Sur quelques problèmes variationnels avec pénalisation d'interfaces

Soutenance d'HDR

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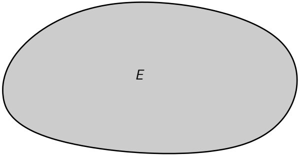
Introduction



General problem: solve

$\min_E P(E) + G(E)$

with or without volume constraint.

Here $P(E) = \mathcal{H}^{d-1}(\partial E)$
G is a (local or non-local) functional depending on the specific problem.

Isoperimetric problem

Fundamental example ($G = 0$):

$$\min_{|E|=V} P(E)$$

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$$\min_{|E|=V} P(E)$$

Solution:

E is a ball

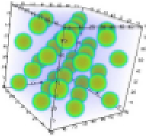
In general competition between P and $G \Longrightarrow$ many possible behaviors:

Can be simple and remain the ball (Gamow, 2 components BEC ...)

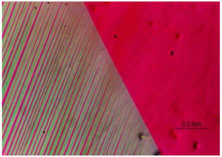
or be more complex

Periodic patterns

- Array of drops (Ohta-Kawasaki)



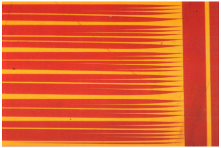
- Stripes (Shape memory alloys, dipolar ferromagnets ...)

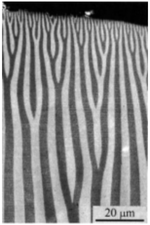


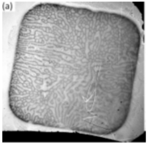
- Others

Branching patterns

Shape memory alloys, uniaxial ferromagnets, type-I superconductors ...







Main Questions

- Give a qualitative/quantitative description of minimizers (when they exist)
- If the model is too complex, derive and study simpler models

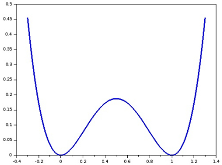
Rk: related question, stability of minimizers (e.g quantitative isoperimetric inequality)

Diffuse interface approximation

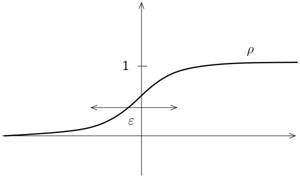
In many physical models/for numerical approximation often $P(E)$ replaced by

$$E_\varepsilon(\rho) = \int |\nabla \rho|^2 + \frac{1}{\varepsilon^2} W(\rho)$$

W is a double well potential



Coherence length $\simeq \varepsilon$



Theorem (Modica-Mortola): $E_\varepsilon \to P$ as $\varepsilon \downarrow 0$.

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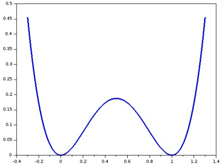

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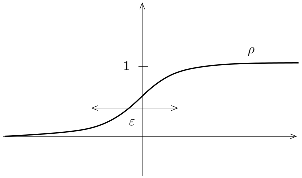


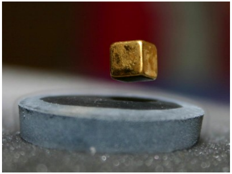
Theorem (Modica-Mortola): $E_\varepsilon \to P$ as $\varepsilon \downarrow 0$.

We focus on 2 problems corresponding to 2 asymptotic limits of the Ginzburg-Landau energy

The Ginzburg-Landau energy

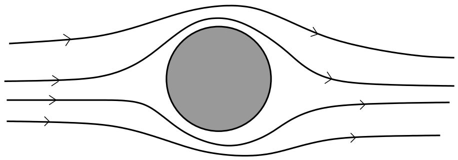
Introduction

Superconductivity was first observed by Onnes in 1911 and has nowadays many applications.



Meissner effect

In 1933, Meissner understood that superconductivity was related to the expulsion of the magnetic field outside the material sample



Ginzburg Landau functional

In the 50's Ginzburg and Landau proposed the model:

$$GL(u, A) = \int_\Omega |\nabla_A u|^2 + \frac{\kappa^2}{2}(1-\rho^2)^2 dx + \int_{\mathbb{R}^3} |\nabla \times A - B_{ex}|^2 dx$$

where $u = \rho e^{i\theta}$ is the order parameter, $B = \nabla \times A$ is the magnetic field, B_{ex} is the external magnetic field, κ is the Ginzburg-Landau constant and

$$\nabla_A u = \nabla u - iAu$$

is the covariant derivative.

