Second-order optimality conditions The case of a state constrained optimal control problem for a parabolic equation

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Motivating example

- The simplest state constrained optimal control problem !
- Joint work with A. Hermant (for this section)

Data of the academic example

$$(\mathcal{P}) \qquad \mathrm{Min} \int_0^1 \left(\tfrac{1}{2}u^2(t) + g(t)y(t)\right) \mathrm{d}t$$

$$\text{s.t.} \qquad \dot{y}(t) = u(t), \qquad y(0) = y(1) = 0, \qquad y(t) \geq h$$

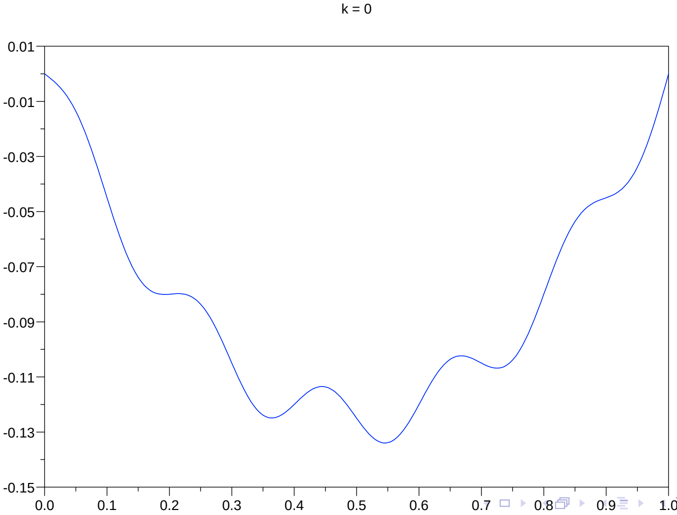
with

$$g(t) := (c - \sin(\alpha t))g_0, \qquad c > 0, \;\; \alpha > 0.$$

Time viewed as second state variable ($\dot{\tau} = 1$)
$\mu = (h - h_0)/(h_1 - h_0)$ homotopy parameter
$h_0 = \min \bar{y}(t)$, where $\bar{y}$ is the solution of unconstrained problem
$h_1 = h$ target value; numerical values are

$$g_0 := 10, \qquad \alpha = 10\pi, \qquad c = 0.1, \qquad h_1 = -0.001.$$

Unconstrained problem: optimal state



Neigborhood of limiting problem: when $\mu > 0$ is small

For $\mu > 0$ the state constraint is active (convex problem)

The contact set could be then for small $\mu > 0$:

1. One point
2. A small interval
3. A non connected set

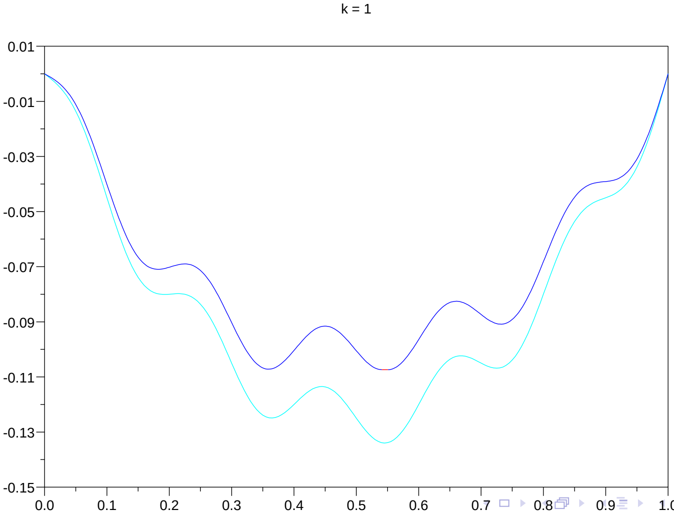
Your guess ?

