Branched transport limit of the Ginzburg-Landau functional

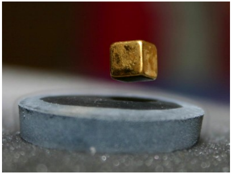
Michael Goldman

CNRS, LJLL, Paris 7

Joint work with S. Conti, F. Otto and S. Serfaty

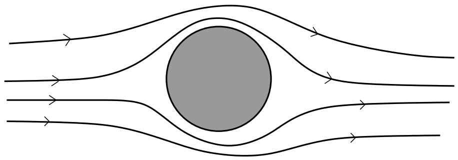
Introduction

Superconductivity was first observed by Onnes in 1911 and has nowadays many applications.



Meissner effect

In 1933, Meissner understood that superconductivity was related to the expulsion of the magnetic field outside the material sample



Ginzburg Landau functional

In the 50's Ginzburg and Landau proposed a phenomenological model (later derived from the BCS theory):

$$E(u, A) = \int_\Omega |\nabla_A u|^2 + \frac{\kappa^2}{2}(1-\rho^2)^2 dx + \int_{\mathbb{R}^3} |\nabla \times A - B_{ex}|^2 dx$$

where $u = \rho e^{i\theta}$ is the order parameter, $B = \nabla \times A$ is the magnetic field, B_{ex} is the external magnetic field, κ is the Ginzburg-Landau constant and

$$\nabla_A u = \nabla u - iAu$$

is the covariant derivative.

$\rho \sim 0$ represents the normal phase and $\rho \sim 1$ the superconducting one.

The various terms in the energy

For $u = \rho e^{i\theta}$, $|\nabla_A u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta - A|^2$.
In $\rho > 0$ first term wants $A = \nabla \theta \Longrightarrow \nabla \times A = 0$
That is

$$\rho^2 B \simeq 0 \qquad \text{(Meissner effect)}$$

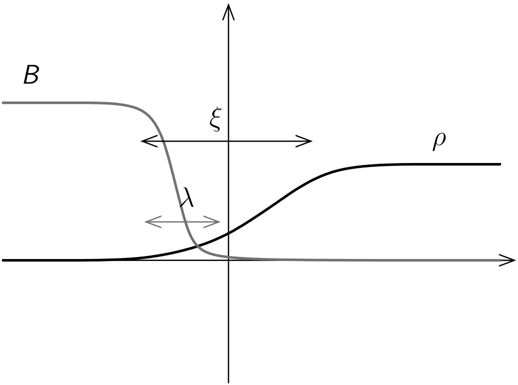
and penalizes fast oscillations of ρ

Second term forces $\rho \simeq 1$ (superconducting phase favored)

Last term wants $B \simeq B_{ex}$. In particular, this should hold outside the sample.

Coherence and penetration length

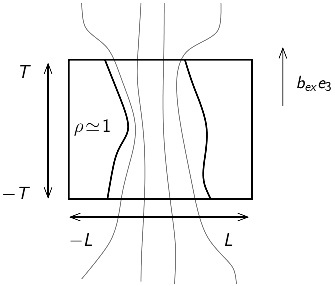
Already two typical lengths, coherence length ξ and penetration length λ.



In our unites, $\lambda = 1$, $\kappa = \frac{1}{\xi}$

Our setting

We consider $\Omega = Q_{L,T} = [-L, L]^2 \times [-T, T]$ with periodic lateral boundary conditions and take $B_{ex} = b_{ex} e_3$.



We want to understand extensive behavior $L \gg 1$.

First rescaling

We let

$$\kappa T = \sqrt{2}\alpha \qquad b_{ex} = \frac{\beta\kappa}{\sqrt{2}}$$

and then

$$\widehat{x} = T^{-1}x \qquad \widehat{u}(\widehat{x}) = u(x)$$
$$\widehat{A}(\widehat{x}) = A(x) \qquad \widehat{B}(\widehat{x}) = \nabla \times \widehat{A}(\widehat{x}) = TB(x)$$

In these units,

$$\text{coherence length} \simeq \alpha^{-1} \quad \text{penetration length} \simeq T^{-1}$$

We are interested in the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$.

