Quelques applications des fonctions à variation bornée en dimension finie et infinie.

Thèse de Doctorat

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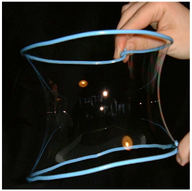
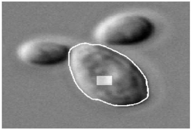
CMAP, Polytechnique

9 décembre 2011

Topic of the Thesis





Introduction

Introduction

Functions of bounded variation have a central position in many problems in the Calculus of Variations.

Definition

Let $u \in L^1(\Omega)$ *then* $u \in BV(\Omega)$ *if*

$$\int_\Omega |Du| := \sup_{\substack{\varphi \in \mathcal{C}^1_c(\Omega) \\ |\varphi|_\infty \leq 1}} \int_\Omega u \, \mathrm{div}\, \varphi < +\infty.$$

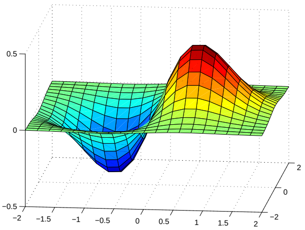
Definition

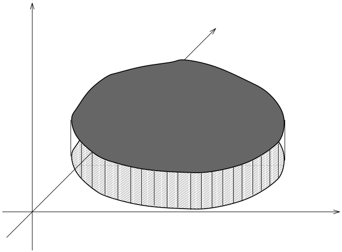
A set $E \subset \mathbb{R}^m$ *is called a set of finite perimeter if*

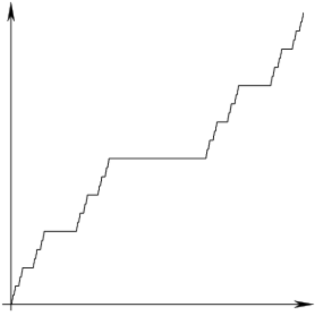
$$P(E) := \int_{\mathbb{R}^m} |D\chi_E| < +\infty.$$

If E is a smooth set then $P(E) = \mathcal{H}^{m-1}(\partial E)$.

Typical functions in BV







Primal-Dual methods in image processing

Examples of problems in image processing

Inpainting

Deblurring

Many problems in image processing can be model as solving a minimization problem :

$$J(u) := \int_\Omega |Du| + G(u)$$

where G is a lsc convex function on L^2.

Example : denoising with ROF corresponds to $G(u) = \frac{\lambda}{2} \int_\Omega |u - f|^2$

It can also be used for inpainting, deblurring, zooming...

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It can also be used for inpainting, deblurring, zooming...

Problem : How to solve the minimization problem ?

Idea of the method

Remind : The total variation is defined as

$$\int_\Omega |Du| = \sup_{\substack{\xi \in \mathcal{C}^1_c(\Omega) \\ |\xi|_\infty \leq 1}} - \int_\Omega u \operatorname{div} \xi$$

Hence the minimization problem rewrites

$$\min_{u \in BV} J(u) = \min_{u \in BV} \sup_{\substack{\xi \in \mathcal{C}^1_c(\Omega) \\ |\xi|_\infty \leq 1}} - \int_\Omega u \operatorname{div} \xi + G(u)$$

$\Rightarrow$ It is thus equivalent to finding a saddle point

The Arrow-Hurwicz method for finding saddle points

For a function K, this method is

$$\begin{cases} \frac{\partial u}{\partial t} = -\nabla_u K(u, \xi) \\ \frac{\partial \xi}{\partial t} = \nabla_\xi K(u, \xi) \end{cases}$$

It is a gradient descent in the primal variable u and a gradient ascent in the dual variable ξ.

When $K(u,\xi) = -\int_\Omega u \operatorname{div} \xi + G(u)$ we find

$$\begin{cases} \nabla_u K = -\operatorname{div} \xi + \partial G(u) \\ \nabla_\xi K = Du \end{cases}$$

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which formally amounts to solve :

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div} \xi - \partial G(u) \\ \frac{\partial \xi}{\partial t} = Du \qquad |\xi|_\infty \leq 1 \end{cases}$$

This method proposed by Appleton and Talbot is the continuous analogous of the method proposed by Chan and Zhu in the discrete setting.