Neigborhood of limiting problem: when $\mu > 0$ is small

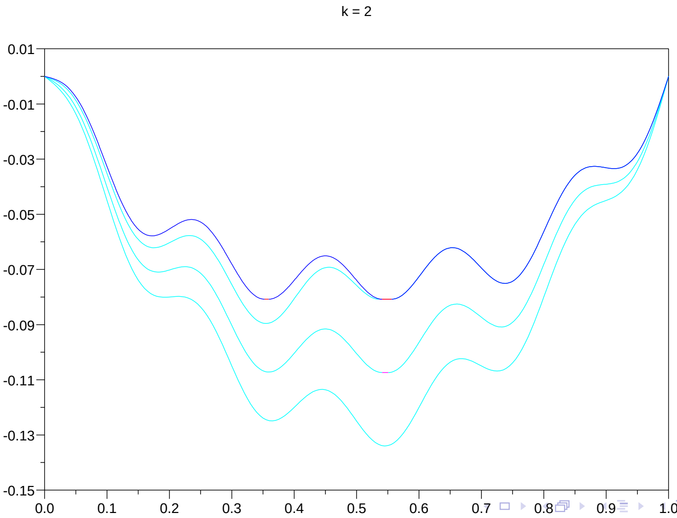
For $\mu > 0$ the state constraint is active (convex problem)

- Structural result: the contact set is an interval
- Quantitative result: first-order expansion of value of extreme points of that interval !