The energy

The energy can be written as

$$E_T(u,A) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla_{TA}u|^2 + \left(B_3 - \alpha(1-\rho^2)\right)^2 + |B'|^2$$
$$+ \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$

- First term: penalizes oscillations + $\rho^2 B \simeq 0$ (Meissner effect)

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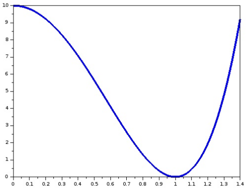
- First term: penalizes oscillations + $\rho^2 B \simeq 0$ (Meissner effect)
- Second term: degenerate double well potential.



If Meissner then:
$\left(B_3 - \alpha(1-\rho^2)\right)^2 \simeq \alpha^2 \chi_{\{\rho>0\}}(1-\rho^2)^2$

Rk: wants $B_3 = \alpha$ in $\{\rho = 0\}$
Similar features in mixtures of BEC
(cf G. Merlet '15)

Crash course on optimal transportation

For ρ_0, ρ_1 probability measures

$$W_2^2(\rho_0,\rho_1) = \inf \left\{ \int_{Q_L \times Q_L} |x-y|^2 d\Pi(x,y) \ : \ \Pi_1 = \rho_0, \ \Pi_2 = \rho_1 \right\}$$

Theorem

- *(Benamou-Brenier)*

$$W_2^2(\rho_0,\rho_1) = \inf_{\mu,B'} \Big\{ \int_0^1 \int_{Q_L} |B'|^2 d\mu \ : \ \partial_3 \mu + \mathrm{div}' B' \mu = 0, \\ \mu(0,\cdot) = \rho_0, \ \mu(1,\cdot) = \rho_1 \Big\}$$

- *(Brenier) If $\rho_0 \ll dx$,*

$$W_2^2(\rho_0,\rho_1) = \min \left\{ \int_{Q_L} |x - T(x)|^2 d\rho_0 \ : \ T\sharp\rho_0 = \rho_1 \right\}$$

The energy continued

$$E_T(u,A) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla_{TA} u|^2 + \left(B_3 - \alpha(1-\rho^2)\right)^2 + |B'|^2$$
$$+ \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$

- Third term: with Meissner and $B_3 \simeq \alpha(1-\rho^2) = \chi$, $\operatorname{div} B = 0$ can be rewritten as

$$\partial_3 \chi + \operatorname{div}' \chi B' = 0$$

Benamou-Brenier $\Longrightarrow$ Wasserstein energy of $x_3 \to \chi(\cdot, x_3)$

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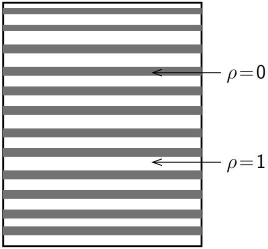
  Benamou-Brenier $\Longrightarrow$ Wasserstein energy of $x_3 \to \chi(\cdot, x_3)$
- Last term: penalizes non uniform distribution on the boundary but negative norm $\Longrightarrow$ allows for oscillations

A non-convex energy regularized by a gradient term

If we forget the kinetic part of the energy, can make $B' = 0$ and

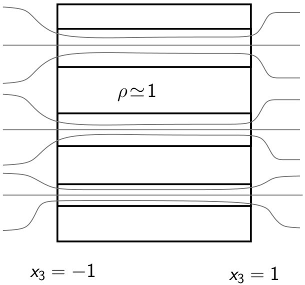
$$E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} \left(B_3 - \alpha(1-\rho^2)\right)^2 + \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$



$\Longrightarrow$ infinitely small oscillations of phases $\{\rho = 0, B_3 = \alpha\}$ and $\{\rho = 1, B_3 = 0\}$ with average volume fraction β.

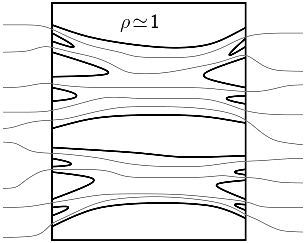
the kinetic term $|\nabla_A u|^2$ fixes the lengthscale.