Theorem

Giving appropriate meaning to the previous system, there exists a unique solution to the Cauchy problem.

Moreover, for $G(u) = \frac{\lambda}{2}|u - f|_{L^2}^2$ *there is convergence towards the minimizer* $\overline{u}$ *of* J *and we have the a posteriori estimation*

$$|u - \overline{u}| \leq \frac{1}{2}\left(\frac{|\partial_t u|}{\lambda} + \sqrt{\frac{|\partial_t u|^2}{\lambda^2} + \frac{8|\Omega|^{\frac{1}{2}}}{\lambda}|\partial_t \xi|}\right)$$

This extends to :

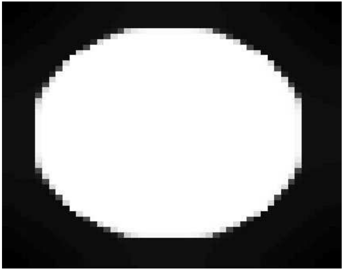
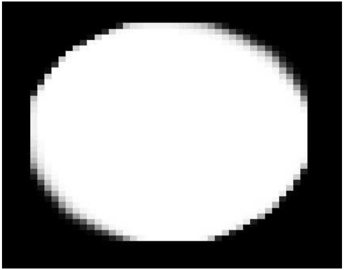
- segmentation with geodesic active contours,

$$J(u) = \int_{\Omega} g(x)|Du|$$

- problems with boundary conditions

Interest of this continuous approach

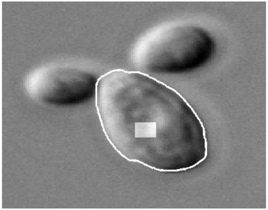
- Give a better understanding of the discrete method
- Lead to new results also in the discrete setting (such as a posteriori estimates)
- Give rise to more isotropic results (absence of discretization bias)

Restauration by AT left and CZ right

Zoom on the top right corner

Numerical results

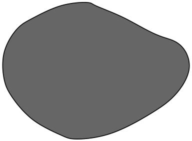


Sets with prescribed mean curvature in periodic media

The Problem

Let $g : \mathbb{R}^m \to \mathbb{R}$ periodic, find a compact set with

$$\kappa = g$$



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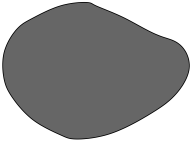
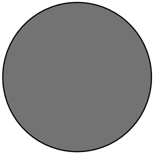


Example : If $g \equiv C > 0$ then a solution is given by a ball.



Main result

In general there is no solution but

Theorem

Let g be periodic with zero mean and sufficiently small norm then for every ε there exists $\varepsilon' \in [0, \varepsilon]$ and a compact solution to

$$\kappa = g + \varepsilon'.$$

Idea of proof

Consider the volume constrained problem

$$f(v) := \min_{|E|=v} P(E) - \int_E g$$

then

Proposition

For every $v > 0$ there exists a compact minimizer E_v of this problem.

E_v satisfies the Euler-Lagrange equation

$$\kappa = g + \lambda$$

we can prove f is Lipschitz and

$$f'(v) = \lambda$$

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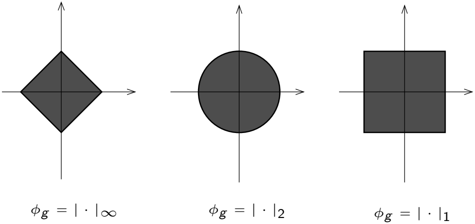
Since $f \approx cv^{\frac{m-1}{m}}$, we can find $v_n \to +\infty$ with $f'(v_n) \to 0$.

Behaviour of large volume minimizers

There exists ϕ_g a one-homogeneous convex function such that letting

$$W_g = \left\{ x \in \mathbb{R}^m : \max_{\phi_g(y) \leq 1} x \cdot y \leq 1 \right\}$$



Theorem

$$\widetilde{E}_v = \left(\frac{|W_g|}{v} \right)^{\frac{1}{m}} E_v + z_v$$

it holds $\lim_{v \to +\infty} \left| \widetilde{E}_v \Delta W_g \right| = 0.$

Variational problems in Wiener spaces

The Wiener space

Definition

A Wiener space is a Banach space X with a Gaussian measure γ.