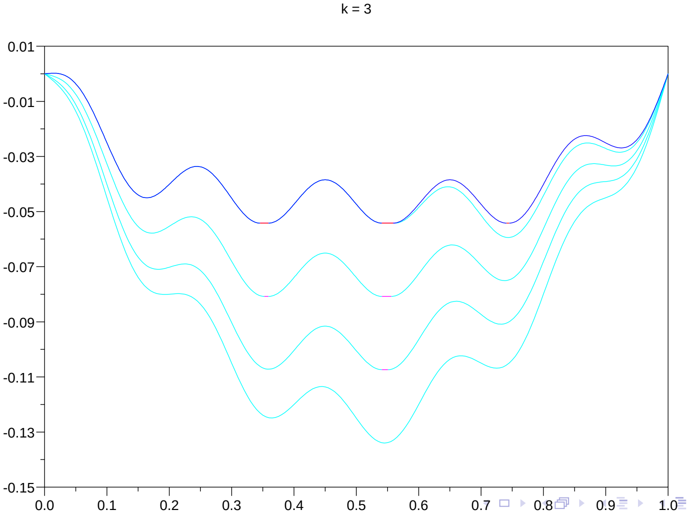
Numerical results II



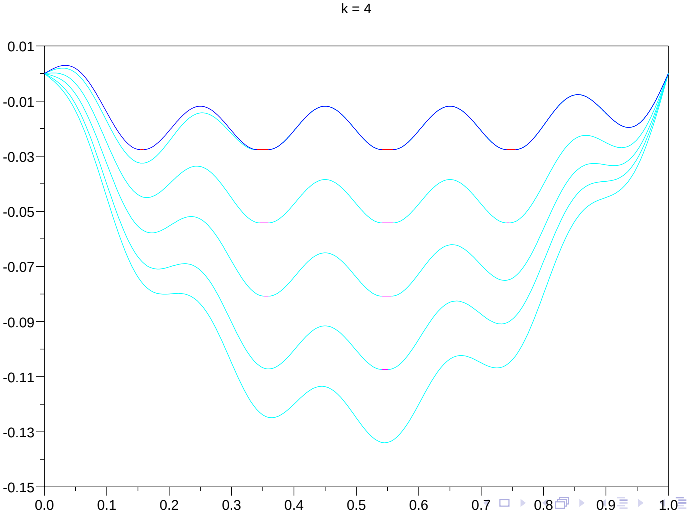
Numerical results III



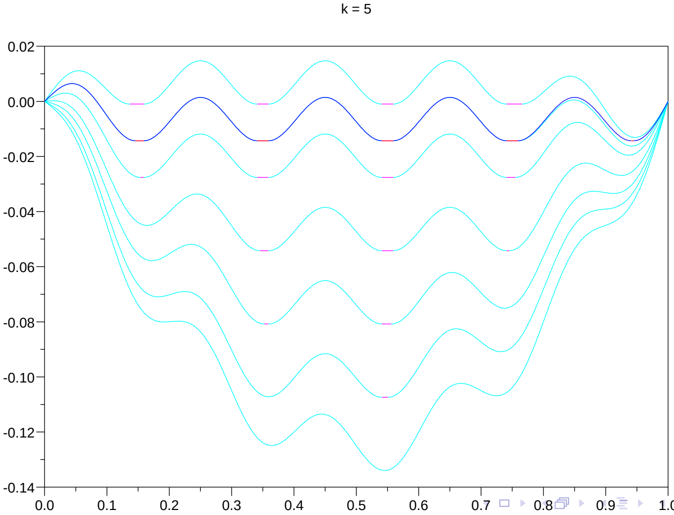
Numerical results IV



Numerical results V



Numerical results VI



State equation

$$
\begin{aligned}
y_t - \Delta y + \gamma y^3 &= i_\omega u \text{ in } Q = \Omega \times [0, T] \\
y &= 0 \text{ over } \Sigma = \partial\Omega \times [0, T], \\
y(\cdot, 0) &= y_0 \text{ over } \Omega,
\end{aligned}
$$

where $\gamma \in \mathbb{R}$, $T > 0$
Ω open bounded subset of $\mathbb{R}^n$, $n \in \{2, 3\}$, with C^2-smooth boundary $\partial\Omega$,
ω open subset of Ω, $Q_\omega = \omega \times [0, T]$, $u \in L^2(Q_\omega)$,
i_ω injection from $L^2(Q_\omega) \to L^2(Q)$
$y_0 \in H^1(\Omega)$.

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Functional spaces

$$H^{2,1}(Q) := \{y \in L^2(0,T,H^2(\Omega));\ y_t \in L^2(Q)\},$$
$$H^{2,1}_\Sigma(Q) := \{y \in H^{2,1}(Q);\ y = 0 \text{ over } \Sigma\}.$$

$$H^{2,1}(Q) \subset C([0,T], H^1_0(\Omega))$$

$$H^1(\Omega) \subset L^6(\Omega), \qquad \text{for } n \leq 3$$

We say that $y \in H^{2,1}(Q)$ is a state associated with $u \in L^2(Q)$ if (y, u) satisfies the state equation.

Well-posedness of the state equation

Refs Bebernes-Kassoy 81, Tartar (Topics in nonlinear analysis, 78)

Lemma

For given $u \in L^2(Q_\omega)$, either the state equation has a unique solution, or there exists a maximal time $\tau \in (0, T]$ such that the state equation with time restricted to $[0, \tau - \varepsilon]$ has, for all $\varepsilon > 0$, a unique solution, and $|y(t)|_6$ is not bounded over $[0, \tau)$.

In any case we denote by y_u the solution.

The implicit function theorem can be applied to the state equation.

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Optimal control problem

Cost function, with $N > 0$:

$$J(u, y) = \frac{1}{2} \int_Q (y(x,t) - y_d(x,t))^2 \mathrm{d}x\mathrm{d}t + \frac{N}{2} \int_{Q_\omega} u^2(x,t)\mathrm{d}x\mathrm{d}t.$$

State constraint

$$g(y(\cdot,t)) := \frac{1}{2} \int_\Omega |y(x,t)|^2 \mathrm{d}x - C \le 0. \qquad (1)$$

$$\operatorname*{Min}_{(u,y)\in L^2(Q_\omega)\times H^{2,1}(Q)} J(u,y) \text{ s.t. the state equation and (1).} \qquad (P)$$

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Existence of a solution

Existence easily obtained when $\gamma \geq 0$

Unclear if $\gamma < 0$.

Quadratic growth

We say that $(\bar{u}, \bar{y})$ is a local solution of (P) that satisfies the *quadratic growth condition* with parameter $\theta \in \mathbb{R}$, if it belongs to $F(P)$ and there exists $\rho > 0$ such that

$$J(\bar{u}, \bar{y}) \geq J(u, y) + \theta |\bar{u} - u|^2_{L^2(Q_\omega)} \text{ if } (u, y) \in F(P) \text{ and } |u - \bar{u}|_{L^2(Q_\omega)} \leq \rho. \tag{2}$$

If this holds for $\theta = 0$, we say that $(\bar{u}, \bar{y})$ is a local solution of (P).

We say that $(\bar{u}, \bar{y}) \in F(P)$ satisfies the quadratic growth condition if (2) holds for some $\theta > 0$ and $\rho > 0$.

