Scaling law and reduced models for epitaxially strained crystalline films

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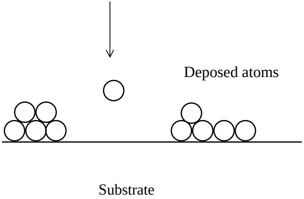
Joint work with B. Zwicknagl

Introduction

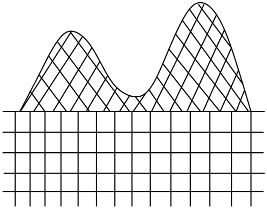
Epitaxially strained crystalline films are obtained by deposing thin layers on a thick substrate



<u>Example</u> : In-GaAs/GaAs or SiGe/Si.

Governing mechanism

There is a mismatch between the lattice parameters of the two crystals



The deposit layer is strained and the atoms try to rearrange for releasing elastic energy but this migration is also energetically expensive

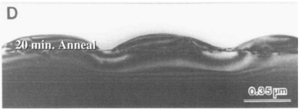
$\Longrightarrow$ interaction between bulk and surface energy.

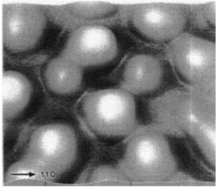
Numerical and experimental observations

Existence of several regimes depending on the volume of the thin layer and of the mismatch

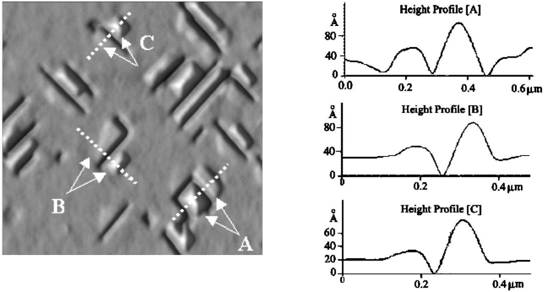
- For small volumes, the flat configuration is favored
- Above a certain threshold, the flat configuration is not stable anymore and the film develops corrugations
- For higher values of the volume/mismatch, there is formation of isolated islands

Goal: Understand these different regimes.



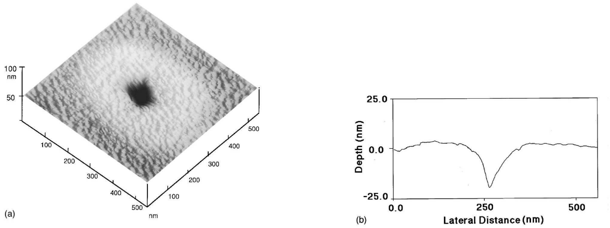


Surface roughening in SiGe/Si, images from Gao, Nix, *Surface roughening of heteroepitaxial thin films*, 1999.



Formation of islands, images from Gray, Hull and Floro *Formation of one-dimensional surface grooves from pit instabilities in annealed SiGe/Si(100) epitaxial films*, 2004.

Experimental results



Example of cusps, images from Chen, Jesson, Pennycook, Thundat, and Warmack, *Cuspidal pit formation during the growth of SixGe1-x strained films*, 1995

Numerical simulations

Numerical simulations from Bonnetier and Chambolle, *Computing the Equilibrium Configuration of Epitaxially Strained Crystalline Films*, 2002.

See also the numerical simulations of University of Cambridge, DoITPoMS,
http://www.doitpoms.ac.uk/tlplib/epitaxial-growth/index.php

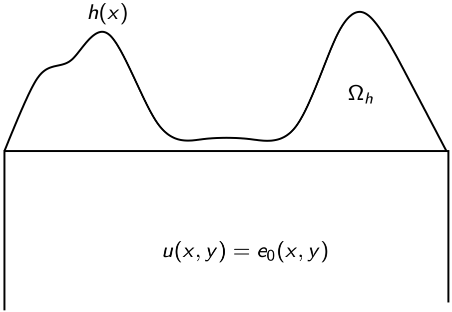
Applications

These epitaxially grown thin films are used for

- Optical and optoelectric devices (quantum dot laser).
- Semiconductors.
- Information storage.
- Nanotechnology.

The Mathematical Model

The film is taken to be the subgraph Ω_h of a function $h : [0,1] \to \mathbb{R}^+$



The substrate is considered as rigid hence in the substrate, the deformation is equal to $e_0(x,y)$ where e_0 is the mismatch.

The energy

Let $W : \mathbb{R}^4 \to \mathbb{R}^+$ be the stored elastic energy then we consider the variational problem:

$$F_{d,e_0}(u,h) := \int_{\Omega_h} W(\nabla u) + \int_0^1 \sqrt{1 + |h'|^2}$$

under the conditions that

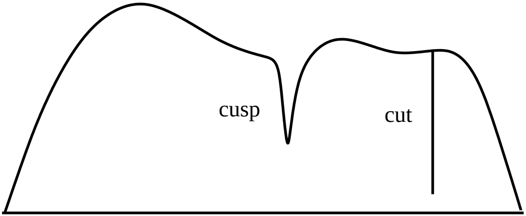
$$u(x,0) = e_0(x,0) \qquad \text{and} \qquad \int_0^1 h = d$$

Remark: most of the works consider energies W depending only on the symmetric part of the gradient.