Branching is energetically favored



$\|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)} \downarrow 0$

but interfacial energy $\uparrow \infty$

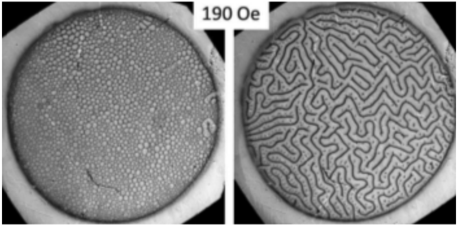


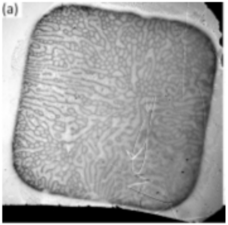
interfacial energy $\downarrow$
but $\int_{Q_{L,1}} |B'|^2 \uparrow$.

Landau '43

Experimental results

Complex patterns at the boundary





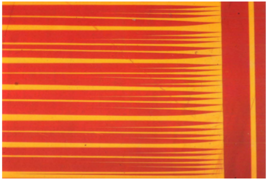
Experimental pictures from Prozorov and al.

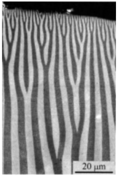
Limitations:

- Difficult to see the pattern inside the sample
- Hysteresis

Branching patterns in other related models

- Shape memory alloys (Kohn-Müller model) (Left, picture from Chu and James)
- Uniaxial ferromagnets (Right, picture from Hubert and Schäffer)





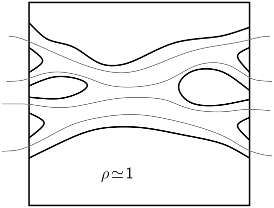
Schematic difference: in our problem W_2^2 replaces H^{-1} norm
See works of Kohn, Müller, Conti, Otto, Choksi ...
Related functional: Ohta-Kawasaki

Scaling law

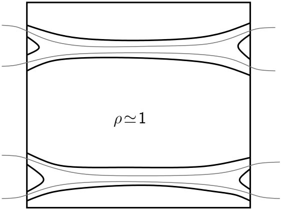
Theorem (Conti, Otto, Serfaty '15, See also Choksi, Conti, Kohn, Otto '08)

In the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$,

$$\min E_T \simeq \min(\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta)$$



First regime: $E_T \sim \alpha^{4/3}\beta^{2/3}$
Uniform branching,
$\|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)} = 0$



Second regime: $E_T \sim \alpha^{10/7}\beta$
Non-Uniform branching,
$\|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)} > 0$
fractal behavior

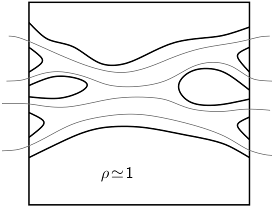
Scaling law

Theorem (Conti, Otto, Serfaty '15, See also Choksi, Conti, Kohn, Otto '08)

In the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$,

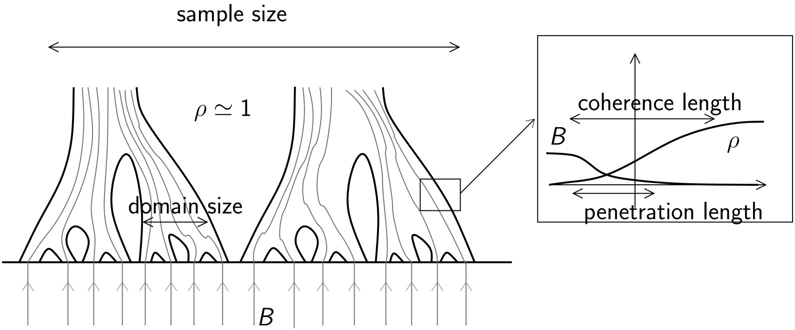
$$\min E_T \simeq \min(\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta)$$