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Definitions

Contact set:

$$I(g(y)) = \{t \in [0, T];\ g(y)(t) = 0\}\,.$$

χ_ω: restriction $L^2(\Omega) \to L^2(\omega)$

Unqualified optimality system

Abstract format:

$$\begin{cases} G : L^2(Q_\omega) \times H^{2,1}_\Sigma(Q) \to L^2(Q) \times H^1_0(\Omega), \\ G(u,y) := \begin{pmatrix} y_t - \Delta y + \gamma y^3 - i_\omega u \\ y(\cdot,0) - y_0 \end{pmatrix}. \end{cases}$$

Linearized state equation (well-posed):

$$z_t - \Delta z + 3\gamma y_u^2 = i_\omega v \text{ in } Q; \quad z = 0 \text{ on } \Sigma, \quad z(\cdot,0) = 0.$$

Cost and constraint expressed as function of control:

$$\mathcal{J}(u) := J(u, y_u); \quad \mathcal{G}(u)(t) := g(y_u(t)) = \tfrac{1}{2}|y_u(t)|^2 - C.$$

Abstract problem, where $K = C([0,T])_-$:

$$\operatorname*{Min}_u \mathcal{J}(u); \quad \mathcal{G}(u) \in K, \qquad (AP)$$

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Abstract optimality conditions

Normal cone to the state constraints

$$N_K(h) = \{\mu \in M(0,T)_+; \ \ \mathrm{supp}(\mu) \subset h^{-1}(0)\}.$$

Generalized Lagrangian $\mathcal{L} : L^2(Q_\omega) \times \mathbb{R} \times M([0,T])$:

$$\mathcal{L}(u,\alpha,\mu) := \alpha \mathcal{J}(u) + \langle \mu, \mathcal{G}(u) \rangle$$

Set of generalized Lagrange multipliers

$$\Lambda_g(u) := \{(\alpha,\mu) \in \mathbb{R}_+ \times N_K(\mathcal{G}(u)); \ (\alpha,\mu) \neq 0; \ D_u\mathcal{L}(u,\alpha,\mu) = 0\}.$$

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Optimality conditions II

Theorem

With a local solution (u, y) of (P) is associated a non empty set of generalized Lagrange multipliers.

explicit form of the constraint:

$$g(y) \leq 0, \quad \mu \geq 0, \quad \int_0^T g(y(t))\mathrm{d}\mu(t) = 0.$$

Use of the costate

Costate equation in the sense of transposition; formally

$$
\begin{aligned}
-p_t - \Delta p + 3\gamma y^2 p &= \alpha(y - y_d) + y\mathrm{d}\mu(t) \text{ in } \mathcal{D}'(Q), \\
p(\cdot, T) &= 0, \\
p &= 0 \text{ on } \Sigma.
\end{aligned}
$$

and

$$
\alpha N u + \chi_\omega p = 0 \quad \text{a.e. over } Q_\omega.
$$

Characterization of the qualification condition

Singular set = set of times for which the constraint is active and the control has no influence on its time derivative:

$$I_s(u) = \{t \in I(g(y_u));\ y_u(\cdot, t) = 0 \text{ a.e. on } \omega\}$$

Theorem

Let (u, y) be a feasible point of (P). Then
The set of singular multipliers is empty iff the singular set is empty.
This happens iff the set of Lagrange multipliers is nonempty and bounded.

In the sequel we assume that the problem is qualified, in the sense that the singular set is empty.

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Alternative formulation

Lemma

Let a and b be two functions of bounded variations in $[0,T]$. Suppose that one is continuous, and the other is right-continuous. Then

$$a(T^-)b(T^-) - a(0^+)b(0^+) = \int_0^T a(t)\mathrm{d}b(t) + \int_0^T b(t)\mathrm{d}a(t). \quad (3)$$

An application:

$$\int_0^T \int_\Omega y(x,t)z(x,t)\mathrm{d}x\mathrm{d}\mu(t) = -\int_0^T \int_\Omega [y_t(x,t)z(x,t)) + y(x,t)z_t(x,t)]$$