$\rho \sim 0$ represents the normal phase and $\rho \sim 1$ the superconducting one.

The various terms in the energy

For $u = \rho e^{i\theta}$, $|\nabla_A u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta - A|^2$.
In $\rho > 0$ first term wants $A = \nabla\theta \Longrightarrow \nabla \times A = 0$
That is

$$\rho^2 B \simeq 0 \qquad \text{(Meissner effect)}$$

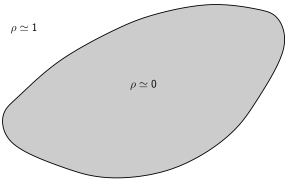
and penalizes fast oscillations of ρ

Second term forces $\rho \simeq 1$ (superconducting phase favored)

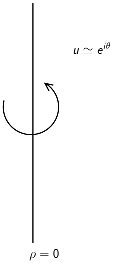
Last term wants $B \simeq B_{ex}$. In particular, this should hold outside the sample.

Two different regimes

$\kappa < 1/\sqrt{2}$ energy penalizes interfaces between normal and superconducting phases (type-I)



$\kappa > 1/\sqrt{2}$ negative surface tension $\Longrightarrow$ formation of vortices (type-II)



A remark about type-II

In the absence of magnetic field ($A = B_{ex} = 0$) and letting $\varepsilon = \kappa^{-1}$, reduces to

$$GL_\varepsilon(u) = \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2$$

In dimension 2 formation of point vortices of energy

$$GL_\varepsilon(u_\varepsilon) = 2\pi |\log \varepsilon| + O(1)$$

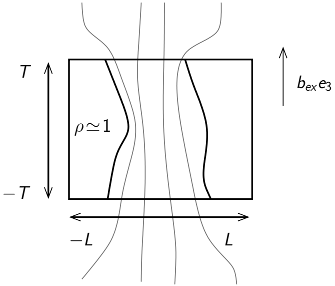
(see BBH, SS, ...)

A branched transport limit for type-I superconductors $(\kappa \downarrow 0)$

Results from CGOS'18, G'18

Our setting

We consider $\Omega = Q_{L,T} = [-L, L]^2 \times [-T, T]$ with periodic lateral boundary conditions and take $B_{ex} = b_{ex} e_3$.



First rescaling

We let

$$\kappa T = \sqrt{2}\alpha \qquad b_{ex} = \frac{\beta\kappa}{\sqrt{2}}$$

and then

$$\widehat{x} = T^{-1}x \qquad \widehat{u}(\widehat{x}) = u(x)$$

$$\widehat{A}(\widehat{x}) = A(x) \qquad \widehat{B}(\widehat{x}) = \nabla \times \widehat{A}(\widehat{x}) = TB(x)$$

In these units,

$$\text{coherence length} \simeq \alpha^{-1} \quad \text{penetration length} \simeq T^{-1}$$

We are interested in the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$.

The energy

The energy can be written as

$$E_T(u,A) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla_{TA} u|^2 + \left(B_3 - \alpha(1-\rho^2)\right)^2 + |B'|^2$$
$$+ \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$

- First term: penalizes oscillations + $\rho^2 B \simeq 0$ (Meissner effect)

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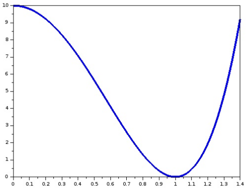
- First term: penalizes oscillations + $\rho^2 B \simeq 0$ (Meissner effect)
- Second term: degenerate double well potential.



If Meissner then:
$\left(B_3 - \alpha(1-\rho^2)\right)^2 \simeq \alpha^2 \chi_{\{\rho>0\}}(1-\rho^2)^2$

Rk: wants $B_3 = \alpha$ in $\{\rho = 0\}$
Similar features in mixtures of BEC (cf GM '15)

Crash course on optimal transportation

For ρ_0, ρ_1 probability measures

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \int_{Q_L \times Q_L} |x - y|^2 d\Pi(x, y) \; : \; \Pi_1 = \rho_0, \; \Pi_2 = \rho_1 \right\}$$