- if $x^* \in X^*$ then $x \to \langle x^*, x \rangle \in L^2_\gamma(X)$.
- $\mathcal{H} := \overline{X^*}^{L^2_\gamma(X)}$.

Canonical cylindrical approximation

Let $(x_i^*)_{i\in\mathbb{N}} \in X^*$ be an orthonormal basis of $\mathcal{H}$. Then

$$\Pi_m(x) := (\langle x_1^*, x\rangle, \dots, \langle x_m^*, x\rangle).$$

Gives a decomposition of $X \cong \mathbb{R}^m \oplus X_m^\perp$ and $\gamma = \gamma_m \otimes \gamma_m^\perp$, with γ_m, $\gamma_m^\perp$ Gaussian measures on $\mathbb{R}^m$, $X_m^\perp$ respectively.

For $u \in L^1_\gamma(X)$,

$$\mathbb{E}_m u(x) := \int_{X_m^\perp} u(\Pi_m(x), y)\, d\gamma_m^\perp(y).$$

Gradient and Divergence

In this context one can define a notion of gradient and divergence such that

Proposition

For smooth enough Φ *and* u,

$$\int_X u \, \mathrm{div}_\gamma \Phi \; d\gamma = -\int_X [\nabla u, \Phi] \; d\gamma.$$

$\Rightarrow$ This gives definitions of Sobolev and BV functions as in the Euclidiean setting.
We say that E is of finite Gaussian perimeter if

$$P_\gamma(E) := \int_X |D_\gamma \chi_E| < +\infty.$$

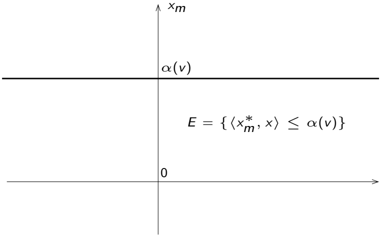
Isoperimetric inequality

Definition

$$\Phi(t) = \gamma(\{\langle x_m^*, x\rangle \le t\})$$

$$\alpha(v) = \Phi^{-1}(v).$$

$$\mathcal{U}(v) = P_\gamma(\{\langle x_m^*, x\rangle \le \alpha(v)\}).$$



Theorem (Isoperimetric inequality)

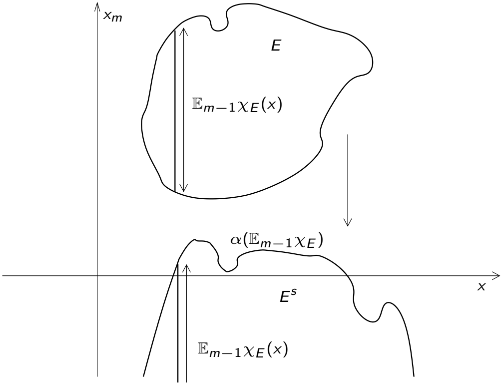
The half spaces are the only isoperimetric sets i.e. for every set E,

$$P_\gamma(E) \ge \mathcal{U}(\gamma(E)).$$

The Ehrhard symetrization

Definition

Let $E \subset X$ and $m \in \mathbb{N}$. The Ehrhard symmetral of E is :



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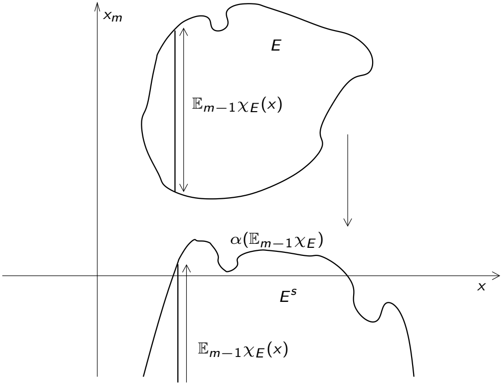


$$P_\gamma(E^s) \leq P_\gamma(E)$$

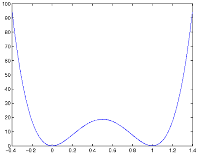
Approximation of the perimeter in Wiener spaces

A Modica-Mortola result

Idea : approximate the perimeter with

$$J_\varepsilon(u) := \int_X \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon} d\gamma$$

where W is a double-well potential.