Alternative costate

$$p^1 := p + g'(y)\mu = p + y\mu \text{ in } L^2(Q).$$

is solution in $H^{2,1}(Q)$ of

$$\begin{aligned} -p^1_t - \Delta p^1 + 3\gamma y^2 p^1 &= y - y_d - (2\Delta y - 6\gamma y^3 + i_\omega u)\mu \quad \text{in } Q, \\ p^1(\cdot, T) &= 0, \\ p^1(\cdot, t) &= 0 \quad \text{on } \Sigma. \end{aligned}$$

Relations with the control:

$$Nu + \chi_\omega(p^1 - \mu y) = 0 \text{ a.e. on } \omega \times [0, T].$$

therefor u has left and right limits in $H^1(\Omega)$.

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Continuity of the control and multiplier

Lemma

Let (u, y) be a regular extremal of (P) and (p, p^1, μ) the classical and alternative costate and the multiplier associated with the state constraint. Then (i) *μ is a continuous function of time and $u \in C([0, T]; H^1(\omega))$,* (ii) *if at time t the state constraint is active and $\int_\omega y^2(x, t)\, dx \neq 0$, then*

$$0 = \frac{d}{dt}(g(y))(t) = -|\nabla y(t)|^2 + \int_\omega u(x, t) y(x, t) \mathrm{d}x - \gamma |y(t)|_4^4,$$

$$\mu(t) = \frac{N|\nabla y(t)|^2 + \gamma N |y(t)|_4^4 + \int_\omega y(x, t) p^1(x, t) \mathrm{d}x}{|\chi_\omega y(t)|_2^2}$$

$$u = \frac{1}{N}\chi_\omega \left(\frac{N|\nabla y(t)|^2 + \gamma N |y(t)|_4^4 + \int_\omega y(x, t) p^1(x, t) \mathrm{d}x}{|\chi_\omega y(t)|_2^2} y - p^1 \right).$$

Proof of continuity

$[\cdot]$ jump function (e.g., $[u](t) := u(t^+) - u(t^-)$),

$$N[u] = [\mu]\chi_\omega y$$

and since g attains a maximum if $[\mu] \neq 0$:

$$N|[u]|^2_{L^2(\omega)} = [\mu]\int_\omega [u] y \mathrm{d}x = [\mu]\left[\frac{d}{dt} g(y)(t)\right] \leq 0$$

Regularity over a boundary arc

Lemma

Let (u, y) be a regular extremal of (P). Assume that the state constraint is active over an interval $[t_1, t_2]$, where $0 \leq t_1 < t_2 \leq T$. Then μ is absolutely continuous over $[t_1, t_2]$.

Lagrangian

$$L(u,y,p,q,\mu) := J(u,y) + \int_Q p\left(\Delta y - \gamma y^3 + i_\omega u - y_t\right) \mathrm{d}x\mathrm{d}t + \int_0^T g(y(t))\mathrm{d}\mu(t) + \int_\Omega q(x)(y(x,0) - y_0(x))\mathrm{d}x.$$

Second-order directional derivative in direction (v,z):

$$\Delta(v,z) := N\|v\|_2^2 + \int_Q \left(1 - 6\gamma p(x,t)y(x,t)\right) z(x,t)^2 \mathrm{d}x\mathrm{d}t + \int_0^T |z(t)|_2^2 d\mu(t).$$

Critical directions

$C(u, y)$ set of critical directions, such that

$$g'(y(t))z(t) \leq 0 \quad \text{over } I(g(y)) \,, \tag{4}$$
$$g'(y(t))z(t) = 0 \quad \text{over } \operatorname{supp}(\mu). \tag{5}$$

The contact set has a *finite structure* if it is a finite union of touch points and boundary arcs.
Strict complementarity holds if the support of $\mathrm{d}\mu$ is the union of the boundary arcs. In that case, a linearized direction (v, z) is critical iff

$$\begin{cases} g'(y(t))z(t) = 0 & \text{over boundary arcs,} \\ g'(y(\tau))z(\tau)) \leq 0 & \text{for each touch point } \tau. \end{cases} \tag{6}$$

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Second-order necessary condition

Theorem

Let (u, y) be a qualified local solution of (P), with associated multiplier μ and costate p. If the contact set has a finite structure and the hypothesis of strict complementarity holds, then

$$\Delta(v, z) \geq 0, \quad \text{for all} \quad (v, z) \in C(u, y). \tag{7}$$

Second-order sufficient conditions

Theorem

Let (u, y) be a regular extremal of (P). Then a sufficient condition for the quadratic growth condition (2) is

$$\Delta(v, z) > 0, \quad \text{for all} \quad (v, z) \in C(u, y) \setminus \{0\}. \tag{8}$$

If, in addition, the contact set has a finite structure, then (8) is a necessary condition for quadratic growth.