Theorem (Benamou-Brenier '00)

$$W_2^2(\rho_0, \rho_1) = \inf_{\mu, B'} \left\{ \int_0^1 \int_{Q_L} |B'|^2 d\mu \; : \; \partial_3 \mu + \mathrm{div}'(B'\mu) = 0, \right.$$

$$\left. \mu(0, \cdot) = \rho_0, \; \mu(1, \cdot) = \rho_1 \right\}$$

The energy continued

$$E_T(u,A) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla_{TA} u|^2 + \big(B_3 - \alpha(1-\rho^2)\big)^2 + |B'|^2$$
$$+ \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$

- Third term: with Meissner and $B_3 \simeq \alpha(1-\rho^2) = \chi$, $\operatorname{div} B = 0$ can be rewritten as

$$\partial_3 \chi + \operatorname{div}'(\chi B') = 0$$

Benamou-Brenier $\Longrightarrow$ Wasserstein energy of $x_3 \to \chi(\cdot, x_3)$

The energy continued

$$E_T(u,A) = \frac{1}{L^2}\int_{Q_{L,1}} |\nabla_{TA} u|^2 + \left(B_3 - \alpha(1-\rho^2)\right)^2 + |B'|^2$$
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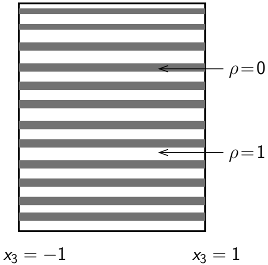
$$\partial_3 \chi + \operatorname{div}' \chi B' = 0$$

  Benamou-Brenier $\Longrightarrow$ Wasserstein energy of $x_3 \to \chi(\cdot, x_3)$
- Last term: penalizes non uniform distribution on the boundary but negative norm $\Longrightarrow$ allows for oscillations

A non-convex energy regularized by a gradient term

If we forget the kinetic part of the energy, can make $B' = 0$ and

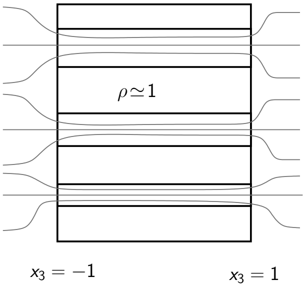
$$E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} \left(B_3 - \alpha(1-\rho^2)\right)^2 + \|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)}$$



$\Longrightarrow$ infinitely small oscillations of phases $\{\rho = 0, B_3 = \alpha\}$ and $\{\rho = 1, B_3 = 0\}$ with average volume fraction β.

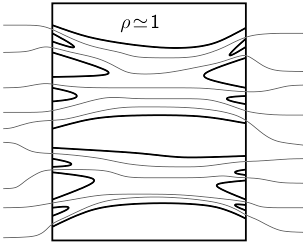
the kinetic term $|\nabla_A u|^2$ fixes the lengthscale.

Branching is energetically favored



$\|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)} \downarrow 0$

but interfacial energy $\uparrow \infty$

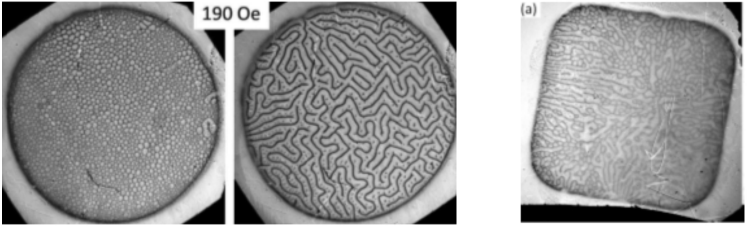


interfacial energy $\downarrow$
but $\int_{Q_{L,1}} |B'|^2 \uparrow$.

Landau '43

Experimental data

Complex patterns at the boundary



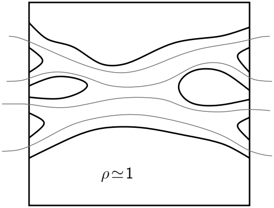
Experimental pictures from Prozorov and al.

Scaling law

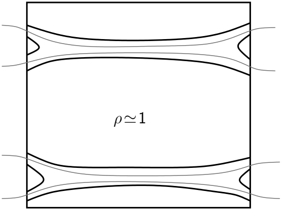
Theorem (COS '15, See also CCKO '08)

In the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$,

$$\min E_T \simeq \min(\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta)$$



First regime: $E_T \sim \alpha^{4/3}\beta^{2/3}$
Uniform branching,
$\|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)} = 0$



Second regime: $E_T \sim \alpha^{10/7}\beta$
Non-Uniform branching,
$\|B_3 - \alpha\beta\|^2_{H^{-1/2}(x_3=\pm 1)} > 0$
fractal behavior