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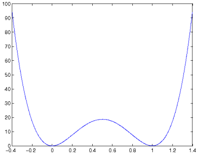

Problem : No compactness in the strong $L^2_\gamma(X)$ topology
$\Rightarrow$ we have to consider the weak topology.

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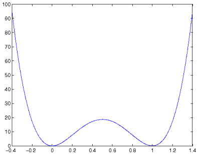

Problem : No compactness in the strong $L^2_\gamma(X)$ topology
$\Rightarrow$ we have to consider the weak topology.

But the perimeter is NOT lsc for this topology...

We thus have to compute the relaxation of the perimeter :

$$\overline{F}(u) := \inf \underline{\lim}\{P_\gamma(E_n) \quad / \quad E_n \rightharpoonup u\}.$$

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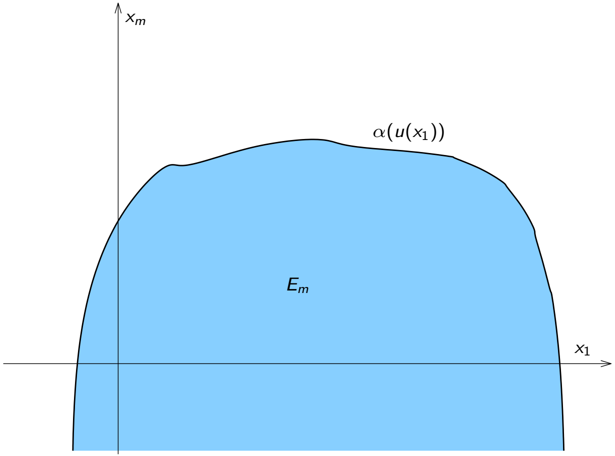
Since $P_\gamma(E^s) \leq P_\gamma(E)$,

$$\overline{F}(u) = \inf \underline{\lim}\{P_\gamma(E_n^s) \quad / \quad E_n^s \rightharpoonup u\}.$$

Relaxation of the perimeter

If $u(x) = u(\langle x_1^*, x\rangle)$ and $E_m = \{\langle x_m^*, x\rangle \leq \alpha \circ u(\langle x_1^*, x\rangle)\}$:



$$E_m \rightharpoonup u \text{ and } P_\gamma(E) = \int_{\mathbb{R}} \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2} d\gamma_1.$$

Main results

Theorem

The relaxation of the perimeter for the weak $L^2_\gamma(X)$ topology is given by

$$\overline{F}(u) = \begin{cases} \int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2}\, d\gamma & \textit{if } 0 \le u \le 1 \\ +\infty & \textit{otherwise.} \end{cases}$$

Theorem

The functionals J_ε Γ-converge for the weak $L^2_\gamma(X)$ topology to $c_W \overline{F}$ where c_W is the usual constant.

Some observations

The functional $\overline{F}$ appears in an alternative proof of the isoperimetric inequality by functional inequalities :

$$\mathcal{U}\left(\int_X u \, d\gamma\right) \leq \overline{F}(u).$$

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If $u := T_t\chi_E$, letting $t \to 0$ we get the isoperimetric inequality.

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If $u := T_t \chi_E$, letting $t \to 0$ we get the isoperimetric inequality.

$\Rightarrow$ the theory of Bakry-Ledoux on functional inequalites and convergence to equilibrium in diffusion process.