Contributions of each term in the energy

- Due to the mismatch, there are no stress free configurations.
- In order to release elastic energy, the bulk term favors creation of singularities.
- On the other hand, the surface term tends to avoid too many oscillations.

Regularity results (in the geometrically linear setting)



Theorem [Chambolle-Larsen 03, Fonseca-Fusco-Leoni-Morini 07]

The profile h is regular out of a finite number of cuts and cuts. Moreover the film satisfies the zero angle condition.

Regularity results continued

Theorem [Fusco-Morini 12]

- For small mismatch, the flat configuration is minimzing (no matter how big is d).
- For greater mismatch, the following holds:
  1. for $d \leq d_0$, the flat configuration is minimizing
  2. for $d \leq d_1$ the flat configuration is locally minimizing
  3. for $d \leq d_2$, the flat configuration is not locally minimizing but every minimizer is smooth

Other results in the litterature

- Physical and engineering: Spencer-Meiron 94, Spencer-Tersoff 10, Gao-Nix 99.
- Regularity, relaxation and approximation: Bonnetier-Chambolle 02, Chambolle-Larsen 03, Fonseca-Fusco-Leoni-Morini 07, Chambolle-Solci 07, Fusco-Morini 12.
- Time evolution: Fonseca-Fusco-Leoni-Morini 12, Piovano 12.

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- Time evolution: Fonseca-Fusco-Leoni-Morini 12, Piovano 12.

No rigourous result on the formation of the islands!

The main result

We will assume that

Hypothesis

(H1) $W \geq 0$

(H2) there exists $C > 0$ and $p > 1$ such that

$$C(|A|^p + 1) \geq W(A) \geq \frac{1}{C}(|A|^p - 1) \qquad \forall A \in \mathbb{R}^{2\times 2}.$$

Theorem

Under these assumptions, for every $e_0 > 0$ and $d > 0$ there holds

$$\min_{u,h} F_{e_0,d}(u,h) \simeq \max(1, d, e_0^{p/3} d^{2/3}).$$

Remark:

- Thanks to (H2), it is enough considering $W(\nabla u) = |\nabla u|^p$.
- Works also in the geometrically linear setting.

Heuristic explanation of the scaling

We consider for simplicity here $p = 2$ so that

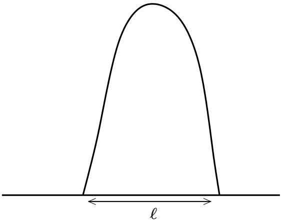
$$F_{e_0,d}(u,h) = \int_{\Omega_h} |\nabla u|^2 + \int_0^1 \sqrt{1+|h'|^2}$$

If $\Omega_h \cap \{y=0\} = [a, a+\ell]$ then since $|\Omega_h| = d$,

$$\int_0^1 \sqrt{1+|h'|^2} \geq \frac{d}{\ell}.$$

On the other hand

$$\min_{u(x,0)=e_0(x,0)} \int_{\Omega_h} |\nabla u|^2 \simeq e_0^2 |u|^2_{H^{1/2}(a,a+\ell)} \simeq e_0^2 \ell^2$$



Putting these together we find that

$$F_{e_0,d}(u,h) \gtrsim e_0^2\ell^2 + \frac{d}{\ell}$$

Optimizing in ℓ, we find that $\ell_{\min} \simeq \min(1, \left(\frac{d}{e_0^2}\right)^{1/3})$. So that two regimes appear:

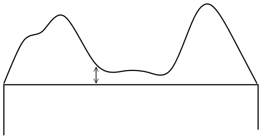
- If $\left(\frac{d}{e_0^2}\right)^{1/3} \leq 1$, we have $\ell_{\min} = \left(\frac{d}{e_0^2}\right)^{1/3}$ and $\min F_{e_0,d} \simeq e_0^{2/3} d^{2/3}$.
- If $\left(\frac{d}{e_0^2}\right)^{1/3} \geq 1$, the flat configuartion is favored and $\min F_{e_0,d} \simeq e_0^2 + d \simeq d$.

Difficulty:

when $h(x) \ll 1$, the constant in

the trace inequality degenerate i.e.

$$\min_{u(x,0)=e_0(x,0)} \int_{\Omega_h} |\nabla u|^2 \not\gtrsim e_0^2 |u|^2_{H^{1/2}(a,a+\ell)}$$



The Strategy

To prove this kind of scaling laws, the general strategy is

- To get the upper bound by construction.
- To prove an ansatz free lower bound.