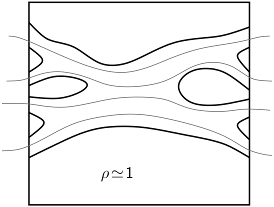
Scaling law

Theorem (COS '15, See also CCKO '08)

In the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$,

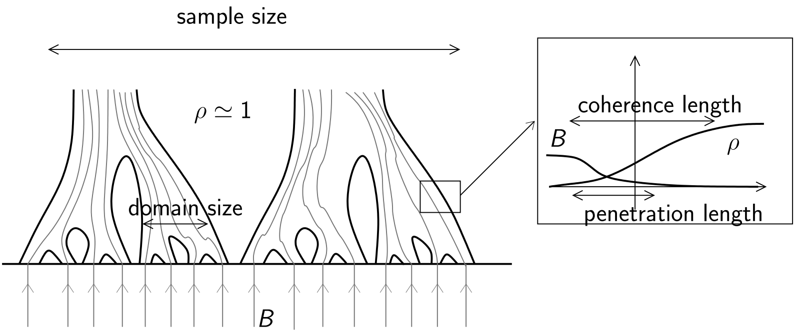
$$\min E_T \simeq \min(\alpha^{4/3}\beta^{2/3}, \alpha^{10/7}\beta)$$

We concentrate on the first regime (uniform branching)



$$\Longrightarrow \alpha^{-2/7} \ll \beta.$$

Multiscale problem



From the upper bound construction, we expect

$$\text{penetration length} \ll \text{coherence length} \ll \text{domain size} \ll \text{sample size}$$

which amounts in our parameters to

$$T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3} \ll L.$$

A Hierarchy of models

From the separation of scales $T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3} \ll L$ we expect formally

Ginzburg-Landau

$$\Downarrow \quad T \uparrow \infty$$

Ginzburg-Landau+Meissner

$$\Downarrow \quad \alpha \uparrow \infty$$

Sharp interface problem : Perimeter + transport

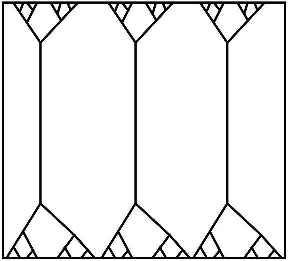
$$\Downarrow \quad \beta \downarrow 0$$

Small volume fraction limit : branched transportation model

The limiting functional

For μ a measure with $\mu_{x_3} = \sum_i \phi_i \delta_{x_i(x_3)}$ for a.e. x_3 and $\mu_{x_3} \rightharpoonup dx'$ when $x_3 \to \pm 1$,

$$I(\mu) = \int_{-1}^{1} \sum_i \frac{8\pi^{1/2}}{3} \phi_i^{1/2} + \phi_i \dot{x}_i^2 \, dx_3$$



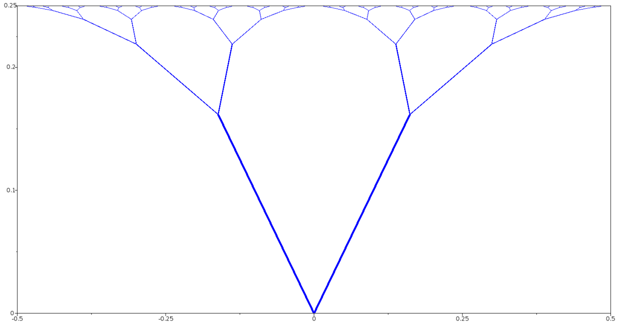
Main theorem

Theorem (CGOS '18)

After appropriate rescaling, E_T converges to $I(\mu)$ in the limit $T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3}\beta^{1/3} \ll L$

Optimal microstructure in 2D

For a related 2D functional, we can prove (G' 18) that the unique minimizer is



Related ongoing work on:

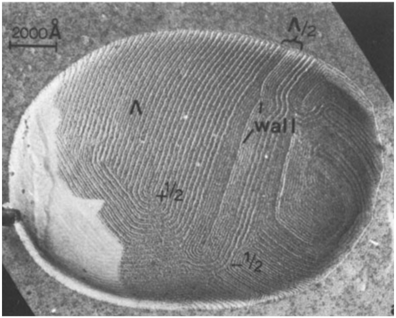
- Non-uniform branching limit, DGR
- Similar questions in micromagnetism, BGZ (see also CDZ '17)