We concentrate on the first regime (uniform branching)



$$\Longrightarrow \alpha^{-2/7} \ll \beta.$$

Multiscale problem



From the upper bound construction, we expect

$$\text{penetration length} \ll \text{coherence length} \ll \text{domain size} \ll \text{sample size}$$

which amounts in our parameters to

$$T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3} \ll L.$$

Crash course in Γ-convergence

F_n sequence of functionals on a metric space (X, d). We say that F_n $\Gamma-$converges to F if

- $\forall x_n \in X,\ F_n(x_n) \leq C \implies$ Compactness +

$$\varliminf_{n \to +\infty} F_n(x_n) \geq F(x)$$

- $\forall x \in X,\ \exists x_n \to x$ with

$$\varlimsup_{n \to +\infty} F_n(x_n) \leq F(x)$$

It implies

- $\inf F_n \to \inf F$
- if x_n are minimizers of $F_n \implies x$ is a minimizer of F.

Compactness and Lower bounds

First limit, $T \to +\infty$

Recall:

$$E_T(u,A) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla_{TA} u|^2 + \left(B_3 - \alpha(1-\rho^2)\right)^2 + |B'|^2 + \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$

Proposition

If $E_T(u_T, A_T) \le C$ *then* $\rho_T = |u_T| \to \rho$, $B_T = \nabla \times A_T \rightharpoonup B$ *and*

- $\rho^2 B = 0$, $\operatorname{div} B = 0$ *(Meissner effect)*
- $\underline{\lim}_T\, E_T(u_T, A_T) \ge F_{\alpha,\beta}(\rho, B)$ *where*

$$F_{\alpha,\beta}(\rho,B) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla \rho|^2 + \left(B_3 - \alpha(1-\rho^2)\right)^2 + |B'|^2 + \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$

Second rescaling

In this limit, penetration length = 0, coherence length $\simeq \alpha^{-1}$, domain size $\alpha^{-1/3}\beta^{1/3}$.
In order to get sharp interface limit with finite domain size, we make the anisotropic rescaling

$$\begin{pmatrix} \hat{x}' \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha^{1/3} x' \\ x_3 \end{pmatrix}, \qquad \widehat{F}_{\alpha,\beta} = \alpha^{-4/3} F_{\alpha,\beta}.$$

$$\begin{pmatrix} \hat{B}' \\ \hat{B}_3 \end{pmatrix} (\hat{x}) = \begin{pmatrix} \alpha^{-2/3} B' \\ \alpha^{-1} B_3 \end{pmatrix} (x), \qquad \hat{\rho}(\hat{x}) = \rho(x),$$

In these variable: coherence length $\simeq \alpha^{-2/3} \ll 1$ and normal domain size $\simeq \beta^{1/3}$

Second limit, $\alpha \to +\infty$

Dropping the hats

$$L^2 F_{\alpha,\beta}(\rho, B) = \int_{Q_{L,1}} \alpha^{-2/3} \left| \begin{pmatrix} \nabla' \rho \\ \alpha^{-1/3} \partial_3 \rho \end{pmatrix} \right|^2 + \alpha^{2/3} |B_3 - (1-\rho^2)|^2 + |B'|^2 + \alpha^{1/3} \| B_3 - \beta \|^2_{H^{-1/2}(x_3 = \pm 1)}$$

and the Meissner condition

$$\operatorname{div} B = 0 \qquad \text{and} \qquad \rho^2 B = 0$$

still holds

Proposition

If $F_{\alpha,\beta}(\rho_\alpha, B_\alpha) \leq C$*, then* $1 - \rho_\alpha^2 \to \chi \in \{0,1\}$ $B'_\alpha \rightharpoonup B'$ *and*

- $\chi(\cdot, \pm 1) = \beta$, $\chi B' = B'$, $\partial_3 \chi + \operatorname{div}' \chi B' = 0$
- $\underline{\lim}_\alpha F_{\alpha,\beta}(\rho_\alpha, B_\alpha) \geq G_\beta(\chi, B')$ *where*

$$G_\beta(\chi, B') = \frac{1}{L^2} \int_{Q_{L,1}} \frac{4}{3} |\nabla' \chi| + |B'|^2$$