Sensitivity analysis

Perturbed state equation:

$$
\begin{aligned}
y_t - \Delta y + \gamma y^3 &= f + i_\omega u \text{ in } Q, && (9)\\
y &= 0 \text{ over } \Sigma, && (10)\\
y(\cdot, 0) &= y_0 \text{ over } \Omega. && (11)
\end{aligned}
$$

Localizing constraint

$$\|u - \bar{u}\|_2 \leq \rho. \tag{12}$$

Perturbed problem

Let $(\bar{u}, \bar{y})$ be a local solution of (P) satisfying the quadratic growth condition (2) for some $\theta > 0$ and $\rho > 0$. Assume that they satisfy the qualification condition, and let $(\bar{p}, \bar{\mu})$ denote the associated costate and Lagrange multiplier.
The perturbed optimal control problem is

$$\operatorname*{Min}_{(u,y)\in L^2(Q_\omega)\times H^{2,1}(Q)} J(u,y) \text{ s.t. (9)-(12); } G(u) \leq 0. \qquad (P_f)$$

Path of perturbations

Denote by $v(f)$ the value of problem (P_f).
Methodology of B., Cominetti, Shapiro:

$$f(\sigma) := \sigma f_1 + \tfrac{1}{2}\sigma^2 f_2 + o(\sigma^2),$$

perturbed linearized equation

$$z_t - \Delta z + 3\gamma \bar{y}^2 z = f_1 + i_\omega v \text{ in } Q; \quad z = 0 \text{ on } \Sigma, \quad z(\cdot, 0) = 0. \quad (13)$$

The related *linearized optimization problem* is

$$\operatorname*{Min}_{(v,z)\in L^2(Q_\omega)\times H^{2,1}(Q)} J'(\bar{u},\bar{y})(v,z); \quad g'(\bar{y}(t))z(t) \le 0 \text{ over } I(g(\bar{y})); \ (13) \qquad (L_{f_1})$$

Quadratic subproblem

$$\operatorname*{Min}_{(v,z)\in S(L_{f_1})} \Delta(v,z) + \int_Q \bar{p}(x,t) f_2(x,t) \mathrm{d}x\mathrm{d}t \qquad (Q)$$

Main result

Theorem

Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P) satisfying the quadratic growth condition (2). Then a) we have the following expansion

$$v(f(\sigma)) = \mathrm{val}(P) + \sigma\,\mathrm{val}(L_{f_1}) + \tfrac{1}{2}\sigma^2\,\mathrm{val}(Q) + o(\sigma). \qquad (14)$$

b) In addition we have that if (u_σ, y_σ) is a path of $o(\sigma^2)$ solutions, then $\|u_\sigma - \bar{u}\|_2 = O(\sigma)$, each weak limit-point in $L^2(Q_\omega)$ is a strong limit-point, and is solution of problem (Q). If the latter has a unique solution $\bar{v}$, then a path u_σ of $o(\sigma^2)$ solutions of $(P_{f(\sigma)})$ satisfies

$$u_\sigma = \bar{u} + \sigma\bar{v} + o(\sigma). \qquad (15)$$

References for alternative formulation

- Bryson Denham, Dreyfus (1963): informal derivation, high-order
- Hager (1979) First-order constraints: Lipschitz stability
- Maurer (1979), unpublished: rigorous derivation, high-order
- Several related works by Maurer and Malanowski
- FB and A. Hermant Second-order Analysis for Optimal Control Problems with Pure State Constraints and Mixed Control-State Constraints. Annals of I.H.P. - Nonlinear Analysis 26 (2009), 561-598.

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