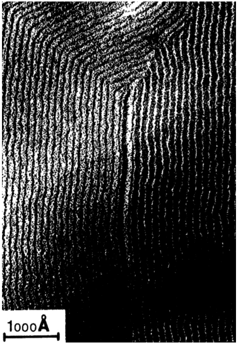
A GL model with topologically induced free discontinuities $(\kappa \uparrow \infty)$

Results from GMM '17

Motivation: ripple phase in lipid bilayers

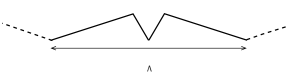
Two types of corrugations



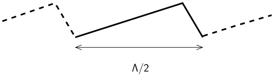


Experimental pictures from Sackmann and al.

Two different profiles



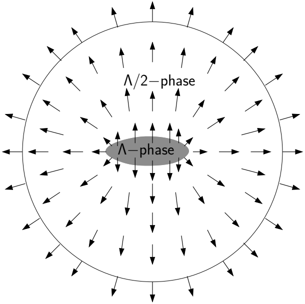
Λ phase symmetric
$\Longrightarrow \pm 1/2$ vortices



$\Lambda/2$ phase asymmetric
$\Longrightarrow \pm 1$ vortices

However, in $\Lambda/2$ phase $\pm 1/2$ vortices connected by line singularity!

Explanation: two $\pm 1/2$ vortices much cheaper than one ± 1
$\Longrightarrow$ phase transition to Λ phase around the singularity

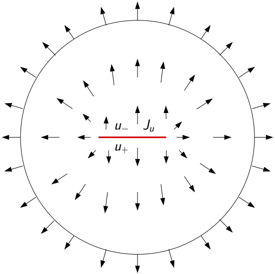


The model (after BFL '91)

$\Omega \subset \mathbb{R}^2$ convex (e.g. $\Omega = B_1$), $u \in SBV(\Omega, \mathbb{C})$ with $Pu \in H^1(\Omega)$ where

$$P : \mathbb{C} \to \mathbb{C}/\{\pm 1\} \text{ is the canonical projection}$$

- $Pu \in H^1 \iff u_+ = -u_-$ on J_u
- if $|u| > \delta$ then $Pu \in H^1(\Omega)$
  $\iff u^2 \in H^1(\Omega)$

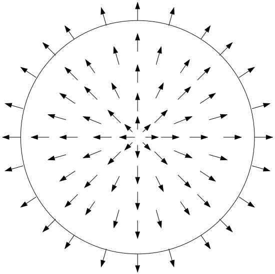


Energy:

$$E_\varepsilon(u) = \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + \mathcal{H}^1(J_u) = GL_\varepsilon(u) + \mathcal{H}^1(J_u)$$

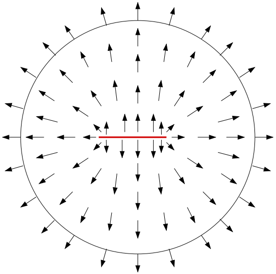
Mix between Ginzburg-Landau and Mumford-Shah

1/2 vortices are favored



Cost of ± 1 vortex:

$2\pi|\log \varepsilon|$



Cost of two $\pm 1/2$ vortices:

$2 \times \left(\frac{1}{2}\right)^2 2\pi|\log \varepsilon|$

Therefore half vortices are indeed energetically favorable.

Important observation

From now on, we fix $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ with $\deg(g, \partial\Omega) = d$ and minimize under the condition $u = g$ on $\partial\Omega$.
Minimizers u_ε satisfy

$$E_\varepsilon(u_\varepsilon) \leq \pi d |\log \varepsilon| + C$$

If $v_\varepsilon = \frac{u_\varepsilon^2}{|u_\varepsilon|} (= re^{2i\theta}) \in H^1(\Omega)$,

$$E_\varepsilon(u_\varepsilon) = \frac{1}{4} GL_\varepsilon(v_\varepsilon) + \frac{3}{4} GL_\varepsilon(|v_\varepsilon|) + \mathcal{H}^1(J_{u_\varepsilon})$$

$\implies GL_\varepsilon(v_\varepsilon) \leq 2\pi(2d)|\log \varepsilon| + C$ and $\deg(v_\varepsilon, \partial\Omega) = 2d$

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$\Longrightarrow GL_\varepsilon(v_\varepsilon) \leq 2\pi(2d)|\log \varepsilon| + C$ and $\deg(v_\varepsilon, \partial\Omega) = 2d$

Therefore classical GL theory applies to v_ε!