Comments on the proof

- Anisotropic rescaling $\Longrightarrow$ control only on the horizontal derivative.
- Thanks to Meissner, double well potential

$$\alpha^{-2/3}\left|\begin{pmatrix}\nabla'\rho\\ \alpha^{-1/3}\partial_3\rho\end{pmatrix}\right|^2+\alpha^{2/3}|B_3-(1-\rho^2)|^2\geq$$

$$\alpha^{-2/3}\left|\nabla'\rho\right|^2+\alpha^{2/3}\chi_{\{\rho>0\}}|(1-\rho^2)|^2$$

Recall Modica-Mortola

$$\int\varepsilon|\nabla'\rho_\varepsilon|^2+\varepsilon^{-1}\rho_\varepsilon^2(1-\rho_\varepsilon^2)\to C\int|\nabla'\chi|$$

Last rescaling

We want to send $\beta \to 0$ and get 1 dimensional trees. We make another anisotropic rescaling:

$$\begin{pmatrix} \hat{x}' \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} \beta^{1/6} x' \\ x_3 \end{pmatrix}, \qquad \hat{\chi}(\hat{x}) = \beta^{-1}\chi(x) \in \{0, \beta^{-1}\}$$

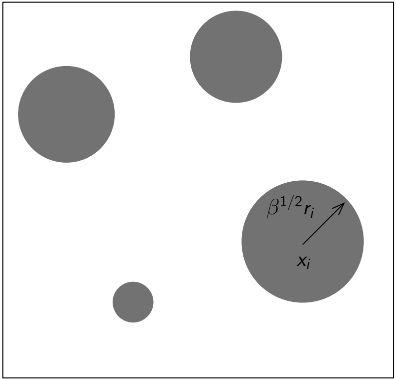
$$\hat{B}'(\hat{x}) = \beta^{1/6} B'(x) \qquad \widehat{G}_\beta = \beta^{-2/3} G_\beta$$

Now, domain width $\simeq \beta^{1/2} \ll 1$, $L = 1$ and (dropping hats)

$$G_\beta(\chi, B') = \int_{Q_{1,1}} \frac{4}{3}\beta^{1/2}|\nabla'\chi| + \chi|B'|^2$$

with $\partial_3\chi + \mathrm{div}'(\chi B') = 0$, $\chi B' = \beta^{-1} B'$ and $\chi(\cdot, x_3) \rightharpoonup dx'$ when $x_3 \to \pm 1$.

Limiting functional



Because of isoperimetric effects, on every slice

$$\chi \simeq \beta^{-1} \sum_i \chi_{B(x_i, \beta^{1/2} r_i)}$$

If $\phi_i = \pi r_i^2$,

$$\int_{[-1,1]^2} \chi \simeq \sum_i \phi_i$$

and

$$\int_{[-1,1]^2} \beta^{1/2} |\nabla' \chi| \simeq 2\pi^{1/2} \sum_i \sqrt{\phi_i}$$

Hence $\chi \rightharpoonup \sum_i \phi_i \delta_{x_i}$

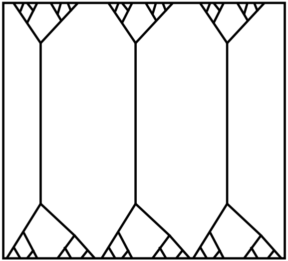
The limiting functional II

For μ a probability measure with $\mu_{x_3} = \sum_i \phi_i \delta_{x_i(x_3)}$ for a.e. x_3 and $\mu_{x_3} \rightharpoonup dx'$ when $x_3 \to \pm 1$, and $m \ll \mu$, with $\partial_3 \mu + \mathrm{div}' m = 0$,

$$I(\mu, m) = \int_{-1}^{1} \frac{8\pi^{1/2}}{3} \sum_{x' \in Q_1} \left(\mu_{x_3}(x')\right)^{1/2} dx_3 + \int_{Q_{1,1}} \left(\frac{dm}{d\mu}\right)^2 d\mu$$