In many related results (see Kohn-Müller, Choksi-Conti-Kohn-Otto, Bella-Kohn, Capella-Otto...) the lower bound is obtained via an interpolation inequality. Here it will not be the case.

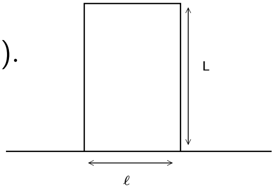
Preamble, playing with rectangles

For $u \in W^{1,p}([0,\ell] \times [0,L])$ let

$\tilde{u}(x,y) = \frac{1}{\ell} u(\ell x, \ell y) \in W^{1,p}([0,1] \times [0, L/\ell])$.

Then $\nabla \tilde{u}(x,y) = \nabla u(x,y)$ and

$$\int_{[0,\ell]\times[0,L]} |\nabla u|^p = \ell^2 \int_{[0,1]\times[0,L/\ell]} |\nabla \tilde{u}|^p$$



Fondamental lemma

$$\min_{u(x,0)=e_0(x,0)} \int_{[0,\ell]\times[0,L]} |\nabla u|^p = e_0^p \ell^2 \min_{u(x,0)=(x,0)} \int_{[0,1]\times[0,L/\ell]} |\nabla u|^p$$

The upper bound

By the considerations above, for the upper bound, it is enough considering a rectangle $[0,\ell]\times[0,d/\ell]$ with $\ell=\min\left(1,\left(\frac{d}{e_0^p}\right)^{1/3}\right)$ and

$$u(x,y)=\begin{cases}\left(e_0x\left(1-\frac{1}{\ell}y\right),0\right) & \text{if } 0\le y\le\ell,\\ 0 & \text{else .}\end{cases}$$

Then $F_{e_0,d}(u,h)\simeq\ell^2e_0^p+1+\frac{d}{\ell}\simeq\max(1,d,e_0^{p/3}d^{2/3})$.

The lower bound: setting the notations

Since $F_{d,e_0}(u,h) \geq 1+d$, we can assume $e_0^{p/3} d^{2/3} \geq \max(1,d)$.
Let :

$$y_0 := \frac{d}{2\sqrt{e_0^{p/3} d^{2/3}}}.$$

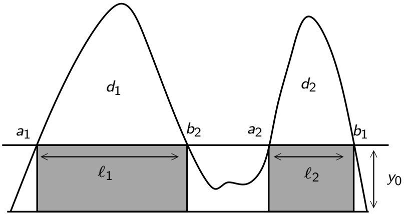
$$\ell := \mathcal{H}^1(\Omega_h \cap (I \times \{y_0\})).$$

$$I_\ell := \Omega_h \cap (I \times \{y_0\}).$$

$$I_\ell = \cup_{i=1}^n [a_i, b_i].$$

$$\ell_i := b_i - a_i.$$

$$d_i := |\Omega_h \cap ([a_i, b_i] \times [y_0, +\infty))|.$$



Then $\sum_{i=1}^n \ell_i = \ell$ and $\sum_{i=1}^n d_i \geq d - y_0 \geq Cd$.

First possibility: $\ell_i \leq \left(\frac{d}{e_0^p}\right)^{1/3}$ for all $i = 1, \ldots, n$

In this case, the surface energy is sufficient to get

$$\begin{aligned} F_{d,e_0}(u,h) &\geq \int_0^1 \sqrt{1+|h'|^2}dx \\ &\geq \sum_{i=1}^n \int_{a_i}^{b_i} \sqrt{1+|h'|^2}dx \\ &\geq \sum_{i=1}^n \frac{d_i}{\ell_i} \geq \left(\frac{e_0^p}{d}\right)^{1/3} \sum_{i=1}^n d_i \\ &\geq C e_0^{p/3} d^{2/3} \ . \end{aligned}$$

And we are done.

Second possibility, $\ell_1 \geq \left(\frac{d}{e_0^p}\right)^{1/3}$

In this case, we focus on the elastic energy and find

$$F_{d,e_0}(u,h) \geq \int_{[a_1,b_1]\times[0,y_0]} |\nabla u|^p dxdy$$
$$\geq \ell_1^2 e_0^p \min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,y_0/\ell_1]} |\nabla u|^p dxdy.$$

Problem: It can happen that $y_0/\ell_1 \ll 1$...

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<u>Problem:</u> It can happen that $y_0/\ell_1 \ll 1$...

$\Longrightarrow$ We have to control how

$$\min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,\varepsilon]} |\nabla u|^p dxdy \to 0$$

when $\varepsilon \to 0$.

Theorem (Dimension reduction)

There holds

$$\lim_{\varepsilon\to 0^+} \min_{u(x,0)=(x,0)} \frac{1}{\varepsilon} \int_{[0,1]\times[0,\varepsilon]} |\nabla u|^p \, dxdy \geq 1.$$

This is a simplified version of the Le Dret-Raoult proof of dimension reduction.