Convexity of minimizers of variational problems in Wiener spaces

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Problem : Let $g \in L^2_\gamma(X)$ be a convex function and $v \in [0,1]$, is the minimizer of

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By the co-area formula, for "large" v, this minimizer is a level-set of the minimizer of

$$\int_X |D_\gamma u| + \frac{1}{2}\int_X (u-g)^2 \; d\gamma.$$

Problem : Let $F : H \to \mathbb{R} \cup \{+\infty\}$ and $g \in L^2_\gamma(X)$ be two convex functions we want to study the convexity of the solution of

$$\min_{u \in L^2_\gamma(X)} J(u) := \int_X F(D_\gamma u) + \frac{1}{2} \int_X (u-g)^2 \, d\gamma.$$

Strategy of proof

- Approximate the infinite dimensional problem by finite dimensional ones.
- Prove the convexity in the finite dimensional case.
- Pass to the limit.

The finite dimensional problem

Theorem

$F : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ and $g \in L^2_\gamma(\mathbb{R}^m)$ convex then the solution of

$$\min_{u \in L^2_\gamma(\mathbb{R}^m)} \int_{\mathbb{R}^m} F(D_\gamma u) + \frac{1}{2} \int_{\mathbb{R}^m} (u-g)^2 d\gamma$$

is convex.

Idea of proof : construct convex sub- and super-solutions and use a result of Alvarez, Lasry and Lions to construct a solution.

Relaxation and Representation formulas

Definition

For $u \in L^2_\gamma(X)$,

$$\int_X F(D_\gamma u) := \sup_{\Phi \in \mathcal{F}\mathcal{C}^1_b(X,H)} \int_X -u \operatorname{div}_\gamma \Phi - F^*(\Phi)\ d\gamma.$$

Theorem

For $u \in BV_\gamma(X)$

$$\int_X F(D_\gamma u) = \int_X F(\nabla u) d\gamma + \int_X F^\infty\left(\frac{dD^s_\gamma u}{d|D^s_\gamma u|}\right) d|D^s_\gamma u|$$

Proposition

Let F be a proper lsc convex function then the functional $\int_X F(D_\gamma u)$ is the relaxation of the functional defined as $\int_X F(\nabla u) d\gamma$ for $u \in W^{1,1}_\gamma(X)$.

If F has p-growth, then it is also the relaxation of the functional $\int_X F(\nabla u) d\gamma$ defined on the smaller class $\mathcal{F}\mathcal{C}^1_b(X)$.

The infinite dimensional problem

Idea of proof : Let $g_m := \mathbb{E}_m g$ and u_m be the minimizer of

$$\min_{u=\mathbb{E}_m u} J_m(u) := \int_X F(D_\gamma u) + \frac{1}{2}\int_X (u-g_m)^2 d\gamma$$

by the finite dimensional Theorem, u_m is convex and $u_m \rightharpoonup \bar{u} \Rightarrow \bar{u}$ is convex.

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$\forall u \in \mathcal{FC}_b^1(X)$,

$$J(u) = \lim_{m\to+\infty} J_m(u) \geq \varliminf_{m\to+\infty} J_m(u_m) \geq J(\bar{u})$$

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$\forall u \in \mathcal{FC}_b^1(X)$,

$$J(u) = \lim_{m\to+\infty} J_m(u) \geq \underline{\lim}_{m\to+\infty} J_m(u_m) \geq J(\bar{u})$$

$\Rightarrow$ By the relaxation Theorem, $\bar{u}$ is the minimizer of J.

For F one homogeneous with linear growth, we can define the anisotropic perimeter

$$P_F(E) := \int_X F(D_\gamma \chi_E)$$

then

Theorem

For v large enough, the minimizer of

$$\min_{\gamma(E)=v} \; P_F(E) + \int_E g \; d\gamma$$

is unique and convex.

Some perspectives

- Look for more accurate algorithms solving the Primal-Dual system.
- Investigate the case of existence of compact solutions to $\kappa = g$ when the mean of g is positive.
- Look for analog problems in quasi-periodic/stochastic media.
- Better understand the link between symmetrizations and functional inequalities.
- Study representation formulas for more complex integrals.
- Define a mean curvature flow in the Wiener space.
- ...

"Il calcolo delle variazioni è [...] una foresta da esplorare, piuttosto che un palazzo da costruire."

Ennio De Giorgi

"La tour de Babel" P. Bruegel

"Les bulles de savon" J.B.S. Chardin

Merci pour votre attention !