Formally,

$$I(\mu) = \inf_m I(\mu, m) = \int_{-1}^{1} \sum_i \frac{8\pi^{1/2}}{3} \phi_i^{1/2} + \phi_i \dot{x}_i^2 dx_3$$



Last lower bound

Proposition

If $G_\beta(\chi_\beta, B'_\beta) \leq C$ $\chi_\beta \rightharpoonup \mu$, $\chi_\beta B'_\beta \rightharpoonup m$ *and*

$$\varliminf_\beta G_\beta(\chi_\beta, B'_\beta) \geq I(\mu, m)$$

The limiting functional

- $I(\mu)$ reminiscent of branched transport models (see Bernot-Caselles-Morel). Our result, similar in spirit to Oudet-Santambrogio '11.
- Minimizers of I, contain no loop, finite number of branching points away from boundary (with quantitative estimate), branches are linear between two branching points
- Every measure can be irrigated

Definition of regular measures

Definition

For $\varepsilon > 0$, we denote by $\mathcal{M}_R^\varepsilon(Q_{1,1})$ the set of regular measures, i.e., of measures such that:

(i) μ is finite polygonal.

(ii) All branching points are triple points. This means that any $x \in Q_{1,1}$ belongs to no more than three segments.

(iii) there is $\varepsilon^{1/2} \gg \lambda_\varepsilon \gg \varepsilon$ with $1/\lambda_\varepsilon \in \mathbb{N}$, such that for all $x_3 \in [1-\varepsilon, 1]$ one has $\mu_{x_3} = \sum_{x' \in \lambda_\varepsilon \mathbb{Z}^2 \cap Q_1} \varphi_{x'} \delta_{x'}$, with $\varphi_{x'} = \lambda_\varepsilon^2$, and the same in $[-1, -1+\varepsilon]$.

A crucial density result

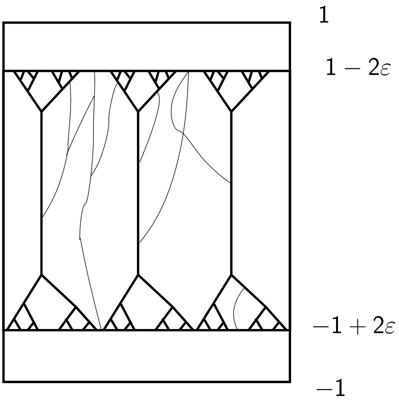
Proposition

For every μ *with* $I(\mu) < \infty$, $\exists \mu_\varepsilon \in \mathcal{M}_R^\varepsilon(Q_{1,1})$ *with* $\mu_\varepsilon \rightharpoonup \mu$ *and* $\overline{\lim}_\varepsilon I(\mu_\varepsilon) \leq I(\mu)$.

$\Longrightarrow \simeq$ enough to make the construction for finite polygonal measures.

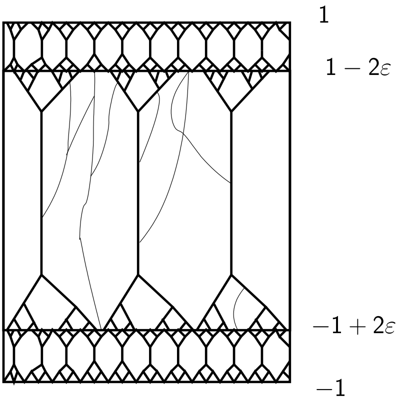
Idea of the proof

- Rescale μ to $[-1+2\varepsilon, 1-2\varepsilon]$



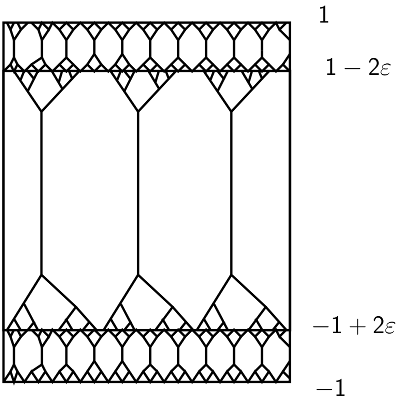
Idea of the proof

- Rescale μ to $[-1+2\varepsilon, 1-2\varepsilon]$
- in the boundary layer plug in a uniform branching construction