Some more notation from *GL* theory

For $x_1, \cdots, x_{2d} \in \Omega$ and $\mu = 2\pi \sum_k \delta_{x_k}$

$$v_\mu = e^{i\varphi_\mu} \prod_k \frac{x - x_k}{|x - x_k|} \qquad \text{with} \qquad \begin{cases} \Delta\varphi_\mu = 0 & \text{in } \Omega \\ v_\mu = g^2 & \text{on } \partial\Omega. \end{cases}$$

Renormalized energy

$$\mathbb{W}(\mu) = \lim_{r \downarrow 0} \left\{ \int_{\Omega \setminus B_r(\mu)} |\nabla v_\mu|^2 - 4\pi d |\log r| \right\}$$

where $B_r(\mu) = \cup_k B_r(x_k)$

Theorem (GMM' 17)

If u_ε *minimizer of* E_ε *and* $v_\varepsilon = \frac{u_\varepsilon^2}{|u_\varepsilon|}$, *there exists* (μ, u) *minimizer of*

$$\min_{\substack{u^2 = v_\mu \\ u = g \ on \ \partial\Omega}} \left\{ \frac{1}{4}\mathbb{W}(\mu) + \mathcal{H}^1(J_u) \right\}$$

with

- $u_\varepsilon \to u$ *in* L^1
- $v_\varepsilon \to v_\mu$ *in* $C^\infty_{loc}(\Omega \setminus \mathrm{Spt}\mu)$

We actually obtain a $\Gamma-$convergence result.
Proof combines ideas from GL (S '98, J '99, JS '02, AP '14 ...) and free discontinuity problems (BCS '07 ...)

Structure of the minimizers for the limit problem

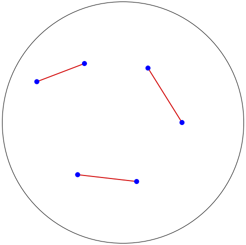
Theorem (GMM' 17)

For every fixed μ, and every minimizer u of

$$\min_{\substack{u \in SBV, u^2 = v_\mu \\ u = g \ on \ \partial\Omega}} \mathcal{H}^1(J_u)$$

J_u is a minimal connection i.e. made of d segments connecting the x_k pairwise with minimal length and $u \in C^\infty(\Omega \setminus J_u)$.

Idea of proof:
if u_1 and u_2 are competitors
$(u_1/u_2)^2 = 1 \implies u_2 = (\chi_E - \chi_{E^c})u_1$
and E is $\simeq$ area minimizing



Structure of the minimizers for $\varepsilon > 0$

Theorem (GMM '17)

For $0 < \varepsilon \ll 1$,

- *The set J_{u_ε} is closed and converges Hausdorff to J_u;*
- *Away from $\mathrm{Spt}\mu$, J_{u_ε} is made of d segments;*
- *Away from J_{u_ε}, u_ε is smooth (and solves the classical GL equation).*

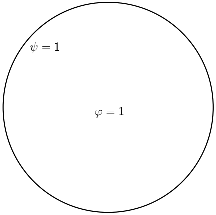
Idea of proof

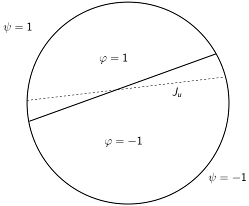
Based on Lassoued-Mironescu trick (write "$u_\varepsilon = \sqrt{v_\varepsilon}\varphi_\varepsilon$") + Wente to reduce to Mumford-Shah functional

$$\min_{\varphi=\psi \text{ on } \partial B_r(x_0)} \int_{B_r(x_0)} |\nabla\varphi|^2 + \mathcal{H}^1(J_\varphi \cap B_r(x_0))$$

with

$\psi = 1$ if $x_0 \notin J_u$ $\qquad$ $\psi = \chi_E - \chi_{E^c}$ if $x_0 \in J_u \backslash \mathrm{Spt}\mu$





Use calibration arguments (ABDM '03) to conclude.

GMM '17 contains extensions to

- vortices of degree $1/m$, $m \in \mathbb{N}$
  $\Longrightarrow$ minimal connections become Steiner type problems;
- diffuse interface version of Ambrosio-Tortorelli type.

"Les bulles de savon" J.B.S. Chardin

Merci pour votre attention!