Remark: For $p = 2$, using Fourier methods, it can be seen that

$$\begin{aligned} \min_{u(x,0)=(x,0)} \int_{[0,1]\times[0,\varepsilon]} |\nabla u|^2 \, dxdy &\simeq \sum_{k=1}^{+\infty} \frac{1}{|k|^3}(1-\exp(-2\pi k\varepsilon)) \\ &\geq 2\pi\varepsilon \sum_{k=1}^{+\infty} \frac{1}{|k|^3} \end{aligned}$$

Conclusion of the lower bound (when $\ell_1 \geq \left(\frac{d}{e_0^p}\right)^{1/3}$)

Remind that $F_{d,e_0}(u,h) \geq \ell_1^2 e_0^p \min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,y_0/\ell_1]} |\nabla u|^p$

Let $c > 0$ s.t. for $\varepsilon < c$,

$$\min_{u(x,0)=(x,0)} \frac{1}{\varepsilon} \int_{[0,1]\times[0,\varepsilon]} |\nabla u|^p \, dxdy \geq 1/2.$$

- If $\left(\frac{d}{e_0^p}\right)^{1/3} \leq \ell_1 \leq y_0/c$ then

$$F_{d,e_0}(u,h) \geq \ell_1^2 e_0^p \min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,c]} |\nabla u|^p$$
$$\geq C e_0^{p/3} d^{2/3}.$$

- If $\ell_1 \geq y_0/c$ then

$$F_{d,e_0}(u,h) \geq \ell_1^2 e_0^p \frac{y_0}{2\ell_1} \geq C e_0^p y_0^2 = C\left(e_0^{p/3} d^{2/3}\right)^2.$$

Reduced models

In order to study the asymptotic behavior of the energy, we rescale the domains and set

- $\widetilde{h} := h/d$
- $\Omega_{\tilde{h}} := \{(x,y) \ : \ (x,dy) \in \Omega_h\}$
- $\widetilde{u}(x,y) = u(x,dy)$

Dropping the tildes, the energy now reads

$$F_{d,e_0}(u,h) = d\left[\int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d}\frac{\partial u}{\partial y}\right) dxdy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx\right]$$

for (u,h) such that $\int_0^1 h dx = 1$, and $u \in W^{1,p}(\Omega_h)$ with

$u(x,0) = e_0(x,0)$.

Γ-convergence

Definition

We say that a sequence of functionals F_n Γ-converges to F if

- $\forall u_n$ with $\sup F_n(u_n) < +\infty$, $\exists u$ such that $u_n \to u$ (up to a subsequence) and

$$\liminf F_n(u_n) \geq F(u)$$

- $\forall u$, $\exists u_n \to u$ with

$$\limsup F_n(u_n) \leq F(u)$$

The trivial regime $F_{d,e_0} \simeq 1$

Proposition

If $\{(u_d, h_d)\}$ is a low energy sequence, i.e. $\sup F_{d,e_0}(u_d, h_d) < +\infty$, then, up to extraction of a subsequence the measures $\mu_d := h_d\, dx$ weak-$*$ converge to a probability measure μ, and

$$\liminf_{d\to 0} F_{d,e_0}(u_d, h_d) \geq 1.$$

Moreover, for every probability measure μ on $[0,1]$ there exists a sequence $\{h_d\}$ of nonnegative Lipschitz functions $h : I \to \mathbb{R}$ with $h_d(0) = h_d(1) = 0$ and $\int_0^1 h_d(x)\ dx = 1$, and a sequence $\{u_d\}$ of functions $u_d \in W^{1,p}(\Omega_{h_d}; \mathbb{R}^2)$, such that $\{h_d\, dx\}$ converges weak-$*$ to μ and

$$\limsup_{d\to 0} F_{d,e_0}(u_d, h_d) \leq 1.$$

The surface dominant regime $F_{d,e_0} \simeq d$

We divide the energy by d and obtain the rescaled energy

$$F_d(u,h) := \int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d}\frac{\partial u}{\partial y}\right) dxdy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx.$$

In this regime, the surface energy is the dominating term, and the limit functional is given by

$$\bar{F}(h) := \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts}).$$

The minimizer of $\bar{F}$ is the flat configuration $h \equiv 1$

Proposition

Suppose $\{(u_d, h_d)\}$ with $\sup_d F_d(u_d, h_d) < +\infty$. Then the sets $\mathbb{R}^2 \backslash \Omega_{h_d}$ converge to $\mathbb{R}^2 \backslash \Omega_h$ where $h(x) := \inf\{\liminf h_d(x_d) \ : \ x_d \to x\}$ and