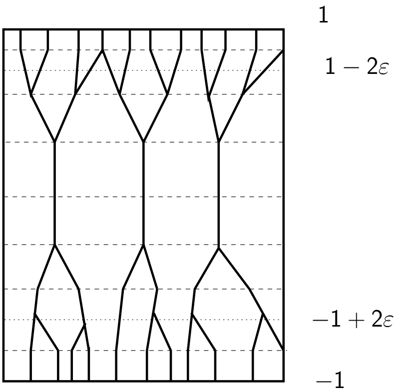
Idea of the proof

- Rescale μ to $[-1+2\varepsilon, 1-2\varepsilon]$
- in the boundary layer plug in a uniform branching construction
- remove small branches. Tool: notion of subsystem cf. Ambrosio-Gigli-Savare



Idea of the proof

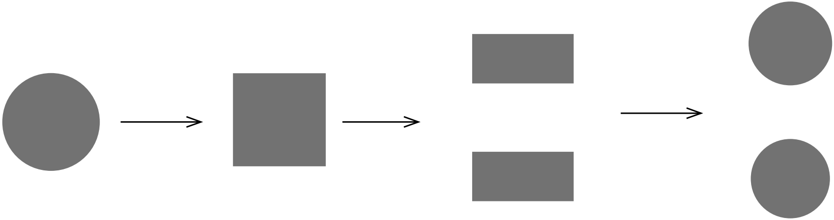
- Rescale μ to $[-1+2\varepsilon, 1-2\varepsilon]$
- in the boundary layer plug in a uniform branching construction
- remove small branches. Tool: notion of subsystem cf. Ambrosio-Gigli-Savare
- discretize and minimize



Recovery sequences

Recovery sequence, from trees to sharp interface

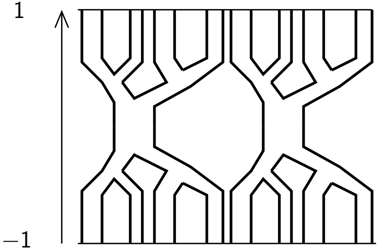
Want to enlarge the 1D trees. Far from branching points, easy (take a tube). At a branching point:



Need to transform a rectangle into disk with controlled energy

Recovery sequence, from sharp to diffuse interface

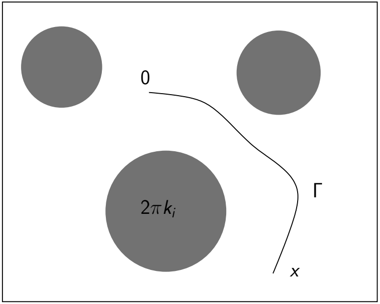
Need to reintroduce a smooth transition + vertical derivative.



To get smooth transition: use optimal profile (keeping Meissner) + careful estimate of the error terms

Recovery sequence, from diffuse interface to full GL

Can define A with $\nabla \times A = B$. Need to define θ. Would be easy if quantization of flux $\phi_i \in 2\pi\mathbb{Z}$.



If Γ $0 \longleftrightarrow x$, $\theta(x) = \int_\Gamma A \cdot \tau$
Quantization+Stokes $\Longrightarrow$
well defined + $\nabla\theta = A$

$\Longrightarrow$ Need to modify the fluxes to get quantization

Ongoing work and perspective

- Understand the cross-over regime $\alpha^{-2/7} \sim \beta$
- Go from GL to sharp interface when coherence $\sim$ penetration (more complex/non-local optimal profile problem)
- Understand minimizers of the limiting functional (self-similarity à la Conti)
- Investigate the non-uniform branching $\rightsquigarrow$ fractal behavior

"What in the name of Sir Isaac H. Newton happened here?"

Dr. Emmett 'Doc' Brown

Thank you! Any Question?