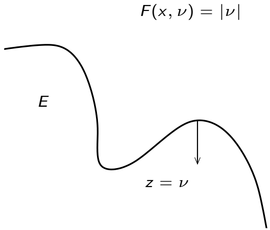
$$z \cdot \nu = F(x, \nu) \quad on\ \partial E.$$

Equivalently, $z = \nabla_p F(x, \nu)$ on ∂E.

Example: half spaces are calibrated by $z \equiv p$.

Proposition

If E is calibrated then E is a Class A Minimizer.



Proposition

For every calibration z in the direction p, every minimizer u and every $s \in \mathbb{R}$, z calibrates

$$\{u + p \cdot x > s\}$$

Proposition

If E and F are calibrated by the same z then either $E \subset F$ or $F \subset E$ and $\partial E \cap \partial F \simeq \emptyset$.

The Birkhoff property

Definition

We say that E has the Strong Birkhoff property if

- $\forall k \in \mathbb{Z}^d,\ k \cdot p \geq 0 \Rightarrow E + k \subset E$
- $\forall k \in \mathbb{Z}^d,\ k \cdot p \leq 0 \Rightarrow E \subset E + k.$

Example: the sets $\{u + p \cdot x > s\}$ are Strong Birkhoff.

Proposition

Every PL with the Strong Birkhoff property is calibrated by every calibration.
Therefore, they form a lamination (possibly with gaps) of the space.

Mañe's Conjecture

Reminder: $\varphi(p) = \min \left\{ \int_{\mathbb{T}} F(x, p + Dv(x)) \,:\, v \in BV(\mathbb{T}) \right\}$

Theorem

For a generic anisotropy F, the minimimum defining φ is attained for a unique measure Du.

See the works of Bernard-Contreras, Bessi-Massart.

Our Main Theorem

Theorem

- *φ^2 is strictly convex,*
- *if there is no gap in the lamination then φ is differentiable at p,*
- *if p is totally irrational then φ is differentiable at p,*
- *if p is not totally irrational and if there is a gap then φ is not differentiable at p.*

Remarks on the differentiability

- If there is no gap, z is prescribed everywhere $\Rightarrow$ the mean is also prescribed,
- if p is totally irrational then the gaps have finite volume $\Rightarrow$ it can be shown that they play no role in the integral (use the cell formula),
- if p is not tot. irr. and there are gaps $\Rightarrow$ using heteroclinic solutions, it is possible to construct two different calibrations with different means.

A concluding observation

Under mild hypothesis, this work extends to functionals of the form

$$\int_{\partial E} F(x,\nu) + \int_E g(x)$$

with g periodic with zero mean.

"Les bulles de savon" J.B.S. Chardin

Thank you!