$$\liminf_{d \to +\infty} F_d(u_d, h_d) \geq \bar{F}(h) \ .$$

Moreover, for every nonnegative lower semicontinuous function h with bounded pointwise variation and $\int_0^1 h(x)\, dx = 1$, there exists a sequence $\{(u_d, h_d)\}$ where $h_d : I \to \mathbb{R}$ are non-negative Lipschitz functions with $\int_0^1 h_d(x)\, dx = 1$, $h_d(0) = h_d(1) = 0$, and $u_d \in W^{1,p}(\Omega_{h_d}; \mathbb{R}^2)$ such that $\mathbb{R}^2 \backslash \Omega_{h_d}$ converge to $\mathbb{R}^2 \backslash \Omega_h$, and

$$\limsup_{d \to +\infty} F_d(u_d, h_d) \leq \bar{F}(h) .$$

Proof

The compactness follows from the uniform bound on $\int_0^1 |h_d'|$ and Blaschke's Theorem.

The lower bound comes from the relaxation result of Bonnetier-Chambolle 02 (see also Fusco-Fonseca-Leoni-Morini 07)

The upper bound is obtained by considering $h_d = h$ (that we can assume Lipschitz) and

$$u_d(x,y) := \begin{cases} e_0\left(x\left(1-dy\right),0\right) & \text{if } y \leq \frac{1}{d}, \\ 0 & \text{if } y \geq \frac{1}{d}. \end{cases}$$

Using the p growth condition we then find

$$\limsup \int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d}\frac{\partial u}{\partial y}\right) dxdy \simeq \limsup \frac{e_0^p}{d} = 0 \text{ as } d \to +\infty.$$

The limit case $e_0^p = d \to +\infty$

The rescaled energy is again

$$F_d(u,h) := \int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d}\frac{\partial u}{\partial y}\right) dxdy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2}\, dx.$$

In this regime, we expect that both elastic and surface part of the energy will contribute.

Notice that from a bound on the energy, we get no good bound on $|u|_{W^{1,p}(\Omega_h)} \cdots$

The boundary layer

We expect that the elastic energy is concentrated in a region of height $O(\frac{1}{d})$ (since it was concentrated in a region of height $O(1)$ in the original domain).

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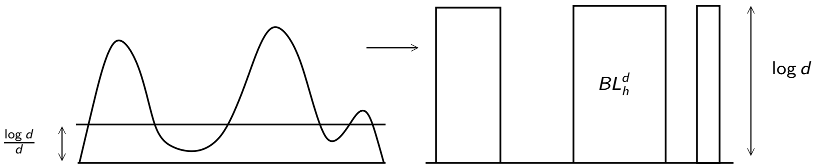
$\Longrightarrow$ we have to rescale back this boundary layer



Set

$$BL_h^d := \left\{(x,y)\ :\ (x,y) \in [0,1] \times \mathbb{R}^+ ,\ y \leq \log(d) < d\ h_d(x)\right\}\ .$$

Compactness

Proposition

If $\sup F_d(u_d, h_d) < +\infty$ then $\left\{\mathbb{R}^2 \backslash \Omega_{h_d}\right\}$ converges in the Hausdorff topology to $\mathbb{R}^2 \backslash \Omega_h$, where $h(x) := \inf\{\liminf h_d(x_d) \ : \ x_d \to x\}$. Set

$$BL_h := \left\{(x, y) \ : \ (x, y) \in [0, 1] \times \mathbb{R}^+ , \, 0 < h(x)\right\}.$$

Then $\left\{BL_h^d\right\}$ converges in the local Hausdorff topology to BL_h. Moreover, if $v_d : BL_h^d \to \mathbb{R}^2$ is defined by $v_d(x, y) := \frac{1}{e_0} u_d\left(x, \frac{y}{d}\right)$ then there exists $v \in W^{1,p}\left(BL_h; \mathbb{R}^2\right)$ with $v(x, 0) = (x, 0)$, $v_d \chi_{BL_h^d} \rightharpoonup v$ locally weakly in $W^{1,p}_{\mathrm{loc}}\left(BL_h; \mathbb{R}^2\right)$.

Proof

The convergence of the sets follows as previously

Regarding the elastic term, changing variables $z = d\, y$ and dividing by e_0, we get

$$\int_{BL_h^d} \frac{W\left(e_0 \nabla v_d\right)}{e_0^p} dxdy \leq C.$$

The p-growth of W then implies a uniform bound on $|v_d|_{W^{1,p}}$

from which the compactness follows.

The recession function

In the previous proof we saw that the quantity which naturally arises is

$$\int_{BL_h^d} \frac{W\left(e_0 \nabla v_d\right)}{e_0^p} dxdy$$

It is thus natural to expect that the recession functional

$$W^{\infty}(A) := \limsup_{t \to +\infty} \frac{W(tA)}{t^p} \quad \text{for } A \in \mathbb{R}^{2\times 2}.$$

will play a role We will assume that

(H3) W is quasiconvex.

(H4) There exist $0 < m < p$, $\gamma > 0$ and $L > 0$ such that for $t\,|A| \geq L$,

$$\left| W^{\infty}(A) - \frac{W(tA)}{t^p} \right| \leq \gamma \frac{|A|^{p-m}}{t^m} .$$

Recession function continued

Lemma

Suppose that W satisfies (H2), (H3) and (H4). If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(t) \to +\infty$ for $t \to +\infty$ then

$$\lim_{e_0 \to +\infty} \min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,f(e_0)]} \frac{W(e_0 \nabla v)}{e_0^p} \, dxdy = \min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,+\infty)} W^\infty(\nabla v) \, dxdy \, .$$

The proof is an adaptation of the proof of semicontinuity of quasiconvex functionals.

Some comments

- Hypothesis (H3) is not necessary.
- The recession function appears naturally in relaxation results for functionals with linear growth (see Fonseca-Müller 93) where a condition similar to (H4) is also needed.
- To the best of our knowledge it is the first time it appears in problems with $p > 1$ growth.

Lower and upper bound

Theorem

Assume W satisfies (H1)-(H4) and that (h_d, u_d) is a sequence of low energy converging in the sense of the previous proposition then

$$\liminf_{d\to+\infty} F_d\left(u_d, h_d\right) \geq \int_{BL_h} W^\infty\left(\nabla v\right) dxdy + \int_0^1 \left|h'\right| + 2\mathcal{H}^1\left(\Gamma_{cuts}\right).$$

Conversely, for every pair (v, h) with $v \in W^{1,p}\left(BL_h; \mathbb{R}^2\right)$, $v\left(x, 0\right) = \left(x, 0\right)$, and $h : I \to \mathbb{R}$ a nonnegative lower semicontinuous function of bounded pointwise variation, $\int_0^1 h\left(x\right) dx = 1$, there exists a sequence $\{u_d\} \in W^{1,p}\left(\Omega_h; \mathbb{R}^2\right)$ with $u_d\left(x, 0\right) = e_0\left(x, 0\right)$, such that $\frac{1}{e_0} u_d\left(x, \frac{y}{d}\right) \rightharpoonup v$ locally weakly in $W^{1,p}\left(BL_h; \mathbb{R}^2\right)$ and

$$\lim_{d\to+\infty} F_d\left(u_d, h\right) = \int_{BL_h} W^\infty\left(\nabla v\right) dxdy + \int_0^1 \left|h'\right| + 2\mathcal{H}^1(\Gamma_{cuts}).$$

Proof

The liminf inequality follows from the previous Lemma.

For the recovery sequence, we can assume that h is Lipschitz and that v has bounded support. Take then

$$u_d(x,y) := \begin{cases} e_0 v(x, dy) & \text{if } y \leq \frac{\log(d)}{d}, \\ e_0 v(x, \log(d)) \left(2 - \frac{d}{\log(d)} y\right) & \text{if } \frac{\log(d)}{d} \leq y \leq \frac{2\log(d)}{d}, \\ 0 & \text{if } y \geq \frac{2\log(d)}{d}. \end{cases}$$

Analysis of the minimizers

Let

$$C_W := \min_{v(x,0)=(x,0)} \int_{[0,1]\times[0,+\infty)} W^\infty(\nabla v)\, dxdy.$$

Proposition

If $C_W \geq 1$ then the minimizer of

$$\int_{BL_h} W^\infty(\nabla v)\, dxdy + \int_0^1 |h'| + 2\mathcal{H}^1(\Gamma_{cuts})$$

corresponds to a rectangle of length $\ell_{\min} = \left(\frac{1}{C_W}\right)^{1/3}$, and v is given by the minimizer of the elastic energy in the corresponding boundary layer. If $C_W < 1$ then the flat configuration is minimizing.

Proof

- h has to be constant on each connected component of $h \neq 0$.
- In the boundary layer, v has to be chosen as the minimizer of the elastic energy.

If $h = \sum d_i \chi_{[a_i, b_i]}$, set $\ell_i = b_i - a_i$ and $h_i = \frac{d_i}{\ell_i}$. Then the minimal energy is given by

$$\min_{\sum_i h_i \ell_i = 1} \frac{C_W}{2} \sum_i \ell_i^2 + \sum_i h_i \ .$$

Assume, for the sake of contradiction, that two of the ℓ_i are non-zero, say $\ell_1 \geq \ell_2 > 0$. For $\eta \in \left[-h_1, \frac{\ell_2}{\ell_1} h_2\right]$ consider $h_1 + \eta$ and $h_2 - \eta \frac{\ell_1}{\ell_2}$.

Since (ℓ_i, h_i) is minimizing and $(h_1+\eta)\,\ell_1 + \left(h_2 - \eta\frac{\ell_1}{\ell_2}\right)\ell_2 = h_1\ell_1 + h_2\ell_2$, we find that

$$\eta - \eta\frac{\ell_1}{\ell_2} \geq 0 \qquad \forall \eta \in \left[-h_1, \frac{\ell_2}{\ell_1}h_2\right]$$

and hence $\ell_1 = \ell_2$ from which we deduce that $\ell_i = \ell$ for every i. The minimization problem then reduces to

$$\min_{\ell \leq 1} \frac{C_W}{2} N\ell^2 + \frac{1}{\ell}$$

where N is the number of intervals where $h \neq 0$. It is then clearly optimal to take $N = 1$ and $\ell_{\min} = \min\left\{1, \left(\frac{1}{C_W}\right)^{1/3}\right\}$.

The elastic dominant regime, $F_{d,e_0} \simeq e_0^{p/3} d^{2/3}$

The relevant parameter is $\eta := \left(\frac{d}{e_0^p}\right)^{1/3} \to 0$, so that the energy scales like $\frac{d}{\eta}$. We thus consider the normalized energy:

$$F_\eta(u,h) := \eta \left[\int_{\Omega_h} W\left(\frac{\partial u}{\partial x}, \frac{1}{d}\frac{\partial u}{\partial y}\right) dxdy + \int_0^1 \sqrt{\frac{1}{d^2} + |h'|^2} dx \right].$$

Notice that in this case, no bound on the total variation of h is available and we expect that the configuration will get more and more irregular.

The convergence result

Theorem

Let $\{(u_\eta, h_\eta)\}$ be such that $\sup F_\eta(u_\eta, h_\eta) \le C$, and set $\mu_\eta := h_\eta \, dx$. Then there exists a subsequence (not relabeled) such that $\{\mu_\eta\}$ weak-$*$ converges to $\mu := \sum_{i=1}^{+\infty} d_i \delta_{c_i}$ where $d_i > 0$ satisfy $\sum_{i=1}^{+\infty} d_i = 1$. Moreover, there holds

$$\liminf_{\eta \to 0} F_\eta(u_\eta, h_\eta) \ge 3 C_W^{1/3} \sum_{i=1}^{+\infty} d_i^{2/3} \, .$$

Conversely, if $\mu := \sum_{i=1}^{+\infty} d_i \, \delta_{c_i}$ then there exist a sequence $\{(u_\eta, h_\eta)\}$ of functions $u_\eta \in W^{1,p}\left(\Omega_{h_\eta}; \mathbb{R}^2\right)$, and nonnegative Lipschitz functions h_η such that $\mu_\eta := h_\eta dx$ are probability measures that weakly-$*$ converge to μ, and

$$\limsup_{\eta \to 0} F_\eta(u_\eta, h_\eta) \le 3 C_W^{1/3} \sum_{i=1}^{+\infty} d_i^{2/3}.$$

Proof

Since μ_η are probability measures, there exists a subsequence and a probability measure μ such that μ_η weakly-$*$ converges to μ.

Let :

$$y_0 := \frac{1}{2\sqrt{e_0^{p/3} d^{2/3}}} \to 0$$

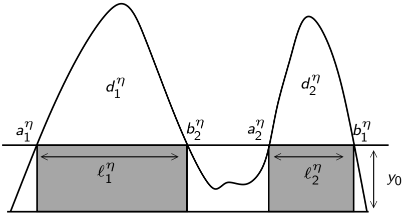
$$\ell := \mathcal{H}^1\left(\Omega_{h_\eta} \cap (I \times \{y_0\})\right).$$

$$I_\ell^\eta := \Omega_{h_\eta} \cap (I \times \{y_0\}).$$

$$I_\ell^\eta = \cup_{i=1}^{+\infty}\left[a_i^\eta, b_i^\eta\right].$$

$$\ell_i^\eta := b_i^\eta - a_i^\eta.$$

$$d_i^\eta := \left|\Omega_{h_\eta} \cap \left(\left[a_i^\eta, b_i^\eta\right] \times [y_0, +\infty)\right)\right| .$$



We assume that the d_i^η are ordered in a decreasing way.

Notice that $1 - \sum_i d_i^\eta \leq y_0$ and thus $\lim_{\eta \to 0} \sum_i d_i^\eta = 1$.

Let finally $d_i := \lim_{\eta \to 0} d_i^\eta$ (which we can assume exists for every $i \in \mathbb{N}$ up to further extraction).

Since $F_{d,e_0} \simeq e_0^{p/3} d^{2/3}$, by the computation in the scaling law, we see that $\max_i \ell_i^\eta \leq C\eta$ and thus $\left(b_i^\eta - a_i^\eta\right) \to 0$ for all i.

Hence we may assume that for some $c_i \in [0,1]$, $a_i^\eta \to c_i$ and $b_i^\eta \to c_i$.

Let $\Omega_i^\eta := \Omega_{h_\eta} \cap \left(\left[a_i^\eta, b_i^\eta\right] \times [0, +\infty)\right)$, then for every i

$$\begin{aligned}
F_\eta\left(u_\eta, h_\eta, \Omega_i^\eta\right) &\geq \eta \int_{a_i^\eta}^{b_i^\eta} \left|h'_\eta\right| dx + \eta \int_{\left[a_i^\eta, b_i^\eta\right]\times[0,y_0]} W\left(\frac{\partial u}{\partial x}, \frac{1}{d}\frac{\partial u}{\partial y}\right) \\
&\geq 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} \min_{u(x,0)=e_0(x,0)} \int_{\left[a_i^\eta, b_i^\eta\right]\times[0,dy_0]} W(\nabla u) \\
&\geq 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} e_0^p \left(\ell_i^\eta\right)^2 \min_{v(x,0)=(x,0)} \int_{[0,1]\times\left[0, \frac{dy_0}{\ell_i^\eta}\right]} \frac{W(e_0 \nabla v)}{e_0^p} \\
&= 2\eta \frac{d_i^\eta}{\ell_i^\eta} + \frac{\eta}{d} e_0^p \left(\ell_i^\eta\right)^2 C_W \left(1 - \psi(\eta)\right) \\
&\geq 3 C_W^{1/3} \left(1 - \psi(\eta)\right)^{1/3} \left(d_i^\eta\right)^{2/3},
\end{aligned}$$

where $\psi(\eta) \to 0$ as $\eta \to 0$ (we used that $dy_0/\ell_i^\eta \geq (e_0^{p/3} d^{2/3})^{1/2}$).

Summing over i and letting $\eta \to 0$, we get the liminf inequality.

Structure of the measure μ

Now for every $\varepsilon > 0$, let $V^\varepsilon := \left\{ i \in \mathbb{N} \ / \ d_i^\eta < \varepsilon \right\}$. Then

$$\begin{aligned}
\sum_{i \in V^\varepsilon} d_i^\eta = \sum_{i \in V^\varepsilon} \left(d_i^\eta\right)^{1/3} \left(d_i^\eta\right)^{2/3} &\leq \varepsilon^{1/3} \sum_{i \in V^\varepsilon} \left(d_i^\eta\right)^{2/3} \\
&\leq C\varepsilon^{1/3} \sum_{i \in V^\varepsilon} F_\eta \left(u_\eta, h_\eta, \Omega_i^\eta\right) \\
&\leq C\varepsilon^{1/3} F_\eta \left(u_\eta, h_\eta\right) \leq C\varepsilon^{1/3} \ .
\end{aligned}$$

Structure of the measure μ continued

The number of islands such that $d_i^\eta > \varepsilon$ is uniformly bounded by some constant $N_\varepsilon \leq \frac{1}{\varepsilon}$. For fixed $\varepsilon > 0$ let $I^\varepsilon := \left(\cup_{i\in V^\varepsilon} \left[a_i^\eta, b_i^\eta\right]\right)^c$ and $\mu_\eta^\varepsilon := h_\eta \chi_{I^\varepsilon}\ dx$. Then $\left\{\mu_\eta^\varepsilon\right\}$ converges weakly-$*$ to $\mu^\varepsilon := \sum_{i=1}^{N_\varepsilon} d_i \delta_{c_i}$. Finally $\mu_\varepsilon \to \mu$ since for every $\phi \in C\left([0,1]\right)$

$$\left|\left(\mu^\varepsilon - \mu\right)(\phi)\right| = \lim_{\eta\to 0} \int_{I^\varepsilon} h_\eta \phi\ dx \leq C \left|\phi\right|_\infty \varepsilon^{1/3} .$$

Since μ_ε weakly-$*$ converges $\sum_{i\in\mathbb{N}} d_i \delta_{c_i}$, this ends the proof.

Upper bound

Every measure $\mu = \sum_{i=1}^{+\infty} d_i \delta_{c_i}$ can be approximated in energy by the measures $\mu_N := \sum_{i=1}^{N} d_i\ \delta_{c_i}$, and by slightly moving the points c_i, we may assume without loss of generality that none of them is 0 or 1.

For these measures, a recovery sequence is easily constructed:

- let $\ell_i := \left(\frac{d_i}{C_W}\right)^{1/3} \eta$,
- let $h_i := \frac{d_i}{\ell_i}$,
- let h_η a Lipschitz function very close to $\sum_{i=1}^{N} h_i \chi_{(c_i - \ell_i/2, c_i + \ell_i/2)}$,
- Finally let u_η be the minimizer of the elastic energy in Ω_{h_η}.

Remark: The minimizer of the limit functional, i.e.

$$\min \left\{ \sum_{i=1}^{+\infty} d_i^{2/3} \ : \ \sum_{i=1}^{+\infty} d_i = 1 \right\}$$

is given by a single Dirac mass, i.e. $d_1 = 1$ and $d_i = 0$ for $i > 1$.

Walfrido 'Morning in the Tropic'

Thank you for your attention!